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Integrals Involving Bessel Functions

By

B. L. Sharma

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Abstract

In this paper, I have evaluated some double integrals involving product of five Bessel functions of different arguments in terms of Appell's function F_4 with the help of operational calculus. The results obtained are believed to be new.

1. The following formulas are to be established here

$$\int_0^\infty\int_0^\infty yx^{\mu_1+1}J_{\mu_1}(ax)K_{\mu_2}(b\sqrt{x^2+y^2})\,I_{\mu_2}(c\sqrt{x^2+y^2})\,J_{\mu_3}(gy)J_{\mu_3}(hy)\,dx\,dy$$

$$=\frac{(a)^{\mu_1}(bc)^{\mu_2}(gh)^{\mu_3}2^{\mu_1+1}\Gamma(\mu_1+\mu_2+\mu_3+2)}{(a^2+b^2+c^2+g^2+h^2)^{\mu_1+\mu_2+\mu_3+2}}\times$$

$$F_4\left[\frac{1}{2}(\mu_1+\mu_2+\mu_3+2),\ \frac{1}{2}(\mu_1+\mu_2+\mu_3+3);\ 1+\mu_2,\ 1+\mu_3;\right.$$

$$\left.\frac{4b^2c^2}{(a^2+b^2+c^2+g^2+h^2)^2},\ \frac{4g^2h^2}{(a^2+b^2+c^2+g^2+h^2)^2}\right],$$

valid for $R(\mu_1+\mu_2+\mu_3+2)>0$, $R(b)>|R(c)|+|I_m(g)|+|I_m(h)|$,

$R(b)>|R(c)|+|I_m(g)|$. ......(1)

$$\int_0^\infty\int_0^\infty yx^{\mu_1+1}J_{\mu_1}(ax)K_{\mu_2}(b\sqrt{x^2+y^2})\,I_{\mu_2}(c\sqrt{x^2+y^2})I_{\mu_3}(gy)I_{\mu_3}(hy)\,dx\,dy$$

$$=\frac{(a)^{\mu_1}(bc)^{\mu_2}(gh)^{\mu_3}2^{\mu_1+1}\Gamma(\mu_1+\mu_2+\mu_3+2)}{(a^2+b^2+c^2-g^2-h^2)^{\mu_1+\mu_2+\mu_3+2}}\times$$

$$F_4\left[\frac{1}{2}(\mu_1+\mu_2+\mu_3+2),\ \frac{1}{2}(\mu_1+\mu_2+\mu_3+3);\ 1+\mu_2,\ 1+\mu_3;\right.$$

$$\left.\frac{4b^2c^2}{(a^2+b^2+c^2-g^2-h^2)^2},\ \frac{4g^2h^2}{(a^2+b^2+c^2-g^2-h^2)^2}\right],$$

valid for

$$R(\mu_1+\mu_2+\mu_3+2)>0, \quad R(b)>|R(c)|+|R(g)|+|R(h)|,$$
$$R(b)>R(c)+|I_m(a)|. \qquad \cdots\cdots(2)$$

$$\int_0^\infty\int_0^\infty x^{\mu_1+1}y^{2\mu_4-1}J_{\mu_1}(ax)K_{\mu_2}(b\sqrt{x^2+y^2})\,I_{\mu_2}(c\sqrt{x^2+y^2})\times$$
$$S_2\left[\frac{1}{2}(\mu_3-1),\ -\frac{1}{2}(\mu_3+1),\ \frac{1}{2}(1-\mu_4),\ -\frac{1}{2}\mu_4;\ \frac{gy^2}{4}\right]dx\,dy$$
$$=\sum_{\mu_3,-\mu_3}\frac{(g)^{\mu_3}(bc)^{\mu_2}(a)^{\mu_1}\Gamma(-\mu_3)\Gamma(\mu_1+\mu_2+\mu_3+\mu_4)2^{\mu_1+\mu_3+3\mu_4-4}}{\sqrt{\pi}\,\Gamma(1+\mu_2)(a^2+b^2+c^2)^{\mu_1+\mu_2+\mu_3+\mu_4}}\times$$
$$F_4\left[\frac{1}{2}(\mu_1+\mu_2+\mu_3+\mu_4),\ \frac{1}{2}(\mu_1+\mu_2+\mu_3+\mu_4+1);\ 1+\mu_2,\ 1+\mu_3;\right.$$
$$\left.\frac{4b^2c^2}{(a^2+b^2+c^2)^2},\ \frac{16g^2}{(a^2+b^2+c^2)^2}\right],$$

valid for

$$R(\mu_1+\mu_2+\mu_3+\mu_4)>0,\ R(b)>|R(c)|,\ R(b)>|R(c)|+|I_m(a)|,\ g>0.\cdots(3)$$

$$\int_0^\infty\int_0^\infty x^{\mu_1+1}y^{2\mu_3+2\mu_4+3}J_{\mu_1}(ax)K_{\mu_2}(b\sqrt{x^2+y^2})\,I_{\mu_2}(c\sqrt{x^2+y^2})$$
$$\times\,{}_1F_2\left(\frac{1}{2}+\mu_3;\ \ 2\mu_3+1,\ \ \mu_3+\mu_4;\ \ -g^2y^2\right)dx\,dy$$
$$=\frac{(a)^{\mu_1}(bc)^{\mu_2}\Gamma(\mu_3+\mu_4)\Gamma(\mu_1+\mu_2+\mu_3+\mu_4+1)2^{2\mu_2+2\mu_3+2\mu_4-5}}{\Gamma(1+\mu_2)(a^2+b^2+c^2+g^2)^{\mu_1+\mu_2+\mu_3+\mu_4+1}}\times$$
$$F_4\left[\frac{1}{2}(\mu_1+\mu_2+\mu_3+\mu_4+1),\ \frac{1}{2}(\mu_1+\mu_2+\mu_3+\mu_4+2);\right.$$
$$\left.1+\mu_2,\ \ 1+\mu_3;\ \frac{4b^2c^2}{(a^2+b^2+c^2+g^2)^2},\ \frac{4g^2}{(a^2+b^2+c^2+g^2)^2}\right],$$

valid for

$$R(\mu_3+\mu_4)>0,\quad R(\mu_1+\mu_2+\mu_3+\mu_4+1)>0,\quad R(b)>|R(c)|+|I_m(g)|,$$
$$R(b)>|R(c)|+|I_m(a)|. \qquad \cdots\cdots(4)$$

For the definition of Appells' function F_4, see [1, p. 224].

2. As usual $\Phi(p) \doteqdot h(t)$ will be used to denote Laplace transform defined by the equation

$$\Phi(p)=p\int_0^\infty e^{-pt}h(t)\,dt, \qquad \ldots\ldots(5)$$

where the integral is convergent and $R(p)>0$.

We prove a theorem of Laplace transform, which will be used in our investigation.

THEOREM: *If $h(t)\in L(0,\infty)$ and if $R(\mu_1)>-1$, $a>0$, $R(p)>0$, then*

$$t^{-\mu_1-1}e^{-\frac{a}{t}}\,h(t)\doteqdot p\,a^{-\frac{1}{2}\mu_1}\int_0^\infty x^{\frac{1}{2}\mu_1}(x+p)^{-1}J_{\mu_1}(2\sqrt{ax})\,\Phi(x+p)\,dx,$$

where $h(t)\doteqdot\Phi(p)$. ......(6)

PROOF: We know that if

$$h(t)\doteqdot\Phi(p)$$

then, Erdélyi [1, p. 129],

$$e^{-\alpha t}h(t)\doteqdot p(p+\alpha)^{-1}\Phi(p+\alpha). \qquad \ldots\ldots(7)$$

Also, Erdélyi [1, p. 245],

$$t^{\frac{1}{2}\mu_1}J_{\mu_1}(2\sqrt{at})\doteqdot p^{-\mu_1}\,a^{\frac{1}{2}\mu_1}\,e^{-a/p}. \qquad \ldots\ldots(8)$$

Using the relations (7) and (8) in Goldstiens' result [2] that if

$$h_1(p)\doteqdot g_1(t) \text{ and } h_2(p)\doteqdot g_2(t),$$

then

$$\int_0^\infty h_1(t)\,g_2(t)\,t^{-1}dt=\int_0^\infty h_2(t)\,g_1(t)\,t^{-1}dt, \qquad \ldots\ldots(9)$$

and replacing α by p, we have

$$\int_0^\infty t^{-\mu_1-1}e^{-pt-\frac{a}{t}}\,h(t)\,dt=a^{-\frac{1}{2}\mu_1}\int_0^\infty x^{\frac{1}{2}\mu_1}(x+p)^{-1}J_{\mu_1}(2\sqrt{ax})\,\Phi(x+p)\,dx. \quad \ldots(10)$$

The theorem follows immediately on interpreting the left side of (10).

To prove (1), we take, Erdélyi [1, p. 284],

$$h(t)=t^{-1}\exp\left[-\frac{b+c}{t}\right]I_{\mu_2}\left(\frac{2\sqrt{bc}}{t}\right)$$

$$\doteqdot 2p\,K_{\mu_2}(2\sqrt{bp})\,I_{\mu_2}(2\sqrt{cp})$$

$$=\Phi(p),\ R(p)>0,\ \ R(b)>0,\ \ R(c)>0. \qquad \ldots\ldots(11)$$

Using (11) in (6), we have

$$t^{-\mu_1-2}\exp\left[-\frac{a+b+c}{t}\right]I_{\mu_2}\left(\frac{2\sqrt{bc}}{t}\right)$$

$$\doteqdot 2p\,a^{-\frac{1}{2}\mu_1}\int_0^\infty x^{\frac{1}{2}\mu_1}J_{\mu_1}(2\sqrt{ax})K_{\mu_2}[2\sqrt{b(x+p)}]\,I_{\mu_2}[2\sqrt{c(x+p)}]\,dx. \qquad \ldots\ldots(12)$$

Also, Erdélyi [1, *p*. 281],

$$J_{\mu_3}(2\sqrt{gt})J_{\mu_3}(2\sqrt{ht})\doteqdot\exp\left[-\frac{g+h}{t}\right]I_{\mu_3}\left(\frac{2\sqrt{gh}}{t}\right), \qquad \ldots\ldots(13)$$

where $R(\mu_3)>-1$.

Applying (12) and (13) in (9), we have

$$\int_0^\infty\int_0^\infty x^{\frac{1}{2}\mu_1}J_{\mu_1}(2\sqrt{ax})K_{\mu_2}[2\sqrt{b(x+t)}]\,I_{\mu_2}[2\sqrt{c(x+t)}]\,J_{\mu_3}(2\sqrt{gt})J_{\mu_3}(2\sqrt{ht})\,dt\ dx$$

$$=\frac{a^{\frac{1}{2}\mu_1}}{2}\int_0^\infty t^{\mu_1+1}\exp[-(a+b+c+g+h)t]I_{\mu_2}(2\sqrt{bc}\,t)\,I_{\mu_3}(2\sqrt{gh}\,t)\,dt.$$

(1) follows immediately on evaluating the single integral with the help of the result, Erdélyi [1, p. 196].

(2), (3), (4) can be proved in the same way by using the results [1, p. 281, 282, 284],

$$I_\rho(2a\sqrt{t})I_\rho(2b\sqrt{t})\doteqdot p^{-1}e^{(a^2+b^2)/p}\,I_\rho\left(\frac{2ab}{p}\right), \qquad \ldots\ldots(14)$$

where $R(\rho)>-1$ and $R(p)>0$,

$$t^{-2\lambda-1}S_2\left(\rho-\frac{1}{2},\ -\rho-\frac{1}{2},\ \lambda+\frac{1}{2},\ \lambda;\ \frac{1}{2}at\right)$$

$$\doteqdot\frac{p^{2\lambda}}{2^{2\lambda}\sqrt{\pi}}K_{2\rho}\left(\frac{2a}{p}\right), \qquad \ldots\ldots(15)$$

where $R(\lambda\pm\rho)<0$, and

$$t^{\lambda+v-1}{}_1F_2\left(v+\frac{1}{2};\ \ 2v+1,\ \ \lambda+v;\ \ -2at\right)$$

$$\doteqdot\frac{\Gamma(v+1)\Gamma(\lambda+v)}{2^{-v}a^v p^\lambda}e^{-\frac{a}{p}}I_v\left(\frac{a}{p}\right), \qquad \ldots\ldots(16)$$

where $R(\lambda+v)>0$, instead of (13) respectively.

Department of Mathematics
Regionalcenter for Post-Graduate studies
Simla-3, India

References

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