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On the Imbedding of the Space of Gödel Type

By

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In 1949 K. Gödel [2][1] proposed a new type of space-time as a solution for Einstein's field equations. It was generalized and discussed by J. L. Synge [6] from the physical viewpoint.

The space of Gödel type is, according to Synge, a 4-dimensional Riemannian space defined by the metric form

$$ds^2=\sum_{\alpha,\beta=1,2} g_{\alpha\beta}\, dx^\alpha\, dx^\beta+\sum_{\gamma,\delta=3,4} g_{\gamma\delta}\, dx^\gamma\, dx^\delta, \tag{0.1}$$ [2]

where $g_{\alpha\beta}$'s are functions of variables (x^3, x^4) and $g_{\gamma\delta}$'s are constants. Since a metric form defining a solution for Einstein's equations must be of signature $+2$, one of the two quadratic forms on the right hand side of (0.1) should be of signature 0 and the other of signature $+2$.

It is well known that an n-dimensional Riemannian space V_n can be imbedded in a flat space of at most $n(n+1)/2$ dimensions. If the lowest dimension of the flat space in which the V_n can be imbedded is $n+p$, the V_n is said to be *of class* p. The purpose of the present paper is to examine a special case of the space of Gödel type under what conditions it is of class 1. The following theorems are essential for this purpose.

Theorem 1. *An n-dimensional Riemannian space is of class* 1 *if and only if its curvature tensor does not vanish and there can be found a value of ε and a set of functions b_{ij} on V_n satisfying the Gauss' equations*

$$\varepsilon R_{ijkl}=b_{ik}b_{jl}-b_{il}b_{jk}, \quad (i, j, k, l=1, 2,\dots, n), \tag{0.2}$$

and the Codazzi's equations

$$b_{ij,k}-b_{ik,j}=0, \quad (i, j, k=1, 2,\cdots, n), \tag{0.3}$$

where R_{ijkl}'s are the components of the curvature tensor and *ε takes the value of $+1$ or -1 so that b_{ij}'s are reals. In equation* (0.3) *comma denotes covariant*

1) Numbers in brackets refer to the references at the end of the paper.
2) Throughout the paper, Greek suffixes α and β take the values 1, 2, and γ and δ the values 3, 4. Roman suffixes take the values 1, 2 and 4 unless otherwise stated.

differentiation.

When the given V_n is of class 1, we can find b_{ij} as a set of solutions for (0.2) and (0.3). If the rank of the matrix (b_{ij}) is 0 or 1, the V_n is said to be *of type* 1 and if the rank is $r \geqq 2$, *of type* r.

THEOREM 2. *If the V_n of class 1 is of type $r \geqq 3$, it is uniquely imbedded in an $(n+1)$-dimensional flat space within a motion.*

In the following sections we consider a special case of the space of Gödel type due to Gödel. The calculation in this paper was suggested by Prof. M. Matsumoto. The author expresses here his gratitude.

§1. Preliminaries

1. Assumptions

Since it is too complicated to consider the space of Gödel type in the most general case, we choose as a special case a V_4 defined by the following metric tensor:

$$(1.1) \qquad (g_{\alpha\beta})=\begin{pmatrix} a & be^f \\ be^f & ce^{2f} \end{pmatrix}, \quad (g_{\gamma\delta})=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where a, b and c are constants and f is a function of x^4 only. Putting $\sigma=b^2-ac$, the signature requirement demands us to assume $\sigma>0$.

It is obvious that in calculating quantities such as the Christoffel's symbols, the components of the curvature tensor etc., the variable x^3 does not concern. So the x^3-direction may be neglected for the present and we may consider the V_3 defined by the metric

$$(1.1') \qquad ds^2=a(dx^1)^2+2be^f dx^1 dx^2+ce^{2f}(dx^2)^2+(dx^4)^2.$$

If the V_3 is of class 1, so is the V_4 defined by (1.1) and vice versa.

The differentiability of the function f is, of course, assumed in sufficient order. And its derivatives will be denoted by primes as f', f'' and so on.

2. Fundamental quantities for the V_3 defined by (1.1′)

Direct calculation gives us the following:

(i) The contravariant components of the metric tensor.

$$(1.2) \qquad (g^{\alpha\beta})=\sigma^{-1}\begin{pmatrix} -c & be^{-f} \\ be^{-f} & -ae^{-2f} \end{pmatrix}, \quad g^{44}=1.$$

(ii) The Christoffel's symbols.

$$
(1.3)\qquad \begin{cases} [14, 2]=[24, 1]=-[12, 4]=\dfrac{b}{2} f' e^{f}, \\ [42, 2]=-[22, 4]=cf' e^{2f}, \\ [ij, k]=0, \quad \text{otherwise.} \end{cases}
$$

$$
(1.4)\qquad \begin{cases} \left\{ {1 \atop 1\ 4} \right\}=\dfrac{b^2}{2\sigma} f', & \left\{ {1 \atop 2\ 4} \right\}=\dfrac{bc}{2} f' e^{f}, \\ \left\{ {2 \atop 1\ 4} \right\}=-\dfrac{ab}{2\sigma} f' e^{-f}, & \left\{ {2 \atop 2\ 4} \right\}=\left(1-\dfrac{b^2}{2\sigma}\right) f', \\ \left\{ {4 \atop 1\ 2} \right\}=-\dfrac{b}{2} f' e^{f}, & \left\{ {4 \atop 2\ 2} \right\}=-cf' e^{2f}, \\ \left\{ {i \atop j\ k} \right\}=0, \quad \text{otherwise.} \end{cases}
$$

(iii) The components of the curvature tensor.

Only four components of the curvature tensor are non-vanishing and independent.

$$
(1.5)\qquad \begin{cases} R_{1212}=\dfrac{1}{4} b^2 f'^2 e^{2f}, \\ R_{1414}=-\dfrac{1}{4} \dfrac{ab^2}{\sigma} f'^2, \\ R_{1424}=-\dfrac{1}{2} b\left(f''+\dfrac{b^2}{2\sigma} f'^2\right) e^{f}, \\ R_{2424}=-c\left\{f''+\left(1+\dfrac{b^2}{4\sigma}\right) f'^2\right\} e^{2f}. \end{cases}
$$

(iv) The components of the Ricci tensor and the scalar curvature.

$$
(1.6)\qquad \begin{cases} R_{11}=\dfrac{ab^2}{2\sigma} f'^2, \\ R_{12}=\dfrac{b}{2}\left(f''+\dfrac{b^2}{\sigma} f'^2\right) e^{f}, \\ R_{22}=c\left\{f''+\left(1+\dfrac{b^2}{2\sigma}\right) f'^2\right\} e^{2f}, \end{cases}
$$

$$\left\{\begin{aligned} &R_{44}=f''+\left(1-\frac{b^2}{2\sigma}\right)f'^2,\\ &R_{14}=R_{24}=0.\end{aligned}\right.$$

$$R=2\left\{f''+\left(1-\frac{b^2}{4\sigma}\right)f'^2\right\}. \tag{1.7}$$

These last results show that the V_4 defined by (1.1) is not an Einstein space. This is reasonable since, according to E. Kasner [4], a 4-dimensional Einstein space cannot be imbedded in a 5-dimensional flat space.

§2. The Gauss' equations

The Jacobi's theorem in the theory of determinants enables us to express b_{ij}'s in the Gauss' equations (0.2) in terms of R_{ijkl}'s.

If we denote

$$\Delta=\begin{vmatrix} \varepsilon R_{2424} & \varepsilon R_{2441} & \varepsilon R_{2412}\\ \varepsilon R_{4124} & \varepsilon R_{4141} & \varepsilon R_{4112}\\ \varepsilon R_{1224} & \varepsilon R_{1241} & \varepsilon R_{1212}\end{vmatrix} \tag{2.1}$$

and regard the right hand side of (0.2) as the minor of order 2 of the matrix (b_{ij}), the Jacobi's theorem assures that

$$b_{ij}=\Delta^{-\frac{1}{2}}B_{ij}, \tag{2.2}$$

where B_{ij} denotes the cofactor of the element in Δ corresponding to b_{ij} in the matrix (b_{ij}), provided that $\Delta\neq0$.

Using the results in the previous section, we obtain

$$\Delta=\frac{\varepsilon}{16}b^4f'^2e^{4f}\left(\frac{ac}{\sigma}Af'^2-B^2\right), \tag{2.3}$$

where $A=f''+\left(1+\frac{b^2}{4\sigma}\right)f'^2$ and $B=f''+\frac{b^2}{2\sigma}f'^2$.

This shows that the Gauss' equations can be solved for b_{ij}'s uniquely except for the cases $b=0, f'=0$ and $\frac{ac}{\sigma}Af'^2-B^2=0$. So the imbedding of the V_3 defined by (1.1′) in a 4-dimensional flat space is unique within a motion if it exists.

In consequence of (2.2) and (2.3) we obtain the set of solutions for b_{ij}'s in the Gauss' equations.

$$
(2.4)\quad \begin{cases}
b_{11}=-\dfrac{ab^2f'^3}{4\sigma\left\{\varepsilon\left(\dfrac{ac}{\sigma}Af'^2-B^2\right)\right\}^{\frac{1}{2}}}, \\[2ex]
b_{12}=-\dfrac{bBf'e^f}{2\left\{\varepsilon\left(\dfrac{ac}{\sigma}Af'^2-B^2\right)\right\}^{\frac{1}{2}}}, \\[2ex]
b_{22}=-\dfrac{cAf'e}{\left\{\varepsilon\left(\dfrac{ac}{\sigma}Af'^2-B^2\right)\right\}^{\frac{1}{2}}}, \\[2ex]
b_{44}=\dfrac{acAf'^2-\sigma B^2}{\sigma f'\left\{\varepsilon\left(\dfrac{ac}{\sigma}Af'^2-B^2\right)\right\}^{\frac{1}{2}}}, \\[2ex]
b_{14}=b_{24}=0.
\end{cases}
$$

§3. The Codazzi's equations

Next we determine the conditions for the function f so that the b_{ij}'s obtained in the previous section satisfy the Codazzi's equations (0.3).

Putting $b_{ijk}=b_{ij,k}-b_{ik,j}$, (0.3) can be written in the form $b_{ijk}=0$. Substituting (1.4) and (2.4) into (0.3), we have the following four equations which are independent:

$$
(3.1)\quad \begin{cases}
b_{114}\equiv\dfrac{\partial b_{11}}{\partial x^4}-\sum\limits_i b_{i1}\left\{{i \atop 1\ 4}\right\}=0, \\[2ex]
b_{124}\equiv\dfrac{\partial b_{12}}{\partial x^4}-\sum\limits_i b_{i2}\left\{{i \atop 1\ 4}\right\}+\sum\limits_i b_{i4}\left\{{i \atop 1\ 2}\right\}=0, \\[2ex]
b_{224}\equiv\dfrac{\partial b_{22}}{\partial x^4}-\sum\limits_i b_{i2}\left\{{i \atop 2\ 4}\right\}+\sum\limits_i b_{i4}\left\{{i \atop 2\ 2}\right\}=0, \\[2ex]
b_{412}\equiv-\sum\limits_i b_{i1}\left\{{i \atop 4\ 2}\right\}+\sum\limits_i b_{i2}\left\{{i \atop 4\ 1}\right\}=0.
\end{cases}
$$

The equation $b_{412}=0$ is written in the form

$$\frac{abcf'^4e^f}{2\sigma\left\{\varepsilon\left(\frac{ac}{\sigma}Af'^2-B^2\right)\right\}^{\frac{1}{2}}}=0. \tag{3.2}$$

Since we have assumed $b\neq 0, f'\neq 0$ and $\frac{ac}{\sigma}Af'^2-B^2\neq 0$, (3.2) means $ac=0$. So we should consider the following three different cases: (i) $a=0$ and $c\neq 0$, (ii) $a\neq 0$ and $c=0$, (iii) $a=c=0$.

(i) The case where $a=0$ and $c\neq 0$.

In this case the non-vanishing Christoffel's symbols are as follows:

$$\left\{{1 \atop 1\ 4}\right\}=\left\{{2 \atop 2\ 4}\right\}=\frac{f'}{2},\ \left\{{1 \atop 2\ 4}\right\}=\frac{c}{2b}f'e^f,\ \left\{{4 \atop 1\ 2}\right\}=-\frac{b}{2}f'e^f,\ \left\{{4 \atop 2\ 2}\right\}=-cf'e^{2f}.$$

$\left\{\varepsilon\left(\frac{ac}{\sigma}Af'^2-B^2\right)\right\}^{\frac{1}{2}}$ in the denominators of the b_{ij}'s reduces to $\left\{\varepsilon\left(-B^2\right)\right\}^{\frac{1}{2}}=B$, if we choose $\varepsilon=-1$ so that the b_{ij}'s to be real. Then the b_{ij}'s can be written in the form,

$$b_{11}=0,\quad b_{12}=-\frac{b}{2}f'e^f,\quad b_{22}=-\frac{cAf'e^{2f}}{B},\quad b_{44}=-\frac{B}{f'},$$

where $A=f''+\frac{5}{4}f'^2$ and $B=f''+\frac{1}{2}f'^2$ in accordance with (2.3).

If we substitute these results into (3.1), the equations $b_{114}=0$ and $b_{124}=0$ are satisfied identically and $b_{224}=0$ becomes

$$b_{224}\equiv\frac{3f'^2e^{2f}}{4B^2}(f'''f'-4f''^2-3f''f'^2-f'^4)=0. \tag{3.3}$$

(ii) The case where $a\neq 0$ and $c=0$.

In this case the non-vanishing Christoffel's symbols and the b_{ij}'s are as follows:

$$\left\{{1 \atop 1\ 4}\right\}=\left\{{2 \atop 2\ 4}\right\}=\frac{f'}{2},\quad \left\{{2 \atop 1\ 4}\right\}=-\frac{a}{2b}f'e^{-f},\quad \left\{{4 \atop 1\ 2}\right\}=-\frac{b}{2}f'e^f,$$

$$b_{11}=-\frac{af'^3}{4B},\quad b_{12}=-\frac{b}{2}f'e^f,\quad b_{22}=0,\quad b_{44}=-\frac{B}{f'},$$

where $B=f''+\frac{1}{2}f'^2$ as in the case (i).

Then the equations $b_{124}=0$ and $b_{224}=0$ of (3.1) are satisfied identically and $b_{114}=0$ becomes

$$b_{114}\equiv\frac{af'^2}{4B^2}(f'''f'-4f''^2-f''f'^2)=0. \tag{3.4}$$

(iii) The case where $a=c=0$.

In this case the non-vanishing Christoffel's symbols and the b_{ij}'s are as follows:

$$\left\{ {1 \atop 1\ 4} \right\}=\left\{ {2 \atop 2\ 4} \right\}=\frac{f'}{2}, \quad \left\{ {4 \atop 2\ 1} \right\}=-\frac{b}{2}f'e^f,$$

$$b_{11}=b_{22}=0, \quad b_{12}=-\frac{b}{2}f'e^f, \quad b_{44}=-\frac{B}{f'},$$

where $B=f''+\frac{1}{2}f'^2$ as in the former two cases.

The Codazzi's equations are satisfied identically by these $\left\{ {i \atop j\ k} \right\}$'s and b_{ij}'s. Now we have obtained the conclusion.

THEOREM A. *In order that the V_4 defined by* (1.1) *is of class* 1, *only three cases for the constants a, c and the function f in g_{ij} are possible if we assume* $b\neq 0$, $f'\neq 0$ and $\frac{ac}{\sigma}Af'^2-B^2\neq 0$, *that is,*

(I) $a=0, \quad c\neq 0$ *and* $f'''f'-4f''^2-3f''f'^2-f'^4=0,$

(II) $a\neq 0, \quad c=0$ *and* $f'''f'-4f''^2-f''f'^2=0,$

(III) $a=c=0$ *and* f *is arbitrary.*

§4. The excepted cases

In the former sections we assumed $b\neq 0$, $f'\neq 0$ and $\frac{ac}{\sigma}Af'^2-B^2\neq 0$. Among these assumptions $f'\neq 0$ is essential, because if $f'=0$, all the components of g_{ij} are constants and the V_4 is itself flat.

Now we consider the cases excepted in the former sections.

(i) The case where $b=0$.

In this case the non-vanishing Christoffel's symbols are

$$[42,2]=-[22,4]=cf'e^{2f}, \quad \left\{ {2 \atop 2\ 4} \right\}=f', \quad \left\{ {4 \atop 2\ 2} \right\}=-cf'e^{2f}.$$

And only one component of the curvature tensor

$$R_{2424}=-c(f''+f'^2)e^{2f},$$

is non-vanishing. Obviously we can assume $R_{2424}\neq 0$.

So the Gauss' equations are expressed as follows:

$$\text{(4.1)}\qquad \begin{cases} b_{11}b_{22}-b_{12}{}^2=0, \\ b_{11}b_{44}-b_{14}{}^2=0, \\ b_{12}b_{44}-b_{14}b_{24}=0, \\ b_{22}b_{44}-b_{24}{}^2=-\varepsilon c(f''+f'^2)e^{2f}, \\ b_{12}b_{24}-b_{22}b_{14}=0, \\ b_{14}b_{12}-b_{24}b_{11}=0. \end{cases}$$

An easy calculation shows $b_{11}=b_{12}=b_{14}=0$, and (4.1) reduces to only one equation

$$\text{(4.1}')\qquad b_{22}b_{44}-b_{24}{}^2=-\varepsilon c(f''+f'^2)e^{2f}.$$

On the other hand the Codazzi's equations lead us to the condition

$$\text{(4.2)}\qquad \begin{cases} \dfrac{\partial b_{22}}{\partial x^1}=\dfrac{\partial b_{24}}{\partial x^1}=\dfrac{\partial b_{44}}{\partial x^1}=0, \\ \dfrac{\partial b_{22}}{\partial x^4}-\dfrac{\partial b_{24}}{\partial x^2}-cf'e^{2f}b_{44}-f'b_{22}=0, \\ \dfrac{\partial b_{24}}{\partial x^4}-\dfrac{\partial b_{44}}{\partial x^2}+f'b_{24}=0. \end{cases}$$

(ii) The case where $\dfrac{ac}{\sigma}Af'^2-B^2=0$ and $b\neq 0$. $\left(A=f''+\left(1+\dfrac{b^2}{4\sigma}\right)f'^2,\ B=f''+\dfrac{b^2}{2\sigma}f'^2\right)$

In this case the Gauss' equations take the following expressions:

$$\text{(4.3)}\qquad \begin{cases} b_{11}b_{22}-b_{12}{}^2=\dfrac{\varepsilon}{4}b^2f'^2e^{2f}, \\ b_{11}b_{44}-b_{14}{}^2=-\dfrac{\varepsilon ab^2}{4\sigma}f'^2, \\ b_{12}b_{44}-b_{14}b_{24}=-\dfrac{\varepsilon b}{2}Be^f, \end{cases}$$

$$\begin{cases} b_{22}b_{44}-b_{24}{}^2=-\varepsilon c\,Ae^{2f}, \\ b_{12}b_{24}-b_{14}b_{22}=0, \\ b_{14}b_{12}-b_{11}b_{24}=0. \end{cases}$$

An elementary calculation shows $b_{14}=b_{24}=b_{44}=0$ and

$$a=0, \quad B=0 \text{ and } cA=0. \tag{4.4}$$

As in the former cases, $a=0$ means $A=f''+\frac{5}{4}f'^2$, $B=f''+\frac{1}{2}f'^2$. But in such a case $A=0$ and $B=0$ cannot be compatible, so we can conclude $c=0$. And the condition (4.4) is rewritten in the form

$$a=c=0, \text{ and } f''+\frac{1}{2}f'^2=0. \tag{4.4$'$}$$

Integrating the last differential equation, we obtain for f

$$f=2\log(\alpha x^4+\beta), \tag{4.5}$$

where α and β are arbitrary constants.

The Codazzi's equations become as follows:

$$\begin{cases} \dfrac{\partial b_{11}}{\partial x^2}-\dfrac{\partial b_{12}}{\partial x^1}=0, \\ \dfrac{\partial b_{12}}{\partial x^2}-\dfrac{\partial b_{22}}{\partial x^1}=0, \\ \dfrac{\partial b_{11}}{\partial x^4}-\dfrac{f'}{2}b_{11}=0, \\ \dfrac{\partial b_{12}}{\partial x^4}-\dfrac{f'}{2}b_{12}=0, \\ \dfrac{\partial b_{22}}{\partial x^4}-\dfrac{f'}{2}b_{22}=0. \end{cases} \tag{4.6}$$

From the last three equations of (4.6) we obtain

$$\begin{cases} b_{11}=\psi(x^1, x^2)e^{\frac{f}{2}}, \\ b_{12}=\varphi(x^1, x^2)e^{\frac{f}{2}}, \\ b_{22}=\xi(x^1, x^2)e^{\frac{f}{2}}. \end{cases} \tag{4.7}$$

Substituting (4.7) and (4.5) into the first equation of (4.3), we obtain

$$\varphi\psi-\xi^2=\varepsilon\alpha^2 b^2. \tag{4.8}$$

Moreover if we put (4.7) into the first and the second equation of (4.6), we obtain the condition for the functions φ, ψ and ξ in (4.7), that is,

$$\frac{\partial\varphi}{\partial x^2}=\frac{\partial\psi}{\partial x^1}, \quad \frac{\partial\psi}{\partial x^2}=\frac{\partial\xi}{\partial x^1}. \tag{4.9}$$

Summing up these results, we have the

THEOREM B. *In order that the V_4 defined by* (1.1) *may be of class* 1, *in the cases* (i) $b=0$ *and* (ii) $\frac{ac}{\sigma}Af'^2-B^2=0$, $b\neq 0$ *where* $A=f''+\left(1+\frac{b^2}{4\sigma}\right)f'^2$, $B=f''+\frac{b^2}{2\sigma}f'^2$, *it is necessary and sufficient that the function f of x^4 and the b_{ij}'s satisfy the following respective conditions*:

(i)
$$c(f''+f'^2)\neq 0, \quad b_{11}=b_{12}=b_{14}=0,$$

$$b_{22}b_{44}-b_{24}{}^2=-\varepsilon c(f''+f'^2)e^{2f},$$

$$\frac{\partial b_{22}}{\partial x^1}=\frac{\partial b_{24}}{\partial x^1}=\frac{\partial b_{44}}{\partial x^1}=0, \quad \frac{\partial b_{24}}{\partial x^4}-\frac{\partial b_{44}}{\partial x^2}+f'b_{24}=0,$$

$$\frac{\partial b_{22}}{\partial x^4}-\frac{\partial b_{24}}{\partial x^2}-cf'e^{2f}b_{44}-f'b_{22}=0,$$

(ii)
$$f=2\log(\alpha x^4+\beta), \quad b_{14}=b_{24}=b_{44}=0,$$

$$b_{11}=\varphi(x^1, x^2)e^{\frac{f}{2}}, \quad b_{12}=\psi(x^1, x^2)e^{\frac{f}{2}}, \quad b_{22}=\xi(x^1, x^2)e^{\frac{f}{2}},$$

where φ, ψ and ξ satisfy (4.8) *and* (4.9).

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