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On the Derivatives of Integral Functions of Several Complex Variables

By

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1. Consider* the double power series

$$f(z_1, z_2)=\sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n \tag{1.1}$$

of complex variables z_1 and z_2, where the coefficients a_{mn} are complex numbers. We say that the power series (1.1) represents an integral function of two variables z_1 and z_2, if it converges for all values of $|z_1|<\infty$ and $|z_2|<\infty$.

Let
$$M(r_1, r_2)=\max_{|z_1|\leqslant r_1, |z_2|\leqslant r_2} |f(z_1, z_2)|$$

be the maximum modulus of the integral function $f(z_1, z_2)$ for $|z_1|\leqslant r_1$, $|z_2|\leqslant r_2$.

From the maximum modulus principle of analytic functions, follows that if the function $f(z_1, z_2)$ is not constant with respect to any one of the variables z_1, z_2 then

(1) $M(r_1', r_2)>M(r_1, r_2)$, for $r_1'>r_1$,

(2) $M(r_1, r_2')>M(r_1, r_2)$, for $r_2'>r_2$,

and consequently

(3) $M(r_1', r_2')>M(r_1, r_2)$, for $r_1'>r_1$, $r_2'>r_2$.

In 1955 M. M. Dzrbasyan has defined that the integral function $f(z_1, z_2)$ has finite order ρ_1 and ρ_2 with respect to variables z_1 and z_2 respectively, if

(1) for any arbitrarily small $\varepsilon>0$ and any $r_2\geqslant 0$, there exists a number $R_1(\varepsilon, r_2)$, such that

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* For simplicity we consider only two variables, though the results can easily be extended to several variables.

$$M(r_1, r_2) < \exp(r_1^{\rho_1+\varepsilon}), \text{ if } r_1 \geqslant R_1(\varepsilon, r_2).$$

In addition there exists at least one value of r_2, say, $r_2^0(\varepsilon)$ and corresponding arbitrarily large values of r_1: $\{r_{1_i}\}$, such that

$$M(r_{1_i}, r_2^0(\varepsilon)) > \exp(r_{1_i}^{\rho_1-\varepsilon});$$

(2) for any arbitrarily small $\varepsilon > 0$ and any $r_1 \geqslant 0$, there exists a number $R_2(\varepsilon, r_1)$, such that

$$M(r_1, r_2) < \exp(r_2^{\rho_2+\varepsilon}), \text{ if } r_2 \geqslant R_2(\varepsilon, r_1).$$

In addition there exists at least one value of r_1, say, $r_1^0(\varepsilon)$ and corresponding arbitrarily large value of r_2: $\{r_{2_k}\}$, such that

$$M(r_1^0(\varepsilon), r_{2_k}) > \exp(r_{2_k}^{\rho_2-\varepsilon}).$$

The assertions (1) and (2) are equivalent to

$$\overline{\lim_{r_2\to\infty}}\left\{\overline{\lim_{r_1\to\infty}}\frac{\log\log M(r_1, r_2)}{\log r_1}\right\} = \rho_1 \tag{1.2}$$

and

$$\overline{\lim_{r_1\to\infty}}\left\{\overline{\lim_{r_2\to\infty}}\frac{\log\log M(r_1, r_2)}{\log r_2}\right\} = \rho_2. \tag{1.3}$$

2. Let

$$M^{(1)}(r_1, r_2) = \max_{|z_1|\leqslant r_1, |z_2|\leqslant r_2}\left|\frac{\partial}{\partial z_1} f(z_1, z_2)\right|,$$

$$M^{(2)}(r_1, r_2) = \max_{|z_1|\leqslant r_1, |z_2|\leqslant r_2}\left|\frac{\partial}{\partial z_2} f(z_1, z_2)\right|.$$

We now prove the following theorem:

THEOREM 1. *If* $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$ *is an integral function of finite order* (ρ_1, ρ_2), ρ_1 *and* ρ_2 *with respect to variables* z_1 *and* z_2 *respectively, then*

$$\overline{\lim_{r_2\to\infty}}\left\{\overline{\lim_{r_1\to\infty}}\frac{\log\left\{r_1\dfrac{M^{(1)}(r_1, r_2)}{M(r_1, r_2)}\right\}}{\log r_1}\right\} \geqslant \rho_1, \tag{2.1}$$

$$\overline{\lim_{r_1\to\infty}}\left\{\overline{\lim_{r_2\to\infty}}\frac{\log\left\{r_2\dfrac{M^{(2)}(r_1, r_2)}{M(r_1, r_2)}\right\}}{\log r_2}\right\} \geqslant \rho_2. \tag{2.2}$$

Further, if $a_{mn} \geqslant 0$, *then*

$$\overline{\lim_{r_2\to\infty}}\left\{\overline{\lim_{r_1\to\infty}}\frac{\log\left\{r_1\dfrac{M^{(1)}(r_1, r_2)}{M(r_1, r_2)}\right\}}{\log r_1}\right\}=\rho_1 \tag{2.3}$$

$$\overline{\lim_{r_1\to\infty}}\left\{\overline{\lim_{r_2\to\infty}}\frac{\log\left\{r_2\dfrac{M^{(2)}(r_1, r_2)}{M(r_1, r_2)}\right\}}{\log r_2}\right\}=\rho_2. \tag{2.4}$$

We shall require the following two lemma's in the proof of the above theorem:

LEMMA 1. *For any fixed value of* $r_2 \geqslant 0$ *there exists a number* $R_1(f, r_2)$, *such that*

$$M^{(1)}(r_1, r_2) \geqslant \frac{M(r_1, r_2)\log M(r_1, r_2)}{r_1 \log r_1} \tag{2.5}$$

for $r_1 \geqslant R_1(f, r_2)$.

LEMMA 2. *For any fixed value of* $r_1 \geqslant 0$ *there exists a number* $R_2(f, r_1)$, *such that*

$$M^{(2)}(r_1, r_2) \geqslant \frac{M(r_1, r_2)\log M(r_1, r_2)}{r_2 \log r_2} \tag{2.6}$$

for $r_2 \geqslant R_2(f, r_1)$.

PROOF OF LEMMA 1. It can easily be shown that for a fixed value of $r_2 \geqslant 0$

$$g(r_1, r_2)=\frac{\log M(r_1, r_2)}{\log r_1}$$

is monotonic increasing, say, for $r_1 \geqslant R_1(f, r_2)$. Let ξ_1 be such that $|\xi_1|=r_1$ and $|f(\xi_1, z_2)|=M(r_1, r_2)$ and let $f_{z_1}(z_1, z_2)=\dfrac{\partial}{\partial z_1}f(z_1, z_2)$. We then have,

$$\begin{aligned} M^{(1)}(r_1, r_2) &\geqslant |f_{\xi_1}(\xi_1, z_2)| \\ &= \left|\lim_{h\to 0}\frac{f(\xi_1, z_2)-f(\xi_1-\xi_1 h, z_2)}{\xi_1 h}\right| \\ &\geqslant \lim_{h\to 0}\frac{M(r_1, r_2)-M(r_1-r_1 h, r_2)}{r_1 h} \end{aligned}$$

$$= \lim_{h\to 0} \frac{r_1^{g(r_1,r_2)} - (r_1 - r_1 h)^{g(r_1 - r_1 h, r_2)}}{r_1 h}$$

$$\geqslant \lim_{h\to 0} \frac{r_1^{g(r_1,r_2)} - (r_1 - r_1 h)^{g(r_1, r_2)}}{r_1 h}$$

$$= \frac{M(r_1, r_2)}{r_1} \frac{\log M(r_1, r_2)}{\log r_1}.$$

The proof of Lemma 2 is similar to that of Lemma 1.

PROOF OF THEOREM 1. From (2.5), we have

$$\overline{\lim_{r_2\to\infty}} \left\{ \overline{\lim_{r_1\to\infty}} \frac{\log\left\{ r_1 \dfrac{M^{(1)}(r_1, r_2)}{M(r_1, r_2)} \right\}}{\log r_1} \right\} \geqslant \overline{\lim_{r_2\to\infty}} \left\{ \overline{\lim_{r_1\to\infty}} \frac{\log\log M(r_1, r_2)}{\log r_1} \right\} = \rho_1. \quad (2.7)$$

We now suppose $a_{mn} \geqslant 0$, then, for any fixed value of $r_2 \geqslant 0$, we have

$$M(r_1, r_2) = f(r_1, r_2);\ M^{(1)}(r_1, r_2) = \frac{\partial}{\partial r_1} M(r_1, r_2);$$

i.e. in this case $M^{(1)}(r_1, r_2)$ coincides with the partial derivative of $M(r_1, r_2)$ w.r.t. r_1 and so is for the higher derivatives.

Further, for any $r_2 \geqslant 0$, $\log M(r_1, r_2)$ is an increasing convex function of $\log r_1$. This enables us to write $\log M(r_1, r_2)$ in the following form:

$$\log M(2r_1, r_2) = \log M(r_1, r_2) + \int_{r_1}^{2r_1} \frac{\dfrac{\partial}{\partial t_1} M(t_1, r_2)}{M(t_1, r_2)} dt_1$$

$$\geqslant r_1 \frac{M^{(1)}(r_1, r_2)}{M(r_1, r_2)} \log 2,$$

and therefore,

$$\overline{\lim_{r_2\to\infty}} \left\{ \overline{\lim_{r_1\to\infty}} \frac{\log\left\{ r_1 \dfrac{M^{(1)}(r_1, r_2)}{M(r_1, r_2)} \right\}}{\log r_1} \right\} \leqslant \rho_1. \quad (2.8)$$

From (2.7) and (2.8) follows

$$\overline{\lim_{r_2\to\infty}} \left\{ \overline{\lim_{r_1\to\infty}} \frac{\log\left\{ r_1 \dfrac{M^{(1)}(r_1, r_2)}{M(r_1, r_2)} \right\}}{\log r_1} \right\} = \rho_1.$$

Similarly, on using Lemma 2, we can prove that (2.2) and (2.4) hold.

3. We shall consider from the family of integral functions of finite order a special subclass of integral functions, i.e. class 'α', which we define as follows:

DEFINITION: We shall say that integral function $f(z_1, z_2)$ of finite order belongs to class 'α', if it always follows

(1) for any fixed value of $r_2 \geqslant 0$, there exists a number $R_1(K_1, \mu_1, r_2)$, such that $(K_1>0, \mu_1>0)$

$$r_1 \frac{M^{(1)}(r_1, r_2)}{M(r_1, r_2)} < K_1 r_1^{\mu_1}, \text{ for } r_1 \geqslant R_1;$$

(2) for any fixed value of $r_1 \geqslant 0$ there exists a number $R_2(K_2, \mu_2, r_1)$, such that $(K_2>0, \mu_2>0)$

$$r_2 \frac{M^{(2)}(r_1, r_2)}{M(r_1, r_2)} < K_2 r_2^{\mu_2}, \text{ for } r_2 \geqslant R_2;$$

and so there exists a number $R(K_1, K_2, \mu_1, \mu_2)$, such that

$$r_1 \frac{M^{(1)}(r_1, r_2)}{M(r_1, r_2)} + r_2 \frac{M^{(2)}(r_1, r_2)}{M(r_1, r_2)} < K r_1^{\mu_1} r_2^{\mu_2}, \text{ for } r_1, r_2 \geqslant R.$$

We prove the following property for the above class of functions:

THEOREM 2. If $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$ *is an integral function of order* $(\rho_1, \rho_2)(0<\rho_1<\infty, 0<\rho_2<\infty)$, *then if* $a_{mn} \geqslant 0$

$$\overline{\lim_{r_1, r_2 \to \infty}} \frac{\log\left\{r_1 \frac{M^{(1)}(r_1, r_2)}{M(r_1, r_2)} + r_2 \frac{M^{(2)}(r_1, r_2)}{M(r_1, r_2)}\right\}}{\rho_1 \log r_1 + \rho_2 \log r_2} = 1. \tag{3.1}$$

PROOF: From (2.3) and (2.4), we have respectively

(1) for any arbitrary $\varepsilon>0$ and any $r_2 \geqslant 0$, there exists a number $R_1(\varepsilon, r_2)$, such that

$$r_1 \frac{M^{(1)}(r_1, r_2)}{M(r_1, r_2)} < r_1^{\rho_1+\varepsilon}, \text{ for } r_1 \geqslant R_1(\varepsilon, r_2);$$

(2) for any arbitrary $\varepsilon>0$ and any $r_1 \geqslant 0$, there exists a number $R_2(\varepsilon, r_1)$, such that

$$r_2 \frac{M^{(2)}(r_1, r_2)}{M(r_1, r_2)} < r_2^{\rho_2+\varepsilon}, \text{ for } r_2 \geqslant R_2(\varepsilon, r_1);$$

and so there exists a number $R(\varepsilon)$, such that

$$r_1\frac{M^{(1)}(r_1, r_2)}{M(r_1, r_2)}+r_2\frac{M^{(2)}(r_1, r_2)}{M(r_1, r_2)}<r_1^{\rho_1+\varepsilon}\, r_2^{\rho_2+\varepsilon},\ \text{for } r_1, r_2 \geqslant R(\varepsilon).$$

From this follows that

$$A=\overline{\lim_{r_1, r_2\to\infty}}\frac{\log\left\{r_1\dfrac{M^{(1)}(r_1, r_2)}{M(r_1, r_2)}+r_2\dfrac{M^{(2)}(r_1, r_2)}{M(r_1, r_2)}\right\}}{\rho_1\log r_1+\rho_2\log r_2}\leqslant 1. \qquad (3.2).$$

Now, let $A<1$ and $A<A'<A''<1$. Then

$$r_1\frac{M^{(1)}(r_1, r_2)}{M(r_1, r_2)}+r_2\frac{M^{(2)}(r_1, r_2)}{M(r_1, r_2)}<r_1^{\rho_1 A'}r_2^{\rho_2 A'},\ \text{for } r_1, r_2\geqslant R. \qquad (3.3)$$

From (3.3), we obtain that for any $r_2\geqslant 0$, there exists a number $R_1(r_2)$, such that

$$r_1\frac{M^{(1)}(r_1, r_2)}{M(r_1, r_2)}<r_1^{\rho_1 A''},\ \text{for } r_1\geqslant R_1(r_2).$$

This contradicts the hypothesis that the integral function $f(z_1, z_2)$ has order ρ_1 with respect to the variable z_1, because for sufficiently small $\varepsilon>0$

$$\rho_1 A''<\rho_1-\varepsilon.$$

Hence $A=1$ and the theorem is proved.

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