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On the Betti Series of Local Rings

By

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Let R be a commutative Noetherian ring and let $\mathfrak{a}$ be an ideal in R. In [6], Tate has shown that it is always possible to construct a free resolution of $R/\mathfrak{a}$ which, at the same time, is a skew commutative differential graded algebra over R, and he successfully applied his "R-algebra resolutions" to the study of the homology theory of Noetherian rings. In the case when R is a local ring with maximal ideal $\mathfrak{m}$, it would be more desirable, however, to construct a "minimal" R-algebra resolution if it is possible.

In § 1, we prove, first of all, that such a resolution always exists. In fact, an R-algebra resolution X of the residue field K, which is constructed in theorem 1 in [6], is actually minimal.

For any integer $p \geqq 0$, the p-th Betti number B_p of R is defined to be the dimension of the vecter space $\mathrm{Tor}_p^R(K, K)$ over K. The power series $\mathscr{B}(R) = \sum B_p Z^p$ is called the Betti series of R. Based on the existence of a minimal R-algebra resolution, we can express $\mathscr{B}(R)$ as a quotient of two power series:

$$(*) \qquad \mathscr{B}(R)=\frac{(1+Z)^n}{(1-Z^2)^{\varepsilon_1}}\cdot\frac{(1+Z^3)^{\varepsilon_2}}{(1-Z^4)^{\varepsilon_3}}\cdot\frac{(1+Z^5)^{\varepsilon_4}}{(1-Z^6)^{\varepsilon_5}}\cdots,$$

where n is the embedding dimension of $R(=\dim_K \mathfrak{m}/\mathfrak{m}^2)$ and ε_i's are non negative integers. In the case when K is of characteristic 0, as was pointed out by Scheja [3], this formula is also obtained by applying the Hopf-Borel structure theorem to the Hopf algebra $\mathrm{Tor}^R(K, K)$ [1], but the formula is true in general as we mentioned above.

In the following sections 2 and 3, we will give alternating proofs of the theorems, due to Scheja [3], by the systematic use of the R-algebra method which would simplify the original arguments in some points. By making use of the formula $(*)$, we investigate in § 2 the relationship between $\mathscr{B}(R)$ and $\mathscr{B}(\bar{R})$, where $\bar{R}$ is the residue ring of R by a non zero divisor of R. In § 3, we represent $\mathscr{B}(R)$ as a rational function in the case $n \leqq 2$, and show that the multiplcative property of the Koszul complex of R gives us an information about the classification of the possible types of $\mathscr{B}(R)$.

Throughout, the terminology and notations are the same as those of [6]. We shall use freely the R-algebra techniques, all of which can be found in

[1] and [6]. By a local ring $(R, \mathfrak{m})$ we mean R is a commutative Noetherian local ring and $\mathfrak{m}$ its maximal ideal.

1. Let $(R, \mathfrak{m})$ be a local ring of embedding dimension n and let K be the residue field $R/\mathfrak{m}$. First, we recall that the limit of the following ascending sequence of R-algebras $X^{(k)}$ $(k=0, 1, 2, \cdots)$ gives us an R-algebra resolution X of K [6].

We take $X^{(0)}=R$ and fix a minimal system of generators $t_1, \cdots, t_n$ of $\mathfrak{m}$. Viewing t_i's as 0-cycles, we adjoin variables $T_1, \cdots, T_n$ of degree 1 to R which kill $t_1, \cdots, t_n$ and put

$$X^{(1)}=R< T_1, \cdots, T_n> ;\ dT_i=t_i.$$

Then, $H_0(X^{(1)})=K$. Denote by ε_1 the dimension of $H_1(X^{(1)})$ over K and choose 1-cycles $s_1, \cdots, s_{\varepsilon_1} \in Z_1(X^{(1)})$ such that whose homology classes generate $H_1(X^{(1)})$, and adjoin variables S_i $(1 \leq i \leq \varepsilon_1)$ of degree 2 to $X^{(1)}$ which kill the cycles s_i. Then we get the next R-algebra

$$X^{(2)}=X^{(1)}< S_1, \cdots, S_{\varepsilon_1}> ;\ dS_i=s_i.$$

Continuing in this way we get a sequence of R-algebras $X^{(k)}$ $(k=0, 1, 2, \cdots)$.

We remark that $X^{(k)}$ $(k=0, 1, 2, \cdots)$ enjoy the following properties:

(1) $X^{(k+1)} \supset X^{(k)}$, and $X_\lambda^{(k+1)}=X_\lambda^{(k)}$ if $\lambda < k+1$.

(2) $H_0(X^{(k)})=K$ and $H_\lambda(X^{(k)})=0$ for $1 \leq \lambda < k$.

(3) $X_{k+1}^{(k+1)}$ is a direct sum of $X_{k+1}^{(k)}$ and ε_k-copies of R, where ε_k is a number of variables adjointed to $X^{(k)}$ which is equal to the dimension of the vector space $H_k(X^{(k)})$ over K. $X^{(1)}$ is nothing but the Koszul complex of R and will be denoted by E. We remark further that ε_i is independent of the choice of the cycles in $Z_i(X^{(i)})$ so that ε_i and, consequently, X are the homological invariants of R. We call ε_i the i-th deflection of R since ε_i's give us an information about the degree of irregularity of R.[1]

A projective resolution P of K is called minimal if it satisfies the condition, $dP \subset \mathfrak{m}P$, where d is the differential operator defined on the complex P. Now, we shall prove that the R-algebra resolution X of K, constructed above, has this additional property. For this we need the following lemma which provides us with the basis of an inductive argument.

LEMMA 1. *Let X be an R-algebra and assume X satisfies the following two*

1) It is well known that $\varepsilon_1=0$ if and only if R is regular, and $\varepsilon_2=0$ if and only if R is a complete intersection [1], [3], [7].

conditions:

(1) $Z_\lambda(X) \subset \mathfrak{m}X_\lambda\ (\lambda \geqq 1)$.

(2) *If* $x \in X_\lambda$ and $dx \in \mathfrak{m}^2 X_{\lambda-1}$, then $x \in \mathfrak{m}X_\lambda\ (\lambda \geqq 1)$.

Now, let $t \in Z_{\rho-1}(X)$ *be a cycle of degree* $\rho-1 (\rho>0)$ *and let* $Y = X<T>$; $dT = t$. *Then*, (1) *and* (2) *also hold in* Y.

Proof. We treat the cases of odd and even ρ separately.

ρ odd: In this case, $Y = X + XT$. Let $y_\lambda = x_\lambda + x_{\lambda-\rho}T$ be an element of $Z_\lambda(Y)$. Since $dy_\lambda = 0$, we have $dx_\lambda + (-1)^{\lambda-\rho} x_{\lambda-\rho} t = 0$ and $dx_{\lambda-\rho} = 0$. From (1), we have $x_{\lambda-\rho} \in \mathfrak{m}X_{\lambda-\rho}$. Hence, $dx_\lambda \in (\mathfrak{m}X_{\lambda-\rho})(\mathfrak{m}X_{\rho-1}) \subset \mathfrak{m}^2 X_{\lambda-1}$. Whence $x_\lambda \in \mathfrak{m}X_\lambda$ by virtue of (2). Consequently, $y_\lambda \in \mathfrak{m}Y_\lambda$. Next, we assume $dy_\lambda \in \mathfrak{m}^2 Y_{\lambda-1}$. Then, $dx_\lambda + (-1)^{\lambda-\rho} x_{\lambda-\rho} t \in \mathfrak{m}^2 X_{\lambda-1}$ and $dx_{\lambda-\rho} \in \mathfrak{m}^2 X_{\lambda-\rho-1}$. Hence, by the similar argument as above, we can easily verify that $y_\lambda \in \mathfrak{m}Y_\lambda$.

ρ even: In this case, $Y = X + XT + XT^{(2)} + \cdots$. Let $y_\lambda = x_\lambda + x_{\lambda-\rho}T + x_{\lambda-2\rho}T^{(2)} + \cdots + x_{\lambda-n\rho}T^{(n)}$ be an element of $Z_\lambda(Y)$. Then, we see at once that $dx_{\lambda-n\rho} = 0$, $dx_{\lambda-(n-1)\rho} + (-1)^\lambda x_{\lambda-n\rho} t = 0, \cdots, dx_\lambda + (-1)^\lambda x_{\lambda-\rho} t = 0$. Therefore, we can prove, step by step, each $x_{\lambda-i\rho} (i = n, n-1, \cdots, 2, 1)$ is contained in $\mathfrak{m}X_{\lambda-i\rho}$, which shows that $y_\lambda \in mY_\lambda$. As for the proof of (2), we will leave it to the reader.

Observe that $X^{(0)} (=R)$ trivially satisfies the condition (1) and (2). Therefore, by the successive applications of lemma 1 to each step of the adjoining variables in the construction of X, we get our following important theorem.

Theorem 1. *A minimal R-algebra resolution of K always exists.*

Another consequence of the particular construction of the minimal resolution X of K is stated in

Theorem 2. *The Betti series* $\mathscr{B}(R)$ *of* R *is given by the following formula*:

$$\mathscr{B}(R) = \frac{(1+Z)^n}{(1-Z^2)^{\varepsilon_1}} \cdot \frac{(1+Z^3)^{\varepsilon_2}}{(1-Z^4)^{\varepsilon_3}} \cdot \frac{(1+Z^5)^{\varepsilon_4}}{(1-Z^6)^{\varepsilon_5}} \cdots,$$

where n *is the embedding dimension of* R *and* ε_i *the i-th deflection of* R. *In particular*,

$$B_1 = n, \quad B_2 = \binom{n}{2} + \varepsilon_1, \quad B_3 = \binom{n}{3} + \binom{n}{1}\varepsilon_1 + \varepsilon_2,$$

$$B_4 = \binom{n}{4} + \binom{n}{2}\varepsilon_1 + \varepsilon_1^2 - \binom{\varepsilon_1}{2} + \binom{n}{1}\varepsilon_2 + \varepsilon_3,$$

$$B_5=\binom{n}{5}+\binom{n}{3}\varepsilon_1+\binom{n}{1}\binom{\varepsilon_1}{2}+\binom{n}{1}\varepsilon_1+\binom{n}{2}\varepsilon_2+\varepsilon_1\varepsilon_2+\binom{n}{1}\varepsilon_3+\varepsilon_4, \cdots$$

PROOF. Since X is minimal, we have $\mathrm{Tor}^R(K, K)=H(X\otimes K)=X\otimes K$. Therefore the p-th Betti number B_p and the number of generators of the free module X_p are exactly the same. Hence, counting the number of generators of X_p, we obtain the desired result.

We point out here that, in view of theorem 2, we see our ε_i coincides with that of Scheja [3] and of Uehara [7] for $i\leq 3$.

2. In this section, as an application of theorem 2, we shall see how the Betti series of R is affected if we pass to its residue ring by a non zero divisor. We begin with the following lemma which has the general character.

LEMMA 2. *Let R and $\bar{R}$ be Noetherian rings and let X and $\bar{X}$ be R- and $\bar{R}$-algebras such that there exists an R-homomorphism φ from X to $\bar{X}$ which induces an isomorphism φ_* of $H(X)$ onto $H(\bar{X})$. Suppose t and $\bar{t}$ are $(\rho-1)$-cycles in X and $\bar{X}$ such that $\varphi(t)=\bar{t}$ and let $Y=X<T>$; $dT=t$ and $\bar{Y}=\bar{X}<\bar{T}>$; $d\bar{T}=\bar{t}$. Then, φ can be extended to an R-homomorphism (again denoted by φ) from Y to $\bar{Y}$ and it induces an isomorphism of $H(Y)$ onto $H(\bar{Y})$.*

PROOF. We again treat the cases of odd and even ρ separately.

ρ odd: In this case, $Y=X+XT$. Consider the exact sequence

$$0\longrightarrow X\xrightarrow{i} Y\xrightarrow{j} X\longrightarrow 0,$$

where i is injective and j is defined by $j(x_1+x_2T)=x_2$. Then, i and j commute with d and φ. Hence, we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_\lambda & \xrightarrow{i} & Y_\lambda & \xrightarrow{j} & X_{\lambda-\rho} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \\ 0 & \longrightarrow & \bar{X}_\lambda & \xrightarrow{\bar{i}} & \bar{Y}_\lambda & \xrightarrow{\bar{j}} & \bar{X}_{\lambda-\rho} & \longrightarrow & 0 \end{array}$$

with exact rows. From this we get a commutative diagram

$$\begin{array}{ccccccccccccc} \cdots\longrightarrow & H_{\lambda-\rho+1}(X) & \xrightarrow{d_*} & H_\lambda(X) & \xrightarrow{i_*} & H_\lambda(Y) & \xrightarrow{j_*} & H_{\lambda-\rho}(X) & \xrightarrow{d_*} & H_{\lambda-1}(X) & \longrightarrow\cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots\longrightarrow & H_{\lambda-\rho+1}(\bar{X}) & \longrightarrow & H_\lambda(\bar{X}) & \longrightarrow & H_\lambda(\bar{Y}) & \longrightarrow & H_{\lambda-\rho}(\bar{X}) & \longrightarrow & H_{\lambda-1}(\bar{X}) & \longrightarrow\cdots, \end{array}$$

where both rows are exact. Therefore $H_\lambda(Y)\approx H_\lambda(\bar{Y})$ $(\lambda=1, 2, 3,\cdots)$ by the "five lemma" [2].

ρ even: In this case, $Y = X + XT + XT^{(2)} + \cdots$, and we have an exact sequence

$$0 \longrightarrow X \xrightarrow{\ i\ } Y \xrightarrow{\ j\ } Y \longrightarrow 0,$$

where i is injective and j is defined by $j(x_0 + x_1 T + x_2 T^{(2)} + \cdots) = x_1 + x_2 T + x_3 T^{(2)} + \cdots$. As in the first case, i and j commute with d and φ, and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_\lambda & \xrightarrow{\ i\ } & Y_\lambda & \xrightarrow{\ j\ } & Y_{\lambda-\rho} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & \bar{X}_\lambda & \xrightarrow{\ \bar{i}\ } & \bar{Y}_\lambda & \xrightarrow{\ \bar{j}\ } & \bar{Y}_{\lambda-\rho} & \longrightarrow & 0 \end{array}$$

is a commutative diagram with exact rows. This yields the following commutative homology diagram

$$\begin{array}{ccccccccccccc} \cdots \longrightarrow & H_1(Y) & \xrightarrow{d_*} & H_\rho(X) & \xrightarrow{i_*} & H_\rho(Y) & \xrightarrow{j_*} & H_0(Y) & \xrightarrow{d_*} & H_{\rho-1}(X) & \xrightarrow{i_*} & H_{\rho-1}(Y) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & H_1(\bar{Y}) & \longrightarrow & H_\rho(\bar{X}) & \longrightarrow & H_\rho(\bar{Y}) & \longrightarrow & H_0(\bar{Y}) & \longrightarrow & H_{\rho-1}(\bar{X}) & \longrightarrow & H_{\rho-1}(\bar{Y}) & \longrightarrow 0 \end{array}$$

where both rows are exact. Since $H_0(Y) \approx H_0(\bar{Y}) \approx K$, we can easily see that $H_{\rho-1}(Y) \approx H_{\rho-1}(\bar{Y})$. On the other hand, from the construction of Y and $\bar{Y}$, we have $H_i(Y) \approx H_i(\bar{Y})$ for $i < \rho - 1$. Thus, applying the five lemma, we have $H_\rho(Y) \approx H_\rho(\bar{Y})$ and similarly $H_\lambda(Y) \approx H_\lambda(\bar{Y})$ for all λ.

Let again $(R, \mathfrak{m})$ be a local ring and let $t_1, \cdots, t_n$ be a minimal system of generotors of $\mathfrak{m}$. Suppose t_n is a non zero divisor in R and not in $\mathfrak{m}^2$. Let $\bar{R} = R/t_n R$ and let $\bar{t}_i$ be the residue class of t_i $(i = 1, 2, 3, \cdots)$. We consider two R- and $\bar{R}$-algebras

$$E = R < T_1, \cdots, T_n > ;\ dT_i = t_i \text{ and } F = \bar{R} < \bar{T}_1, \cdots, \bar{T}_{n-1} > ;\ d\bar{T}_i = \bar{t}_i.$$

If x is a homogeneous element of degree ρ in E, then x can be written as $x = x_1 + x_2 T_n$, where x_1 and x_2 are homogeneous elements in $E' = R < T_1, \cdots, T_{n-1} >$ of degree ρ and $\rho - 1$ respectively. Then, the canonical map φ: $E \rightarrow F$ defined by $\varphi(x) = \bar{x}_1$ induces a homomorphism φ_*: $H(E) \rightarrow H(F)$.

Lemma 3. *In the situation just described, φ_* is an isomorphism and $\varepsilon_1 = \bar{\varepsilon}_1$ where $\bar{\varepsilon}_1$ is the first deflection of $\bar{R}$.*

Proof. First we show that φ induces an R-homomorphism of $Z_\rho(E)$ onto $Z_\rho(F)$. Take $\bar{x}_1 \in Z_\rho(F)$. Since $d\bar{x}_1 = 0$, we can write $dx_1 = y_1 t_n + y_2 T_n$, with y_1

and y_2 in E'. From the relation

$$0=d^2x_1=(dy_1)t_n+(dy_2)T_n+(-1)^{\rho-2}y_2t_n,$$

we get $dy_1=(-1)^{\rho-1}y_2$ and $dy_2=0$ since t_n is a non zero divisor. Now, by a direct calculation, we easily see that the element $x=x_1+(-1)^{\rho}y_1T_n$ belongs to $Z_\rho(E)$ such that $\varphi(x)=\bar{x}_1$. By the similar argument we can show $\varphi^{-1}(B_\rho(F))=B_\rho(E)$. Therefore φ_* is an isomorphism of $H(E)$ onto $H(F)$ as we asserted.

THEOREM 3. Let $(R, \mathfrak{m})$ *be a local ring and let* x *be an element of* $\mathfrak{m}$, *which is not a zero divisor in R. Put* $\bar{R}=R/xR$ *and denote by* $\mathscr{B}(\bar{R})$ *and* $\bar{\varepsilon}_i$ *the Betti series and the i-th deflection of* $\bar{R}$ *respectively. Then*:

(i) *If* $x\notin\mathfrak{m}^2$, *we have* $\bar{\varepsilon}_i=\varepsilon_i$ $(i=1, 2,\cdots)$ *and* $\mathscr{B}(R)=\mathscr{B}(\bar{R})(1+Z)$ [3].

(ii) *If* $x\in\mathfrak{m}^2$, *we have* $\bar{\varepsilon}_1=\varepsilon_1+1$, $\bar{\varepsilon}_i=\varepsilon_i$ $(i=2, 3,\cdots)$ *and* $\mathscr{B}(R)=\mathscr{B}(\bar{R})(1-Z^2)$.

PROOF. (i) If $x\notin\mathfrak{m}^2$, we can take x as a member of a minimal generating system of $\mathfrak{m}$. Observe that dim $\bar{R}$[2)]$=$dim $R-1$. Hence, by lemma 2 and 3, combining with the formula of Betti series in theorem 2, we have our assertion.

(ii) We remark first that $\bar{E}=E/xE$ is the Koszul complex of $\bar{R}$ since $x\in\mathfrak{m}^2$. Write $x=\sum a_it_i$. Then, $s=\sum a_iT_i$ is in E_1 and satisfies $ds=x$. The residue class $\bar{s}$ (mod xE) is a 1-cycle in $\bar{E}$, whose homology class we denote by σ. The canonical map j: $E\to\bar{E}$ induces an isomorphism j_* of $H(E)$ into $H(\bar{E})$ and $H(\bar{E})=(j_*H(E))<\sigma>$ [6, theorem 3]. Hence, $\bar{\varepsilon}_1=\varepsilon_1+1$.

We adjoin a variable S of degree 2 to $\bar{E}$ which kills $\bar{s}$ and obtain

$$E'=\bar{E}<S>;\ dS=\bar{s}.$$

Since σ is a skew non zero divisor in $j_*(H(E))$, we have

$$H(E')\approx H(\bar{E})/\sigma H(\bar{E})\approx H(E)$$

by theorem 2 in [6]. Now, we can conclude our proof by applying lemma 2 to the R-algebra E and the $\bar{R}$-algebra E'.

COROLLARY. (i) *If R is a regular local ring of dimension n, then* $\mathscr{B}(R)=(1+Z)^n$ [3], [4].

(ii) *If R is a complete intersection of embedding dimension n,* $\mathscr{B}(R)=\dfrac{(1+Z)^n}{(1-Z^2)^{n-d}}$, *where* $d=$dim R [6].

3. As we stated in the introduction, the results concerning the struc-

2) We denote by dim R the dimension of R in the sense of Krull.

ture of Betti series in this section were originally obtained by Scheja [3]. He used the Syzygy theory of modules and his method was rather ideal theoretic. We shall present here a simplified proof which is based on the theory of R-algebras.

We shall use the same notations as in §1 and the Koszul complex of R will be denoted by E. For the obvious reason we treat only non regular case.

THEOREM 4. *If $(R, \mathfrak{m})$ is a local ring of embedding dimension* 1 *and if $H_1(E)\neq 0$, then we have*

$$\varepsilon_1=1,\ \varepsilon_i=0\ (i\geq 2) \text{ and } \mathscr{B}(R)=\frac{1+Z}{1-Z^2}.$$

PROOF. In this case, we have $H_0(E)=K$, $H_1(E)=(0:\mathfrak{m})T$ and $H_i(E)=0$ for $i\geq 2$. And, moreover, R is a principal ideal ring. Hence our hypothesis implies that there is an element $a\neq 0$ in R such that $0:\mathfrak{m}=aR$.

Now, we adjoin a variable S of degree 2 which kills 1-cycle aT, and obtain an R-algebra $X=E<S>$; $dS=aT$. Consider the exact sequence

$$0\longrightarrow E\xrightarrow{\ i\ }X\xrightarrow{\ j\ }X\longrightarrow 0,$$

where i is the injective map and j is defined by $j(x_0+x_1S+x_2S^{(2)}+\cdots)=x_1+x_2S+\cdots$. i and j commute with d and we get the following exact sequence:

$$\cdots\longrightarrow H_3(E)\longrightarrow H_3(X)\longrightarrow H_1(X)\longrightarrow H_2(E)\longrightarrow H_2(X)\longrightarrow K$$

$$\xrightarrow{d_{0*}} H_1(E)\longrightarrow H_1(X)=0.$$

Since $H_1(E)\approx K$, d_{0*} is an isomorphism. Hence, $H_2(X)=0$ by virtue of $H_2(E)=0$. In the same way, the relations $H_3(E)=0$ and $H_1(X)=0$ imply that $H_3(X)=0$. Thus, step by step, we have $H_i(X)=0$ for $i=1, 2, 3,\cdots$, and hence $\varepsilon_1=1$ and $\varepsilon_i=0$ for $i\geq 2$. Consequently, $\mathscr{B}(R)=\dfrac{1+Z}{1-Z^2}$ in view of theorem 2.

THEOREM 5. *Let $(R, \mathfrak{m})$ be a local ring of embedding dimension* 2 *and suppose that R is not regular. Then:*

(i) *If $H_1(E)^2=0$, then $\varepsilon_1\geq 1$, $\varepsilon_2=\varepsilon_1-1$, $\varepsilon_3=\binom{\varepsilon_1}{2}$ and $\mathscr{B}(R)=\dfrac{(1+Z)^2}{1-\varepsilon_1Z^2-\varepsilon_2Z^3}$.*

(ii) *If $H_1(E)^2\neq 0$, then $\varepsilon_1=2$, $\varepsilon_2=0$ and $\mathscr{B}(R)=\dfrac{(1+Z)^2}{(1-Z^2)^2}$.*

PROOF. First we remark that the vector space $0:\mathfrak{m}$ over K has dimension ε_1-1. In fact, since dim $R<2$, the Euler-Poincaré characteristic of the Koszul complex E is equal to zero [5], i.e.,

$$\dim_K H_2(E) - \dim_K H_1(E) + \dim_K H_0(E) = 0.$$

Since $H_2(E) \approx 0 : \mathfrak{m}$, $\dim_K H_1(E) = \varepsilon_1$ and $\dim_K H_0(E) = 1$, we have our assertion.

From this remark and from the facts $H_2(X^{(2)}) \approx H_2(E)/H_1(E)^2$ [1, Proposition 2.5] and $\varepsilon_2 = \dim_K H_2(X^{(2)})$, we find $\varepsilon_2 = \varepsilon_1 - 1$ if $H_1(E)^2 = 0$, and $\varepsilon_2 \leq \varepsilon_1 - 2$ if $H_1(E)^2 \neq 0$.

Now we consider the first case, $H_1(E)^2 = 0$. Let X be a minimal R-algebra resolution of K constructed in §1:

$$X: \cdots \longrightarrow X_p \longrightarrow X_{p-1} \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \xrightarrow{\varepsilon} K \longrightarrow 0$$

where $X_0 = R$ and ε is the augmentation homomorphism. Then, by the construction of X, X_3 has the following form:

$$X_3 = \sum_{j=1}^{\varepsilon_1} (RT_1 + RT_2) S_j + \sum_{k=1}^{\varepsilon_2} RU_k,$$

where $S_j (1 \leq j \leq \varepsilon_1)$ and $U_k (1 \leq k \leq \varepsilon_2)$ are variables of degree 2 and 3 which kill cycles s_j and u_k respectively. We remark that, since $0 : \mathfrak{m} \approx H_2(E) \approx H_2(X^{(2)})$, we can take $c_i T_1 T_2$ $(i = 1, \cdots, \varepsilon_1)$ as u_i, where $c_1, \cdots, c_{\varepsilon_1}$ is a minimal generating system of the ideal $0 : \mathfrak{m}$.

Let $M = dX_3$ and write $M = M_1 + M_2$, where M_1 (resp. M_2) is an R-module generated by $t_1 S_j - T_1 s_j$ and $t_2 S_j - T_2 s_j$ $(1 \leq j \leq \varepsilon_1)$ (resp. $c_k T_1 T_2$ $(1 \leq k \leq \varepsilon_2)$). We contend first that M_1 is isomorphic to the direct sum of ε_1-copies of $\mathfrak{m}$. To see this, it is enough to prove that the projection $\varphi: M_1 \rightarrow \sum (Rt_1 + Rt_2) S_j = \sum \mathfrak{m} S_j$ defined by

$$\varphi\left(\sum \lambda_j (t_1 S_j - T_1 s_j) + \sum \mu_j (t_2 S_j - T_2 s_j)\right) = \sum (\lambda_j t_1 + \mu_j t_2) S_j$$

is an isomorphism. Assume $\alpha = \sum \lambda_j (t_1 S_j - T_1 s_j) + \sum \mu_j (t_2 S_j - T_2 s_j) \in \operatorname{Ker} \varphi$. Then, we have $\lambda_j t_1 + \mu_j t_2 = 0$ and hence $\lambda_j T_1 + \mu_j T_2 \in Z_1(E)$ for $j = 1, \cdots, \varepsilon_1$. Therefore

$$\alpha = -\left(\sum \lambda_j s_j T_1 + \sum \mu_j s_j T_2\right) = -\sum (\lambda_j T_1 + \mu_j T_2) s_j = 0$$

in view of $H_1(E)^2 = 0$. Hence φ is injective, and whence bijective because clearly it is surjective. By a similar argument, M_2 is isomorphic to the direct sum of ε_2-copies of K. In this case, we consider the free module $RZ_1 + \cdots + RZ_{\varepsilon_2}$ and consider the map $\psi: RZ_1 + \cdots + RZ_{\varepsilon_2} \rightarrow 0 : \mathfrak{m}$ defined by

$$\psi\left(\sum \nu_i Z_i\right) = \sum \nu_i c_i.$$

Then, ψ induces, obviously, an isomorphism between $\bigoplus^{\varepsilon_2} K$ and $0 : \mathfrak{m}$, since $c_1, \cdots, c_{\varepsilon_2}$ is a minimal generating system of $0 : \mathfrak{m}$. Finally, we mention that

M is actually the direct sum of M_1 and M_2 in view of $H_1(E)^2=0$. Summarizing, we obtain

$$M\approx(\overset{\varepsilon_1}{\oplus}\mathfrak{m})\oplus(\overset{\varepsilon_2}{\oplus}K).$$

Now, since the torsion functer has an additive property, we have

$$\mathrm{Tor}_p^R(M, K)=(\overset{\varepsilon_1}{\oplus}\ \mathrm{Tor}_p^R(\mathfrak{m}, K))\oplus(\overset{\varepsilon_2}{\oplus}\ \mathrm{Tor}_p^R(K, K))$$

for $p\geq 0$. But, clearly $\mathrm{Tor}_p^R(M, K)=\mathrm{Tor}_{p+3}^R(K, K)$ and $\mathrm{Tor}_p^R(\mathfrak{m}, K)=\mathrm{Tor}_{p+1}^R(K, K)$. Hence, we obtain the following recurrence relation of Betti numbers:

$$B_{p+3}=\varepsilon_1 B_{p+1}+\varepsilon_2 B_p\ (p\geq 0).$$

Combining this with the fact $B_0=1$, $B_1=2$ and $B_2=1+\varepsilon_1$, we obtain

$$\mathscr{B}(R)=\frac{(1+Z)^2}{1-\varepsilon_1 Z^2-\varepsilon_2 Z^3}.$$

The fact $\varepsilon_3=\binom{\varepsilon_1}{2}$ follows from theorem 2 by a direct computation.

Next we consider the case when $H_1(E)^2\neq 0$. In this case, $0\leq\varepsilon_2\leq\varepsilon_1-2$, as we already mentioned, and hence we shall have $\varepsilon_2=0$ if we show $\varepsilon_1\leq 2$.

Observe that everything is unchanged when we pass to the completion of R. Therefore, we can assume R is complete. By the structure theorem of Cohen, there exists a minimal embedding of R, that is, there exists a regular local ring $\tilde{R}$ of dimension 2 and an ideal $\tilde{\mathfrak{a}}$ of $\tilde{R}$ such that $R=\tilde{R}/\tilde{\mathfrak{a}}$, $\tilde{\mathfrak{a}}\subset\tilde{\mathfrak{m}}^2$, where $\tilde{\mathfrak{m}}$ is the maximal ideal of $\tilde{R}$. Denote by h: $\tilde{R}\rightarrow R$ the canonical map and let $\tilde{t}_i$ be an element of $\tilde{R}$ such that $h(\tilde{t}_i)=t_i$. Then, obviously $\tilde{\mathfrak{m}}=(\tilde{t}_1, \tilde{t}_2)\tilde{R}$. Let $\tilde{a}_1,\cdots, \tilde{a}_r$ be a minimal system of generators of $\tilde{\mathfrak{a}}$ and write $\tilde{a}_i=\tilde{\lambda}_i\tilde{t}_1+\tilde{\mu}_i\tilde{t}_2$. Then, $s_i=\lambda_i T_1+\mu_i T_2$ $(1\leq i\leq r)$ constitutes a minimal generating system of $Z_1(E)$ modulo $B_1(E)$ [1, p. 196] where $\lambda_i=h(\tilde{\lambda}_i)$ and $\mu_i=h(\tilde{\mu}_i)$. Since $H_1(E)^2\neq 0$, there exist at least two elements, say $\tilde{a}_1$ and $\tilde{a}_2$, in $\tilde{a}_1,\cdots, \tilde{a}_r$ such that $(\lambda_1\mu_2-\lambda_2\mu_1)\,T_1T_2\neq 0$.

Let $\mathfrak{a}_1=(\tilde{a}_1, \tilde{a}_2)\tilde{R}$, $\bar{R}=\tilde{R}/\mathfrak{a}_1$ and $\bar{\mathfrak{m}}=\tilde{\mathfrak{m}}/\mathfrak{a}_1$. Since $\tilde{a}_1$, $\tilde{a}_2$ is a minimal system of generators of $\mathfrak{a}_1$, we have $\dim_K H_1(\bar{E})=2$, where $\bar{E}$ is the Koszul complex of $\bar{R}$. Hence, by the remark at the first paragraph of the proof, we have $\dim_K 0:\bar{\mathfrak{m}}=1$, since $\bar{R}$ is not regular. Therefore, it follows that $0:\bar{\mathfrak{m}}=(\bar{\lambda}_1\bar{\mu}_2-\bar{\lambda}_2\bar{\mu}_1)\bar{R}$, where $\bar{\lambda}_i$ and $\bar{\mu}_i$ are the residue classes of $\tilde{\lambda}_i$ and $\tilde{\mu}_i$ in $\bar{R}$ respectively. Thus $H_2(\bar{E})=H_1(\bar{E})^2$ and $\bar{R}$ is a complete intersection and of dimension 0 [1, theorem 2.7]. Hence the zero ideal of $\bar{R}$ is irreducible [8, IV theorem 34].

Suppose $\varepsilon_1>2$, then $\tilde{\mathfrak{a}}\supsetneqq\mathfrak{a}_1$. Hence $0:\bar{\mathfrak{m}}\subset\tilde{\mathfrak{a}}/\mathfrak{a}_1$ [8, IV theorem 34] and therefore $\lambda_1\mu_2-\lambda_2\mu_1=0$. But, this contradicts the choice of $\tilde{a}_1$ and $\tilde{a}_2$.

As for the Betti series of R, it is enough to mention that $\tilde{\mathfrak{a}}=\mathfrak{a}_1$ is generated by an $\tilde{R}$-sequence and hence coroll. of theorem 3 can be applied to R.

COROLLARY. *Let* $(R, \mathfrak{m})$ *be a local ring of embedding dimension* 2, *and assume that R is not regular. If* $\mathfrak{m}$ *contains at least one non zero divisor, then we have* $\mathscr{B}(R)=\dfrac{(1+Z)^2}{1-Z^2}$.

PROOF. Our hypothesis implies that $H_2(E)\approx 0:\mathfrak{m}=0$. Therefore $H_1(E)^2=0$. Hence, we have the corollary in view of theorem 5.

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