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Integral Involving Kampé-de-Feriét's Function

By

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1. The object of this paper is to evaluate an integral involving Kampé-de-Feriét's function (also called the hypergeometric function of two variables of superior order) defined by him [1] as

$$(1.1)\qquad F\left(\begin{array}{c|c|c} \mu & \alpha_1, \cdots\cdots, \alpha_\mu & \\ \nu & \beta_1, \beta_1'; \cdots; \beta_\nu, \beta_\nu' & \\ & & x, y \\ \rho & \gamma_1, \cdots\cdots, \gamma_\rho & \\ \sigma & \delta_1, \delta_1'; \cdots; \delta_\sigma, \delta_\sigma' & \end{array}\right)$$

$$= \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{\mu}(\alpha_j, m+n)\prod_{j=1}^{\nu}(\beta_j, m)(\beta_j', n)x^m y^n}{\prod_{j=1}^{\rho}(\gamma_j, m+n)\prod_{j=1}^{\sigma}(\delta_j, m)(\delta_j', n)(1, m)(1, n)},$$

absolutely convergent if $\mu+\nu\leqslant\rho+\sigma+1$ and to deduce a number of interesting integrals, say, the product of two Jacobi's polynomials, four Bessel functions and four Laguerre polynomials. Many other integrals can also be evaluated by specializing the parameters. This is due to the fact that the function (1.1) reduces to the product of two generalized functions by putting $\mu=\rho=0$. It also reduces to Appell's function by giving special values to the parameters.

The results to be established are of general nature and reduces to many other interesting integrals involving special functions.

For the sake of brevity and to economize space, I will denote the left hand side of (1.1) as

$${}^{\mu,\nu}f_{\rho,\sigma}\left(\begin{matrix}\alpha_1, \cdots, \alpha_\mu: \beta_1, \beta_1'; \cdots; \beta_\nu, \beta_\nu' \\ \gamma_1, \cdots, \gamma_\rho: \delta_1, \delta_1'; \cdots; \delta_\sigma, \delta_\sigma'\end{matrix}\quad x, y\right)$$

2. The result to be proved is

$$\text{(2.1)}\qquad \int_0^t u^{\alpha-1}(t-u)^{\beta-1}\, {}^{\mu,\nu}f_{\rho,\sigma}\begin{pmatrix} A_\mu : a_\nu \\ & xu^K(t-u)^S,\ yu^K(t-u)^S \\ B_\rho : b_\sigma \end{pmatrix} du$$

$$= B(\alpha, \beta)\, t^{\alpha+\beta-1}\, {}^{\mu+K+S,\nu}f_{\rho+K+S,\sigma}\begin{pmatrix} A_\mu,\ c_j,\ c'_r : a_\nu \\ & xT,\ yT \\ B_\rho,\ d_l : b_\sigma \end{pmatrix},$$

where $A_\mu = \alpha_1, \cdots, \alpha_\mu$ $\qquad c_j = \dfrac{\alpha+j-1}{K}\ (j=1, 2, \cdots, K)$

$a_\nu = \beta_1, \beta'_1; \cdots, \beta_\nu, \beta'_\nu$ $\qquad c'_r = \dfrac{\beta+r-1}{S}\ (r=1, 2, \cdots, S)$

$B_\rho = \gamma_1, \cdots, \gamma_\rho$ $\qquad d_l = \dfrac{\alpha+\beta+l-1}{K+S}\ (l=1, 2, \cdots, K+S)$

$b_\sigma = \delta_1, \delta'_1; \cdots; \delta_\sigma, \delta'_\sigma$ $\qquad T = \dfrac{t^{K+S} K^K S^S}{(K+S)^{K+S}}$.

under the conditions that

i) $Rl(\alpha) > 0,\quad Rl(\beta) > 0.$

ii) K and S are non-negative integers and not zero simultaneously and none of the denominator parameters is a negative integer or zero.

Proof:— Consider the integral

$$\int_0^t u^{\alpha-1}(t-u)^{\beta-1}\, {}^{\mu,\nu}f_{\rho,\sigma}\begin{pmatrix} A_\mu : a_\nu \\ & xu^K(t-u)^S,\ yu^K(t-u)^S \\ B_\rho : b_\sigma \end{pmatrix} du$$

$$= \int_0^t u^{\alpha-1}(t-u)^{\beta-1} \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{\mu}(\alpha_j, m+n) \prod_{j=1}^{\nu}(\beta_j, m)(\beta'_j, n)\, x^m y^n u^{K(m+n)}(t-u)^{S(m+n)}}{\prod_{j=1}^{\rho}(\gamma_j, m+n) \prod_{j=1}^{\sigma}(\delta_j, m)(\delta'_j, n)(1, m)(1, n)}\, du.$$

Changing the order of integration and summation (which we assume to be permissible) we get

$$\sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{\mu}(\alpha_j, m+n) \prod_{j=1}^{\nu}(\beta_j, m)(\beta'_j, n)\, x^m y^n}{\prod_{j=1}^{\rho}(\gamma_j, m+n) \prod_{j=1}^{\sigma}(\delta_j, m)(\delta'_j, n)(1, m)(1, n)} \int_0^t u^{\alpha+(m+n)K-1}(t-u)^{\beta+(m+n)S-1}\, du.$$

Now evaluating the U-integral by the formula

$$\int_0^t x^{\alpha-1}(t-x)^{\beta-1}\,dx=t^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

we get

$$\sum_{m,n=0}^{\infty}\frac{\prod_{j=1}^{\mu}(\alpha_j,\,m+n)\prod_{j=1}^{\nu}(\beta_j,\,m)(\beta_j',\,n)\,x^m y^n}{\prod_{j=1}^{\rho}(\gamma_j,\,m+n)\prod_{j=1}^{\sigma}(\delta_j,\,m)(\delta_j',\,n)(1,\,m)(1,\,n)}$$

$$\times\, t^{\alpha+\beta+(K+S)(m+n)-1}\frac{\Gamma[\alpha+(m+n)K]\Gamma[\beta+(m+n)S]}{\Gamma[\alpha+\beta+(m+n)(K+S)]}.$$

Now using the Lemma 6[2, 22], viz.

$$(\alpha)_{Kn}=K^{nK}\Big(\frac{\alpha}{K}\Big)_n\Big(\frac{\alpha+1}{K}\Big)_n\cdots\Big(\frac{\alpha+K-1}{K}\Big)_n,$$

(2.1) is obtained.

3. Partictlar Cases:— We now evaluate a number of interesting integrals by proper choosing the parameters.

Let $\mu=\rho=0$, then (2.1) reduces to

(3.1) $$\int_0^t u^{\alpha-1}(t-u)^{\beta-1}\;{}^{\nu}F_{\sigma}\begin{pmatrix}\beta_1,\,\cdots,\,\beta_\nu & \\ & xu^K(t-u)^S\\ \delta_1,\,\cdots,\,\delta_\sigma & \end{pmatrix}{}^{\nu}F_{\sigma}\begin{pmatrix}\beta_1',\,\cdots,\,\beta_\nu' & \\ & yu^K(t-u)^S\\ \delta_1',\,\cdots,\,\delta_\sigma' & \end{pmatrix}du$$

$$=B(\alpha,\,\beta)t^{\alpha+\beta-1}\;{}^{K+S,\nu}f_{K+S,\sigma}\begin{pmatrix}c_j,\,c_r':\,a_\nu & \\ & xT,\;yT\\ d_l:\,b_\sigma & \end{pmatrix},$$

using proper conditions for convergence.

Now putting $t=1$, $\nu=2$, $\sigma=1$, $K=0$, $S=1$, $x=1$, $y=1$; $\beta_1=-m$, $\beta_2=m+\lambda_1+\lambda_2+1$, $\delta_1=1+\lambda_1$, $\beta_1'=-n$, $\beta_2'=n+\lambda_1'+\lambda_2'+1$, $\delta_1'=1+\lambda_1'$ and multiplying both sides by $\dfrac{\Gamma(1+\lambda_1+m)\Gamma(1+\lambda_1'+n)}{\Gamma(1+m)\Gamma(1+n)\Gamma(1+\lambda_1)\Gamma(1+\lambda_1')}$ in (3.1) we get an interesting integral

(3.11) $$\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}p_m^{\lambda_1,\lambda_2}(2u-1)p_n^{\lambda_1',\lambda_2'}(2u-1)du$$

$$=\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(1+\lambda_1+m)\Gamma(1+\lambda_1'+n)}{\Gamma(1+m)\Gamma(1+n)\Gamma(\alpha+\beta)\Gamma(1+\lambda_1)\Gamma(1+\lambda_1')}$$

$$\times\, {}^{1,2}f_{1,1}\begin{pmatrix}\beta:\ -m,\ -n;\ \lambda_1+\lambda_2+m+1,\ \lambda_1'+\lambda_2'+n+1 & \\ & 1,1 \\ \alpha+\beta:\ 1+\lambda_1,\ 1+\lambda_1' & \end{pmatrix}.$$

Using the following result [3] to sum the right hand side, viz.

$$\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-m,r)(-n,S)(\alpha,r+S)(\beta,r)(\beta',S)}{(1,r)(1,S)(\delta,r)(\delta',S)(\gamma,r+S)}$$

$$=\frac{(\beta+\beta'-\alpha,\ m+n)(\beta,m)(\beta',n)}{(\beta+\beta',\ m+n)(\beta'-\alpha,n)(\beta-\alpha,n)},$$

under the conditions $\gamma=\beta+\beta'$, $\gamma+\delta=\alpha+\beta-m+1$, $\gamma+\delta'=\alpha+\beta'-n+1$, we get, after putting $u=\cos^2\theta$,

(3.12)
$$\int_0^{\frac{\pi}{2}}\cos^{2\alpha-1}\theta\sin^{2\beta-1}\theta p_m^{\lambda_1,\alpha-1}(\cos 2\theta)p_n^{\lambda_1',\alpha-1}(\cos 2\theta)\,d\theta$$

$$=\frac{\Gamma(\alpha+m+n)\Gamma(\alpha+\lambda_1+m+n)\Gamma(\alpha+\lambda_1'+m+n)\Gamma(1+\lambda_1+m)\Gamma(1+\lambda_1'+n)}{2\Gamma(\alpha+\beta+m+n)\Gamma(1+\lambda_1-\beta+m+n)\Gamma(1+\lambda_1'-\beta+m+n)\Gamma(\alpha+\lambda_1+m)}$$

$$\times\frac{\Gamma(1+\lambda_1-\beta+m)\Gamma(1+\lambda_1'-\beta+n)\Gamma(\beta)}{\Gamma(\alpha+\lambda_1'+n)\Gamma(1+\lambda_1)\Gamma(1+\lambda_1')\Gamma(1+m)\Gamma(1+n)},$$

under the conditions $\beta-\alpha=\lambda_1+\lambda_1'+m+n$.

If we put $x=y=\frac{1}{2}$ in (3.1) and proceeding as (3.11) we get

(3.13)
$$\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}p_m^{\lambda_1,\lambda_2}(u)p_n^{\lambda_1',\lambda_2'}(u)\,du$$

$$=\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(1+\lambda_1+m)\Gamma(1+\lambda_1'+n)}{\Gamma(1+m)\Gamma(1+n)\Gamma(\alpha+\beta)\Gamma(1+\lambda_1)\Gamma(1+\lambda_1')}$$

$$\times\, {}^{1,2}f_{1,1}\begin{pmatrix}\beta:\ -m,\ -n;\ \lambda_1+\lambda_2+m+1,\ \lambda_1'+\lambda_2'+n+1 & \\ & \frac{1}{2},\ \frac{1}{2} \\ \alpha+\beta:\ 1+\lambda_1,\ 1+\lambda_1' & \end{pmatrix}.$$

Again, if we put $\nu=1$, $\sigma=1$, $\beta_1=-m$, $\delta_1=1+\lambda$, $\beta_1'=-m$, $\delta_1'=1+\lambda'$ in (3.1) and multiply both sides by $\frac{\Gamma(1+\lambda+m)\Gamma(1+\lambda'+n)}{\Gamma(1+m)\Gamma(1+n)\Gamma(1+\lambda)\Gamma(1+\lambda')}$, we get

(3.14)
$$\int_0^t u^{\alpha-1}(t-u)^{\beta-1}L_m^{\lambda}[xu^K(t-u)^S]L_n^{\lambda'}[yu^K(t-u)^S]\,du$$

$$=\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(1+\lambda+m)\Gamma(1+\lambda'+n)}{\Gamma(1+m)\Gamma(1+n)\Gamma(\alpha+\beta)\Gamma(1+\lambda)\Gamma(1+\lambda')}\, t^{\alpha+\beta-1}$$

$$\times\, {}^{K+S,1}f_{K+S,1}\begin{pmatrix} c_j,\ c'_r:\ -m,\ -n & \\ & xT,\ yT \\ d_l:\ 1+\lambda,\ 1+\lambda' & \end{pmatrix}.$$

If we put $\nu=0$, $\sigma=1$, $\delta_1=1+\nu_1$, $\delta'_1=1+\nu'_1$, and use the relation

$$I_\nu(z)=\frac{\left(\frac{1}{2}z\right)^\nu}{\Gamma(1+\nu)}\,{}_0F_1\left(\nu+1;\ \frac{1}{4}z^2\right),$$

we get

$$\text{(3.15)}\quad \int_0^t u^{\alpha'-1}(t-u)^{\beta'-1}I_{\nu_1}(2\sqrt{xu^K(t-u)^S})\,I_{\nu'_1}(2\sqrt{yu^K(t-u)^S})\,du$$

$$=\frac{x^{\frac{\nu_1}{2}}y^{\frac{\nu'_1}{2}}\Gamma(\alpha)\Gamma(\beta)}{\Gamma(1+\nu_1)\Gamma(1+\nu'_1)\Gamma(\alpha+\beta)}\,t^{\alpha+\beta-1}\ {}^{K+S,0}f_{K+S,1}\begin{pmatrix} c_j,\ c'_r: & \\ & xT,\ yT \\ d_l:\ 1+\nu_1,\ 1+\nu'_1 & \end{pmatrix},$$

where $\alpha'=\alpha-\frac{K}{2}(\nu_1+\nu'_1)$ and $\beta'=\beta-\frac{K}{2}(\nu_1+\nu'_1)$; $Rl(\alpha')>0$, $Rl(\beta')>0$.

Again in (3.1), using Ramanujan's theorem [2,106] viz,

$${}_1F_1\begin{pmatrix}\alpha; & \\ & x\\ \beta; & \end{pmatrix}{}_1F_1\begin{pmatrix}\alpha; & \\ & -x\\ \beta; & \end{pmatrix}={}_2F_3\begin{pmatrix}\alpha,\ \beta-\alpha & \\ & \frac{x^2}{4}\\ \beta,\ \frac{1}{2}\beta,\ \frac{1}{2}\beta+\frac{1}{2} & \end{pmatrix},$$

we get

$$\text{(3.16)}\quad \int_0^t u^{\alpha-1}(t-u)^{\beta-1}{}_1F_1\begin{pmatrix}\beta_1; & \\ & 2\sqrt{x\nu}\\ \delta_1; & \end{pmatrix}{}_1F_1\begin{pmatrix}\beta_1; & \\ & -2\sqrt{x\nu}\\ \delta_1; & \end{pmatrix}{}_1F_1\begin{pmatrix}\beta'_1; & \\ & 2\sqrt{y\nu}\\ \delta'_1; & \end{pmatrix}{}_1F_1\begin{pmatrix}\beta'_1; & \\ & -2\sqrt{y\nu}\\ \delta'_1; & \end{pmatrix}du$$

$$=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,t^{\alpha+\beta-1}\ {}^{K+S,2}f_{K+S,3}\begin{pmatrix} c_j,\ c'_r:\ \beta_1,\ \beta'_1;\ \delta_1-\beta_1,\ \delta'_1-\beta'_1 & \\ & xT,\ yT\\ d_l:\ \delta_1,\ \delta'_1;\ \frac{\delta_1}{2},\ \frac{\delta'_1}{2};\ \frac{1}{2}(\delta_1+1),\ \frac{1}{2}(\delta'_1+1) & \end{pmatrix},$$

where $\nu=u^K(t-u)^S$ and $Rl(\alpha)>0$, $Rl(\beta)>0$.

This (3.16) integral takes the interesting form by reducing ${}_1F_1$ to Whittakar function $M_{K,m}(x)$, generalized Laguerre polynomial $L_n^{(\alpha)}(x)$ and Weber's parabolic cylinder function $D_n(x)$.

Also, using the result [2,105] in (3.1)

$$ {}_0F_1\left[\begin{matrix} -; \\ a; \end{matrix} x\right] {}_0F_1\left[\begin{matrix} -; \\ b; \end{matrix} x\right] = {}_2F_3\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}b, \frac{1}{2}a+\frac{1}{2}b-\frac{1}{2} \\ a,\ b,\ a+b-1 \end{matrix} 4x\right], $$

we get

$$ (3.17) \qquad \int_0^t u^{\alpha-1}(t-u)^{\beta-1}{}_0F_1(\delta_1;\ x\nu){}_0F_1(\delta_2;\ x\nu){}_0F_1(\delta_1';\ y\nu){}_0F_1(\delta_2';\ y\nu)\,du $$

$$ =\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\, t^{\alpha+\beta-1}\ {}^{K+S,2}f_{K+S,3}\left(\begin{matrix} c_j,\ c_r':\ \delta,\ \delta';\ \delta-\frac{1}{2},\ \delta'-\frac{1}{2} \\ d_l:\ \delta_1,\ \delta_1';\ \delta_2,\ \delta_2';\ 2\delta-1,\ 2\delta'-1 \end{matrix}\ xT,\ yT\right), $$

where $\nu=\frac{u^K(t-u)^S}{4}$, $\delta=\frac{1}{2}(\delta_1+\delta_2)$ and $\delta'=\frac{1}{2}(\delta_1'+\delta_2')$.

Here ${}_0F_1$ can be reduced to Bessel function $I_\nu(z)$.

Since the function (1.1) reduces to Appell's function $F^{(1)}(\mu=\nu=\rho=1,\ \sigma=0)$, $F^{(2)}(\mu=\nu=\sigma=1,\ \rho=0)$, $F^{(3)}(\mu=\sigma=0,\ \nu=2,\ \sigma=1)$ and $F^{(4)}(\mu=2,\ \nu=\sigma=0,\ \sigma=1)$, we can write down the integrals involving these functions.

For instance, taking $\mu=\rho=\nu=1$, $\sigma=0$ in (2.1), we get

$$ (3.18) \qquad \int_0^t u^{\alpha-1}(t-u)^{\beta-1}\, F^{(1)}[\alpha_1;\ \beta_1,\ \beta_1';\ \gamma_1;\ xu^K(t-u)^S,\ yu^K(t-u)^S]\,du $$

$$ =\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\, t^{\alpha+\beta-1}\ {}^{K+S+1,1}f_{K+S+1,0}\left(\begin{matrix} \alpha_1,\ c_j,\ c_r':\ \beta_1,\ \beta_1' \\ \gamma_1,\ d_l: \end{matrix}\ xT,\ yT\right). $$

Further the integral (2.1) can be reduced to Rainville's integral [2,104] by putting $y=0$.

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