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THE STUDENT'S DISTRIBUTION FOR A UNIVERSE BOUNDED AT ONE OR BOTH SIDES (Continued)

By

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(Received September 30, 1964)

In the present note the author attempts to assume the existence of central limit theorem about the sampling mean taken from a universe with non-negative argument first without proof, but rather whence conversely to discover some provisional approximation of the correction factor $\mathfrak{h}_n$ for the exact sampling distribution he had previously in the foregoing note reported most generally on the net integral form, which however being too much complicate, no connection between them is yet made for the present.

21. *The Sampling $x\tau$-Joint Distribution taken from a T.N.D. as Universe.* Let the parent T.N.D. be[1)]

(2. 1) $$f(x)=\frac{1}{\sqrt{2\pi}\Phi(a)}\exp\left(-\frac{1}{2}(x-a)^2\right),\quad x>0$$

with the parent mean $m=a+\lambda$, where λ denotes $\varphi(a)/\Phi(a)$, the logarithmic derivative of untruncated N.D. $\Phi(a)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a}e^{-t^2/2}dt$. Now from (1) a n-sized sample with mean $\bar{x}$ and S.D. s being drawn, the $\bar{x}s$-joint sampling f.f. is given by

(21.2) $$f_n(\bar{x},s)\,dV_n=c_n\exp\left[-\frac{n}{2}(\bar{x}-a)^2-\frac{n}{2}s^2\right]s^{n-2}\mathfrak{h}_n\left(\frac{s}{\bar{x}}\right)ds\,d\bar{x},\quad\text{where}$$

(21.3) $$c_n=\frac{2\sqrt{\pi}^{\,n-1}\sqrt{n}^{\,n}}{\sqrt{2\pi}^{\,n}\Phi^n(a)\,\Gamma((n-1)/2)}\simeq\frac{ne^{n/2}}{\pi\sqrt{2}\,\Phi^n(a)},\quad\text{as } n \text{ is large.}$$

Hence, on writing $\tau=s/\bar{x}$ or $z=\bar{x}/s$, the total probability yields

(21.4) $$1=c_n\iint_G\exp\left[-\frac{n}{2}(\bar{x}-a)^2-\frac{n}{2}\bar{x}^2\tau^2\right]\bar{x}^{n-1}\tau^{n-2}\mathfrak{h}_n(\tau)\,d\tau\,d\bar{x},$$

where G denotes the whole domain of integration: $0\leq\bar{x}\leq\infty$, $0\leq\tau\leq b=\sqrt{n-1}$. First we begin by recognizing solely the well-defined property that $\mathfrak{h}_n(\tau)$ is non-negative continuous and monotonically decreases from 1 to 0. To obtain an asymptotic value of the integral, we compute after Laplace method[2)]. Rewriting (4) conveniently

(21.5) $$1=c_n\iint_G\mathfrak{f}^{n-2}(\bar{x},\tau)\,\mathfrak{g}(\bar{x},\tau)\,d\tau\,d\bar{x}\equiv c_nJ_n,$$

1) H. Cramér, Mathematical Methods of Statistics, p. 248.

2) Cf. Polya und Szegö : Aufgaben und Lehrsätze, Bd. I, S. 78 and S. 244. Also compare Y. Ichijô : Ueber die Laplacesche asymptotische Formel für das Integral von Potenze mit grossem Indexe, this Journal vol. VI (1955), p. 63, and Y. Watanabe u. Y. Ichijô : Zur Laplaceschen asymptotishen Formel, ibid. vol. IX (1958), p. 1, which are cited below as [I]*, [II]*, and besides as [I] : Y. Watanabe : Some exceptional example to Student's distribution, ibid. vol. X (1959), p. 11, and [II]-[V], Y. Watanabe, the same topics with the present, ibid. vol. XI-XIV (1960-63).

where $\mathfrak{f}=\bar{x}\tau E,\quad \mathfrak{g}=\bar{x}E^2\mathfrak{h}_n(\tau),\quad E=\exp\left(-\frac{1}{2}(\bar{x}-a)^2-\frac{1}{2}x^2\tau^2\right),$

and we call $\mathfrak{f}$, the base of large power, the main part and $\mathfrak{g}$ the subsidiary factor. Or putting $F=\log\mathfrak{f}=\log\bar{x}\tau-\frac{1}{2}(\bar{x}-a)^2-\frac{1}{2}\bar{x}^2\tau^2$, we have to evaluate

$$J_n=\iint_G \exp(n-2)F(x,\tau)\cdot\mathfrak{g}(\bar{x},\tau)\,d\tau dx.$$

The maximum of F or $\mathfrak{f}$ is found from

(21.6) $F_x=1/\bar{x}-(\bar{x}-a)-\bar{x}\tau^2=0,$

(21.7) $F_\tau=1/\tau-\bar{x}^2\tau=0,$

which expressions are both continuous inside G. The former, the positive root being taken, yields

(21.8) $\bar{x}=\dfrac{a+\sqrt{a^2+4(1+\tau^2)}}{2(1+\tau^2)}$

or $\tau^2=(1+ax-x^2)/x^2$

we call Agnesi or Ag by its resemblance to the Witch of Agnesi, which lies rightside its asymptote $\bar{x}=0$ (Fig. 1), while the latter denotes simply an ordinary hyperbola H

(21.9) $\bar{x}\tau=1$ in the first quadrant.

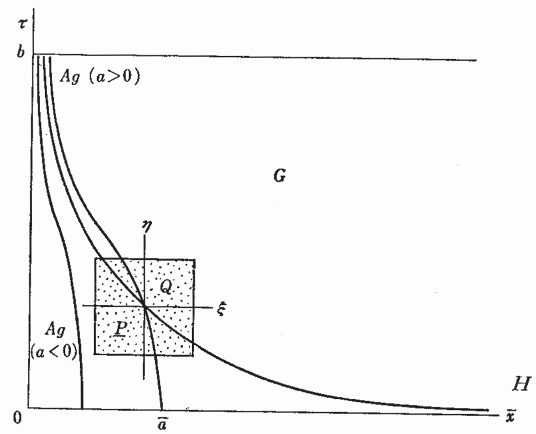

Fig. 1

The upper boundary line $\tau=b=\sqrt{n-1}$ cuts the curves at $x_H=1/\sqrt{n-1}\simeq\frac{1}{\sqrt{n}}\left(1+\frac{1}{2n}\right)$ and $x_A\simeq\frac{1}{\sqrt{n}}\left(1+\frac{a}{2\sqrt{n}}\right)$, so that if $a>0$, $x_H<x_A$ follows. On the otherhand for the lower boundary $\tau=0$, $x_A=\frac{1}{2}(a+\sqrt{a^2+4})=\bar{a}$ remains finite against $x_H\to\infty$. Hence, if $a>0$, the 2 curves intersect at a point $P(a,\ 1/a)$, at which hold relations (6) (7) and besides $F_{xx}=-1-\tau^2-1/\bar{x}^2$, $F_{x\tau}=-2\bar{x}\tau$, $F_{\tau\tau}=-\bar{x}^2-1/\tau^2$, with determinant $(F_{xx}F_{\tau\tau}-F_{x\tau}^2)_P=2a^2>0$. Therefore F and $\mathfrak{f}$ become maximum at P. Describe a quadrate Q with center P and side 2δ, so small that it lies wholly inside G. Take the new $\xi\eta$-coordinates so as $\bar{x}=a+\xi=a(1+u/N)$, $\tau=1/a+\eta=\left(1+\frac{v}{N}\right)/a$ with $N=\sqrt{n-2}$ which tends ∞ as n. Now conceive the integral (cf. [I]* loc. cit.).

(21.10) $$\iint_G \exp(n-2)[F(\bar{x},\tau)-F(P)]\cdot\mathfrak{g}\,d\tau d\bar{x}=\iint_Q+\iint_{G-Q=R}=\text{(i)}+\text{(ii)},$$

where $F(P)=F(a,1/a)=-\frac{1}{2}$ being the max. F in G, it holds that $\exp(F \text{ in } R-F(P))=(\mathfrak{f}\text{ in } R)/\mathfrak{f}(P)=\rho<1$. But $\mathfrak{g}$ being integrable in G

$$\text{(ii)}<\rho^{n-2}\iint_G \mathfrak{g}\,d\tau d\bar{x}=0(1/n^\omega),\ \text{however great } \omega \text{ may be.}$$

Accordingly (ii) becomes negligibly small and we have only to compute (i). Expanding the integrand of (10) in powers of N and neglecting those terms with negative power, we get the approximate value of (i), when n is sufficiently large :

$$\iint_{-N\delta\sim-\infty}^{N\delta\simeq\infty} \exp\left[-\frac{1}{2}a^2u^2-(u+v)^2\right]\cdot\frac{a}{e}\mathfrak{h}_n\left(\frac{1}{a}\left(1+\frac{v}{N}\right)\right)\frac{dudv}{N^2}\simeq\frac{\sqrt{2}\,\pi}{ne}\mathfrak{h}_n\left(\frac{1}{a}\right),$$

which multiplied by (3) and the factor divided out in advance, i.e. $\exp\left(-\frac{1}{2}(n-2)\right)$, we see that the required integral (5) becomes

$$1\simeq\frac{ne^{n/2}}{\pi\sqrt{2}\,\Phi^n(a)}\cdot\frac{\sqrt{2}\,\pi}{ne}\mathfrak{h}_n\left(\frac{1}{a}\right)\exp\left(1-\frac{n}{2}\right)\simeq\mathfrak{h}_n\left(\frac{1}{a}\right)\Big/\Phi^n(a). \tag{21.11}$$

But, a being any positive quantity, on writing $1/a=\tau$, we get the first approximation

$$\mathfrak{h}_n(\tau)\simeq\Phi^n(1/\tau). \tag{21.12}$$

Really, when $\tau\to0$, $1/\tau\to\infty$ and $\Phi^n(1/\tau)$ tends 1, while, if $\tau\to\sqrt{n-1}\simeq\infty$, $1/\tau\to0$ and $\Phi^n(0)\simeq1/2^n$ becomes sufficiently near 0, as $n\to\infty$. Thus, the asymptotic approximation (12) endures the well-defined properties of $\mathfrak{h}_n(\tau)$, although the order of zero at $\tau=\sqrt{n-1}$ compared with (21) below, cannot be said enough satisfactory. At any rate, if (12) be granted, the identity $E(\bar{x}^0)\simeq1$ would follow approximately. Similarly by multiplying $\bar{x}=a(1+u/N)$ to the intergrand of J_n, we obtain also the identity $E(\bar{x})=a$, which however conflicts with $E(\bar{x})=m$, what the C.L.T. designates.

In fact, when (12) hold, we had to compute J_n more exactly by transferring the factor $\mathfrak{h}_n(\tau)=\Phi^n(1/\tau)$ under the main part. Putting $\tau=1/z$ for convenience, we have to replace $\mathfrak{h}_n(1/z)$ by the asymptotic approximation

$$\mathfrak{h}_n(1/z)=\Phi^n(z)q^n(z)r(z), \tag{21.13}$$

where the factor q and r are inserted in order to make the final result$=1$, and besides to lighten calculations, it is postulated to be q' nearly 0, i.e. q almost constant. Under these trial assumptions we have to recompute the integral

$$c_nJ_n=c_n\int_0^\infty\int_{1/b}^\infty(\bar{x}\Phi qE/z)^n\,r(z)/\bar{x}dzd\bar{x}=c_n\iint_G\exp(n-1)\log\mathfrak{f}\cdot\mathfrak{g}\,dzd\bar{x}, \tag{21.14}$$

where $\quad\mathfrak{f}=\bar{x}\Phi qE/z,\quad\mathfrak{g}=\Phi qEr/z,\quad E=\exp\left(-\frac{1}{2}(\bar{x}-a)^2-\frac{1}{2}\bar{x}^2/z^2\right).$

Again putting

$$F=\log\mathfrak{f}=\log\left(\frac{\bar{x}}{z}\Phi q\right)-\frac{1}{2}(\bar{x}-a)^2-\frac{1}{2}\frac{\bar{x}^2}{z^2},$$

we obtain

$$F_x=1/\bar{x}-(\bar{x}-a)-\bar{x}/z^2=0, \tag{21.15}$$

$$F_z=-1/z+\lambda(z)+x^2/z^3=0. \tag{21.16}$$

The former is the same Ag as (8):

$$\bar{x}=\frac{a+\sqrt{a^2+4(1+1/z^2)}}{2(1+1/z^2)} \tag{21.17}$$

or $\quad\bar{x}^2/z^2=1-\bar{x}(\bar{x}-a)$.

But the latter now becomes (Fig. 2)

$$\bar{x}^2/z^2=1-\mu(z),\ \mu(z)=z\lambda(z), \tag{21.18}$$

which being a deformed hyperbola, may be called a pseud-hyperbola PH and its positive branch $\bar{x}=z\sqrt{1-\mu(z)}$ is only con-

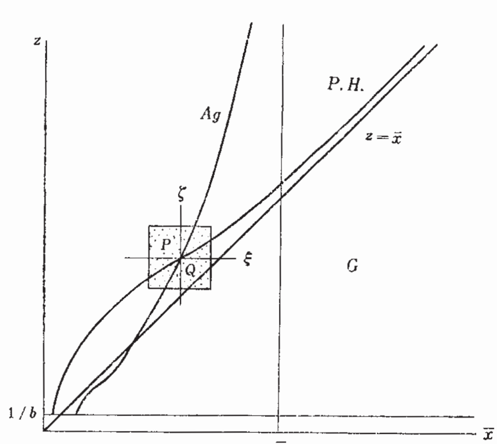

Fig. 2

sidered. This time the lower boundary $z=1/b$ cuts the 2 curves at $x_A\simeq\frac{1}{\sqrt{n}}\left(1+\frac{a}{2\sqrt{n}}\right)$, $x_P\simeq\frac{1}{\sqrt{n}}\left(1-\frac{1}{\sqrt{2\pi n}}\right)$, respectively, so that if $a>-\sqrt{2/\pi}=-.7979$, $x_A>x_P$. Now that Ag lies leftside its vertical asymptote $\bar{x}=\bar{a}=\frac{1}{2}(a+\sqrt{a^2+4})$ and $x_A<\bar{a}$, while PH has its oblique asymptote $z=\bar{x}$ along whose leftside the curves spreads up to infinity and accordingly extends rightside $\bar{x}=\bar{a}$ ultimately. Hence if $a>-\sqrt{2/\pi}$, e.g. when $a=0$ or ± 0.6745, the 2 curves surely intersect at a point $P(x_0, z_0)$, and at which (15) (16) as well as (20) below hold, so that F becomes maximum. Hence, again describing a small quadrate Q with center P and side 2δ, and putting $\bar{x}=x_0+\xi=x_0(1+u/N)$, $z=z_0+\xi=z_0(1+v/N)$, $N=\sqrt{n-1}$, we may compute J_n by integrating only over Q, similarly as in (10):

(21.19) $$c_nJ_n=\frac{z_0r(z_0)}{\sqrt{2\pi}}\left[\frac{x_0\Phi(z_0)q}{z_0\Phi(a)}e^{1/2}E(x_0, z_0)\right]^n\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}(Au^2+2Buv+Cv^2)\right]dudv,$$

where $A=1+x_0^2+x_0^2/z_0^2>0$, $B=-2x_0^2/z_0^2$, $C=-1+\mu_0^2+\mu_0z_0^2+3x_0^2/z_0^2$, $\mu_0=z_0\lambda(z_0)>0$ with the determinant

(21.20) $$D=AC-B^2=2(1-\mu_0)x_0^2+(1-\mu_0+x_0^2)(\mu_0^2+\mu_0z_0^2)+\mu_0(z_0^2-x_0^2)>0,$$

because of $0<x_0^2/z_0^2=1-\mu_0<1$ after (18), $0<\mu_0<1$, $x_0^2<z_0^2$. Consequently the above double integral reduces to $2\pi/\sqrt{D}$. Besides, in view of $\bar{x}_0^2/z_0^2=1-\mu_0=1-x_0(x_0-a)$, after (17), not only x_0 but a also being expressible by z_0, we get after all

$$c_nJ_n\simeq z_0\sqrt{\frac{2}{D}}r(z_0)\cdot(Qq)^n,\text{ where } Q=\frac{\sqrt{2\pi}}{z_0}\Phi(z_0)x_0\exp\left(\frac{1}{2}ax_0\right)\cdot\lambda(a),$$

$$\text{with}\quad x_0=z_0\sqrt{1-\mu_0},\ a=z_0\sqrt{1-\mu_0}-\mu_0/z_0\sqrt{1-\mu_0},$$

So that $c_nJ_n\simeq 1$, if $q=Q^{-1}$, $r=\sqrt{D/2}/z_0$. Thus we get $E(\bar{x}_0)\simeq 1$ and similarly $E(\bar{x})\simeq x_0$.

The actual maximal point $P(x_0, z_0)$ shall be found by Newton's method of successive approximations. E.g. in case 1° $a=0$ we get equations $x=z/\sqrt{1+z^2}=z\sqrt{1-z\lambda(z)}$ and whence $z_0=.6225$, $x_0=.5619$. In case $a=\pm.6745$, we have $\lambda(z)=(z^3+.7725z\mp.6745\sqrt{1+1.1137z^2})/(1+z^2)^2$, which gives 2° $z_0=1.1275$, $x_0=.8474$; 3° $z=.095$, $x_0=.0967$. However, the C.L.T. insists 1° $m=.7979$, 2° $m=1.2098$ and 3° $m=.5966$. Thus here obtained new values x_0 are all still too small than the theoretical true value m although they are far better reformed compared with those obtained before, e.g. in 2° $x_0=.8474>a=.6745$, &c. All these discrepancies would be revised by adjusting the asymptotic estimation for $\mathfrak{h}_n$ more suitably, e.g. raising the power excelsior, so that the pseud-hyperbola spreads farther leftward $z=x$ to cut Agnesi at a higher position and make x_0 increase.

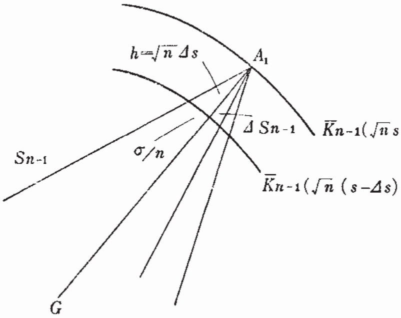

Fig. 3

Really the author has obtained an exact asymptotic formula for $\mathfrak{h}_n$, as $\tau=s/\bar{x}$ is $\simeq b=\sqrt{n-1}$. Let A_1 and G be a vertex and centroid of the simplex S_{n-1} (Fig. 3). If the s-sphere $\overline{K}_{n-1}$ passes through A_1, i.e. its radius becomes $GA_1=\sqrt{n(n-1)}\bar{x}$, then $\tau=b$ and the common area $\sigma=S_{-1}\cap\overline{K}_{n-1}$ reduces to naught. However when the radius is only a little smaller, say, $\sqrt{n}\,(s-\Delta s)=\sqrt{n}\,(b-\Delta\tau)\bar{x}$, the sphere cuts out from S_{n-1} an infinitesimal simplex ΔS_{n-1} of height $h=\sqrt{n}\,\Delta s=\sqrt{n}\,\bar{x}\Delta\tau$ with base-area σ/n by simmetry. Since the volume of $\Delta S_{n-1}=\dfrac{\sigma}{n}\dfrac{h}{n-1}=\left(h\sqrt{\dfrac{n-1}{n}}\right)^{n-1}\dfrac{\sqrt{n}}{\Gamma(n)}$ (cf. (1.3) and (1.7) in [I] loc. cit.), we get $\sigma=\sqrt{n-1}^{\,n+1}h^{n-2}/\sqrt{n}^{\,n-4}\Gamma(n)$. On the otherhand the general surface is $\sigma=F_{n-2}(s-\Delta s)\,\mathfrak{h}_n(\tau)=\dfrac{2\sqrt{\pi}^{\,n-1}\sqrt{n}^{\,n}(s-\Delta s)^{n-2}}{\Gamma((n-1)/2)}\mathfrak{h}_n(\tau)$. On eliminating σ, h among them, we obtain

$$\mathfrak{h}_n(\tau)\simeq\left(\frac{\bar{x}\Delta\tau}{s-\Delta s}\right)^{n-2}\sqrt{\frac{n-1}{n}}^{\,n}\frac{\sqrt{n}\,(n-1)}{2\sqrt{\pi}^{\,n-1}}\frac{\Gamma((n-1)/2)}{\Gamma(n)},$$

in which the multiplication theorem of gamma function $\Gamma(n)=\dfrac{2^{n-1}}{\sqrt{\pi}}\Gamma\left(\dfrac{n}{2}\right)\Gamma\left(\dfrac{n+1}{2}\right)$ and asymptotic relations $\sqrt{(n-1)/n}^{\,n}\simeq e^{-1/2}$, $\Gamma(n)\simeq\sqrt{2\pi}n^{n-1/2}e^{-n}$ being applied, we find that

(21.21) $$\mathfrak{h}_n(z\simeq 1/b)\simeq\frac{1}{\sqrt{2}}\left(\sqrt{\frac{e}{2\pi}}\right)^{n-1}\left(\frac{\Delta\tau}{\sqrt{n}\,\tau}\right)^{n-2},$$

where $b=\sqrt{n-1}$ is enough large, and $\Delta\tau=b-\tau$ may also be pretty large yet sufficiently small compared with b, say $<b^{\varepsilon}\,(0<\varepsilon<1)$, so that

$$\frac{\mathfrak{h}_n(\tau\simeq b)}{\Phi^n(0)}<\frac{2^n}{\sqrt{2}}\sqrt{\frac{e}{2\pi}}^{\,n-1}\left(\frac{\Delta\tau}{b\tau}\right)^{n-2}<\frac{4}{\sqrt{2}}\sqrt{\frac{e}{2\pi}}^{\,n-1}\left(\frac{2}{b(b^{1-\varepsilon}-1)}\right)^{n-2},$$

and thus the correction factor $\mathfrak{h}_n(\tau)$ is far smaller than $\Phi^n(1/\tau)\simeq 1/2^n$. Or, if τ and $\Delta\tau$ be replaced by $1/z$ and $b\Delta z/z$, $\Delta\tau=z-1/b$, we get

(22.22) $$\mathfrak{h}_n(z\simeq 1/b)\simeq\frac{1}{\sqrt{2e}}\sqrt{\frac{e}{2\pi}}^{\,n-1}(\Delta z)^{n-2},\quad \frac{1}{z^n}\mathfrak{h}_n\left(\frac{1}{z}\right)=\frac{1}{\sqrt{2e}}\sqrt{\frac{e}{2\pi}}^{\,n-1}\frac{(\Delta\tau)^{n-2}}{z^n},$$

in which as small enough z is, yet $\Delta z=z\Delta\tau/b$ becomes furthermore small, and indeed

(21.23) $\dfrac{1}{z^n}\mathfrak{h}_n\left(\dfrac{1}{z}\right)$ is integrably small, as z tends $1/b=1/\sqrt{n-1}\sim 0$.

We require to reconstruct the asymptotic formula for $\mathfrak{h}_n$ more suitablly. However, to perform it, we ought to treat prelusively

22. *Some Corollaries concerning $\lambda(a)$, $m(a)$ and Their Allied Functions,* as these seem to be somewhat important even apart the pressing needs. If the T.N.D.

(22.1) $$f(x)=\frac{1}{\sqrt{2\pi}\Phi(a)}\exp\left(-\frac{1}{2}(x-a)^2\right)\text{ for }x\geq 0$$

be taken as universe, its mean (parent mean) is

(22.2) $$m=\frac{1}{\sqrt{2\pi}\Phi(a)}\int_0^{\infty}x\,\exp\left(-\frac{1}{2}(x-a)^2\right)dx=a+\lambda,\text{ where}$$

(22.3) $$\lambda=\varphi(a)/\Phi(a),\quad \varphi(a)=e^{-a^2/2}/\sqrt{2\pi}$$

is the logarithmic derivative of $\Phi(a)=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{a}e^{-t^2/2}dt$. Besides, the parent variance is

(22.4) $$\sigma^2=\frac{1}{\sqrt{2\pi}\Phi(a)}\int_0^{\infty}x^2\exp\left(-\frac{1}{2}(x-a)^2\right)dx-m^2=1-\lambda m>0.$$

Thus, the T.N.D. having its mean and variance, the concerned C.L.T. holds in all probability. Also it is clear that both of $m=a+\lambda$ and λ are essentially positive for all finite $a>=<0$; even for $a<0$ holds $a+\lambda>0$, so that $\lambda(a)>-a>0$. Furthermore, as

(22.5) $\lim_{a\to\pm\infty} a^{\omega}\varphi(a)=0$ holds for however great ω, so also

(22.6) $\lim_{a\to+\infty} a^{\omega}\lambda^{\alpha}(a)=0$ for $\omega\geq 0$, $\alpha>0$. Besides we have $\lim_{a\to+\infty}\lambda m^{\alpha}=\lim\lambda(a+\lambda)^{\alpha}=$

$$=\lim\lambda a^{\alpha}\left(1+\frac{\lambda}{a}\right)^{\alpha}=\lim\sum_{\nu}\binom{\alpha}{\nu}a^{\alpha-\nu}\lambda^{\nu+1}=0. \quad \text{Notwithstanding}$$

(22.7) $\lim_{a\to-\infty}\lambda(a)=\lim\varphi(a)/\phi(a)=\lim -a\varphi/\varphi=\lim(-a)=+\infty$ by l'Hospital.

In fact, although both $y=\varphi$, $y=\Phi$ tend 0 as $a\to-\infty$, the latter tends 0 far rapidly than the former (Fig. 4), and lies below the former already beyond the point (−0.3026,

a	−1	−2	−3
φ	.24197	.00443	.00014
Φ	.15866	.00135	.00003

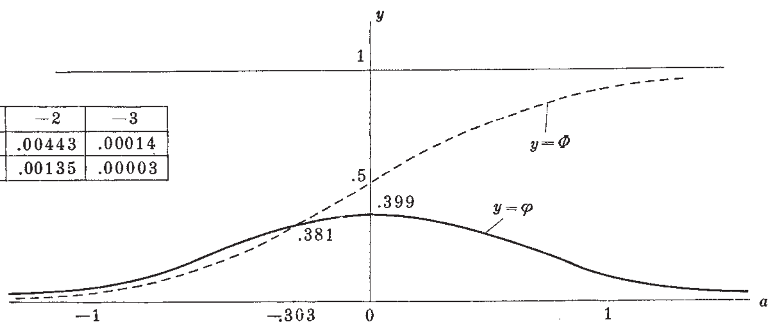

Fig. 4

0.3810). However for any prescribed small $\alpha>0$, we have again by l'hospital $\lim_{a\to-\infty}\varphi^{1+\alpha}/\Phi=\lim-(1+\alpha)a\varphi^{\alpha}=(1+\alpha)\lim(-a\varphi^{\alpha})=(1+\alpha)\lim(-a)e^{-\alpha^2/2}/\sqrt{2\pi}^{\alpha}=+0^{\omega}$, i.e.

(22.8) $\varphi^{1+\alpha}<\Phi<\varphi$, as $a\to-\infty$ for whatsoever small $\alpha>0$, a delicate relation between φ and Φ. After (8) and (5) $a^{\omega}\Phi$ tends 0 as $a\to-\infty$ also. However, if we consider

(22.9) $m=\lambda+a=(\varphi+a\Phi)/\Phi\equiv\Psi/\Phi$, say, the ratio $a\Psi/\Phi=am$ tends

(22.10) $$\lim_{a\to-\infty}\frac{a\Psi}{\Phi}=\lim\frac{a\Phi+\Psi}{\varphi}=\lim\frac{2\Phi+a\varphi}{-a\varphi}=-1-\lim\frac{2\Phi}{a\varphi}=-1-\lim\frac{2\varphi}{\varphi-a^2\varphi}$$
$$=-1+\lim\frac{1}{a^2-1}=-1+0^2,$$

so that $\lim_{a\to-\infty}-a\Psi/\Phi=1-0^2$, namely $\Phi\simeq-a\Psi$, as $a\to-\infty$. Reterning to λ, we have

(22.11) $$\lim_{a\to-\infty}\frac{\lambda}{a}=\lim\frac{\varphi}{a\Phi}=\lim\frac{-a\varphi}{\Phi+a\varphi}=\lim\frac{a^2-1}{2-a^2}=-1-\lim\frac{1}{a^2-2}=-1-0^2.$$

A little more generally: $\lim_{a\to-\infty}\lambda/|-a|^{\alpha}=+\infty$, 1, 0 according as $(0<)\alpha<=>1$. Also

(22.12) $\lim_{a\to-\infty}m=\lim(\lambda+a)=\lim\Psi/\Phi=\lim\Phi/\varphi=\lim 1/(-a)=+0^1$.

Or else, as with large $|a|$, $a<0$, it holds asymptotically, because of $\varphi'(a)=-a\varphi(a)$,

$$\Phi(a)=\int_{-\infty}^{a}\varphi(a)\,da=\int_{-\infty}^{a}\frac{\varphi'(a)}{-a}da=\frac{\varphi(a)}{-a}-\int_{-\infty}^{a}\frac{\varphi(a)}{a^2}da\simeq\frac{\varphi(a)}{-a}\times\left[1-\frac{1}{a^2}+\cdots\right],$$

so also follows

$$\lambda(a<0)=\frac{\varphi(a)}{\Phi(a)}\simeq-a\left(1-\frac{1}{a^2}+\cdots\right)^{-1}=-a-\frac{1}{a}+\cdots, \text{ and } \lim_{a\to-\infty}(\lambda+a)\simeq-\frac{1}{a}+\cdots\simeq+0 \text{ again.}$$

In view of (7) and (12) $y=\lambda(a)$ has its asymptote $y=-a$ besides $y=0$ Quite similary.

(22.13) $\lim_{a\to+\infty} m = \lim a = +\infty,\quad \lim_{a\to+\infty} \frac{m}{a} = 1 + \lim \frac{\lambda}{a} = 1+0,\quad \lim_{a\to+\infty} (m-a) = \lim \lambda = 0^{\omega}$

(22.14) $\lim_{a\to-\infty} m = \lim(\lambda+a) = \lim -1/a = 0,$

$$\lim_{a\to-\infty} am = \lim a(a+\lambda) \simeq \lim a\left(\frac{-1}{a}\right) \simeq -1 \text{ after (12).}$$

Thus $y=a$ and $y=0$ are also the asymptotes of $y=m$. Besides $y=m$, $y=\lambda$ meet at $(0, \sqrt{2/\pi}=.7979)$. Nevertheless these two curves are not symmetrical with respect to y-axis, as shown in Fig. 5:

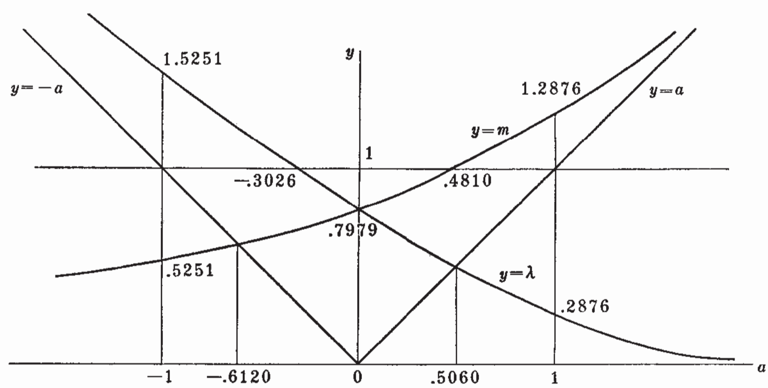

Fig. 5

The monotonic decreasing property of $\lambda(a)$ is clarified by the fact that

(22.15) $\lambda' = \frac{d\lambda}{da} = -\lambda(\lambda+a) = -\lambda m < 0.$ And whence follows

(22.16) $\lambda'' = \frac{d^2\lambda}{da^2} = -\frac{d\lambda m}{da} = (2\lambda+a)(\lambda+a)\lambda - \lambda = 2\lambda(\lambda-\alpha)(\lambda-\beta) > 0,$

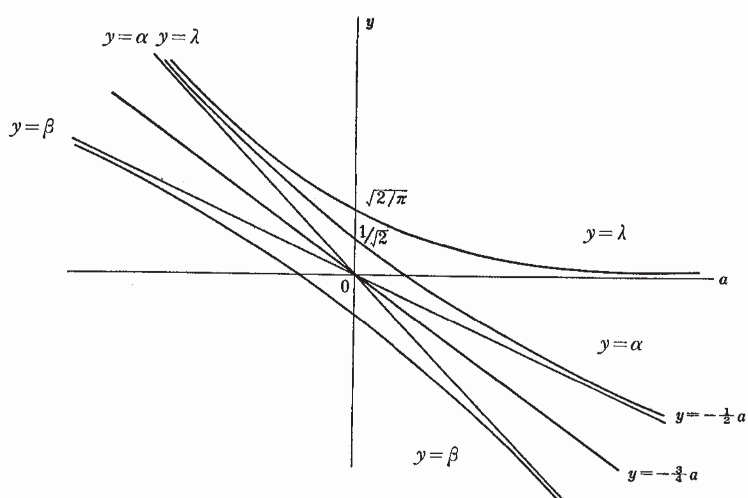

Fig. 5a

where $\alpha, \beta = \frac{1}{4}(-3a \pm \sqrt{a^2+8})$. For, the quadratic $(y-\alpha)(y-\beta) = 2y^2+3ay+a^2-1=0$ denotes a hyperbola, whose 2 branches $y=\alpha$ and $y=\beta$ extend above and below its diameter $y=-\frac{3}{4}a$ symmetrically, with 2 asymptotes $y=-a$ and $y=-a/2$ (Fig. 5*a*). The λ-curve being wholly above the hyperbola, $\lambda > \alpha$ as well as $\lambda > \beta$, so that $\lambda'' > 0$ hold and the λ-curve is concave upward in the whole domain $-\infty < a < \infty$ throughout.

Further, in view of (16), the curves

(22.17) $y = \lambda m = 1-\sigma^2$ (complementary variance) and $y = 1-\lambda m = \sigma^2$ (variance)

are monotonic decreasing and increasing respectively with the properties

(22.18) $\lim_{a\to+\infty} \lambda m = \lim \frac{a\varphi}{\Phi}\cdot\frac{m}{a} = 0^\omega$ after (5) and (13); but $\lim_{a\to-\infty} \lambda m = \lim \frac{(a\Phi+\varphi)\lambda}{\Phi^2}$ which becomes after (10) and (11) $= \lim \frac{a\Psi}{\Phi}\cdot\frac{\lambda}{a} = \lim\left(1-\frac{1}{a^2-1}\right) = 1-0^2$.

(22.19) $\lim_{a\to+\infty} \sigma^2 = \lim(1-\lambda m) = 1-0^\omega$, $\lim_{a\to+\infty} \sigma^{\pm 1} = \lim(1-\lambda m)^{\pm 1/2} \simeq 1 \mp 0^\omega$.

(22.20) $\lim_{a\to-\infty} \sigma^2 = \lim(1-\lambda m) = \lim 1/a^2 = 0^2$ in virtue of (18), and $\lim_{a\to-\infty} \sigma^{\pm 1} = 0^1$ or ∞.

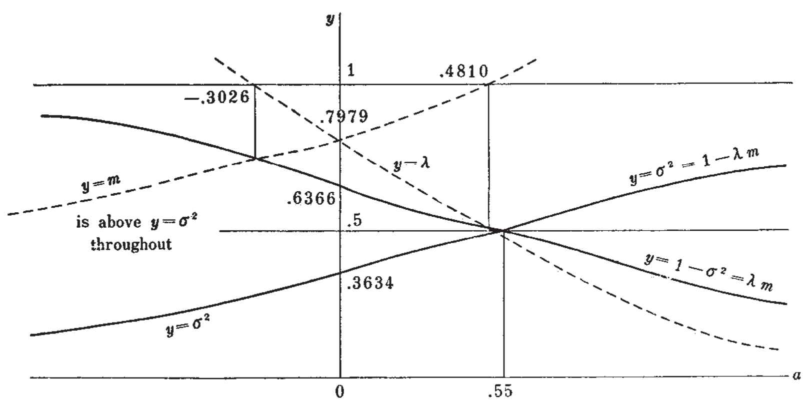

Fig. 6

Thus both curves extend between 2 parallels $y=0$, $y=1$ and symmetrically about $y=0.5$, intersecting with each other at (0.55, 0.5), and their point of inflections yield at $a=-0.05$ (Fig. 6). Whence we see that $m' = 1+\lambda' = 1-\lambda m > 0$ and $m'' = \lambda'' = (-\lambda m)' > 0$, after (16), so that the m-curve monotonic increasing and concave upward also.

Besides, not only the m-curve lies above the variance $y=\sigma^2$ when $a(<0)$ is finite, but also when $a \to -\infty$, the same still endures. For, as seen from the asymptotic expansion (27) below, it holds that

$$\lim(m-\sigma^2) = \lim(m-(1-\lambda m)) \simeq -\frac{1}{a} + \frac{2}{a^3} - \left(\frac{1}{a^2} + \cdots\right) \simeq -\frac{1}{a} > 0.$$

On the otherhand the λ-curve intersects already with the complementary variance $y=\lambda m$ when $m=1$, i.e. at $a=.4810$, and for $a>.4810$, the λ-curve undergos below $y=\lambda m$, because, as $a\to+\infty$, $\lambda \simeq 0^\omega$, while $\lambda m \simeq 0^2$ (Fig. 6). These facts show that m- and λ-curves are never symmetrical about y-axis ($a=0$), while variance and its complementary are exactly symmetrical about the y-parallel $y=\frac{1}{2}$, since $(1-\lambda m)-\frac{1}{2}=\frac{1}{2}-\lambda m$.

Also the curve (Fig. 7)

(22.21) $\qquad y=\mu(a)=a\lambda(a)=a\varphi(a)/\Phi(a)$

is of frequent use which becomes maximum at (0.84, 0.294), so that $\mu(a)<1$. It intersects with λ-curve at (1, 0.2876) and μ is $\lesseqgtr\lambda$ according as $a\lesseqgtr 1$, but both λ, μ tend 0^{∞} when $a\to+\infty$. However, when $a\to-\infty$, μ behaves as the parabola $y=-a^2$

When the curve $y=\mu=a\lambda$ is superposed with the parabola $y=a^2$, the resultant-curve becomes (Fig. 7).

(22.22) $\qquad y=a\lambda+a^2=am(a)$,

which touches $y=\mu$ upside at origin having $y_0'=\sqrt{2/\pi}$ in common, and downward has its asymptote $y=-1$ at left, since after (10) $\lim\limits_{a\to-\infty} am=\lim a(\varphi+a\Phi)/\Phi=\lim a\Psi/\Phi=-1+0^2$ holds, while, when $a\to+\infty$, $y=am\simeq a^2$, so that $y=a^2$ is the asymptotic parabola of $y=am$ at right.

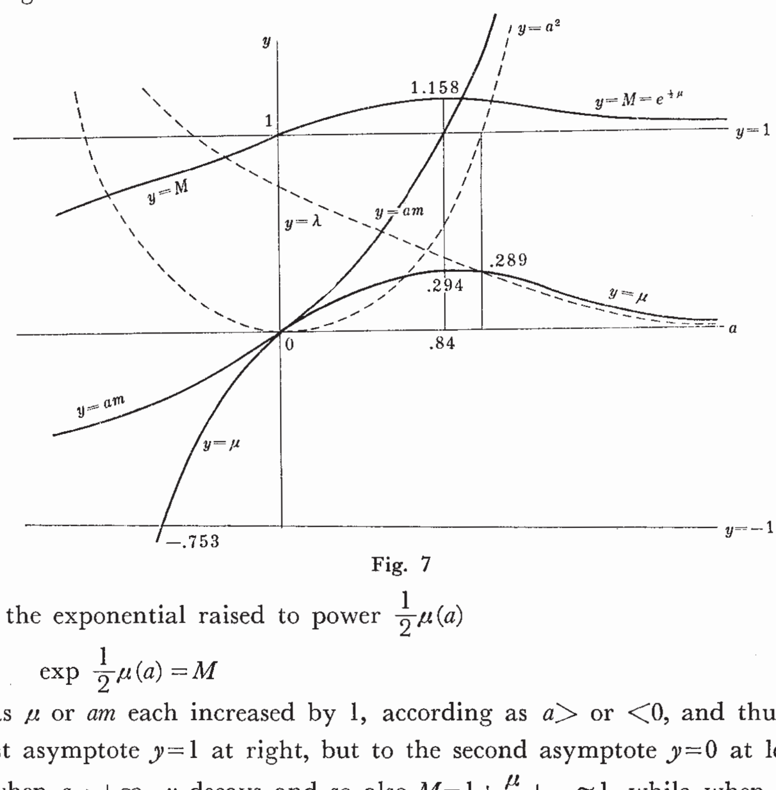

Fig. 7

Also, the exponential raised to power $\frac{1}{2}\mu(a)$

(22.23) $\qquad \exp\frac{1}{2}\mu(a)=M$

behaves as μ or am each increased by 1, according as $a>$ or <0, and thus it touches to the first asymptote $y=1$ at right, but to the second asymptote $y=0$ at left (Fig. 7). In fact, when $a\to+\infty$, μ decays and so also $M=1+\frac{\mu}{2}+\cdots\simeq 1$, while, when $a\to-\infty$, $\mu\to -a^2$, $am\simeq-1$ and $M\simeq\exp\left(\frac{1}{2}\mu\right)\simeq\exp\left(-\frac{1}{2}a^2\right)\simeq 0^{\infty}$. Naturally M's maximum takes places at the same time with μ. The maximum of μ is obtained from $\frac{d\mu}{da}=\frac{\varphi}{\Phi}\left(1-a^2-\frac{a\varphi}{\Phi}\right)=0$. Solving $f(a)\equiv 1-a^2-a\varphi/\Phi=0$ with $f'(a)=-2a-f(a)\varphi/\Phi$ by Newton's method, we get $a_0=$ 0.8400, so that max. $\mu=a_0\lambda(a_0)=0.2945$ and max. $M=\exp.\left(\frac{1}{2}\times.2945\right)=1.1585$ (Fig. 7).

Now we consider the variable which plays very important role later on:

(22.24) $z = m/\sigma = m/\sqrt{1-\lambda m} \quad (>0)$,

for which, by (13) and (19) or (14) and (20), hold

(22.25) $\lim\limits_{a\to+\infty} z = \lim m \simeq \lim a \to +\infty$,

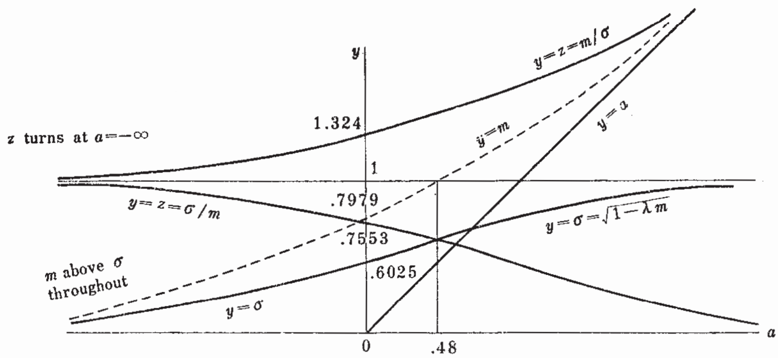

Fig. 8

(22.26) $\lim\limits_{a\to-\infty} z^2 = \lim m^2/\sigma^2 = \lim a^2/\sigma^2 = 1, \qquad \lim\limits_{a\to-\infty} z = 1.$

Hence the z-curve has 2 asymptotes $y=a$ and $y=1$ (Fig. 8).

To prove (26) directly, we may proceed by successive applications of l'Hospital: $\lim\limits_{a\to-\infty} z^2 = \lim m^2/(1-\lambda m) = \lim \Psi^2/(\Phi^2 - \Psi\varphi) = \cdots = \lim (2a^4-13a^2+7)/(2a^4-17a^2+17) \simeq 1+0^2$.

However, more briefly, it can be shown by asmptotic expansions in a^{-1}, as $a \sim -\infty$:

(22.27)
$$\Phi(a) \simeq \frac{\varphi(a)}{-a}\Big(1 - \frac{1}{a^2} + \frac{3}{a^4} - \frac{15}{a^6} + \frac{105}{a^8} - \frac{945}{a^{10}} + \frac{10395}{a^{12}} - \cdots\Big),$$
$$1/\Phi(a) \simeq \frac{-a}{\varphi(a)}\Big(1 + \frac{1}{a^2} - \frac{2}{a^4} + \frac{10}{a^6} - \frac{74}{a^8} + \frac{706}{a^{10}} - \frac{92}{a^{12}} + \cdots\Big),$$
$$\lambda(a) = \varphi/\Phi \simeq -a\Big(1 + \frac{1}{a^2} - \frac{2}{a^4} + \frac{10}{a^6} - \quad // \quad // \quad \Big),$$
$$m(a) = a + \lambda \simeq \frac{1}{-a}\Big(1 - \frac{2}{a^2} + \frac{10}{a^4} - \frac{74}{a^6} + \frac{706}{a^8} - \frac{92}{a^{10}} + \cdots\Big),$$
$$m^2(a) \simeq \frac{1}{a^2}\Big(1 - \frac{4}{a^2} + \frac{24}{a^4} - \frac{188}{a^6} + \frac{1808}{a^8} - \frac{3488}{a^{10}} + \cdots\Big),$$
$$\lambda m \simeq 1 - \frac{1}{a^2} + \frac{6}{a^4} - \frac{50}{a^6} + \frac{518}{a^8} - \frac{1716}{a^{10}} + \cdots,$$
$$\sigma^2 = 1 - \lambda m \simeq \frac{1}{a^2}\Big(1 - \frac{6}{a^2} + \frac{50}{a^4} - \frac{518}{a^6} + \frac{1716}{a^8} - \cdots\Big),$$
$$\sigma = \sqrt{1-\lambda m} \simeq \frac{1}{-a}\Big(1 - \frac{3}{a^2} + \frac{41}{2a^4} - \frac{395}{2a^6} + \frac{443}{8a^8} - \cdots\Big),$$
$$1/\sigma = 1/\sqrt{1-\lambda m} \simeq -a\Big(1 + \frac{3}{a^2} - \frac{23}{2a^4} + \frac{230}{2a^6} - \frac{1981}{4a^8} + \cdots\Big),$$

These hold also for $a \to +\infty$, i.e. if $\Phi, -a$ be replaced by $1-\Phi$ and a, e.g. $1-\Phi \simeq \varphi(a)/a$, or $\Phi \simeq 1 - \varphi/a$, &c.

Therefore we attain e.g. to prove (26)

$$\lim_{a\to-\infty} z = \lim \frac{m}{\sigma} = \lim\Big(1 - \frac{2}{a^2} + \cdots\Big)\Big(1 + \frac{3}{a^2} - \cdots\Big) = 1 + \lim \frac{1}{a^2} = 1 + 0^2.$$

Next, as $\lambda'=-\lambda m$, $m'=1-\lambda m=\sigma^2$, we get for $z=m/\sqrt{1-\lambda m}$ (Fig. 8).

(22.28) $$\frac{dz}{da}=\frac{1}{\sqrt{1-\lambda m}}\frac{dm}{da}+\frac{m}{2\sqrt{1-\lambda m^3}}\left(\lambda\frac{dm}{da}+m\frac{d\lambda}{da}\right)=\frac{1}{2\sigma^3}[\sigma^2(1+\sigma^2)-m^2(1-\sigma^2)].$$

So that $z'_{-\infty}=+0$ and $z'_{\infty}\simeq 1-\frac{1}{2}\lambda m^3\simeq 1-0$, because of (6).

By the way we observe that both $y=m$, $y=\sigma$ osculate the negative a-axis, when $a\to-\infty$, but

(22.29) $$m>\sigma,\quad m^2>\sigma^2$$

hold in the whole domain throughout.

Although the members m and σ in $z=m/\sigma$ are both monotonic increasing, yet the former varies far greater than the latter, what is the more remarkable as a is the larger. Consequently the ratio m/σ becomes monotonic increasing and the derivative is so also. Thus, when a increases from $-\infty$ to $+\infty$, z as its function increases monotonic as $1\leq z<\infty$, and really (29) hold. Hence inversely a can be also considered as a monotonic function of z in $(1, \infty)$, so that any function of a may be also defined as a function of z, and vice versa. If the middle point $a=0$ be considered, when a runs from ∞ into 0, z goes from ∞ into

(22.30) $$z_{a\to 0}=\sqrt{\frac{2}{\pi}}\Big/\sqrt{1-\frac{2}{\pi}}=\sqrt{\frac{2}{\pi-2}}=1.3236$$

and thenceforth a runs further from 0 into $-\infty$, and z restarts from z_0 to end at 1.

Now, it is very desirous to continue z furthermore into the internal $(1\sim 0)$, which can be naturally done by taking the reciprocal of the original z defined by (24)

(22.31) $$\bar{z}=\sigma/m=\sqrt{1-\lambda m}/m,$$ so that z and $\bar{z}$ are both essentially positive.

To distinguish them, the hitherto considered original z in $(\infty, 1)$ may be said to be proper and the continued $\bar{z}$ in $(1, 0)$ to be improper. Analytic character of the improper z follows from that of the proper z. When the proper z goes from ∞ into 1, the variable a runs already over its whole course from ∞ into $-\infty$, while, when the improper z continues from 1 into 0, the variable a goes again over its whole course in a backway from $-\infty$ into ∞, as shown by the arrowed piles annexed on $z\bar{z}$ curves in Fig. 8. Or we may put them altogether, and call $z=1$ to be the turn or turning point; to say more precisely, the proper z turns in 1, and the improper z turns out 1.

We may also conceive $\varphi(z)$, $\Phi(z)$, $\lambda(z)$, $\mu(z)$ &c. and e.g. $\Phi(z)$ takes either $\Phi(\infty)=1$ or $\Phi(0)=1/2$ when $a=\infty$, but $\Phi(1+0)=\Phi(1-0)=.841345$ when $a=-\infty$. However they are uniquely determined for the proper z in $(1, \infty)$ as well as for the improper z in $(0, 1)$, so that e.g. $\mu(z)$ is wholly positive, and properly monotonic decreasing from 0.2876 into 0, while, improperly starting from 0 ends at 0.2876, taking in the midway the maximum value 0.298, when $a=0.84$, $\bar{z}=0.64$ (cf. Fig. 7). Thus

(22.32) $$z(a)>\bar{z}(a)>0,\quad \Phi(z)>\Phi(\bar{z}),\quad \varphi(z)<\varphi(\bar{z}),\quad \lambda(z)<\lambda(\bar{z}),\ \&c.$$

We notice that, as $z=m(a)/\sigma(a)$ is readily computable for any prescribed a, but inversely to find a from a given z it is somewhat troublesome, which, however, can be interpolated by means of Tables in Sect. 23. Still to be added, we have

(22.33) $$\frac{dz}{da}\geq 0,\quad \text{but}\quad \frac{d\bar{z}}{da}=\frac{-1}{z^2(a)}\frac{dz}{da}<0.$$

Hence, the proper z is, so to speak, semisynchronous with a: i.e. when a increases in $(-\infty, \infty)$, so also z increases in $(1, \infty)$, although their speeds are different. On the otherhand, the improper $\bar{z}$ is contrasynchronous to a, as they move with contrary sense: when a increases in $(-\infty, \infty)$, the continued $\bar{z}$ decreases in $(1, 0)$. We shall write below simply Φ, φ, λ, &c. omitting the argument, when the parameter is a, or its value is obvious, however for z it is explicitely written as $\Phi(z)$, $\varphi(z)$, $\lambda(z)$ &c.

We construct below several special functions of z, where z being defined by m/σ properly, but by σ/m improperly. First we define, just likewise as $z=m/\sigma$,

(22.34) $Z=Z(z,a)=\Phi/\sigma=z\Phi/m=\Phi/\sqrt{1-\lambda m}$, properly, so that, improperly

(22.35) $\bar{Z}=\bar{z}\Phi/m=\sigma\Phi/m^2=(\sigma/m)^2 Z=\sqrt{1-\lambda m}\,\Phi/m^2.$

It starts properly from $\lim\limits_{a\to\infty,\ z\to\infty} Z=\lim \Phi/\sqrt{1-\lambda m}=\lim 1/\sqrt{1-a\varphi}=1+0^\omega$ because, as $a\to\infty$, $\lambda m\equiv\Psi\varphi/\Phi^2\simeq a\varphi$ while $\Phi>1-\varphi$ (cf. (22.8),) and becomes at the turning point $\lim\limits_{a\to-\infty,\ z\to1} Z=\lim\frac{\varphi(a)}{-a}\left(1-\frac{1}{a^2}+\cdots\right)(-a)\left(1+\frac{3}{a^2}-\cdots\right)=\lim\varphi(a)=0^\omega$; then restarting improperly ends again with $\lim\limits_{a\to+\infty,\ z\to0} Z=\lim 1/m^2\simeq 0^2$ (Fig. 9). Intermediately for $a=0$, $Z=\frac{1}{2}\Big/\sqrt{1-\frac{2}{\pi}}=$ $=0.8294$, and $\bar{Z}=\frac{1}{2}\sqrt{1-\frac{2}{\pi}}\Big/\frac{2}{\pi}=0.4734$. To find the maximum of Z, putting

$$\frac{dZ}{da}=\frac{\varphi}{\sigma^3}\left(\frac{3}{2}-\frac{m^2}{\sigma^2}\right)=\frac{\varphi}{2\sigma^3}(3-3\lambda m-m^2)=0,\quad \text{i.e.}\quad 4\lambda^2+5a\lambda+a^2-3=0\ \ (\lambda=\varphi(a)/\Phi(a))$$

which solved by Newton, affords the root $a_0=1.245$, $\lambda(a_0)=.2057$, $Z=1.067$. Also, as to the improper $\bar{Z}$, we have $\bar{Z}'=\Phi f/2m^3\sigma$, $f(a)\equiv m^2(1-\sigma^2)+\sigma^2-5\sigma^4$, so that max. $\bar{Z}_0$ is obtained by solving $f=0$. Really we get $a_0=.2458$, $\bar{Z}_0=.48425$, again by Newton.

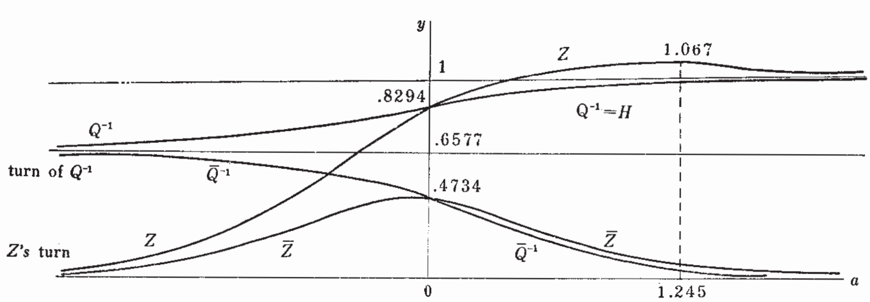

Fig. 9

Next, consider the quotient formed by (23) and (34): $Q=M/Z$ or its reciprocal

(22.36) $Q^{-1}=Z/M=(\Phi/\sigma)\exp\left(-\frac{1}{2}\mu(a)\right)$ properly, and

(22.37) $\bar{Q}^{-1}=\bar{Z}/M=(\sigma\Phi/m^2)\exp\left(-\frac{a}{2}\lambda(a)\right)$ improperly. Hence we have firstly

$\lim\limits_{a\to\infty,z\to\infty} Q^{-1}=\lim\left(1-\frac{\varphi}{a}\right)\left(1+\frac{a}{2}\varphi\right)\left(1-\frac{a\varphi}{2}\right)=\lim\left(1-\frac{\varphi}{a}\right)=1-0^\omega$; next as $\mu=a\lambda=a(m-a)$, availing (27) for Z and am, $\lim_{a\to-\infty, z\to 1+0} Q = \lim\varphi(a)\left(1+\frac{3}{a^2}\right)\exp\left[-\frac{1}{2}am+\frac{1}{2}a^2\right] = \lim\sqrt{\frac{e}{2\pi}}\left(1+\frac{1}{a^2}\right)$ $=.6577+0^2$; but this being multiplied by $\sigma^2/m^2 \simeq 1-2/a^2$ yields $\lim_{z\to 1-0} \bar{Q}^{-1} = .6577-0^2$. Lastly $\lim_{a\to\infty, z\to 0} \bar{Q} = \lim \bar{Z}/\lim M = +0^2$. But, as $\lim \bar{Q}/\bar{Z} = \lim 1/M = 1-0^\omega$, $\bar{Q}$ lies below ultimately. Thus Q^{-1} extends between $y=1$ and $y=\sqrt{e/2\pi}$, while $\bar{Q}^{-1}$ between $y=\sqrt{e/2\pi}$ and $y=0$. The turning-in and turning-out value are $.6577\pm0^2$ (Fig. 9), respectively.

Further we define a power index

(22.38) $\quad p = m\lambda/\mu(z) = m\lambda/z\lambda(z) = \sigma\lambda/\lambda(z)$ properly, but

$\bar{p} = m^2\lambda/\sigma\lambda(\bar{z}) = m\lambda/\mu(\bar{z})$ improperly (Fig. 10).

At first in order to find the starting value $\lim_{z\to\infty, a\to\infty} p = \lim(m/z)\frac{\lambda}{\lambda(z)}$, we observe first $m/z = \sigma = \sqrt{1-\lambda m} \simeq 1-\frac{1}{2}a\varphi$, so that $z \simeq m\left(1+\frac{1}{2}a\varphi\right) \simeq a(1+\varphi/a)\left(1+\frac{1}{2}a\varphi\right) \simeq a+\frac{1}{2}a^2\varphi$. Second $\frac{\lambda}{\lambda(z)} \simeq \frac{\varphi/(1-\varphi/a)}{\varphi(z)/(1-\varphi(z)/z)} \simeq \frac{\varphi}{\varphi(z)}\left(1-\frac{\varphi}{a}\right)\left(1+\frac{\varphi(z)}{z}\right)$. Third $\varphi/\varphi(z) \simeq \exp\frac{1}{2}(z^2-a^2) \simeq \exp\frac{1}{2}a^3\varphi$ $\simeq 1+\frac{1}{2}a^3\varphi$. Hence, $p(z\to\infty) \simeq \left(1-\frac{1}{2}a\varphi\right)(1+\varphi(z)/z)\left(1+\frac{1}{2}a^3\varphi\right)$. But $\varphi(z)/z \simeq \varphi\left(1-\frac{1}{2}a^3\varphi\right)\Big/ a\left(1+\frac{1}{2}a\varphi\right) \simeq \varphi/a$. Therefore we obtain as the starting value

$$p(+\infty) \simeq \left(1-\frac{1}{2}a\varphi\right)\left(1-\varphi/a\right)\left(1+\frac{1}{2}a^3\varphi\right) \simeq 1+\frac{1}{2}a^3\varphi \simeq 1+0^\omega.$$

Next, as to the turning-in value $\lim_{z\to 1, a\to-\infty} p$, after (27) holds $\lambda\sigma \simeq (1+1/a^2)(1-3/a^2) \simeq 1$ $-2/a^2$, and since $z = m/\sigma \simeq 1+1/a^2$, we have $\lambda(1+1/a^2) \simeq \frac{\varphi(z)}{\Phi(z)} = \frac{\varphi(1)-\varphi(1)/a^2}{\Phi(1)+\varphi(1)/a^2} \simeq \frac{1-1/a^2}{1+\lambda_1/a^2}\lambda_1$ $\simeq \lambda_1\left(1-\frac{1+\lambda_1}{a^2}\right)$, where $\lambda_1 = \varphi(1)/\Phi(1) = .2876$. Therefore the turning-in value is

$$p(z=1+0) \simeq (1-2/a^2)(1+(1+\lambda_1)/a^2)/\lambda_1 \simeq (1-(1-\lambda_1)/a)^2/\lambda_1 \simeq 1/\lambda_1 - 0^2 = 3.47705-0^2,$$

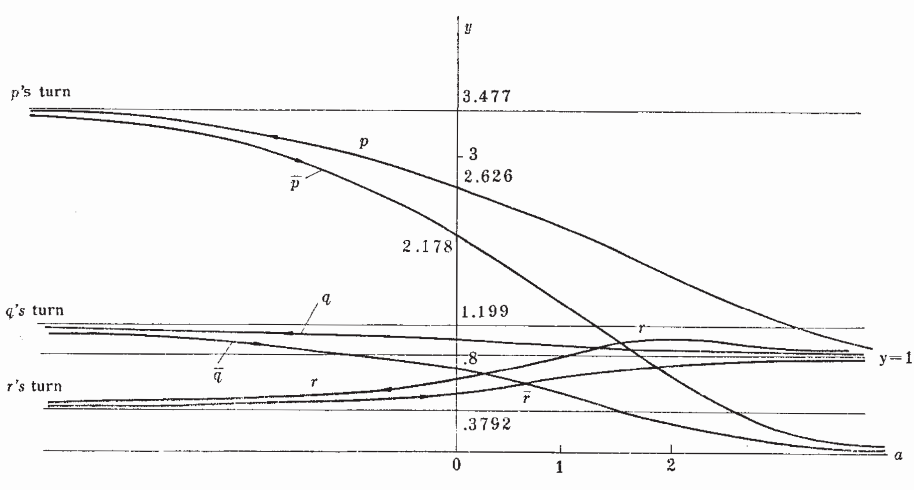

Fig. 10

which also gives the turning-out value. Intermediately, when $a=0$, it hold either $z_0=(m/\sigma)_0=\sqrt{2/(\pi-2)}=1.3236$ and $p(z_0)=\sigma(0)\lambda(0)/\lambda(z_0)=2.626$, or $\bar{z}_0=(\sigma/m)_0=0.7555$, $p(\bar{z}_0)=m^2(0)\lambda(0)/\sigma(0)\lambda(\bar{z}_0)=2.178$. The ultimate ending value is $\lim_{z\to 0,\ a\to=\infty}\bar{p}=\lim_{a\to\infty} m^2\lambda/\sigma\lambda(0)$ $=\lim(a+\lambda)^2\lambda\sqrt{\pi/2}=+0$ (Fig. 10).

Now considered $\Phi(z)$ raised to power p and multiplied by Q

(22.39) $\Phi^p(z)Q$ properly, or $\Phi^p(\bar{z})\bar{Q}$ improperly, ((36), (38) and Fig. 9, 10),

of which the proper value starting from $\lim_{z\to\infty,\ a\to\infty}=1$ decreases to the turning-in value $\lim_{z\to 1,\ a\to-\infty}=.841345^{3\cdot 477}\times.6577=.8335$, and then increases to end with $\lim_{z\to 0,\ a\to-\infty}\left(\frac{1}{2}\right)^0\bar{Q}(z)$ $=\infty$. Therefore, its reciprocal

(22.40) $q=1/Q\Phi^p(z)$ or $\bar{q}=1/\bar{Q}\Phi^p(\bar{z})$

beginning with $q(\infty)=1$ increases up to $q(1)=1.199$ very slowly and then decreases to end at $q(0)=0$ (Fig. 10). Consequently we get finally

(22.41) $q\Phi^p(z)=Q^{-1}=H(z)$

which starting from $H(\infty)=1$ properly, monotonic decreases to the turning value $H(1)=.6577$ and continues decreasing up to $H(0)=0$, as shown before in Fig. 9.

Our ultimate idea is to approximate the correction-factor $\mathfrak{h}_n(1/z)$ by the combination of the above members:

(22.42) $\mathfrak{h}_n(1/z)=(\Phi^p(z)q(z))^n r(z)=Q^{-n}r=H^n r,$

where $H=Q^{-1}=z\Phi^p(a)\exp\left(-\frac{1}{2}\mu(a)\right)/m(z)$, in which a is defined inversely by $z=m(a)/\sigma(a)$. The full reasoning process would be developped in section 24. But, it should be touched on the factor standing outside the power

(22.43) $r(z)=\frac{1}{z}\sqrt{\mathfrak{D}/2}$, where $\mathfrak{D}$ denotes the determinant of a certain quadratic:

(22.44) $\mathfrak{D}(z,a)=(2-p\mu(z)+m^2)[2-3p\mu(z)+p\mu(z)(\mu(z)+z^2)]-4(1-p\mu(z))^2,$

in which $p\mu=\lambda m$ after (38), so that $0<p\mu<1$ and properly $z>1$, so that $0<\mu(z)<.2876$ hold (Fig. 7). Hence it can be readily seen that $\mathfrak{D}>0$. For, on rewriting, we get

(22.44)′ $\mathfrak{D}=(2-\lambda m+m^2)\lambda m(\mu(z)+z^2)+m^2(2-3\lambda m-\lambda^2)$

$=(1-\lambda m)[\lambda m(\mu(z)+z^2)+2m^2]+\lambda m[\mu(z)+z^2-\lambda m+m^2(\mu(z)+z^2-1)]>0,$

because of $0\leq\lambda m=p(\mu)\leq 1\leq z^2$. Besides, so far a is finite, so also $\mathfrak{D}$ is finite. But, when $a\to\infty$, $z\to\infty$, we have $\lambda m\simeq o^\omega$ and $\mu(z)\simeq 0^\omega$, so that $\lim_{z\to\infty}\sqrt{\mathfrak{D}/2}=\lim_{a\to\infty}m\simeq\infty$. However, this being divided by $z=m/\sigma$, we obtain $\lim_{z\to\infty}r=\lim\sigma=1$. Or more indetail, since, for $a\to\infty$, $\Phi\sim 1-\varphi/a\simeq 1$, $\lambda\sim\varphi$, $m\sim a+\varphi$, $\sigma\simeq 1-\frac{1}{2}a\varphi\simeq 1$ hold, we get $z=m/\sigma\simeq a\left(1+\frac{1}{2}a\varphi\right)$, $\mu(z)\simeq a\varphi$, so that $\mathfrak{D}=a^5\varphi+a^2(2-3a\varphi)\simeq 2a^2\left(1+\frac{1}{2}a^5\varphi\right)$. Hence $r=\frac{1}{z}\sqrt{\mathfrak{D}/2}\simeq\left(1+\frac{1}{4}a^5\varphi\right)$ $\left(1-\frac{1}{2}a\varphi\right)\simeq 1+\frac{1}{4}a^5\varphi=1+0^\omega$. On the otherhand, when $a\to-\infty$, $z=1+0$, we have after (27), $z=\frac{m}{\sigma}\simeq 1+\frac{1}{a^2}$, $\lambda(z)\simeq\lambda_1\left(1-\frac{1+\lambda_1}{a^2}\right)$, $\lambda_1=\lambda(1)$, $\mu(z)\simeq\lambda_1\left(1-\frac{\lambda_1}{a^2}\right)$ and $\mathfrak{D}\simeq\lambda_1+(4+\lambda_1$

$-\lambda_1^3)/a^2$. So at length $r=\frac{1}{z}\sqrt{\mathfrak{D}/2}\simeq\sqrt{\lambda_1/2}+(3+\lambda_1-\lambda_1^3)\sqrt{2}a^2=.3792+0$. In fact r starting from $r(\infty)=1+0$, after taking a maximum midway, decreases up to the turning-in value $r=0.3792$.

Lastly, to continue the function $r(z)=\frac{1}{z}\sqrt{\mathfrak{D}/2}$ into the region $z<1$, we have to take $\bar{z}=1/z$, so that

(22.45) $\bar{r}=\bar{z}\sqrt{\mathfrak{D}/2}$, where $\bar{z}=\sigma/m$ and

(22.46) $\mathfrak{D}=(2-\lambda m+m^2)[2-3\lambda m+\lambda m(\mu(\bar{z})+\bar{z}^2)-4(1-\lambda m)^2$
$= (2-\lambda m+m^2)\lambda m[\mu(z)+z^2]+m^2[2-3\lambda m-\lambda^2)$

the same as (44) and its positivity still holds. The turning-out value becomes $\bar{r}\simeq\sqrt{\frac{\lambda_1}{2}}\left(1+\frac{3+\lambda_1^2}{2a^2}\right)$, the same as the turning-in value. Finally it ends when $a\to+\infty$, $z\to 0$. Here again $m\simeq a(1+\varphi/a)\sim\infty$, $\lambda m\sim a\varphi$, $\sigma\simeq 1-a\varphi/2$ and $\bar{z}=\sigma/m\simeq\frac{1}{a}\left(1-\frac{a\varphi}{2}\right)$, $\lambda(\bar{z})\sim\lambda_0=\sqrt{2/\pi}$, $\mu(\bar{z})\simeq\lambda_0/a$, so that $\mathfrak{D}\simeq 2a^2\left(1-\frac{3}{2}a\varphi\right)\simeq\infty$, but $\bar{r}=\frac{\sigma}{m}\sqrt{\mathfrak{D}/2}\simeq 1-\frac{5}{4}a\varphi=1-0^{\omega}$.

Thus r remains finite>0 throughout (Fig. 10).

N.B. Below we shall frequently neglect the derivatives of p and q in regard to z, partly because of simplifying the computations. It is clear that it is permissible about q', as may be seen from Fig. 10. Also, as to the proper p, it is $\frac{dp}{da}$ is rather flat. Really assumed that $a>0$ or $|a|$ is not so large if $a<0$, $\frac{dz}{da}$ is also finite, and $p'=\frac{dp}{dz}=\frac{dp}{da}\Big/\frac{dz}{da}$ is tolerably small. In addition, it appears in the derivative $(p\Phi(z))'=p\lambda(z)+p'\log\Phi(z)$ and as $z>z(a_0)>1$, $\log\Phi(z)$ becomes small so that the last term may be neglected. Also r' is negligiblly small by the flatness of the r-curve (Fig. 10).

When p' be preserved without omitting, we should find somehow the value of p, e.g. by means of an infinite series. We have clearly

(22.47) $$\varphi(a)=\frac{1}{\sqrt{2\pi}}e^{-a^2/2}=\frac{1}{\sqrt{2\pi}}\Sigma(-1)^k\frac{a^{2k}}{2^k k!}\equiv\Sigma\varphi_0^{(k)}\frac{a^{2k}}{2k}=\frac{d}{da}\int_{-\infty}^{a}\varphi(a)\,da=\Phi'.$$

But, since $\Phi_0^{(2k+1)}=\varphi_0^{(2k)}$ and $\Phi_0^{(2k)}=\varphi_0^{(2k+1)}=0$ except $k=0$, we get

(22.48) $$\Phi(a)=\Phi_0+\frac{1}{\sqrt{2\pi}}\Sigma\frac{(-1)^k a^{2k+1}}{2^k(2k+1)k!}=\frac{1}{2}+\frac{1}{\sqrt{2\pi}}\left[a-\frac{a^3}{6}+\frac{a^5}{40}-\frac{a^7}{336}+\frac{a^9}{3456}-\frac{a^{11}}{42240}\right.$$
$$\left.+\frac{a^{13}}{599040}-\frac{a^{15}}{9676800}+\frac{a^{17}}{165150720}-\frac{a^{19}}{3158507520}+\frac{a^{21}}{70601932800}-\cdots\right.,$$

whence, e.g. for $a=1$, $\Phi(1)=0.8413447438$, exactly.

Both φ, Φ being entire (besides 'Einheits'), the radius of convergence$=\infty$, and the above hold steadily for any finite $a\lesseqgtr 0$. However, the convergence becomes slow when $|a|$ is somewhat large, it is rather preferable to use the asymtotic expansion, i.e. if $a\sim+\infty$, taking

$$\Phi(a)=1+\frac{\varphi(a)}{-a}\left[1-\frac{1}{a^2}+\frac{1.3}{a^4}-\frac{1.35}{a^6}+\cdots\right]$$

stopping at the term before that which becomes absolutely minimum, while, if $a\sim-\infty$, the above -1 will do. In general, for moderate a, denoting $\sqrt{2/\pi}$ by c, we obtain from (47) (48)

(22.49) $$\lambda(a)=\frac{\varphi(a)}{\Phi(a)}=c\left[1-ca+\left(c^2-\frac{1}{2}\right)a^2-\left(c^3-\frac{2}{3}\right)a^3+\left(c^4-\frac{5}{6}c^2+\frac{1}{8}\right)a^4+\cdots\right],$$

and whence furthermore $m(a)=a+\lambda(a)$, $\sigma^2=1-\lambda m$, &c. can be also given by infinite series, which all converge for any $a>0$.

As is shown later on, the maximum of $F=\log(x\Phi^p(z)q(z)/z)-\frac{1}{2}(x-a)^2-\frac{1}{2}x^2/z^2$ yields when $F_x=[1-x(x-a)-x^2z^2]/x=0$, $F_z=\frac{d}{dz}p\log\Phi(z)-\frac{1}{z}\left(1-\frac{x^2}{z^2}\right)=0$. Hence, if the C.L.T. be affirmed, the first equation affords already $x_0=m$, $z_0=m/\sigma$ for the coordinates of the muximum point, and accordingly the second equation reduces to $\frac{d}{dz}(p\log\Phi(z))$ $=p'\log\Phi(z)+p\lambda(z)=\lambda\sigma$. Here however the term $p'\log\Phi(z)$ has been neglected in the text, for the sake of simplicity. If this term be preserved we obtain by integration

(22.50) $$p\log\Phi\left(\frac{m(a)}{\sigma(a)}\right)-p_0\log\Phi\left(\frac{m(0)}{\sigma(0)}\right)=\int_0^a\lambda(a)\,\sigma(a)\frac{dz}{da}da=\frac{1}{2}\int_0^a\lambda\left[1+\sigma^2-\left(\frac{1}{\sigma^2}-1\right)m^2\right]da,$$

which integral can be found by means of series in a, if a moderate, or else, more generally after Gauss' method of numerical integrations. In particular, if $a\to\infty$, the corresponding $z=m(\infty)/\sigma(\infty)$ becomes also $+\infty$, so that $\log\Phi(z)=0$ and (50) reduces to

(22.51) $$p_0\log 1/\Phi\left(\frac{m(0)}{\sigma(0)}\right)=\frac{1}{2}\int_0^\infty\lambda\left[1+\sigma^2-\left(\frac{1}{\sigma^2}-1\right)m^2\right]da$$
$$=\int_0^{\pi/2}\quad{}'' \quad \sec^2\theta d\theta,\ \text{if } a=\tan\theta,$$

which is again capable to use Gauss' method. Hence, combining (50) and (51), the values of p, p_0 can be determined, whenever a is prescribed. Lastly, taking

(22.52) $$q=\frac{\Phi(a)e^{-1\mu(a)/2}}{\sigma(a)\Phi^p(z_0)}=\frac{Z}{M}\Phi^{-p}(z_0)=1/Q(z_0)\Phi^p(z_0),$$ the same form as (40),

the further process continued in the same way as in the text, leads just to the same result.

23. *Numerical Tables for Several Values of $\Phi(a)$, $\lambda(a)$ etc. computed for Assigned Values of a.* They are inserted here partly in order to explain the foregoing theoretical results and partly to make use in the subsequent sections. As the original calculations were made with many figured numbers, they are generally contracted into 4 or 5 efficient figures for brevity. Naturally the approximations being roughly of the first order, it is quite nonsence to treat so many figured numbers. Notwithstanding, since ultimately we concern with so large n-sized sample, as $n=100$, 1000, &c., which become power-indices, it requires rather tolerably many figures.

TABLE I

No.	(1) a	(2) $\Phi(a)$	(3) $\varphi(a)$	(4) $\lambda(a)=\varphi/\Phi$	(5) $\mu(a)=a\lambda$	(6) $m=a+\lambda$	(7) $m\lambda=1-\sigma^2$	(8) $\sigma=\sqrt{1-\lambda m}$
1	∞	1	0^ω	0^ω	0^ω	∞^1	0^ω	1
2	5	$.9^6 71335$	$.0^5 148672$	φ	$.0^5 743360$	$5+\varphi$	$.0^5 743360$	$.9^5 62832$
3	4	$.9^4 683$	$.0^3 133830$	$.0^3 133834$	$.0^3 535336$	$4+\varphi$	$.0^3 535354$	$.9^3 73232$
4	3	$.9^2 8650$	$.0^2 44318$	$.0^2 443784$	.0133135	$3.0^2 4438$	.013333	.993311
5	2	.977250	.539910	.055248	.110496	2.05525	.113548	.941516
6	1	.841345	.241971	.287600	.287600	1.28760	.370314	.793528
7	1.64488	.95	.10313	.10856	.17857	1.75344	.19035	.89980
8	1.28156	.9	.17550	.19500	.24990	1.47656	.28793	.84384
9	1.03645	.85	.23316	.27431	.28431	1.31076	.35955	.80028
10	.84163	.8	.27996	.34995	.29453	1.19158	.41699	.76355
11	.67449	.75	.31777	.42370	.28578	1.09819	.46530	.73123
12	52441	.7	.34769	.49670	.26047	1.02111	.50718	.70201
13	.38532	.65	.37039	.56983	.21957	.95516	.54428	.67507
14	.28335	.6	.38634	.64390	.16313	.87725	.57774	.64982
15	.12567	.55	.39580	.71964	.090437	.84531	.60832	.62584
16	0	.5	.39894	.79788	0	.79788	.63662	.60281
17	−.12567	.45	.39580	.87956	−.11053	.75389	.66308	.58044
18	−.25335	.4	.38634	.96585	−.24470	.71250	.68817	.55842
19	−.38532	.35	.37039	1.0583	−.40777	.67294	.71214	.53652
20	−.52441	.3	.34769	1.1590	−.60778	.63456	.73544	.51435
21	−.67449	.25	.31778	1.2711	−.85734	.59663	.75838	.49155
22	−.84163	.2	.27996	1.3998	−1.1781	.55817	.78133	.46762
23	−1.0364	.15	.23316	1.5544	−1.6110	.51796	.80512	.44145
24	−1.2816	.1	.17550	1.7550	−2.2491	.47344	.83089	.41134
25	−1.6440	.05	.10313	2.0626	−3.3929	.41778	.86174	.37183
26	$-\infty$	0	0	∞	$-\infty$	0	1	0
21′	−.67449	do. to No. 21						
16′	0	〃 〃 16						
11′	.67449	〃 〃 11						
6′	1	〃 〃 6						
1′	∞	〃 〃 1						

TABLE II

No.	(9) $z=m/\sigma$	(10) $\Phi(z)$	(11) $\varphi(z)$	(12) $\lambda(z)$	(13) $\mu(z)$	(14) $M=e^{\mu(\alpha)/2}$	(15) $Z=\frac{\Phi(a)}{\sigma}=\frac{z\Phi}{m}$	(16) $Q=M/Z$
1	∞	1	0^{∞}	0^{∞}	0^{∞}	1	1	1
2	$5.0^{4}20071$	$.9^{6}71338$	$.0^{5}74329$	$.0^{5}74329$	$.0^{5}74329$	$1.0^{5}37168$	$1.0^{5}34301$	$1.0^{6}28665$
3	$4.0^{2}1205$	$.9^{4}6846$	$.0^{3}13366$	$.0^{3}1337$	$.0^{3}53292$	$1.0^{3}26770$	$1.0^{3}23604$	$1.0^{4}31656$
4	3.02467	$.9^{2}8705$	$.0^{2}41144$	$.0^{2}4567$	$.0^{2}12461$	$1.0^{2}66813$	$1.0^{2}53745$	$1.0^{2}12998$
5	2.18291	.985478	.0368281	.03737	.0815773	1.056803	1.03745	1.01816
6	1.62266	.947663	.10695	.11286	.18313	1.15465	1.06026	1.08908
7	1.9487	.974334	.05974	.06131	.11948	1.0934	1.0558	1.03562
8	1.7498	.959924	.08648	.09010	.15764	1.1331	1.0666	1.04382
9	1.6379	.949277	.10399	.10955	.17943	1.1523	1.0621	1.08533
10	1.5606	.940690	.11671	.12407	.19362	1.1587	1.0477	1.10587
11	1.5018	.933424	.12917	.13838	.20782	1.1531	1.0257	1.12473
12	1.4546	.927101	.13852	.14941	.21732	1.1391	.99714	1.14236
13	1.4149	.921448	.14871	.16139	.22835	1.1604	.96286	1.15909
14	1.3808	.916329	.15378	.16782	.23173	1.0850	.92333	1.17507
15	1.3507	.911603	.16023	.17577	.23741	1.0463	.87882	1.19053
16	1.3236	.907179	.16615	.18315	.24242	1.	.82945	1.20562
17	1.2988	.902913	.17164	.19010	.24686	.94623	.77527	1.22052
18	1.2759	.899002	.17678	.19664	.25089	.88484	.71631	1.23528
19	1.2543	.895131	.18167	.20295	.25456	.81556	.65235	1.25019
20	1.2337	.891340	.18639	.20911	.25798	.73795	.58326	1.26520
21	1.2138	.887586	.19098	.21517	.26117	.65137	.50860	1.28074
22	1.1936	.883680	.19568	.22144	.26431	.55485	.42770	1.29729
23	1.1733	.879660	.20043	.22785	.26734	.44686	.33979	1.31511
24	1.1510	.875133	.20570	.23505	.27054	.32480	.24311	1.33602
25	1.1236	.869406	.21221	.24409	.27426	.18333	.13447	1.36335
26	1.	.841345	.24197	.28760	.28760	0	0	1.52035
21′	.82385	.794985	.28414	.35742	.29446	.65135	.34520	1.88698
16′	.75551	.775026	.29989	.38694	.29234	1	.47344	2.11220
11′	.66587	.747250	.31962	.42773	.28481	1.1531	.45475	2.53678
6′	.61628	.731144	.32793	.44852	.27810	1.15465	.21782	2.86732
1′	0	.5	.39894	.79788	0	1	0	∞

TABLE III

No.	(17) $p=\dfrac{\lambda m}{\mu(z)}$	(18) $\Phi^p(a)$	(19) $q=1/\Phi^p Q$	(20) $r=\dfrac{1}{z}\sqrt{\mathfrak{D}/2}$	(21) $\log_{10}Q^{-1}$ $=\log q\Phi^p$	(22) $n\log Q^{-1}$ for the example $n=100$	(23) Q^{-n} for the example $n=100$	(24) $\mathfrak{h}_n=Q^{-n}r$ for the example $n=100$
1	1	1	1	1	0	0	1	1
2	$1.0^{4}955$	$.9^{6}71335$	$1.0^{9}26$	$1.0^{4}83$	$-.0^{6}660036$	$\bar{1}.9999340$	.999848	.99993
3	$1.0^{2}4573$	$.9^{4}68316$	$1.0^{7}28$	$1.0^{2}20$	$-.0^{4}72956$	$\bar{1}.9927044$	.983342	.98531
4	1.06948	$.9^{2}86155$	$1.0^{4}865$	1.0264	$-.0^{3}56414$	$\bar{1}.943586$	.878185	.90158
5	1.39191	.979844	$1.0^{2}237$	1.0431	$-0.{}^{2}78160$	$\bar{1}.21840$	.16535	.63086
6	2.02214	.89700	1.0237	.95125	−0.370383	$\bar{4}.29617$	$.0^{3}1978$	$.0^{3}1881$
7	1.5932	.95942	1.0065	1.0050	−.0152004	$\bar{2}.47996$	.030197	.03035
8	1.8264	.92797	1.0143	.9776	−.0186256	$\bar{2}.13744$	.013723	.01342
9	2.0038	.90095	1.0227	.9566	−.0355618	$\bar{4}.44382$	$.0^{3}2779$	$.0^{3}266$
10	2.1536	.87662	1.0315	.9285	−.0437041	$\bar{5}.62959$	$.0^{4}4262$	$.0^{4}396$
11	2.2390	.85705	1.0379	.8999	−.0510483	$\bar{6}.8952$	$.0^{5}7856$	$.0^{5}707$
12	2.3338	.83807	1.0445	.8753	−.0578030	$\bar{6}.2197$	$.0^{5}1658$	$.0^{5}145$
13	2.3835	.82284	1.0485	.8530	−.0641171	$\bar{7}.5883$	$.0^{6}3875$	$.0^{6}331$
14	2.4932	.80424	1.0581	.8414	−.0700637	$\bar{8}.9932$	$.0^{7}9845$	$.0^{7}828$
15	2.5623	.78888	1.0648	.8105	−.0757404	$\bar{8}.4260$	$.0^{7}2667$	$.0^{7}216$
16	2.6261	.77428	1.0713	.7908	−.0811744	$\bar{9}.8826$	$.0^{8}7631$	$.0^{8}603$
17	2.6861	.76008	1.0780	.7713	−.0865449	$\bar{9}.3455$	$.0^{8}2216$	$.0^{8}171$
18	2.7429	.74675	1.0841	.7530	−.0917654	$\overline{10}.8235$	$.0^{9}6660$	$.0^{9}501$
19	2.7976	.73350	1.0905	.7317	−.0969760	$\overline{10}.3024$	$.0^{9}2006$	$.0^{9}147$
20	2.8508	.72042	1.0971	.7156	−.1021592	$\overline{11}.7841$	$.0^{10}6083$	$.0^{10}435$
21	2.9038	.70731	1.1039	.6962	−.1074610	$\overline{11}.2539$	$.0^{10}179$	$.0^{10}124$
22	2.9561	.69381	1.1110	.6759	−.1130370	$\overline{12}.6963$	$.0^{11}497$	$.0^{11}34$
23	3.0116	.68077	1.1170	.6684	−.1189621	$\overline{12}.1038$	$.0^{11}127$	$.0^{12}85$
24	3.0712	.66746	1.1181	.6263	−.1258130	$\overline{13}.4187$	$.0^{12}262$	$.0^{12}16$
25	3.1421	.64329	1.1402	.5883	−.1346074	$\overline{14}.5393$	$.0^{13}346$	$.0^{13}20$
26	3.4770	.54844	1.1993	.3792	−.1819436	$\overline{19}.8056$	$.0^{18}64$	$.0^{18}24$
21′	2.5755	.55383	.9564	.4144	−.2757673	$\overline{28}.4233$	$.0^{27}26$	$.0^{27}1$
16′	2.1777	.57407	.8247	.4664	−.3247350	$\overline{35}.5265$	$.0^{34}34$	$.0^{34}2$
11′	1.5916	.62127	.6348	.5656	−.4042828	$\overline{41}.5717$	$.0^{40}37$	$.0^{40}22$
6′	1.3315	.65903	.5292	.6371	−.4574761	$\overline{46}.2524$	$.0^{45}18$	$.0^{45}1$
1′	0	1	0	1	$-\infty$	$-\infty$	0	0

TABLE IV

No.	(25) a	(26) $z=\sigma(a)/m(a)$	(27) $\Phi(z)$	(28) $\varphi(z)$	(29) $\lambda(z)=\varphi(z)/\phi(z)$	(30) $\mu(z)=z\lambda(z)$	(31) $p\mu(z)=\lambda(a)m(a)$	(32) p	(33) $p\lambda(z)$
1′	∞	0	.5	.39894	.79789	0	0^{ω}	0	0
2′	5	$.19^5$	.579260	.39104	.67507	.13501	$.0^5 74336$	$.0^4 55060$	$.0^4 37169$
3′	4	.24992	.598319	.38676	.64641	.16154	$.0^3 53535$	$.0^2 33140$	$.0^2 21422$
4′	3	.33061	.639526	.37772	.60001	.19836	.013333	.067216	.040330
5′	2	.45810	.676559	.35970	.53166	.24355	.113548	.46622	.24787
6′	1	.61627	.731149	.32993	.45125	.27811	.37031	1.3315	.60084
7′	1.64488	.51316	.696092	.34971	.50239	.25783	.19035	0.73828	.37091
8′	1.28156	.57149	.716168	.33883	.47312	.27039	.28793	1.0649	.50382
9′	1.03645	.61054	.729234	.33111	.45405	.27720	.35955	1.2971	.58895
10′	1.84163	.64078	.739173	.32489	.43953	.28165	.41699	1.4805	.65072
11′	.67449	.66587	.747260	.31961	.42771	.28480	.46530	1.6338	.69879
12′	.52441	.68747	.754114	.31497	.41767	.28714	.50718	1.7663	.73773
13′	.38532	.70676	.760152	.31076	.40881	.28893	.54428	1.8838	.77012
14′	.28335	.72422	.765526	.30692	.40098	.29036	.57774	1.9716	.79047
15′	.12567	.74036	.770471	.30330	.39366	.29145	.60832	2.0872	.82165
16′	0	.75552	.775023	.29989	.38694	.29234	.63662	2.1777	.84264
17′	−.12567	.76994	.779050	.29661	.38073	.29314	.66308	2.2620	.86121
18′	−.25335	.76376	.783419	.29343	.37455	.29356	.68817	2.3442	.87802
19′	−.38532	.79726	.787360	.29031	.36871	.29396	.71214	2.4226	.89324
20′	−.52441	.81057	.791202	.28723	.36303	.29426	.73544	2.4993	.90732
21′	−.67449	.82386	.794988	.28413	.35741	.29446	.75838	2.5755	.92051
22′	−.84163	.83780	.798927	.28086	.35155	.29453	.78133	2.6528	.93259
23′	−1.0364	.85230	.802975	.27744	.34552	.29449	.80512	2.7339	.94462
24′	−1.2816	.86881	.807521	.27353	.33873	.29429	.83089	2.8234	.95637
25′	−1.6449	.89000	.813267	.26848	.33013	.29382	.86174	2.9329	.96824
26′	$-\infty$	1	.841345	.24197	.28760	.28760	1	3.4770	1

In the same way as above Table III (22)–(24) made for $n=100$, we may calculate $\mathfrak{h}_n$ for several values of n. The results are given in Table V and Fig. 11. In particular, when $n\to\infty$, $\mathfrak{h}_n(\tau)$ behaves just as Dirak's δ-Function: $\mathfrak{h}_\infty(\tau)=1$ for $\tau=0$, but$=0$ for $\tau\neq 0$, which corresponds to what the central limit theorem enunciates, i.e. the sampling mean of a sufficiently large size concentrates about its parent mean with almost vanishing S.D., so that it hits the true parent mean.

TABLE V

No.	(34) z	(36) $\tau=1/z$	(36) $\mathfrak{h}_n(\tau)$ for $n=10$	$n=25$	$n=50$	$n=500$	$n=10^3$	$n=10^4$	$n=10^6$	$n=\infty$
1	∞	0	1	1	1	1	1	1	1	1
2	$5.0^{4}20$	.200	$1.0^{4}7$	$1.0^{4}6$	$1.0^{5}3$	.9998	.9986	.9850	.4572	0
3	$4.0^{2}12$	.245	$1.0^{3}32$	.9978	.9936	.9213	.8471	.1868	$.0^{72}11$	0
4	3.0247	.331	1.0132	.9936	.9518	.5361	.2800	$.0^{7}23$	$.0^{564}7$	0
5	2.1829	.458	.8713	.9199	.8112	$.0^{3}107$	$.0^{7}16$	$.0^{77}10$	$.0^{7815}1$	0
(7)	1.949	.513	.7082	.4123	.1746	$.0^{7}252$	$.0^{15}6$	$.0^{151}1$	$.0^{15200}4$	0
(8)	1.750	.571	.6367	.3347	.1145	$.0^{9}476$	$.0^{18}2$	$.0^{186}5$		0
(9)	1.638	.610	.4218	.1235	.0159	$.0^{17}16$	$.0^{35}2$	$.0^{355}2$		0
(6)	1.623	.616	.4054	.1128	.0134	$.0^{18}29$	$.0^{37}9$	$.0^{370}6$		0
10	1.561	.639	.3395	.0750	$.0^{2}61$	$.0^{21}13$	$.0^{43}2$	$.0^{437}8$		0
11	1.502	.666	.2778	.0476	$.0^{2}25$	$.0^{25}27$	$.0^{31}8$	$.0^{510}3$		0
12	1.455	.687	.2312	.0314	$.0^{2}11$	$.0^{28}11$	$.0^{57}1$			0
13	1.415	.707	.1949	.0213	$.0^{3}56$	$.0^{32}75$	$.0^{64}7$			0
14	1.381	.724	.1696	.0149	$.0^{3}26$	$.0^{35}78$	$.0^{70}7$			0
15	1.351	.740	.1417	.0104	$.0^{3}13$	$.0^{37}11$	$.0^{75}1$			0
16	1.324	.755	.1220	$.0^{2}74$	$.0^{4}7$	$.0^{40}2$	$.0^{81}7$			0
17	1.299	.769	.1052	$.0^{2}53$	$.0^{4}4$	$.0^{43}4$				0
18	1.276	.783	.0910	$.0^{2}38$	$.0^{4}1$	$.0^{45}1$				0
19	1.254	.797	.0784	$.0^{2}28$	$.0^{5}6$	$.0^{48}2$				0
20	1.234	.811	.0681	$.0^{2}20$	$.0^{5}3$	$.0^{51}6$				0
21	1.214	.823	.0586	$.0^{2}14$	$.0^{5}2$	$.0^{53}1$				0
22	1.194	.838	.0501	$.0^{2}10$	$.0^{6}8$					0
23	1.173	.852	.0432	$.0^{3}7$	$.0^{6}7$					0
24	1.151	.869	.0346	$.0^{3}4$	$.0^{6}3$					0
25	1.124	.890	.0265	$.0^{3}2$	$.0^{6}1$					0
26	1	1	$.0^{2}58$	$.0^{4}1$	$.0^{9}3$					0
21′	.824	1.12	$.0^{3}7$	$.0^{7}5$						0
16′	.756	1.32	$.0^{3}2$	$.0^{8}1$						0
11′	.666	1.50	$.0^{4}5$	$.0^{10}4$						0
6′	.616	1.62	$.0^{4}2$	$.0^{11}2$	$.0^{24}8$	$.0^{228}1$	$.0^{457}2$	$.0^{4574}1$	$.0^{457475}1$	0
1′	0	∞	0	0	0	0	0	0	0	0

N.B. Since $0<\tau<\sqrt{n-1}$, $\infty>z>1/\sqrt{n-1}$ truly, it must be already $\mathfrak{h}_n(\tau)=0$ for $\tau=\sqrt{n-1}$ in reality; e.g. $0=\mathfrak{h}_{10}(3)=\mathfrak{h}_{25}(4.9)=\mathfrak{h}_{50}(7)$ ……&c.

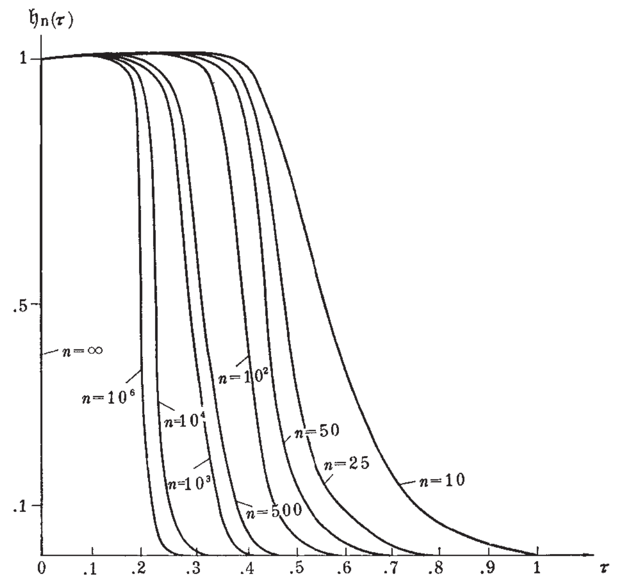

Fig. 11

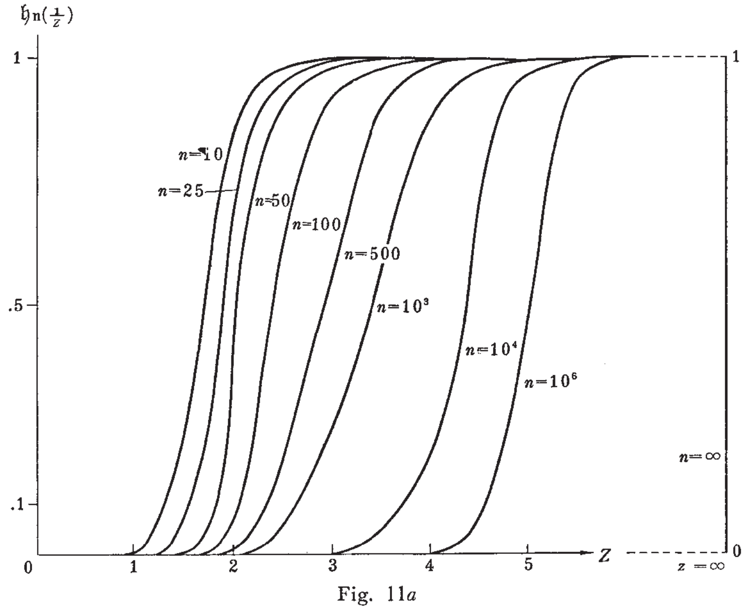

Fig. 11a

24. *A Trial Determination of* $\mathfrak{h}_n$ *under Affirmation of the* C.L.T. *about the Sampling Distribution taken from a* T.N.D. Now we are to explain under the assumption that the C.L.T. exists about our sampling distribution, how reasonably it is determined the asymptotic approximation for $\mathfrak{h}_n(1/z)$ in the form (22.42), i.e.

(24.1) $\mathfrak{h}_n(1/z) \simeq (\Phi^p(z)q(z))^n r(z) \equiv Q^{-n}r,$

where p, q, r denote some positive functions of z, however, as have been seen in Sect. 22, 23, the variations of p and q being slight, their derivatives are neglected.

We consider the total $\bar{x}z$-joint probability taken in the whole domain G: $0 \leq \bar{x} < \infty$, $1/b = 1/\sqrt{n-1} \leq z < \infty$:

(24.2) $$\Pr = c_n \iint_G \exp\left[-\frac{n}{2}(\bar{x}-a)^2 - \frac{n}{2}\frac{x^2}{z^2}\right] \frac{x^{n-1}}{z^n} \Phi^{pn}(z) q^n(z) r(z)\, dz d\bar{x}$$

$$= c_n \iint_G \mathfrak{f}^n(\bar{x},z) \cdot \mathfrak{g}(\bar{x},z)\, dz dx,$$

where

(24.3) $$\begin{cases} \mathfrak{f} = \dfrac{x}{z}\Phi^p(z) q(z) E, \\ \mathfrak{g} = \dfrac{r(z)}{x}, \\ E = \exp\left(-\dfrac{1}{2}(x-a)^2 - \dfrac{1}{2}\dfrac{x^2}{z^2}\right), \\ c_n \simeq n e^{n/2}/\pi\sqrt{2}\,\Phi^n(a), \quad a >=< 0. \end{cases}$$

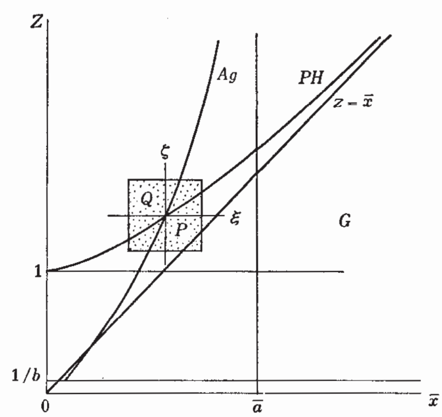

Fig. 12

To compute the integral after Laplace, putting

(24.4) $F = \log \mathfrak{f}$

$$= \log \frac{x}{z}\Phi^p(z) q(z) - \frac{1}{2}(x-a)^2 - \frac{1}{2}\frac{x^2}{z^2},$$

we have to find the point $P_0(x_0, z_0)$, where F becomes maximum, and x_0 is presupposed to be the sample mean $E(\bar{x})$. First writing

(24.5) $F_x = 1/\bar{x} - (\bar{x}-a) - \bar{x}/z^2 = 0,$

we obtain the same Agnesi as defined in (21.17):

(24.6) $$\frac{x^2}{z^2} = 1 - \bar{x}(\bar{x}-a), \text{ i.e. } z = \frac{\bar{x}}{\sqrt{1-\bar{x}(\bar{x}-a)}}, \text{ or } \bar{x} = \frac{a+\sqrt{a^2+4(1+1/z^2)}}{2(1+1/z^2)}$$

with the vertical asymptote $\bar{x} = \bar{a} = \frac{1}{2}(a+\sqrt{a^2+4}) = 2/(\sqrt{a^2+4}-a) > 0$, which tends $\bar{x}=0$, when $a \to -\infty$, but extends indefinitely remote when $a \to +\infty$ (Fig. 12).

The sample-size being sufficiently large, we may assume that the C.L.T. exists, so

(24.7) $x_0 \simeq m = a + \lambda,$

holds, where m denotes the parent mean of the T.N.D., and which substituted in (6) yields immediately the coresponding z

(24.8) $z_0 = m/\sqrt{1-\lambda m} = m/\sigma,$

that coincides with the proper $z(\geq 1)$ in (22.24). Thus already (7) combined with (8) afford the coordinates of the maximum point P_0. Further

(24.9) $F_z = -1/z + p\lambda(z) + x^2/z^3 = 0$

holds under neglections of p', q', which reduces to

(24.10) $\bar{x}^2/z^2=1-p\mu(z)$, a pseud-hyperbola, P.H., whose positive branch only being taken.

Equated (10) and (6), yields

(24.11) $p\mu(z)=\bar{x}(\bar{x}-a)=m\lambda$ after (7), so that $p=m\lambda/\mu(z)$

coincident with (22.38). And when $a\to-\infty$, $z\to 1$, we get $p\to 1/\mu(1)=1/.2876=3.744$, $m\to 0$, $\lambda\to\infty$, but when $a\to+\infty$, $z\to\infty$, it hold $p\to 1$, $m\to\infty$, $\lambda\to 0$. Hence a point on P.H. starting from $(0,1)$ extends indefinitely along its asymptote $z=x$. Since (5) yields already $E_0=\exp\left[-\frac{1}{2}(1-a\lambda)\right]$, the max $\mathfrak{f}$ is attained by

(24.12) $\exp F(m,z_0)=(m\Phi^p(z_0)q(z_0)/z_0)\exp\left(-\frac{1}{2}(1-\mu(a)\right)$, where $\mu(a)<1$ after (22.21).

Now, to integrate (2), writing as before

(24.13) $\bar{x}=m+\xi=m(1+u/N),\ z=z_0+\zeta=z_0(1+v/N),\ N=\sqrt{n},$

we see that, by the same reasoning made in (21.10), we have only to integrate over the small quadrate Q with center P and side 2δ (Fig. 12): As $N=\sqrt{n}$ is sufficiently large, we may expand several terms in power of N and neglect those with negative indices, and obtain approximately

(24.14) $\Pr\simeq c_n\exp(nF(m,z_0))\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\exp\left[n(F(\bar{x},z)-F(m,z_0))\right]\frac{r(z)}{\bar{x}}d\xi d\zeta$

$$\simeq\frac{ne^{n/2}}{\pi\sqrt{2}\,\Phi^n(a)}\left(\frac{m\Phi^p q}{z_0}\exp-\left(\frac{1}{2}-\mu(a)\right)\right)^n\int_{-N\delta/m}^{N\delta/m}\int_{-N\delta/z_0}^{N\delta/z_0}\exp n\left\{\log\left[\left(1+\frac{u}{N}\right)\left(1+\frac{v}{N}\right)^{-1}\Phi^p\left(z_0\left(1+\frac{v}{N}\right)\right)\Big/\right.\right.$$
$$\left.\left.\Phi^p(z_0)\right]-\frac{1}{2}\left(m\left(1+\frac{u}{N}\right)-a\right)^2+\frac{1}{2}(m-a)^2-\frac{1}{2}\frac{m^2}{z_0^2}\left(\left(1+\frac{u}{N}\right)^2\left(1+\frac{v}{N}\right)^{-2}-1\right)\right\}\ \frac{r(z_0)mz_0dudv}{\bar{x}n}\ (\equiv J_n),$$

whose coefficient after (22, 23, 34, 39 and 36) reduces to

(24.15) $$\frac{1}{\pi\sqrt{2}}\left[\frac{m\Phi^p(z_0)q(z_0)}{z_0\Phi(a)}\exp\frac{1}{2}\mu(a)\right]^n=\frac{1}{\pi\sqrt{2}}\left(\frac{1}{Q}\frac{M}{Z}\right)^n=\frac{1}{\pi\sqrt{2}},$$

by using the notations in the previous section. Executing the integration we get

(24.16) $$J_n\simeq z_0r(z_0)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}(Au^2+2Buv+Cv^2)\right]dudv,$$

where coefficients of linear terms Nu, Nv in the brackets have reduced to naught in view of (6) (10) (11), and

(24.17) $A=1+m^2+\sigma^2>0,\quad B=-2\sigma^2,\quad C=p\mu_0(\mu_0+z_0^2)+3\sigma^2-1,\quad z_0=m/\sigma,\quad \mu_0=\mu(z_0),$

whose determinant $D=AC-B^2$ being nothing but (22.44) becomes positive. So we get

(24.18) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}=\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\frac{D}{C}u^2\right)du\int_{-\infty}^{\infty}\exp\left(-\frac{C}{2}\left(v-\frac{B}{C}u\right)^2\right)dv=\frac{2\pi}{\sqrt{D}}.$$

Therefore, the total $\bar{x}z$-joint probability arrives after all

(24.19) $\Pr\simeq E(x_0)\simeq z_0r(z_0)\sqrt{\frac{2}{D}}=1$ by definition of r in (22.45).

Similarly, by multiplying $\bar{x}=m(1-u/N)$ to the integrand of (14) or (16) and integrating, we obtain

(24.20) $E(\bar{x})\simeq m,$

what the C.L.T. claims. Thus our approximation (1) stands in good stead.

However, we have above neglected all negative powers of $N=\sqrt{n}$. Let us seek a more precise expression taking some negative terms, up to $O(1/n)$, say, into account. Writing in short

(24.21) $$\varDelta F=F(m+\xi,\ z_0+\zeta)-F(m,\ z_0)=\sum F_k(\xi,\ \zeta)$$

where $$F_k=\sum_{i+j=k}\frac{a^{ij}}{\lfloor i \ \lfloor j}\xi^i\zeta^j \text{ with } a_{ij}=\frac{\partial^{i+j}F(\bar{x},\ z)}{\partial x^i\partial z^j}\Big|_{\bar{x}=m,\ z=z0=m/\sigma}$$

and to point out the order of magnitudes in evidence, ξ, ζ are expressed in u, v by

(24.22) $\xi=mu/N,\quad \zeta=z_0v/N,\quad N=\sqrt{n},\quad$ as mz_0 is $\neq 0$.

It was already $a_{10}=a_{01}=F_1=0$ and there remains the quadratic

(24.23) $$\begin{aligned} nF_2&=\frac{n}{2}(a_{20}\xi^2+2a_{11}\xi\zeta+a_{02}\zeta^2)\\ &=-\frac{1}{2}[(2+am)u^2-2\sigma^2uv+(p\mu_0(\mu_0+z_0^2)+3\sigma^2-1)v^2]\\ &=-\frac{1}{2}(Au^2+2Buv+Cv^2)=-\mathbf{Q},\text{ say.}\end{aligned}$$

To compute up to $0(1/n)$, we are still further to take $nR=nF_3+nF_4$, but no more:

(24.24) $$\begin{aligned}\exp nR&=1+nF_3(\xi,\ \zeta)+nF_4(\xi,\ \zeta)+\frac{1}{2}n^2F_3^2(\xi,\ \zeta)+0(1/n)\\ &=1+\frac{1}{\sqrt{n}}f_3(u,\ v)+\frac{1}{n}f_4(u,\ v)+0\left(\frac{1}{n}\right),\end{aligned}$$

where F_3, F_4 being homogeneous in $\xi=mu/\sqrt{n}$ and $\zeta=z_0v/\sqrt{n}$ with degree 3 and 4 about $1/\sqrt{n}$, f_3 and f_4 are independent of n. In detail e.g.

(24.25) $$f_3=\frac{1}{3}\sigma^6u^3+\sigma^2(u^2v-3uv^2)+\frac{1}{6}[p\mu_0(2\mu_0+(3\mu_0-1)z_0^2+z_0^4)+12\sigma^2-2]v^3,\ \&\text{c.}$$

The hitherto used determinant

$$D=AC-B^2=(-m^2a_{20})(-z_0^2a_{02})-(-mz_0a_{11})^2=m^2z_0^2(F_{xx}F_{zz}-F_{xz}^2)|_{x=m,\ z=z_0}$$

was a particular one pertaining to the quadratic $\mathbf{Q}$ about u, v. Now it needs to treat the general determinant

$$\mathfrak{D}=\mathfrak{D}(x,\ z)=m^2z_0^2(F_{xx}F_{zz}-F_{xz}^2)|_{x=m+\xi,\ z=z_0+\zeta}.$$

With their general arguments it holds

$$\begin{aligned} m^2F_{xx}&=-A+m^2(a_{30}\xi+a_{21}\zeta)+\frac{1}{2}m^2(a_{40}\xi^2+2a_{31}\xi\zeta+a_{22}\zeta^2)+0(1/n),\\ mz_0F_{xz}&=-B+mz_0(a_{21}\xi+a_{12}\zeta)+\frac{1}{2}mz_0(a_{31}\xi^2+2a_{22}\xi\zeta+a_{13}\xi^2)+0(1/n),\\ z_0^2F_{zz}&=-C+z_0^2(a_{12}\xi+a_{03}\xi)+\frac{1}{2}z_0^2(a_{22}\xi^2+2a_{13}\xi\zeta+a_{04}\zeta^2)+0(1/n).\end{aligned}$$

Therefore we obtain

(24.26) $$\begin{aligned}\mathfrak{D}(x=m+\zeta,\ z=z_0+\zeta)&=m^2z_0^2(F_{xx}F_{zz}-F_{xz}^2)_{x=m+\xi,\ z=z_0+\zeta}\\ =D&-(Az_0^2a_{12}-2Bmz_0a_{21}+Cm^2a_{30})\xi-(Az_0^2a_{03}-2Bmz_0a_{12}+Cm^2a_{21})\zeta\\ &-\left(\frac{1}{2}Az_0^2a_{22}-Bmz_0a_{31}+\frac{1}{2}Cm^2a_{40}-m^2z_0^2(a_{12}a_{30}-a_{21}^2)\right)\xi^2\\ &-\left(\frac{1}{2}Az_0^2a_{13}-2Bmz_0a_{22}+Ca_{31}a_{31}-m^2z_0^2(a_{30}a_{03}-a_{12}a_{21})\right)\xi\zeta\end{aligned}$$

$$-\left(\frac{1}{2}Az_0^2a_{04}-Bmz_0a_{13}+\frac{1}{2}Cm^2a_{22}-m^2z_0^2(a_{21}a_{03}-a_{12}^2)\right)\zeta^2$$
$$=D-L(u,v)/\sqrt{n}-M(u,v)/n+0(1/n),$$

where

(24.27) $L(u,v)=(Az_0^2a_{12}-2Bmz_0a_{21}+Cm^2a_{30})mu+(Az_0^2a_{03}-2Bmz_0a_{12}+Cm^2a_{21})z_0v$, &c.

Consequently

(24.28) $$\sqrt{\frac{\mathfrak{D}}{2}}=\sqrt{\frac{D}{2}}\left[1-\frac{L(u,v)}{2D\sqrt{n}}-\frac{M(u,v)}{2Dn}\right]=\sqrt{\frac{D}{2}}\left[1-\frac{d_1(u,v)}{\sqrt{n}}-\frac{d_2(u,v)}{n}+0\left(\frac{1}{n}\right)\right],$$

where d_1, d_2 are of degree 1, 2 with regard to u, v. In detail,

(24.29) $$d_1=\frac{u}{D}[p\mu_0(\mu_0+z_0^2)+\sigma^4-3m^2\sigma^2]+\frac{v}{D}\left[\frac{1}{2}p\mu_0(2\mu_0+3\mu_0z_0^2+z_0^2(z^2-1)(1+\sigma^2+m^2)\right.$$
$$\left.+p\mu_0(\mu_0+z_0^2)\sigma^2+m^2(6\sigma^2-1)+9\sigma^4-8\sigma^2-1\right],\quad \text{\&c.}$$

In view of (14)-(19) and (21)-(27) the expectation of $\bar{x}^\nu=m^\nu(1+u/N)^\nu$ expanded up to $0(1/n)$ is

(24.30) $$E(\bar{x}^\nu)=\frac{m^\nu}{2\pi}\sqrt{D}\iint_{\text{Quadrate}}e^{-Q/2}\left(1+\frac{f_3}{\sqrt{n}}+\frac{f_4}{n}\right)\left(1-\frac{d_1}{\sqrt{n}}-\frac{d_2}{n}\right)\left(1+\frac{u}{N}\right)^\nu\frac{dudv}{1+(u+v)/N+uv/N^2}$$
$$=\frac{m^\nu}{2\pi}\sqrt{D}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-Q/2}\left[1+\frac{H_\nu}{\sqrt{n}}+\frac{K_\nu}{n}+0\left(\frac{1}{n}\right)\right]dudv,$$

where

(24.31) $H_\nu=\nu u-u-v-d_1+f_3,$

$$K_\nu=\frac{1}{2}\nu(\nu-1)u^2+u^2+uv+v^2+d_1(\nu u-u-v)-d_2+(\nu u-u-v-d_1)f_3+f_4.$$

But H_ν being of odd degree about u, v, the corresponding integral reduces to naught. So that on writing

(24.32) $$\frac{\sqrt{D}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-Q/2}K_\nu(u,v)\,dudv=k_\nu,\qquad \frac{\sqrt{D}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-Q/2}dudv=1,$$

we obtain finally

(24.33) $E(\bar{x}^\nu)=m^\nu(1+k_\nu/n+0(1/n)).$

In particular for $\nu=0$, $1+k_0/n=C_0^{-1}$ say $(\neq 1)$, which discrepancy from 1 appears because in our foregoing approximation for $\mathfrak{h}_n$ the magnitude $0\ (1/n)$ has been ignored. Really, e.g. $r(z)$ should have been multiplied by

(24.34) $C_0=1-k_0/n+0(1/n).$

In fact, if this factor multiplied throughout, we obtain

(24.35) $E(\bar{x}^\nu)=m^\nu(1+k_\nu/n)(1-k_0/n)=m^\nu(1+(k_\nu-k_0)/n)+0(1/n)$ with

(24.36) $$k_\nu-k_0=\frac{\sqrt{D}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-Q/2}\left[\frac{1}{2}\nu(\nu-1)u^2+\nu u(f_3-d_1)\right]dudv$$
$$=\frac{1}{2}\nu(\nu-1)C/D+\nu K,$$

where

(24.37) $$K=\frac{\sqrt{D}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(f_3-d_1)e^{-Q/2}dudv,$$

which value if wanted may be found from (25) (29) by aid of the following table for

(24.38) $$I_{ij}=\frac{\sqrt{D}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-Q/2}u^i v^j\,dudv.$$

(i, j)	(2, 0)	(1, 1)	(4, 0)	(3, 1)	(2, 2)	(1, 3)
I_{ij}	C/D	$-B/D$	$(C/D)^2$	$-B/D$	$1/D$	$-B/D-B^2/D^2$

Thus we get ultimately

(24.39) $$E(\bar{x}^\nu)=m^\nu\Big(1+\frac{\nu}{n}\Big(\frac{\nu-1}{2}\frac{C}{D}+K\Big)\Big).$$ In particular,

(24.40) $$\left.\begin{aligned} &E(\bar{x}^0)=1,\\ &E(\bar{x})=m(1+K/n)\simeq m\\ &E(\bar{x}^2)=m^2\Big(1+\frac{1}{n}\Big(\frac{C}{D}+2K\Big)\Big),\end{aligned}\right\}\text{ for } n\to\infty,$$

(24.41) Variance $D^2(\bar{x})=E(\bar{x}^2)-E(\bar{x})^2=m^2C/nD$,

(24.42) S.D. $\sigma_{\bar{x}}=\frac{m}{\sqrt{n}}\sqrt{\frac{C}{D}}$.

The distribution of $u=\sqrt{n}(\bar{x}-m)/m$, or $u\Big/\sqrt{\frac{C}{D}}=(\bar{x}-m)/\sigma_x$ is given by (30) with $\nu=0$:

$$f(u)=\frac{\sqrt{D}}{2\pi}\exp\Big[-\frac{1}{2}\frac{D}{C}u^2\Big]\int_{-\infty}^{\infty}\exp\Big[-\frac{C}{2}\Big(v+\frac{B}{C}u\Big)^2\Big]\Big(1+\frac{H_0(u,v)}{\sqrt{n}}+\frac{K_0(u,v)}{n}\Big)dv$$
$$=\frac{1}{\sqrt{2\pi}}\sqrt{\frac{D}{C}}\exp\Big(-\frac{1}{2}\frac{D}{C}u^2\Big)\Big(1+O\Big(\frac{1}{n}\Big)\Big).$$

But

(24.43) $$\frac{u}{\sqrt{\frac{C}{D}}}=\frac{\bar{x}-m}{\frac{m}{\sqrt{n}}\sqrt{\frac{C}{D}}}=\frac{\bar{x}-E(\bar{x})}{\sigma_x}$$

being nothing but the standardized $\bar{x}$, the C.L.T. concerned with the sample mean $\bar{x}$ taken from a T.N.D. has already been thereby reassured.

25. *The Simplified Student Ratio $z=\bar{x}/s$ (or $\tau=s/\bar{x}$) as a Random Variable, and Its Probability Function.* First we consider the truncated Laplace distribution $f(x)=e^{-x}(x>0)$ as universe. The $\bar{x}s$-joint sampling f.f. being

(25.1) $$e^{-n\bar{x}}dV_n=l_n e^{-n\bar{x}}s^{n-2}\mathfrak{h}_n(s/\bar{x})d\bar{x}ds=l_n e^{-n\bar{x}}\bar{x}^{n-1}\mathfrak{h}_n(\tau)\tau^{n-2}d\bar{x}d\tau$$

with $l_n=2\sqrt{\pi}^{\,n-1}\sqrt{n}^{\,n}/\Gamma((n-1)/2)$, the total probability becomes

(25.2) $$1=l_n\int_0^\infty \bar{x}^{n-1}e^{-n\bar{x}}d\bar{x}\int_0^{b=\sqrt{n-1}}\mathfrak{h}_n(\tau)\tau^{n-2}d\tau.$$

Thus $\bar{x}$ and τ (or z) being independent, we may integrate the first half about $\bar{x}$ or $\xi=n\bar{x}$ and obtain a constant

(25.3) $$k_n=\frac{l_n}{n^n}\int_0^\infty \xi^{n-1}e^{-\xi}\,d\xi=\frac{2}{\sqrt{\pi}}\sqrt{\frac{\pi}{n}}^{\,n}\frac{\Gamma(n)}{\Gamma((n-1)/2)}=\frac{2^{n-1}(n-1)}{\pi}\sqrt{\frac{\pi}{n}}^{\,n}\Gamma\Big(\frac{n}{2}\Big),$$

which however may be approximated asymptotically as

$$\simeq\sqrt{\frac{n}{\pi}}\sqrt{\frac{2\pi}{e}}^{\,n}\equiv\sqrt{\frac{n}{\pi}}A^{-n}\text{ say.}$$

However, the ratios of the true k_n to the last approximation are e.g. 1.102, 1.019, 1.009, $\cdots$, $1+0$ for $n=10$, 50, 100, $\cdots\infty$, so that, unless n is quite large, rather the true expression is to be recommended. We have therefore

$$(25.4)\qquad 1=k_n\int_0^{b=\sqrt{n-1}}\mathfrak{h}_n(\tau)\,\tau^{n-2}d\tau=k_n\int_{1/b}^{\infty}\mathfrak{h}_n\left(\frac{1}{z}\right)\frac{dz}{z^n}.$$

This indentity shows indeed that the simplified Student ratio $\tau=\bar{x}/s$ or $z=s/\bar{x}$ conceived quite apart from several concrete distributions, Laplace or T.N.D. and such like, may be seen as an independent random variable with the f.f.

$$(25.5)\qquad f(\tau)=k_n\mathfrak{h}_n(\tau)\,\tau^{n-2}\quad\text{or}\quad f(z)=k_n\mathfrak{h}_n(1/z)/z^n.$$

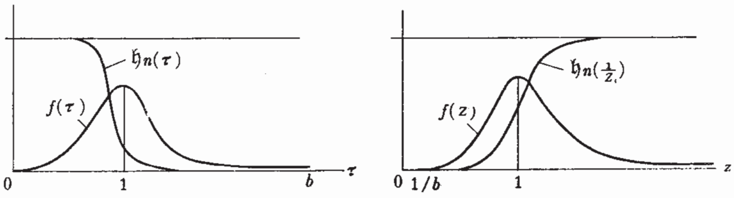

Fig. 13

The depression in the first half interval $0<\tau<1$ or the last half interval $1<z<\infty$ is caused by the introduction of the factor τ^{n-2} or z^{-n}, while in the other half, $1<\tau=b=\sqrt{n-1}$ or $0<1/b<z<1$ the appearance of this factor influses so to speak the dying frequency not to damp so suddenly; really there the amplitude falls into a strong decay, as was seen in the end of Sect. 21. Consequently the whole f.f. is reduced to a usual bell-shaped configuration (Fig. 13).

Truly, if our asymptotic approximation for $\mathfrak{h}_n(1/z)$ be applied as in (24.1), the expectation $E(z)^\nu$ ($\nu=0$, 1, 2) should be

$$(25.6)\qquad E(z^\nu)=\int_{1/b}^{\infty}z^\nu f(z)\,dz=k_n\int_{1/b}^{\infty}\left(\frac{1}{z}\Phi^p(z)\,q(z)\right)^n z^\nu r(z)\,dz\equiv k_nJ_n.$$

In order to make Laplace method applicable here, we decompose the integrand to the main and two subsidiary factors as follows:

$$(25.7)\qquad J_n=\int_{1/b}^{1}\mathfrak{f}^n g(z)dz+\int_1^{\infty}\mathfrak{f}^{n-4}h(z)\,dz,\ \text{where}$$

$$(25.8)\qquad \mathfrak{f}=\Phi^p(z)\,q(z)/z,\quad g=z^\nu r(z),\quad h=\mathfrak{f}^4z^\nu r(z),\quad \nu=0,\ 1,\ 2.$$

These factors are absolutely integrable in the respective subinterval $1/b<z<1$, $1<z<\infty$, since, we have $\mathfrak{f}=1/zQ(z)$ (cf. (22.41)), $Q^{-1}(z)<1$ after Fig. 9 and $r(z)=\frac{1}{z}\sqrt{\mathfrak{D}/2}<1.2$ by (22.43) and Fig. 10. Writing as before

$$(25.9)\qquad F=\log\mathfrak{f}=p\log\Phi(z)+\log q(z)-\log z,$$

we get the derivative about z

$$F'=p\lambda(z)-1/z$$

under assumption that p', q' are negligibly small. Hence, making

$$F'=p\lambda(z)-1/z=0,\quad\text{i.e.}\quad p\mu(z)=\lambda(a)m(a)=1,$$

it is seen that the parameter $a\sim-\infty$ and $z\simeq 1$ already. In consequence we get

(25.10) $$F_1'=0,\quad z\simeq 1,\quad \lambda(1)=\lambda_1=\mu_1=0.2876,\quad p=1/\lambda_1=3.477,$$
$$F_1''=-p\lambda(\lambda+z)+1/z^2\,|_{z=1}=-\lambda_1<0,$$
$$F_1'''=2\lambda_1^2+3\lambda_1-2=-0.9718,$$
$$F_1^{\mathrm{IV}}=8-3\lambda_1(1+4\lambda_1+2\lambda_1^2)=6.002,\ \&\text{c.}$$

Accordingly F becomes maximum at $z=1$, $a\sim-\infty$. Take a small interval with center at $z=1$ and breadth 2δ, and put

(25.11) $$J_n=\int_{1/b}^{1-\delta}+\int_{1-\delta}^{1+\delta}+\int_{1+\delta}^{\infty}=\text{(i)}+\text{(ii)}+\text{(iii)}.$$

Now that in (i) and (iii) the inequalities $0<\mathfrak{f}/\mathfrak{f}_1<e^{-\varepsilon}<1$, i.e. $F(z)-F(z_1)<-\varepsilon<0$ hold,

$$\text{(i)}<\exp nF_1\int_{1/b}^{1-\delta}\exp n(F(z)-F_1)\cdot g dz<\mathfrak{f}_1^{\,n}\cdot e^{-n\varepsilon}\int_{1/b}^{1}g dz=0(1/n^{\omega})$$

and similarly $\text{(iii)}=0(1/n^{\omega})$, so that they are all negligible for sufficiently large n. Hence we have only to treat

(25.12) $$\text{(ii)}=\mathfrak{f}_1^{\,n}\int_{1-\delta}^{1+\delta}\exp\ n(F(z)-F_1)\ \cdot g\,dz.$$

First to evaluate $\mathfrak{f}_1$ we remind formula (22.27). As $z=\lim\limits_{a=-\infty}\dfrac{m(a)}{\sigma(a)}=1$, we obtain

$$\mathfrak{f}(z)=\frac{1}{z}\Phi^{p}(z)q(z)=\frac{1}{zQ}=\frac{Z}{z}\exp\Big(-\frac{1}{2}\mu(a)\Big)=\frac{\Phi(a)}{m(a)}\exp\Big(-\frac{1}{2}a\lambda(a)\Big)$$
$$=\varphi(a)\Big(1-\frac{1}{a^2}+\frac{3}{a^4}\Big)\Big(1+\frac{2}{a^2}-\frac{6}{a^4}\Big)\exp\Big(\frac{a^2}{2}+\frac{1}{2}-\frac{1}{a^2}+\frac{5}{a^4}\Big)$$
$$\simeq\sqrt{\frac{e}{2\pi}}\Big(1+\frac{1}{a^2}-\frac{5}{a^4}\Big)\Big(1-\frac{1}{a^2}+\frac{11}{2a^4}\Big)\simeq\sqrt{\frac{e}{2\pi}}\Big(1-\frac{1}{2a^4}\Big),$$

and consequently

(25.13) $$\mathfrak{f}_1=\lim_{a\to-\infty}\mathfrak{f}(z(a))=\sqrt{\frac{e}{2\pi}}=.6577=A\ \ (\text{cf. }(3)).$$

Further we require to calculate (ii) up to $0(1/n)$. For this purpose we put

(25.14) $$z=1+\zeta=1+v/N\ \text{with}\ N=\sqrt{\frac{1}{2}n\lambda_1}=0.3792\sqrt{n},\quad v^2=N^2\zeta^2=\frac{1}{2}n\lambda_1\zeta^2,$$

and expand every factor in the integrand of (12) up to $1/N^2$. Now in view of (10) we get

(25.15) $$J_n\simeq\text{(ii)}=A^n\int_{-\delta}^{\delta}\exp\Big[-\frac{n}{2}\lambda_1\zeta^2+\frac{n}{6}F_1'''\zeta^3+\frac{n}{24}F_1^{\mathrm{IV}}\zeta^4\Big]\cdot(1+\zeta)^{\nu-1}\sqrt{\mathfrak{D}(1+\zeta)/2}\ d\zeta$$

whose main factor becomes

(25.16) $$e^{-v^2}\Big[1+\frac{\alpha}{\sqrt{n}}v^3+\frac{\beta}{n}v^4+\frac{\alpha^2}{2n}v^6\Big],\quad\text{where}$$
$$\alpha=\frac{F'''}{6}\sqrt{\frac{2}{\lambda_1}}^{\,3}=-2.970,\quad \beta=\frac{F_1^{\mathrm{IV}}}{6\lambda_1^2}=12.094.$$

We should further compute subfactors. Really the first subfactor becomes

(25.17) $$(1+\zeta)^{\nu-1}=1+(\nu-1)\zeta+\frac{1}{2}(\nu-1)(\nu-2)\zeta^2+\cdots$$
$$=1-3+\zeta^2,\ 1,\ 1+\zeta,\quad\text{according as }\nu=0,\ 1,\ 2.$$

The second subfactor is after (22.44)

(25.18) $$\sqrt{\mathfrak{D}(z)/2}=\frac{1}{\sqrt{2}}[(z^2+\mu(z))(1+m^2+\sigma^2)(1-\sigma^2)-m^2(1-3\sigma^2)-(1-\sigma^2)^2]^{1/2},$$

which is developable in ζ as follows: Since in the vicinity of $z=1$ we have by (22.27)

$$1<z=\frac{m(a)}{\sigma(a)}=1+\frac{1}{a^2}-\frac{15}{2a^4},\quad \text{as well as}\quad 1>z=\frac{\sigma(a)}{m(a)}=1-\frac{1}{a^2}+\frac{17}{2a^4},$$

so that the deviation of z from 1 is to be seen as almost $\zeta\simeq\pm(1/a^2-8/a^4)$ for $z\gtrless 1$. But the values of m^2 and σ^2 corresponding to these $z=1+\zeta$ are obtained again by (22.27) and those terms rearranged after power of ζ yield

$$m^2=\frac{1}{a^2}-\frac{8}{a^4}+4\Big(\frac{1}{a^2}-\frac{8}{a^4}\Big)^2+88\Big(\frac{1}{a^2}+\cdots\Big)^3\simeq|\zeta|+4\zeta^2+\cdots \quad \text{as well as}$$

$$\sigma^2=\frac{1}{a^2}-\frac{8}{a^4}+2\Big(\frac{1}{a^2}-\frac{8}{a^4}\Big)^2+82\Big(\frac{1}{a^2}+\cdots\Big)^3\simeq|\zeta|+2\zeta^2+\cdots.$$

On the other hand

$$\lambda(z)=\lambda(1+\zeta)\simeq\lambda_1-(\lambda_1+\lambda_1^2)\zeta+\Big(\frac{3}{2}\lambda_1^2+\lambda_1^3\Big)\zeta^2+\cdots,$$

$$\mu(z)=z\lambda(z)\simeq\lambda_1-\lambda_1^2\zeta-\Big(\lambda_1-\frac{1}{2}\lambda_1^2-\lambda_1^3\Big)\zeta^2+\cdots,\quad z^2=1+2\zeta+\zeta^2.$$

All these substituted in (18), we obtain

(25.19) $$\mathfrak{D}(1+\zeta)\simeq\lambda_1+(4+\lambda_1-\lambda_1^2)\zeta+\Big(11+\lambda_1-\frac{1}{2}\lambda_1^2+\lambda_1^3\Big)\zeta^2 \qquad \text{for } \zeta>0,$$

$$\simeq\lambda_1-(\lambda_1+\lambda_1^2)\zeta+\Big(7+\lambda_1+\frac{3}{2}\lambda_1^2+\lambda_1^2\Big)\zeta^2 \qquad \text{for } \zeta>0.$$

And in these expressions $\zeta=v/N$ being substituted, we get finally

(25.20) $$\sqrt{\mathfrak{D}(1+\zeta)/2}\simeq\sqrt{\frac{\lambda_1}{2}}\Big[1+\frac{Lv}{N}+\frac{Mv^2}{N^2}+0\Big(\frac{1}{n}\Big)\Big],\quad \text{where}$$

$$L=\frac{2}{\lambda_1}+\frac{1-\lambda_1}{2},\qquad M=-\frac{2}{\lambda_1^2}+\frac{9}{2\lambda_1}+\frac{11}{8}+\frac{3}{8}\lambda_1^2 \qquad \text{for } v>0,$$

$$L'=-\frac{1}{2}(1+\lambda_1),\qquad M'=\frac{7}{2\lambda_1}+\frac{3}{8}+\frac{\lambda_1}{2}+\frac{3}{8}\lambda_1^2 \qquad \text{for } v>0.$$

Now we can compute $E(z^\nu)$: First for $\nu=1$, we get from (3) (6) (15)-(20)

(25.21) $$E(z)=k_nJ_n=\frac{1}{\sqrt{\pi}}\int_0^{N\delta\sim\infty}e^{-v^2}\Big[1+\frac{L}{N}v+\frac{\alpha}{\sqrt{n}}v^3+\frac{M}{N^2}v^2+\Big(\frac{L\alpha}{\sqrt{n}N}+\frac{\beta}{n}\Big)v^4+\frac{\alpha^2}{2n}v^6\Big]dv$$

$$+\frac{1}{\sqrt{\pi}}\int_{-N\delta\sim-\infty}^{0} \;\;〃\;(L,\,M \text{ in } \;〃\;\;〃\; \text{ replaced by } L',M'),$$

in which however we need the identities:

$$j_{2p+1}=\int_0^\infty v^{2p+1}e^{-v^2}dv=-\int_{-\infty}^0 \;〃\; =\frac{1}{2}\Gamma(p+1),\quad \text{e.g. } j_1=j_3=1/2;$$

as well as

$$j_{2p}=\int_0^\infty v^{2p}e^{-v^2}dv=\int_{-\infty}^0 \;〃\; =\frac{1}{2}\Gamma\Big(p+\frac{1}{2}\Big),\ \text{e.g. } j_0=\frac{\sqrt{\pi}}{2}\ j_2=\frac{\sqrt{\pi}}{4},\ j_4=\frac{3}{8}\sqrt{\pi},\ j_6=\frac{15\sqrt{\pi}}{16}.$$

Hence, executing integration (21), we obtain

(25.22) $$E(z)=1+\frac{L-L'}{2N\sqrt{\pi}}+\frac{M+M'}{4N^2}+\frac{3}{8}\Big(\frac{(L+L')\alpha}{\sqrt{n}N}+\frac{2\beta}{n}\Big)+\frac{15}{16}\frac{\alpha^2}{n}$$

$$=1+A/\sqrt{n}+B_1/n,\ \text{say}.$$

Similarly we get for $\nu=2$ and 0

(25.23) $$E(z^2)=E(z)+\frac{L+L'}{4N^2}+\frac{3\alpha}{4N\sqrt{n}}=1+\frac{A}{\sqrt{n}}+\frac{B_2}{n},$$

(25.24) $$E(z^0)=E(1)=-\frac{L+L'}{4N^2}-\frac{3\alpha}{4N\sqrt{n}}+\frac{1}{2N^2}=1+\frac{A}{\sqrt{n}}+\frac{B_0}{n}=C_0^{-1}\text{ say.}$$

However it should hold $E(z^0)=1$. This apparent discrepancy arised because in our previous approximation no negative power of n or N has been regarded. To take those terms into account, it must be multiplied by

(25.25) $$C_0=1-\frac{A}{\sqrt{n}}+\frac{A^2-B_0}{n}.$$ And thus we get exactly up to $0(1/n)$

(25.26) $E(z^0)=1$ exactly; $E(z)=1+(B_1-B_0)/n$; $E(z^2)=1+(B_2-B_0)/n$.

So that the variance and S.D. are

(25.27) $$D^2(z)=E(z^2)-E(z)^2=(B_0+B_2-2B_1)/n=1/2N^2=1/n\lambda_1,$$

(25.28) $$\sigma_z=1/\sqrt{n\lambda_1}.$$

Therefore the exact sample mean is

(25.29) $$E(z)=1+\frac{B_1-B_0}{n}=1+\frac{L+L'}{4N^2}+\frac{3\alpha}{4N\sqrt{n}}-\frac{1}{2N^2}$$
$$=1+\frac{1}{n\lambda_1^2}\left(1-\lambda_1-\frac{1}{2}\lambda_1^2+\frac{1}{2}F_1'''\right)=1+\frac{2.238}{n},$$

which is larger than 1, yet tends 1 for tolerably large n. Hence the standardized z is

(25.30) $$x=\frac{z-1}{1/\sqrt{n\lambda_1}}\ (=v\sqrt{2}).$$

But, after (21) the f.f. of v for large n being $\frac{1}{\sqrt{\pi}}e^{-v^2}$, that of x tends $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, which shows the existence of the C.L.T. for the variable z.

26. *The Student's Distribution for the Large Sample from* T.N.D. The n-sized sampling $\bar{x}s$-joint distribution taken from a T.N.D. with the parent mean m and variance σ^2 is given in the domain $0<\bar{x}<\infty$, $0<s<bx$, $b=\sqrt{n-1}$, by

$$f_n(\bar{x},\ s)\,d\bar{x}\,ds=c_n\exp\left[-\frac{n}{2}(\bar{x}-a)^2-\frac{n}{2}s^2\right]s^{n-2}\mathfrak{h}_n(s/\bar{x})\,d\bar{x}ds,$$

where n being pretty large, $\mathfrak{h}_n$ is approximated by (24.1), and the coefficient

$$c_n=\frac{2}{\sqrt{\pi}\,\Gamma((n-1)/2)}\left(\sqrt{\frac{n}{2}}\frac{1}{\Phi(a)}\right)^n=\frac{2^{n-1}(n-1)\Gamma(n/2)}{\pi\Gamma(n)}\left(\sqrt{\frac{n}{2}}\frac{1}{\Phi(a)}\right)^n\simeq\frac{ne^{n/2}}{\pi\sqrt{2}\,\Phi^n(a)},$$

in which the ratio of the last approximation to the true c_n, e.g. in case $n=10$, 50, 100, $\cdots$ are .9917, .9983, .9992, $\cdots$, but it tends $1-0$ for $n\to\infty$. Or, replaced s by Student's $t=b(\bar{x}-m)/s$ in the $\bar{x}t$-domain $0<\bar{x}<\infty$, $-\infty<t<\infty$ yields

$$f_n(\bar{x},\ t)\,d\bar{x}dt=c_nb^{n-1}\exp\left[-\frac{n}{2}(\bar{x}-a)^2-\frac{n}{2}\left(\frac{b(\bar{x}-m)}{t}\right)^2\right]\cdot\mathfrak{g}_n\left(\frac{b(\bar{x}-m)}{\bar{x}t}\right)\frac{|\bar{x}-m|^{n-1}}{|t|^n}d\bar{x}dt.$$

Lastly, when the argument of $\mathfrak{g}_n$ is transformed back into $1/z=b(\bar{x}-m)/\bar{x}t$, namely $\bar{x}=mbz/(bz-t)$, we obtain the zt-joint distribution:

$$f_n(z,\ t)\,dzdt=c_nb^nm^n\exp\left[-\frac{n}{2}\frac{(b\lambda z+at)^2+b^2m^2}{(bz-t)^2}\right]\cdot\mathfrak{g}_n\left(\frac{1}{z}\right)\frac{dzdt}{|bz-t|^{n+1}},$$

in the zt-domain $1/b<z<\infty$, $-\infty<t<bz$, and Student's f.f. by adopting (24.1)

(26.1) $$s_n(t)\,dt = c_n m^n dy \int_{1/b}^{\infty} \exp\left[-\frac{n}{2}\,\frac{(\lambda z+ay)^2+m^2}{(z-y)^2}\right] \Phi^{np}(z)\,q^n(z)\,\frac{r(z)\,dz}{|z-y|^{n+1}},$$

where $y=t/b$ $(-\infty<y<\infty)$ is written for the sake of brevity, and finally the d.f.

(26.2) $$S_n(t_\alpha) = c_n m^n \int_{-\infty}^{t\alpha/b} dy \int_{1/b}^{\infty} \left[\exp\left(-\frac{1}{2}\,\frac{(\lambda z+ay)^2+m^2}{(z-y)^2}\right)\frac{\Phi^p(z)\,q(z)}{|z-y|}\right]^n \frac{r(z)}{|z-y|}dz.$$

The whole domain is G: $1/b\leq z<\infty$, $-\infty<t<bz$ and as we are concerned with large samples, the initial boundary $z=1/b=1/\sqrt{n-1}$ is nearly zero. Hence on this boundary also $\Phi^p(z)\,q(z)=\bar{Q}^{-1}(z)$ is almost vanishing (cf. (22.41) and Fig. 8, 9). Clearly our integrand behaves continuous everywhere in G, since not only the negative exponential is bounded under 1, but also $\Phi^p(z)\,q(z)$ is a continuous positive fraction $1/Q(z)$, so that the whole bracketed expression $=\mathfrak{f}$ say, remains finite, and $r(z)$ is so also.

We call conveniently the loci on which the integrand $\mathfrak{f}$ tends to naught, the null lines, and truly it occurs on the line $y=z$ finitely, in virtue of $\exp[-A/(y-z)^2]/(z-y)\simeq 0^\omega$, for $A>0$ and however great ω, besides on the lines at infinity $z=\infty$, $y=-\infty$. Also the initial boundary line $z=1/b$ for large sample may be seen almost a null line as said above. Noticing that the null line $y=z$ constitutes at the same time a boundary, as $-\infty<t<bz$ means $-\infty<y<z$, we see that the whole domain G is surrounded by null lines. Therefore the continuous function $\mathfrak{f}$ defined in it should have a maximum inside G. We are interested to show that for $z\to\infty$, $t_\alpha\to\infty$, the total probability $S\,(t_\alpha)$ tends 1, which might be a matter of course, but to ascertain the validity of our approximation (24.1), and further to obtain the expectation $E(y^\nu)=E(t^\nu)/b^\nu$ $(\nu=0, 1, 2)$:

(26.3) $$E(y^\nu) = d_n \int_{-\infty}^{\infty} dy \int_{1/b}^{\infty} \mathfrak{f}^{n-\nu-1}(y,z)\cdot\mathfrak{g}_\nu(y,z)\,dz \equiv d_n J_\nu, \text{ where } d_n=c_n m^n \text{ and}$$

(26.4) $$\mathfrak{f} = \frac{\Phi^p(z)\,q(z)}{|z-y|}\exp\left[-\frac{1}{2}\,\frac{(\lambda z+ay)^2+m^2}{(z-y)^2}\right], \quad \mathfrak{g}_\nu = \frac{\mathfrak{f}^{\nu+1}y^\nu r(z)}{|z-y|},$$

so that $\mathfrak{g}_\nu$ is finitely integrable in the whole domain G, since it becomes at least 0^2 in the vicinity of null lines (cf. (22.37)). Writing as before

(26.5) $$F(y,z) = \log\mathfrak{f} = p\log\Phi(z) + \log q(z) - \log|z-y| - \frac{(\lambda z+ay)^2+m^2}{2(z-y)^2},$$

and differentiating about y, z, we get

(26.6) $$F_y = [y^2-(2+am)yz+\sigma^2 z^2-m^2]/(z-y)^3 \equiv H(y,z)/(z-y)^3,$$

and under assumption that p', q' are negligibly small,

(26.7) $$F_z = p\lambda(z) - [z^2-(2+\lambda m)yz+(1-am)y^2-m^2)]/(z-y)^3 = p\lambda(z) - K(y,z)/(z-y)^3.$$

Both $H=0$, $K=0$ denote ordinary hyperbola OH. As F_y is $=H$ divided by $(z-y)^3$, the locus $F_y=0$ coincides with the hyperbola $H=0$. However, the locus $F_z=0$ being somewhat deformed from the hyperbola $K=0$, call it a pseud-hyperbola PH.

Now to obtain the maximum point of F in G, we have to solve $F_y=0$, $F_z=0$, which for $y=0$, reduce to

(26.8) $$F_y = (\sigma^2 z^2-m^2)/z^3 = 0, \quad F_z = p\lambda(z) - (z^2-m)/z^3 = 0.$$

The first equation gives $z=m/\sigma$, so that $p\mu(z)=\lambda m$ by (22.38) and $p\lambda(z)=\lambda\sigma$. These substituted in the second equation, yields $\lambda\sigma-\frac{\sigma}{m}+\frac{\sigma^3}{m}=\frac{\sigma}{m}(\lambda m-1+\sigma^2)=0$. Hence the point $P_0(y=0,\ z=m/\sigma=z_0)$ may afford the required maximum. But, then it holds

(26.9) $$F_{yy}\Big|_0=\frac{2y-(2+am)z}{(z-y)^3}+\frac{3H}{(z-y)^4}\Big|_0=-\frac{\sigma^2}{m^2}(1+m^2+\sigma^2)\equiv -A(<0),$$

$$F_{yz}\Big|_0=\frac{2\sigma^2 z-(2+am)y}{(z-y)^3}-\frac{3H}{(z-y)^4}\Big|_0=\frac{2\sigma^4}{m^2}\equiv -B.$$

$$F_{zz}\Big|_0=-p\lambda(z)(\lambda(z)+z)-\frac{2z-(2+\lambda m)y}{(z-y)^3}+\frac{3K}{(z-y)^4}\Big|_0=-\frac{\sigma^2}{m^2}[\lambda m(\mu_0+z_0^2)+2-3\lambda m]\equiv -C, \text{ say,}$$

and whose determinant becomes

(26.10) $$D=AC-B^2=\frac{\sigma^4}{m^4}[(2+am)\lambda m(\mu_0+z_0^2)+m^2(2-3\lambda m-\lambda^2)]$$

$$\equiv\frac{1}{z_0^4}\mathfrak{D}(z_0,\ a)>0 \text{ (cf. (22.44'))},$$

So that $F(0,\ z_0=m/\sigma)$ furnishes certainly maximum at P_0. It remains only to repeat the process before done: We describe a small open quadrate Q with center P_0 and side 2δ, δ being small enough (Fig. 14), and rewrite the integral of (3)

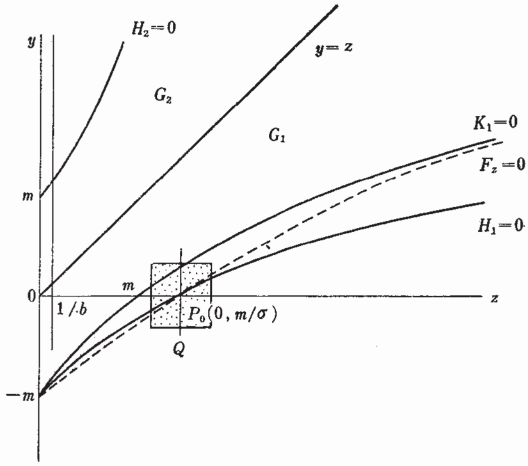

Fig. 14

(26.11) $$J_\nu=\int_{-\infty}^{\infty}\int_{1/b}^{\infty}\exp(n-\nu-1)F(y,z))\cdot\mathfrak{g}_\nu(y,z)\,dzdy$$

$$=\exp(n-\nu-1)F(0,z_0)\iint_G\exp(n-\nu-1)[F(y,z)-F(0,z_0)]\ \mathfrak{g}_\nu\ dzdy\ (=J_G).$$

We decompose the integral as

$$J_G=\iint_G=\iint_Q+\iint_{R=G-Q}=\text{(i)}+\text{(ii)}$$

The max. $F(y,z)$ in the remainder-domain R exists on its closed boundary (Q's open boundary) at which satisfies $\exp\{\max F(y,z) \text{ in } R-F(0,z_0)\}<e^{-\varepsilon}$ with finite positive ε, and much more $\exp\{\text{any } F(y,z) \text{ in } R-F(0,z_0)\}<e^{-\varepsilon}<1$ stands. Hence it holds

$$\text{(ii)}<\exp(-(n-\nu-1)\varepsilon\iint_R\mathfrak{g}_\nu\ dydz=O(n^{-\omega}), \text{ howsoever great } \omega \text{ may be.}$$

Therefore (ii) is negligibly small compared with (i), and (i) is only to be treated.

Now we transform the coordinates so as $z=z_0+\zeta=z+v/N$, $y=\eta=u/N$, $N=\sqrt{n}$, and expand the integrand in powers of N, neglecting those with negative indices, as a first approximation. First for $\nu=0$ we obtain

$$(26.12)\qquad E(y^0)=d_nJ_0=d_n\mathfrak{f}^n(0,z_0)\frac{r(z_0)}{z_0}\iint_Q \exp\ n[F(y,z)-F(0,z_0)]d\eta d\zeta=K_nj_0,$$

where the coefficient K_n is the product of the factors put out beforehand from $\mathfrak{f}^{n-1}\mathfrak{g}_0$ and obtained after neglection of negative powered terms. But, after formulas in Sect. 22 we get

$$(26.13)\qquad \mathfrak{f}\left(0,\ z_0=\frac{m}{\sigma}\right)=\frac{1}{z_0}\Phi^p(z_0)q(z_0)\exp\left(-\frac{1}{2}(\lambda^2+\sigma^2)\right)=\frac{1}{z_0Q(z_0)}\exp-\frac{1}{2}(\lambda^2+1-\lambda m)$$
$$=\frac{\sigma Z}{m}e^{-\mu(a)/2}\exp\left(-\frac{1}{2}(1-a\lambda)\right)=\frac{\Phi(a)}{m}e^{-1/2},$$

and by (22.43) and (10)

$$(26.14)\qquad \frac{1}{z_0}r(z_0)=\frac{1}{z_0^{\ 2}}\sqrt{\mathfrak{D}(z_0)/2}=\sqrt{D/2}.$$

So that the adjoined coefficient becomes

$$(26.15)\qquad K_n=c_nm^n\mathfrak{f}^n(0,z_0)\frac{r(z_0)}{z_0}=\frac{ne^{n/2}m^n}{\pi\sqrt{2}\,\Phi^n(a)}\left(\frac{\Phi(a)}{m}e^{-1/2}\right)^n\sqrt{\frac{D}{2}}=\frac{n}{2\pi}\sqrt{D}.$$

As to the main integral, we obtain in view of (8) and (9)

$$(26.16)\qquad j_0=\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\exp n\left[-\frac{1}{2}(A\eta^2+2B\eta\zeta+C\zeta^2)\right]d\eta d\zeta$$
$$\simeq\int_{-N\delta}^{N\delta}\int_{-N\delta}^{N\delta}\exp\left[-\frac{1}{2}(Au^2+2Buv+Cv^2)\right]\frac{dudv}{N^2}.$$

Now that $N=\sqrt{n}$ is sufficiently large, this integral may be approximated after Laplace method by the infinite integral:

$$(26.17)\qquad j_0\simeq\frac{1}{n}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\frac{D}{C}u^2\right)du\int_{-\infty}^{\infty}\exp\left[-\frac{C}{2}\left(v+\frac{B}{C}u\right)^2\right]dv$$
$$=\frac{1}{n}\sqrt{\frac{2\pi}{C}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\frac{D}{C}u^2\right)du=\frac{2\pi}{n\sqrt{D}},$$

and we get at length

$$(26.18)\qquad S_n(\infty)=\lim_{t_\alpha\to\infty}\int_{-\infty}^{t_\alpha}s_n(t)\ dt=E(y^0)=d_nJ_0=K_nj_0\simeq 1.$$

By the way, if the upper limit t_α be made 0, we have the lower half plane as the domain of integration G_-. Yet the max. F in G_- being still the same as before, we have only to take the lower half of the quadrate Q chosen above, so that the domain of integration is now $-N\delta<u<0$, $-N\delta<v<N\delta$. Consequently the results becomes just the half of the preceding, and we obtain

$$(26.19)\qquad S_n(0)=\int_{-\infty}^{0}s_n(t)\,dt=\frac{1}{2}=\overline{S}_n(0)=\int_0^{\infty}s_n(t)\,dt.$$

Next, upon computing similarly for $\nu=1$, we get

$$E(y)=d_nJ_1=K_nj_1,\qquad\text{where}$$

$$j_1=\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\exp\ n(F(\eta,z_0+\zeta)-F(0,z_0))\eta d\eta d\zeta\simeq\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}(Au^2+2Buv+Cv^2)\right]\frac{ududv}{N^3}$$

$$=\frac{1}{n\sqrt{n}}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\frac{D}{C}u^2\right]udu\sqrt{\frac{2\pi}{C}}=\frac{1}{n}\sqrt{\frac{2\pi}{nC}}\frac{C}{D}\left[-\exp\left(-\frac{1}{2}\frac{D}{C}u^2\right)\right]_{-\infty}^{\infty}=0\,;$$

and therefore

(26.20) $$E(y)=\int_{-\infty}^{\infty}yf(y)\,dy=0 \quad \text{as well as} \quad E(t)=\int_{-\infty}^{\infty}ts_n(t)\,dt=0.$$

Lastly for $\nu=2$ we obtain

$$d_nJ_2=K_n\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\frac{D}{C}u^2\right)\frac{u^2du}{N^4}\sqrt{\frac{2\pi}{C}}=\frac{1}{n}\frac{C}{D}.$$

Thus

(26.21) $$E(y^2)=\int_{-\infty}^{\infty}y^2f(y)\,dy=\frac{1}{n}\frac{C}{D}=\frac{1}{b^2}E(t^2),\qquad E(t^2)\simeq\frac{C}{D}.$$

Hence the variance and S.D. are

(26.22) $D^2(t)=E(t^2)-E(t)^2=C/D$, as well as

(26.23) $\sigma_t=\sqrt{C/D}$.

We wish thereby to prove that our Student's ratio would also satisfy the central limit theorem: Standardizing Student ratio t after (20) and (23), we have

(26.24) $$x=\frac{t-E(t)}{\sigma_t}=(t-0)\bigg/\sqrt{\frac{C}{D}},\quad \text{or}\quad t=\sqrt{\frac{C}{D}}x.$$

But $y=t/b\simeq u/N$, so that $t\simeq u$ and

(26.25) $$u=\sqrt{\frac{C}{D}}x.$$

This being substituted in (17) and multiplied by e^{iTx}, we obtain

(26.26) $$j=\frac{1}{n}\sqrt{\frac{2\pi}{C}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}x^2\right)e^{iTx}\sqrt{\frac{C}{D}}dx=\frac{2\pi}{n\sqrt{D}}\int_{-\infty}^{\infty}e^{iTx-x^2/2}\frac{dx}{\sqrt{2\pi}}=\frac{2\pi}{n\sqrt{D}}e^{-T^2/2},$$

after Cramér[1]. The coefficient $2\pi/n\sqrt{D}\,(=j_0)$ multiplied by the adjoined coefficient K_n reduces to 1, as shown in (18). Hence $E(e^{iTx})=e^{-T^2/2}$ holds and the central limit theorem concerning Student's f.f. $s_n(t)$ made from T.N.D. has been thus proved.

Further, it is very desirous to treat the problem concerning lower and upper critical points of the exact sampling distribution with sizes, which are neither so small nor so large, say $n=5\sim25$ &c. Yet, to obtain somewhat reliable results in these intermediate cases, it becomes necessary to compute several figures at least up to $O(1/n^2)$, or desirably to $O(1/n^3)$, which work however is a pretty cumbersome one. So that those investigations are postponed as a future task.

1) H. Cramér, loc. cit. p. 100, (10. 5. 4).

ON A CRITERION FOR ANALYTICALLY UNRAMIFICATION OF A LOCAL RING

By

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1. Let $\mathfrak{o}$ be a local ring with maximal ideal $\mathfrak{m}$. $\mathfrak{o}$ is called analytically unramified in case the completion of $\mathfrak{o}$ has no nilpotent element. A criterion that $\mathfrak{o}$ is analytically unramified, obtained by D. Rees in [5], is stated in terms of the integral closures of ideals: $\mathfrak{o}$ is analytically unramified in case there is an integer k such that $\mathfrak{q}_n \subset \mathfrak{q}^{n-k}$ for all $n \geqq k$ where $\mathfrak{q}_n$ is the integral closure of the n-th power of a zero dimensional ideal $\mathfrak{q}$. Moreover, if this condition is satisfied $\mathfrak{a}_n \subset \mathfrak{a}^{n-k}$ for all $n \geqq k$, where $\mathfrak{a}$ is any ideal and k is an integer depending on $\mathfrak{a}$.

In this note we shall show that this result of Rees can be translated in the finiteness condition of the integral closure of the Rees' ring associated with $\mathfrak{q}$. In the course of proof we shall also show $l(\mathfrak{q}_n)$, the length of $\mathfrak{q}_n$, can be expressed as the Hilbert function $\mu(n)$ if n is large. This is a theorem due to Muhly. He discussed it under some additional restrictions placing on $\mathfrak{o}$ [1].

2. Let $\mathfrak{a}$ be an ideal in a commutative Noetherian ring $\mathfrak{o}$. An element $x \in \mathfrak{o}$ is called integral over $\mathfrak{a}$ in case x satisfies the equation of the form, $x^{\rho} + c_1 x^{\rho-1} + \dots + c_{\rho} = 0$, where $c_i \in \mathfrak{a}^i$. The set of elements which are integral over $\mathfrak{a}$ forms an ideal [4]. We call it the integral closure of $\mathfrak{a}$, and is denoted by $\mathfrak{a}_a$. We write $\mathfrak{a}_n$ in place of $(\mathfrak{a}^n)_a$. If the set $a_1, \dots, a_r$ is a basis of $\mathfrak{a}$ we associate with $\mathfrak{a}$ the Ree's ring $\mathfrak{o}(\mathfrak{a})$ defined by $\mathfrak{o}(\mathfrak{a}) = \mathfrak{o}[a_1 t, \dots, a_r t, t^{-1}]$ where t is an indeterminate over $\mathfrak{o}$. Obviously, $\mathfrak{o}(\mathfrak{a})$ is a graded subring of $\mathfrak{o}[t, t^{-1}]$. Consider the integral closure $\mathfrak{o}^*(\mathfrak{a})$ of $\mathfrak{o}(\mathfrak{a})$ in $\mathfrak{o}[t, t^{-1}]$. Then, it is immediately seen that $\mathfrak{o}^*(\mathfrak{a})$ is a graded ring. Moreover, if $x \in \mathfrak{o}$, then $xt^n \in \mathfrak{o}^*(\mathfrak{a})$ if and only if $x \in \mathfrak{a}_n$.

LEMMA 1. *Suppose there is an integer k such that $\mathfrak{a}_n \subset \mathfrak{a}^{n-k}$ for all $n \geqq k$. Then $\mathfrak{o}^*(\mathfrak{a})$ is a finite $\mathfrak{o}(\mathfrak{a})$-module* (Rees [6]).

PROOF. If $xt^n \in \mathfrak{o}^*(\mathfrak{a})$ and if $n \geqq k$, then $xt^n \in \mathfrak{a}_n t^n \subset \mathfrak{a}^{n-k} t^n = (\mathfrak{a}^{n-k} t^{n-k}) t^k \subset t^k \mathfrak{o}(\mathfrak{a})$. If $n < k$, we also have $xt^n = x(t^{-1})^{k-n} t^k \in t^k \mathfrak{o}(\mathfrak{a})$. Hence in either case $xt^n \in t^k \mathfrak{o}^*(\mathfrak{a})$.

Since $\mathfrak{o}^*(\mathfrak{a})$ is graded we can consider homogeneous ideals in $\mathfrak{o}^*(\mathfrak{a})$. For such ideal A, we associate ideals A_n in $\mathfrak{o}$ defined by $A_n=\{x\in\mathfrak{o};\ xt^n\in A\}$. Then, as the converse of lemma 1, we get

LEMMA 2. *If A is a homogeneous ideal in $\mathfrak{o}^*(\mathfrak{a})$ and if $\mathfrak{o}^*(\mathfrak{a})$ is finite over $\mathfrak{o}(\mathfrak{a})$, then there is an integer k such that $A_n=\mathfrak{a}^{n-k}A_k$ for all integer $n\geqq k$. In particular, we have $A_n\subset\mathfrak{a}^{n-k}$.*

PROOF. We show first $\mathfrak{a}^pA_q\subset A_{p+q}$. In fact, if $a\in\mathfrak{a}^p$ and $b\in A_q$, then $at^p\in\mathfrak{o}^*(\mathfrak{a})$ and $bt^q\in A$. Hence $abt^{p+q}\in A$. Therefore $ab\in A_{p+q}$. Now, since an $\mathfrak{o}(\mathfrak{a})$-module $\mathfrak{o}^*(\mathfrak{a})$ is generated by homogeneous elements, A is also generated by homogeneous elements as an $\mathfrak{o}(\mathfrak{a})$-module. Let

$$A=\mathfrak{o}(\mathfrak{a})\omega_1+\ \dots\ +\mathfrak{o}(\mathfrak{a})\omega_m,$$

with $\omega_i=x_it^{\lambda_i}$ and $x_i\in A_{\lambda_i}$ $(i=1,\ \dots,\ m)$. If k is an integer such that $k\geqq\operatorname{Max}\lambda_i$ and if $x\in A_n$ $(n\geqq k)$, then $xt^n\in A$ and can be written as

$$xt^n=(y_1t^{n-\lambda_1})(x_1t^{\lambda_1})+\ \dots\ +(y_mt^{n-\lambda_m})(x_mt^{\lambda_m})$$

with $y_i\in\mathfrak{a}^{n-\lambda_i}$. Therefore we have

$$\begin{aligned}x\in\mathfrak{a}^{n-\lambda_1}A_{\lambda_1}+\ \dots\ +\mathfrak{a}^{n-\lambda_m}A_{\lambda_m}&=\mathfrak{a}^{n-k}\mathfrak{a}^{k-\lambda_1}A_{\lambda_1}+\ \dots\ +\mathfrak{a}^{n-k}\mathfrak{a}^{k-\lambda_m}A_{\lambda_m}\\ \subset\mathfrak{a}^{n-k}A_{(k-\lambda_1)+\lambda_1}+\ \dots\ +\mathfrak{a}^{n-k}A_{(k-\lambda_m)+\lambda_m}&=\mathfrak{a}^{n-k}A_k.\end{aligned}$$

In case $A=\mathfrak{o}^*(\mathfrak{a})$, we have $A_n=\mathfrak{a}_n$. Hence

COROLLARY. *The converse of lemma 1 is true. Moreover we have $\mathfrak{a}_n=\mathfrak{a}^{n-k}\mathfrak{a}_k$ for all $n\geqq k$.*

Now, recall that an ideal $\mathfrak{b}$ in $\mathfrak{o}$ is called normal in case $\mathfrak{b}^n=\mathfrak{b}_n$ for all integers n [2]. Then, summarizing lemma 1 and 2, we have the following

THEOREM 1. *$\mathfrak{o}^*(\mathfrak{a})$ is a finite module over $\mathfrak{o}(\mathfrak{a})$ if and only if there exists an integer k such that $\mathfrak{a}_n\subset\mathfrak{a}^{n-k}$ for all $n\geqq k$ and when this is so $\mathfrak{a}_k$ is a normal ideal.*

PROOF. Put $\mathfrak{b}=\mathfrak{a}_k$. Then the last part of the theorem follows from the relation; $\mathfrak{b}_n=\mathfrak{a}_{nk}=\mathfrak{a}^{nk-k}\mathfrak{a}_k=(\mathfrak{a}^k)^{n-1}\mathfrak{a}_k\subset\mathfrak{b}^{n-1}\mathfrak{b}=\mathfrak{b}^n$.

3. In this section we put the restriction on $\mathfrak{o}$ and $\mathfrak{o}$ is assumed to be a semi-local ring with maximal ideals $\mathfrak{p}_1,\ \dots,\ \mathfrak{p}_r$. Then the theorem of Rees can be stated as follows:

LEMMA 3. *Let $\mathfrak{v}$ be a defining ideal in $\mathfrak{o}$. Then $\mathfrak{o}$ is analytically unramified if and only if we can find an integer k such that $\mathfrak{v}_n\subset\mathfrak{v}^{n-k}$ for $n\geqq k$. If this condition is satisfied we have $\mathfrak{a}_n\subset\mathfrak{a}^{n-k}$ for any ideal $\mathfrak{a}$ where k is an integer depending on $\mathfrak{a}$.*

PROOF. Let $\hat{\mathfrak{o}}$ be the completion of $\mathfrak{o}$. Then $\hat{\mathfrak{o}}$ is a direct sum of complete local rings $\mathfrak{o}_i$ ($i=1, \ldots, r$) and $\mathfrak{o}_i$ is isomorphic to the completion $\mathfrak{o}_{\mathfrak{p}_i}$ [3]. Hence if $\hat{\mathfrak{o}}$ has no nilpotent element, then each $\mathfrak{o}_{\mathfrak{p}_i}$ is analytically unramified and we can apply the Rees' result to the pair of rings $\mathfrak{o}_{\mathfrak{p}_i}$ and $\mathfrak{o}_i$. Whence we can find an integer k such that $(\mathfrak{a}\mathfrak{o}_{\mathfrak{p}_i})_n \subset (\mathfrak{a}\mathfrak{o}_{\mathfrak{p}_i})^{n-k}$ for $n \geqq k$ ($i=1, \ldots, r$). Since $\mathfrak{b}_a\mathfrak{o}_s = (\mathfrak{b}\mathfrak{o}_s)_a$ for any multiplicatively closed set S, $0 \notin$ S, and since $\mathfrak{b} = \cap_{i=1}^{r} (\mathfrak{b}\mathfrak{o}_{\mathfrak{p}_i} \cap \mathfrak{o})$ holds for any ideal $\mathfrak{b}$ [3], we get $\mathfrak{a}_n \subset \mathfrak{a}^{n-k}$ if we contract the above relation back to $\mathfrak{o}$. As for the converse, it is enough to mention that the proof of lemma 1 of [5] is still true without any change.

Now, from theorem 1 and lemma 3, we obtain immediately our main theorem:

THEOREM 2. *In a semi-local ring the following three conditions are equivalent.*

(1) $\mathfrak{o}$ *is analytically unramified.*

(2) $\mathfrak{o}^*(\mathfrak{v})$ *is finite over* $\mathfrak{o}(\mathfrak{v})$ *for some defining ideal* $\mathfrak{v}$.

(3) *Existence of a normal defining ideal.*

Moreover, when this is so, for any ideal $\mathfrak{a}$, (2) *is still true and* $\mathfrak{a}_k$ *is normal for some* k.

If E is a finite module over $\mathfrak{o}$ and $\mathfrak{v}$ is a defining ideal of $\mathfrak{o}$, then it is well known that the length of $E/\mathfrak{v}^n E$ is expressed as a polynomial if n is large [3]. Therefore from corollary of lemma 2, jointly with theorem 2, we have the following.

COROLLARY *If* $\mathfrak{v}$ *is a defining ideal of an analytically unramified semi-local ring, then* $l(\mathfrak{v}_n)$, *the length of the integral closure of* $\mathfrak{v}^n$, *is represented as a Hilbert function* $\mu(n)$ *if* n *is sufficiently large.*

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NOTE ON RING EXTENSIONS OF REGULAR LOCAL RINGS

By

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1. STATEMENT.

Let R be a regular local ring with maximal ideal $\mathfrak{m}$ and S be an over ring of R such that (i) S is a finite R-module and (ii) no non-zero element in R is a zero divisor in S. Such a pair of rings sometimes occurs in the theory of rings so that it might be interesting, under this situation, to find a condition that the ideal $\mathfrak{m}S$ coincides with the J-radical[1)] of S.

Our theorem will be stated as:

The ideal $\mathfrak{m}S$ coincides with the J-radical of S if and only if the equality

(a) $$[S:R]=\sum[S/\mathfrak{P}:R/\mathfrak{m}]$$

holds, where $[S:R]$ is the maximum number of linearly independent elements of S over R[2)] *and the sum runs over all maximal ideals $\mathfrak{P}$ of S.*

And as a corollary we have :

If (a) *holds, then the quotient ring $S_{\mathfrak{P}}$ is a regular local ring for any maximal ideal $\mathfrak{P}$ of S.*

2. PROOF OF THE THEOREM.

Let R be a (commutative Noetherian) semi-local ring and S be an over ring of R. And assume the pair of rings (R, S) satisfies the conditions (i) and (ii). Then it is well known that S is a semi-local ring and $\dim S=\dim R$. Moreover, if R is a normal local ring (i.e. integrally closed in its quotient field), then $\dim S_{\mathfrak{P}}=\dim S$ for any maximal ideal $\mathfrak{P}$ of S (cf. [1] and [2]).

We say that a semi-local ring R is unmixed if $\dim \hat{R}/\mathfrak{p}=\dim R$ for any prime divisor $\mathfrak{p}$ of the zero ideal in $\hat{R}$ where $\hat{R}$ is the completion of R (cf. [1]).

PROPOSITION. *If R is unmixed, then S is also unmixed.*

PROOF. Let $\hat{R}$ and $\hat{S}$ be completions of R and S respectively. Then $\hat{S}$ is an over ring of $\hat{R}$ and is a finite $\hat{R}$-module. Let $\mathfrak{P}$ be any prime divisor of the zero ideal in $\hat{S}$. Putting $\mathfrak{p}=\hat{R}\cap\mathfrak{P}$, then $\mathfrak{p}$ is a prime ideal of $\hat{R}$ and any element of $\mathfrak{p}$ is a zero divisor in $\hat{S}$. Therefore any element of $\mathfrak{p}$ is a zero divisor in $\hat{R}$ (cf. [2]). Hence $\mathfrak{p}$ is contained in some prime divisor of the zero ideal in $\hat{R}$. Since R is unmixed, the zero

1) We mean by the J-radical the intersection of all maximal ideals of S.

2) It is equal to the dimension of the total quotient ring of S considered as a vector space over the quotient field of R.

ideal of $\hat{R}$ has no imbeded prime divisor. Whence $\mathfrak{p}$ is a prime divisor of the zero ideal in $\hat{R}$, and we have $dim\ \hat{R}/\mathfrak{p}=dim\ R$. On the otherhand, $\hat{S}/\mathfrak{P}$ is an over ring of $\hat{R}/\mathfrak{p}$ and is a finite $\hat{R}/\mathfrak{p}$-module. This shows that $dim\ \hat{S}/\mathfrak{P}=dim\ \hat{R}/\mathfrak{p}$. Therefore we have $dim\ \hat{S}/\mathfrak{P}=dim\ S$, and the proof is complete.

LEMMA. *If R is a regular local ring with maximal ideal $\mathfrak{m}$ and if $\mathfrak{m}S$ coincides with the J-radical of S, then $S_{\mathfrak{P}}$ is a regular local ring for any maximal ideal $\mathfrak{P}$ of S.*

The proof is easy and we omit it.

Now, assume that R is a regular local ring with maximal ideal $\mathfrak{m}$, and we prove the theorem stated in the paragraph 1. Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_t$ be the set of all maximal ideals of S and put $\mathfrak{M}=\mathfrak{P}_1\cap\ldots\cap\mathfrak{P}_t$. Let $\mathfrak{m}S=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_t$ be the irredundant primary decomposition of $\mathfrak{m}S$. Then we have

(b) $$[S:R]=\sum_{i=1}^{t}[S/\mathfrak{P}_i:R/\mathfrak{m}]e(\mathfrak{q}_i),$$

where $e(\mathfrak{q}_i)$ is the multiplicity of the primary ideal $\mathfrak{q}_iS_{\mathfrak{P}_i}$ (cf. [2]). Now we assume that (a) is satisfied. Then, from (b) we have $e(\mathfrak{q}_i)=1$, and hence $e(\mathfrak{P}_i)=1$. Therefore we have $e(\mathfrak{M})=t$, that is, the multiplicity of the semi-local ring S is equal to the number of maximal ideals of S. On the otherhand, since a regular local ring is unmixed (cf. [1]), S is unmixed by virtue of the preceding proposition. Therefore $S_{\mathfrak{P}_i}$ is a regular local ring (cf. [3]). Recall that in a regular local ring the maximal ideal is the only primary ideal (belonging to the maximal ideal) whose multiplicity is equal to 1. This shows that $\mathfrak{q}_i=\mathfrak{P}_i$ and consequently $\mathfrak{m}S=\mathfrak{M}$. The only if part is an easy consequence of the preceding lemma and the equality (b).

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ESTIMATION OF COEFFICIENTS OF ENTIRE FUNCTIONS REPRESENTED BY DIRICHLET SERIES

By

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1. Let $f(s)=\sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ be an entire function represented by Dirichlet series, where $0=\lambda_0<\lambda_1<\cdots<\lambda_n\to\infty$ as $n\to\infty$, and

(1.1) $\overline{\lim}_{n\to\infty} \frac{n}{\lambda_n}=D;$ (1.2) $\underline{\lim}_{n\to\infty} \frac{n}{\lambda_n}=d,\ 0<d<D<\infty;$

(1.3) $\underline{\lim}_{n\to\infty} (\lambda_{n+1}-\lambda_n)=h;$ (1.4) $\overline{\lim}_{n\to\infty} (\lambda_{n+1}-\lambda_n)=m,\ 0<h\leqslant m<\infty;$

where $h\leqslant D^{-1}$, $m\leqslant d^{-1}$.

I wish to prove certain results involving the coefficients a_n of $f(s)$. Throughout, it is supposed that λ_n's satisfy the above relations, unless specified.

2. Define:

$$\theta(n)=\frac{\log|a_n|^{-1}}{\lambda_n\log\lambda_n};\quad \varphi(n)=\frac{\log|a_n/a_{n+1}|}{\log\lambda_n},$$

and let

$$\overline{\underline{\lim}}_{n\to\infty}\,\varphi(n)=\begin{matrix}\beta\\ \alpha\end{matrix};\qquad \overline{\underline{\lim}}_{n\to\infty}[\varphi(n)]^{-1}=\begin{matrix}\delta\\ \gamma\end{matrix};$$

$$\overline{\underline{\lim}}_{n\to\infty}\,\theta(n)=\begin{matrix}B\\ A\end{matrix};\qquad \overline{\underline{\lim}}_{n\to\infty}[\theta(n)]^{-1}=\begin{matrix}\Delta\\ C\end{matrix};$$

Then we have the following

THEOREM 1: *The following relations will hold:*

$$\text{(i)}\quad \alpha d\leqslant A=\frac{1}{\Delta};\qquad \text{(ii)}\quad \frac{1}{C}=B\leqslant D\beta,$$

$$\text{(iii)}\quad \frac{\gamma}{D}\leqslant C\leqslant\frac{\gamma}{d};\qquad \text{(iv)}\quad \alpha=\frac{1}{\delta}.$$

Further, if $x_n=\log|a_{n-1}/a_n|/(\lambda_n-\lambda_{n-1})$ *is a non-decreasing function of* n *for* $n>n_0$, *then*

$$\text{(v)}\quad B\geqslant\beta/m;\qquad \text{(vi)}\quad \Delta\geqslant h\delta.$$

PROOF: (i) We have:

$$\log\left|\frac{a_n}{a_{n+1}}\right|>(\alpha-\varepsilon)\log\lambda_n,\ n\geqslant n_0.$$

But

$$\log|a_n|^{-1}=\log\left\{\left|\frac{a_{n_0}}{a_{n_0+1}}\cdots\frac{a_{N-1}}{a_N}\frac{a_N}{a_{N+1}}\cdots\frac{a_{n-1}}{a_n}\right|\right\}+O(1) \tag{2. 1}$$

$$>(\alpha-\varepsilon)[(N-n_0)\log\lambda_{n_0}+\{(d-\varepsilon)-(D+\varepsilon)\lambda_N\}\log\lambda_N]+O(1),$$

where N is larger than n_0. Let

$$\lambda_n=[\lambda_N(\log\lambda_N)^2]+1,$$

then $\log\lambda_n\sim\log\lambda_N$ as $N\to\infty$. Hence for sufficiently large n,

$$\log|a_n|^{-1}>(\alpha-\varepsilon)(d-\varepsilon)(1+o(1))\lambda_n\log\lambda_n,$$

or,

$$A\geqslant\alpha d.$$

The proofs of (ii) and (iii) are similar and so omitted; the proof of (iv) being straight forward.

PROOF (v): Let $0<\beta<\infty$. Then

$$x_n=\frac{\log|a_{n-1}/a_n|}{\lambda_n-\lambda_{n-1}}>\frac{(\beta-\varepsilon)\log\lambda_{n-1}}{(\lambda_n-\lambda_{n-1})},$$

for a sequence of n, say $n=N_p+1$ $(p=1,2,\cdots)$, $N_1>n_0$, $N_p\to\infty$ as $p\to\infty$. Then as in (2.1),

$$\log|a_n|^{-1}=\log\left\{\left|\frac{a_{N_1-1}}{a_{N_1}}\cdots\frac{a_{N_p-1}}{a_{N_p}}\cdots\frac{a_{n-1}}{a_n}\right|\right\}+O(1)$$

$$=(\lambda_{N_1}-\lambda_{N_1-1})x_{N_1}+\cdots+(\lambda_{N_p}-\lambda_{N_p-1})x_{N_p}+\cdots+(\lambda_n-\lambda_{n-1})x_n+O(1)$$

$$\geqslant(N_p-N_1)x_{N_1}+(\lambda_n-\lambda_{N_p-1})x_{N_p}+O(1).$$

Let

$$\lambda_{n-1}=[\lambda_{N_p}(\log\lambda_{N_p})^{\eta}]+1,\ \eta>0.$$

Then $\log\lambda_{n-1}\sim\log\lambda_{N_p}$ as n and $p\to\infty$. Then

$$\log|a_n|^{-1}>(N_p-N_1)x_{N_1}+\frac{(\lambda_n-\lambda_{N_p})(\beta-\varepsilon)\log\lambda_{N_p}}{(\lambda_{N_p}-\lambda_{N_p-1})}+O(1)$$

$$>\frac{(1+o(1))(\beta-\varepsilon)\lambda_n\log\lambda_{N_p}}{(m+\varepsilon)}.$$

Therefore $B\geqslant\beta/m$. The proof of (vi) being on the same lines, is therefore omitted.

REMARK: It is to be noted that under some less strictly followed conditions on λ_n's, Δ is equal to order (R) of $f(s)$ (see for example [8], p. 217) and C is equal to the lower order (R) of $f(s)$ (see [6]).

3. Let $1/h=D=d$. Then, if ρ denotes the order $(R)\rho$ of $f(s)$, we have from (i), (iv) and (vi) of the previous theorem:

$$D\rho=D\overline{\lim_{n\to\infty}}\frac{\lambda_n\log\lambda_n}{\log|a_n|^{-1}}=\overline{\lim_{n\to\infty}}\frac{\log\lambda_n}{\log|a_n/a_{n+1}|}. \tag{3. 1}$$

In the following theorem we suppose $\lambda_n=n$. Then

THEOREM 2: *If* $a_n>0$ *for all* n *and*

$$\lim_{n\to\infty}n\left(\frac{a_n^2}{a_{n-1}\,a_{n+1}}-1\right)=\frac{1}{\rho}, \tag{3. 2}$$

then (i) $f(s)=\sum_{n=1}^{\infty}a_ne^{ns}$ *is an entire function of order* $(R)\rho$ *and* (ii) $\log\mu(\sigma)\sim\frac{\lambda_{\nu(\sigma)}}{\rho}$.

PROOF: The proof follows from (3. 1) and from the method adopted by Pólya and Szegö ([9], p. 13).

The converse of the theorem is not necessarily true. Consider

$$f(s) = \exp(e^{2s}) + \exp(e^{s}) = \sum_{n=0}^{\infty} e^{2ns}\left\{\frac{1}{n!} + \frac{1}{(2n)!}\right\} + \sum_{n\in E}\frac{e^{ns}}{n!},$$

where E is the set of all positive odd integeres except 1. This is an entire function of order (R) equal to 2. Here the limit in (3. 2) does not exist (see for example [10]). Also

$$\log \mu(\sigma) \sim e^{2\sigma};\ \lambda_{\nu(\sigma)} \sim 2e^{2\sigma}.$$

Thus (ii) follows.

We can construct examples to show that the above theorem holds good, in its converse sense; for example we have ([1], p. 27–28).

$$f(s) = \sum_{n=1}^{\infty} \frac{e^{ns}}{\exp(x_1 + \cdots + x_n)};$$

$$x_n = \frac{1}{\rho}\log n + S_n,\ n \geqslant n_0;\ x_n = 0,\ n < n_0,\ \varliminf_{n\to\infty} S_n = -\frac{1}{\rho}\log \gamma,$$

$$\varlimsup_{n\to\infty} S_n = -\frac{1}{\rho}\log \delta,\ \ (0 < \delta \leqslant \gamma < \infty);\ (S_{n+1} - S_n) = O(1/\log n);$$

$$\left(S_n - \frac{S_1 + \cdots + S_n}{n}\right) = O(1/\log n).$$

Then we have ([5], p. 76–77) omitting the details:

$$\lim_{\sigma\to\infty} \frac{\log \mu(\sigma)}{\lambda_{\nu(\sigma)}} = \frac{1}{\rho},$$

and clearly (3. 2) holds.

4. Here I prove a theorem furnishing a systematic study of the results obtained in the preceding two theorems. λ_n's satisfy (1. 1)–(1. 4). Let us introduce the following notation:

$$k(n+1,\ n) = (\lambda_{n+1} - \lambda_n)\, x_{n+1} - (\lambda_n - \lambda_{n-1})\, x_n$$

THEOREN 3: *Let $f(s)$ in the usual series form be an entire function of order $(R)\,\rho < \infty$ and if*

$$\varlimsup_{n\to\infty} \lambda_n\left(\frac{|a_n|^2}{|a_{n-1}a_{n+1}|} - 1\right) = \frac{L}{l};$$

then

(4. 1) $$ld \leqslant \varliminf_{n\to\infty} \frac{\log|a_n/a_{n+1}|}{\log \lambda_n} \leqslant LD;$$

and if $\lambda_n = n$, then

(4. 2) $$\varliminf_{n\to\infty} n\ k(n+1,\ n) = \max(o,\ l);$$

and further

(4. 3) $$\overline{\lim_{n\to\infty}}\, \lambda_n\, k(n+1,n) \leqslant L.$$

PROOF: It is quite clear that if

$$\overline{\lim_{n\to\infty}}\, \lambda_n \log\left\{\frac{|a_n|^2}{|a_{n-1}\, a_{n+1}|}\right\} = \frac{L_1}{l_1}\ ;$$

then $L=L_1$; $l=l_1$. We now prove the results. Let $-\infty < l < \infty$, then

$$\left|\frac{a_n^2}{a_{n+1}\ a_{n-1}}\right| > \exp\left\{\frac{l-\varepsilon}{\lambda_n}\right\},\ n \geqslant N.$$

Now

$$\left|\frac{a_n}{a_{n+1}}\right| = |k|\left|\frac{a_{N+1}^2}{a_{N+2}\ a_N}\cdots\frac{a_n^2}{a_{n+1}\, a_{n-1}}\right|.$$

Therefore

$$\log\left|\frac{a_n}{a_{n+1}}\right| > \log|k| + (l-\varepsilon)\sum_{p=N+1}^{n}\lambda_p^{-1}$$
$$\sim (l-\varepsilon)\ (d-\varepsilon)\log\lambda_n,$$

and (4. 1) follows.

To prove (4. 2), we take $\lambda_n=n$. Either $\log|a_n/a_{n+1}|>\log|a_{n-1}/a_n|$ for all $n>n_0$, in which case for all large n,

$$n\ k(n+1,\ n) = n\log\left|\frac{a_n^2}{a_{n+1}\, a_{n-1}}\right|,$$

hence

(4. 4) $$l \geqslant \underline{\lim}_{n\to\infty}\, n\ k(n+1,\ n) \geqslant l,\ l \geqslant 0,$$

or, we shall have $\log\left|\dfrac{a_n}{a_{n+1}}\right| \leqslant \log\left|\dfrac{a_{n-1}}{a_n}\right|$ for an infinity of n say $n=m_1,\ m_2,\cdots$. Then $|a_n|e^{\sigma\lambda_n}$ $(n=m_1,\ m_2,\cdots)$ does not become the maximum term for any σ, and $x_{n+1} < x_n$ infinitely often, which is contrary to our hypothesis. Therefore

(4. 5) $$\underline{\lim}_{n\to\infty}\, n\ k(n+1,\ n) = 0 \geqslant l,$$

and so (4. 2) follows.

We now prove (4. 3). We may suppose that $L<\infty$. Let $|a_m|e^{\sigma\lambda_m}$, $|a_n|e^{\sigma\lambda_n}$ and $|a_p|e^{\sigma\lambda_p}$ be three consecutive maximum terms $(m \leqslant n-1 \leqslant p-2)$. Now if $|a_m|e^{\sigma\lambda_m}$ is to become the maximum term, then $x_m \leqslant \sigma < x_{m+1}$ (for exhaustive arguments see [5], Ch. I, Part I). Similarly $|a_{m+2}|e^{\sigma\lambda_{m+2}}$, $|a_{m+3}|e^{\sigma\lambda_{m+3}}$, $\cdots$, $|a_{n-1}|e^{\sigma\lambda_{n-1}}$ are all maximum terms if

$$x_{m+2} \leqslant \sigma < x_{m+3},$$
$$\cdots\quad \cdots\quad \cdots$$
$$x_{n-1} \leqslant \sigma < x_n.$$

Thus $x_m < x_{m+1} = x_{m+2} = \cdots = x_{n-1} = \sigma < x_n$. But by hypothesis $|a_n|e^{\sigma\lambda_n}$ is a maximum term, hence $x_n \leqslant \sigma < x_{n+1}$. Therefore

(4. 6) $$x_{m+1} = x_{m+2} = \cdots = x_n.$$

Similarly we have:

(4. 7) $$x_{n+1}=x_{n+2}=\cdots=x_p.$$

We require a

LEMMA: *Let m be a positive integer such that* $\lambda_m=\lambda_{\nu(\sigma)}$, $\lambda_m>\lambda_{\nu(0)}$, *then*

$$x_m=\max\left(\frac{\log\left|\frac{a_0}{a_m}\right|}{\lambda_m-\lambda_0}, \frac{\log\left|\frac{a_1}{a_m}\right|}{\lambda_m-\lambda_1},\cdots,\frac{\log\left|\frac{a_{m-1}}{a_m}\right|}{\lambda_m-\lambda_{m-1}}\right).$$

PROOF OF THE LEMMA: With the smallest value of σ if all the following inequalities hold, viz.,

$$|a_0|\leqslant|a_m|e^{\sigma\lambda_m},\ |a_1|e^{\sigma\lambda_1}\leqslant|a_m|e^{\sigma\lambda_m},\cdots,\ |a_{m-1}|e^{\sigma\lambda_{m-1}}\leqslant|a_m|e^{\sigma\lambda_m},$$

then $|a_m|e^{\sigma\lambda_m}$ becomes the maximum term. This value of σ is, therefore, equal to x_m. Then if we refer to the convex polygon construction for the maximum term (see for instance [5], fig. 1), it is clear that

$$\frac{-\log|a_m|+\log|a_0|}{\lambda_m-\lambda_0},\quad \frac{-\log|a_m|+\log|a_1|}{\lambda_m-\lambda_1},\cdots,\quad \frac{-\log|a_m|+\log|a_{m-1}|}{\lambda_m-\lambda_{m-1}}$$

are all $\leqslant$ the slope x_m and therefore the lemma follows.

From the lemma, it follows that

$$x_n=\max\left\{\frac{\log\left|\frac{a_{n-1}}{a_n}\right|}{\lambda_n-\lambda_{n-1}},\cdots,\frac{\log\left|\frac{a_m}{a_n}\right|}{\lambda_n-\lambda_m},\cdots\right\}$$

or,

(4. 8) $$x_n=\frac{\log\left|\frac{a_m}{a_n}\right|}{\lambda_n-\lambda_m},\ n>m;$$

since the maximum term goes from $|a_m|e^{\sigma\lambda_m}$ to $|a_n|e^{\sigma\lambda_n}$. Similarly

$$x_p=\max\left\{\frac{\log\left|\frac{a_{p-1}}{a_p}\right|}{\lambda_p-\lambda_{p-1}},\cdots,\frac{\log\left|\frac{a_n}{a_p}\right|}{\lambda_p-\lambda_n},\cdots\right\}$$

(4. 9) $$=\frac{\log\left|\frac{a_n}{a_p}\right|}{\lambda_p-\lambda_n},\ p>n.$$

Hence from (4. 6) and (4. 7), we have:

$$\lambda_n k(n+1,\ n)=\lambda_n\{(\lambda_{n+1}-\lambda_n)x_p-(\lambda_n-\lambda_{n-1})x_{m+1}\}$$

(4. 10) $$=\lambda_n\left\{\left(\frac{\lambda_{n+1}-\lambda_n}{\lambda_p-\lambda_n}\right)\log\left|\frac{a_n}{a_p}\right|-\left(\frac{\lambda_n-\lambda_{n-1}}{\lambda_n-\lambda_m}\right)\log\left|\frac{a_m}{a_n}\right|\right\}.$$

From (4. 8), we have:

(4. 11) $$\log\left|\frac{a_m}{a_n}\right|\geqslant\left(\frac{\lambda_n-\lambda_m}{\lambda_n-\lambda_{n-1}}\right)\log\left|\frac{a_{n-1}}{a_n}\right|,$$

and from (4. 7) and (4. 9) we have:

(4. 12) $$\log\left|\frac{a_n}{a_p}\right|=\left(\frac{\lambda_p-\lambda_n}{\lambda_{n+1}-\lambda_n}\right)\log\left|\frac{a_n}{a_{n+1}}\right|.$$

Combining (4. 10), (4. 11) and (4. 12) we obtain

$$\lambda_n k(n+1,\ n) \leqslant \lambda_n\left\{\log\left|\frac{a_n}{a_{n+1}}\right| - \log\left|\frac{a_{n-1}}{a_n}\right|\right\}$$
$$= \lambda_n \log\left|\frac{a_n^2}{a_{n+1}a_{n-1}}\right|.$$
$$\therefore\quad \lambda_n k(n+1,\ n) < (L+\varepsilon),\ n > n_0,$$

and so (4. 3) follows.

5. If $f(s)$ is an entire function of order $(R)\rho$ and lower order $(R)\lambda$, $0<\lambda\leqslant\rho<\infty$, then I have shown ([3], p. 45; equa. (5))

$$\overline{\lim_{\sigma\to\infty}}\ \frac{\log\mu(\sigma)}{\lambda_{\nu(\sigma)}\log\lambda_{\nu(\sigma)}} \leqslant \frac{1}{\lambda}-\frac{1}{\rho}.$$

I wish to prove now a result of the above type in terms of the coefficient a_n. We have then the following.

THEOREM 4: *If* $f(s)=\sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ *is an entire function of finite order* $(R)\rho<\infty$, $\rho>0$ *finite lower order* $(R)\lambda$, $\lambda>0$ *and finite type* T $(T>0)$, *where* (i) $\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n)=h$ *and* (ii) $\log|a_{n-1}/a_n|/(\lambda_n-\lambda_{n-1})$ *is a non-decreasing function of* n *for* $n>n_0$, *then*

$$\lim_{n\to\infty}\frac{\log|a_n/a_{n+1}|}{\lambda_n\log\lambda_n} \leqslant \frac{1}{\lambda}-\frac{1}{\rho}.$$

PROOF: If order $(R)\rho$ of $f(s)$ is finite and λ_n's satisfy (i), then we have from a result on type ([4], p. 276), (taking $\rho(\sigma)=\rho$ in particular):

(5. 1) $$|a_n| < \left\{\frac{(T+\varepsilon)\rho e}{\lambda_n}\right\}^{\lambda_n/\rho},\ n>n_0;\ \varepsilon>0.$$

Again, as ρ is finite and (i) holds good, we have ([2], Th. 2):

$$\log m(\sigma) \sim \log\mu(\sigma),$$

and so ([6], following after (1.6)), for all $n>n_0$, and $\varepsilon>0$,

(5. 2) $$|a_{n+1}|^{-1} < \{\lambda_{n+1}\}^{\lambda_{n+1}/(\lambda-\varepsilon)}.$$

Combining (5. 1) and (5. 2)

$$\log\left|\frac{a_n}{a_{n+1}}\right| < \frac{\lambda_n}{\rho}\log((T+\varepsilon)e\rho) - \frac{\lambda_n}{\rho}\log\lambda_n + \frac{\lambda_{n+1}\log\lambda_{n+1}}{\lambda-\varepsilon}$$

(5. 3) $$< \frac{\lambda_n}{\rho}\log((T+\varepsilon)e\rho) - \frac{\lambda_n}{\rho}\log\lambda_n + \frac{\lambda_n+h+\varepsilon}{\lambda-\varepsilon}\log(\lambda_n+h+\varepsilon),$$

for arbitrarily large n and the result follows.

REMARK: In case we restrict λ_n's further, that is if in addition to (i), λ_n's also satisfy: $\overline{\lim_{n\to\infty}}(\lambda_{n+1}-\lambda_n)=m$, we get a stronger result:

$$\overline{\lim_{n\to\infty}}\ \frac{\log|a_n/a_{n+1}|}{\lambda_n\log\lambda_n} \leqslant \frac{1}{\lambda}-\frac{1}{\rho},$$

but it is weker in the sense that λ_n's are subject to more stringent conditions. The proof is almost the same, except in (5. 3) we get on the right-hand side $m+\varepsilon$ in place of $h+\varepsilon$ and this being true for all $n > n_0$.

NOTE: Theorem 1 can be applied to provide numerous results involving orders (R) and lower orders (R) for two and more entire functions, for instance results of the type of Theorems 3, 4 & 5 in [7].

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ON THE ORDER, TYPE AND THE ZEROS OF AN ENTIRE FUNCTION (II)

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1. Let $f=f(z)=\sum_{n=0}^{\infty}c_n z^n$ be an entire function, then $\varliminf_{n\to\infty}|c_n|^{-1/n}=\infty$. The order ρ, lower order λ; type T and lower type t of $f(z)$ are well-known in terms of the l.u.b. of $|f(z)|$ when arg z ranges over the circle $|z|=r$. Their respective analogues in terms of the coefficients c_n are also well-known and so we write

(1) $$\rho=\varlimsup_{n\to\infty}\frac{n\log n}{\log|c_n|^{-1}}\,;\ e\rho T=\varlimsup_{n\to\infty} n|c_n|^{-1/n},\ \text{if } 0<\rho<\infty.$$

Further, if $x=x(n)=|c_n/c_{n+1}|$ is a non-decreasing function of n, at least for $n\geqslant n_0$, n_0 being some large but fixed positive integer, then

(2) $$\lambda=\varliminf_{n\to\infty}\frac{n\log n}{\log|c_n|^{-1}}\,;\ e\rho t=\varliminf_{n\to\infty} n|c_n|^{-1/n},\ \text{if } 0<\rho<\infty.$$

My main aim in this paper is to find out certain relationships between ρ, λ, T and t for two or more entire functions. For the sake of brevity I introduce the following abbreviations:

$$f_1=f_1(z)=\sum_{n=0}^{\infty}a_n z^n\,;\ f_2=f_2(z)=\sum_{n=0}^{\infty}b_n z^n,$$
$$x_1=x_1(n)=|a_n/a_{n+1}|\,;\ x_2=x_2(n)=|b_n/b_{n+1}|.$$

2. I state and prove the following theorems*:

THEOREM 1: *Let* (i) f_1 *and* f_2 *be two entire functions of orders* $\rho_1(0<\rho_1<\infty)$ *and* $\rho_2(0<\rho_2<\infty)$; *lower types* t_1 $(0<t_1<\infty)$, $t_2(0<t_2<\infty)$ *respectively, and* x_1 *and* x_2 *be non-decreasing functions of* n *for* $n>n_0$; *and that*

(ii) $$\log|c_n|^{-1}\sim(\log|a_n|^{-1})^{\alpha}(\log|b_n|^{-1})^{\beta},\ 0<\alpha<1,\ 0<\beta<1,\ \alpha+\beta=1.$$

Then f *is also an entire function of order* ρ *and lower type* t, *such that*

(2. 1) $$\rho\leqslant\rho_1^{\alpha}\rho_2^{\beta};$$

(2. 2) $$t\geqslant t_1^{\alpha}t_2^{\beta}.$$

PROOF: First we show that f is an entire function; since f_1 and f_2 are entire functions, we have:

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* Most of the results cited and proved in this paper were prepared in 1961–62 and submitted in the form of a thesis to Raj. Uni. (1963). The author thanks Dr. S.C. Mitra for his kind encouragement and encouraging criticism.

$$\varliminf_{n\to\infty} |a_n|^{-1/n} = \infty\,; \tag{2. 3}$$

$$\varliminf_{n\to\infty} |b_n|^{-1/n} = \infty. \tag{2 .4}$$

Therefore for every $\varepsilon > 0$ and every arbitrarily large R,

$$(\log|a_n|^{-1})^{\alpha} > (n\log(R-\varepsilon))^{\alpha},\ n > n_1\,; \tag{2. 5}$$

$$(\log|b_n|^{-1})^{\beta} > (n\log(R-\varepsilon))^{\beta},\ n > n_2. \tag{2. 6}$$

Making use of (ii) of the theorem, we get for sufficiently large n:

$$\log|c_n|^{-1} > n\log(R-\varepsilon),$$

which means that f is an entire function.

Using (i) for f_1, we have for every $\varepsilon > 0$,

$$\frac{\log|a_n|^{-1}}{n\log n} > (\rho_1+\varepsilon)^{-1},\ n > n_1$$

or, we have:

$$(\log|a_n|^{-1})^{\alpha} > \{(\rho_1+\varepsilon)^{-1} n\log n\}^{\alpha},\ n > n_1.$$

Similarly for f_2, we get:

$$(\log|b_n|^{-1})^{\beta} > \{(\rho_2+\varepsilon)^{-1} n\log n\}^{\beta},\ n > n_2.$$

Therefore for $n > \max\ (n_1, n_2)$ and $\varepsilon > 0$,

$$(\log|a_n|^{-1})^{\alpha}(\log|b_n|^{-1})^{\beta} > \frac{n\log n}{(\rho_1+\varepsilon)^{\alpha}(\rho_2+\varepsilon)^{\beta}}. \tag{2. 7}$$

Hence making use of (ii) of the theorem and (2. 7), we find for sufficiently large n.

$$\frac{\log|c_n|^{-1}}{n\log n} > \frac{1}{(\rho_1+\varepsilon)^{\alpha}(\rho_2+\varepsilon)^{\beta}}$$

and so taking limit inferior of the preceding inequality, we get (2. 1).

We next prove (2. 2). Using (i) for f_1 and f_2, we have:

$$n|a_n|^{\rho_1/n} > (t_1-\varepsilon)e\rho_1,\ n > n_1,\ \varepsilon > 0\,; \tag{2. 8}$$

$$n|b_n|^{\rho_2/n} > (t_2-\varepsilon)e\rho_2,\ n > n_2,\ \varepsilon > 0. \tag{2. 9}$$

Inequalities (2. 8) and (2. 9) lead to

$$(\log|a_n|^{-1})^{\alpha} < \left[\frac{n}{\rho_1}\log\{n/(t_1-\varepsilon)e\rho_1\}\right]^{\alpha},\ n > n_2\,;$$

$$(\log|b_n|^{-1})^{\beta} < \left[\frac{n}{\rho_2}\log\{n/(t_2-\varepsilon)e\rho_2\}\right]^{\beta},\ n > n_2\,,$$

which when combined yield

$$(\log|a_n|^{-1})^{\alpha}(\log|b_n|^{-1})^{\beta} < \frac{n}{\rho_1^{\alpha}\rho_2^{\beta}}\left\{\log\left(\frac{n}{A}\right)\right\}^{\alpha}\left\{\log\left(\frac{n}{B}\right)\right\}^{\beta}, \tag{2. 10}$$

where

$$A = (t_1-\varepsilon)e\rho_1\,;\ B = (t_2-\varepsilon)e\rho_2.$$

So in accordance with (ii), we find from (2.10) for sufficiently large n,

$$\log|c_n|^{-1} < \frac{n}{\rho_1^{\alpha}\rho_2^{\beta}}\left\{\log\left(\frac{n}{A}\right)\right\}^{\alpha}\left\{\log\left(\frac{n}{B}\right)\right\}^{\beta}, \tag{2. 11}$$

and as $\rho_1^{-\alpha}\rho_2^{-\beta} \leqslant \rho^{-1}$ from (2. 1), we see that for sufficiently large n,

$$\log|c_n|^{-1} < \frac{n}{\rho}\ (\log n - \log A)^{\alpha}(\log n - \log B)^{\beta}$$
$$= \frac{n}{\rho}\left\{1 - \frac{\alpha \log A}{\log n} + O((\log n)^{-2})\right\}\left\{1 - \frac{\beta \log B}{\log n} + O((\log n)^{-2})\right\}\log n$$
$$= \frac{n}{\rho}\left\{1 - \frac{\log(A^{\alpha}B^{\beta})}{\log n} + O((\log n)^{-2})\right\}\log n.$$

Therefore for large n,

$$|c_n|^{-\rho/n} < n^{\left[1 - \frac{\log A^{\alpha}B^{\beta}}{\log n} + O((\log n)^{-2})\right]},$$

or,
$$\frac{\rho en|c_n|^{\rho/n}}{\rho e} > n^{[\log(A^{\alpha}B^{\beta})/\log n + O((\log n)^{-2})]},$$

and as the power of n on the right-hand side of the preceding inequality tends to $A_1^{\alpha}B_1^{\beta}$, $A_1 = e\rho_1 t_1$, $B_1 = e\rho_2 t_2$ we find $\rho e t \geqslant e^{\alpha+\beta} t_1^{\alpha} t_2^{\beta} \rho_1^{\alpha} \rho_2^{\beta}$, and using (2.1) again we finally have (2.2).

THEOREM 2:* *If f_1 and f_2 be two entire functions having the same order $\rho\,(0 < \rho < \infty)$, lower type $t_1\,(0 < t_1 < \infty)$; $t_2\,(0 < t_2 < \infty)$ respectively and if x_1 and x_2 be non-decreasing functions of n for $n > n_0$, then f, where* (i) $|c_n| \sim |a_n|^{\alpha} |b_n|^{\beta}$, *$\alpha$ and β satisfying the condition of Th.* 1, *is also an entire function of order* ρ *and lower type t such that*

(2. 12) $$t \geqslant t_1^{\alpha} t_2^{\beta}$$

Further, if $T_1\,(0 < T_1 < \infty)$ and $T_2\,(0 < T_2 < \infty)$ are types of f_1 and f_2, then the type T of f is given by

(2. 13) $$T \leqslant T_1^{\alpha} T_2^{\beta}.$$

PROOF: We have: for every $\varepsilon > 0$,

(2. 14) $$\left(\frac{n}{\rho e}\right)^{\alpha} |a_n|^{\alpha\rho/n} > (t_1 - \varepsilon)^{\alpha},\ n > n_1;$$

(2. 15) $$\left(\frac{n}{\rho e}\right)^{\beta} |b_n|^{\beta\rho/n} > (t_2 - \varepsilon)^{\beta},\ n > n_2.$$

On multiplying (2. 14) and (2. 15) and then using (i) of the theorem, we find for $n >$ max (n_1, n_2)

$$\frac{n}{\rho e}|c_n|^{\rho/n} > (t_1 - \varepsilon)^{\alpha}(t_2 - \varepsilon)^{\beta},$$

and so (2. 12) follows. Similarly, making use of (i) we can prove (2.13). The proof that f is an entire function can be obtained on making an appeal to the proof of Th. 1.

COR. If $t_1 = T_1$, $t_2 = T_2$, then $t = T = T_1^{\alpha} T_2^{\beta}$

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* We are interested in those $f(z)$'s for which the order, under the condition (i), is ρ. Even if the order of f is $\leq \rho$, (2. 12) will hold ; but for (2. 13) only such f's are to be considered for which the order is equal to ρ.

REMARK: We can obtained a result similar to Theorem 1 relating to types T_1 and T_2 and connecting them to the type T of f, provided we assume that the order of ρ under the conditions of Theorem 1 is given as: $\rho=\rho_1^{\alpha}\rho_2^{\beta}$ (which has to be considered as hypothesis in case we wish to obtain such results); in fact we have:

THEOREM 1′: *If f_1 and f_2 be two entire functions satisfying all the conditions of Th. 1, excluding that imposed on x_1 and x_2 then f is also an entire function of order ρ. If T_1, T_2, T are types of f_1, f_2, f respectively and if $\rho=\rho_1^{\alpha}\rho_2^{\beta}$, then $T\leqslant T_1^{\alpha}T_2^{\beta}$, the types being non-zero finite.*

PROOF: We omit the proof as it is based on the proof of Theorem 1. We also omit numerous obvious corollaries.

3. THEOREM 3: *Let* (i) *f_1 and f_2 be two entire functions of orders $\rho_1\,(0<\rho_1<\infty)$, $\rho_2\;(0<\rho_2<\infty)$; lower orders $\lambda_1\,(0<\lambda_1<\infty)$, $\lambda_2\,(0<\lambda_2<\infty)$ respectively and that x_1 and x_2 are non-decreasing functions of n for $n>n_0$.* (ii) $\log|c_n/c_{n+1}|\sim(\log|a_n/a_{n+1}|)^{1/r}(\log|b_n/b_{n+1}|)^{1/\delta}$, *where $1<\delta<\infty$; $1<r<\infty$; $\delta^{-1}+r^{-1}=1$. Then f is also an entire function of order ρ and lower order λ, such that*

(3. 1) $\rho\leqslant\rho_1^{1/r}\rho_2^{1/\delta}$, (3. 2) $\lambda\geqslant\lambda_1^{1/r}\lambda^{1/\delta}$.

PROOF: Shah [2] has proved if $F(z)=\sum_{n=0}^{\infty}A_nz^n$ is an entire function then

(A) $$\text{Order of } F=\overline{\lim_{n\to\infty}}\frac{\log n}{\log|A_n/A_{n+1}|}$$

and if $|A_n/A_{n+1}|$ is non-decreasing then

(B) $$\text{Lower order of } F=\underline{\lim_{n\to\infty}}\frac{\log n}{\log|A_n/A_{n+1}|}$$

Hence making use of (B) and (A), since x_1 is non-decreasing, we get:

(3. 3) $$\frac{\log|a_n/a_{n+1}|}{\log n}<\frac{1}{\lambda_1}+\varepsilon,\ n>n_1;$$

(3. 4) $$\frac{\log|a_n/a_{n+1}|}{\log n}>\frac{1}{\rho_1}-\varepsilon,\ n>n_2$$

for every $\varepsilon>0$. Similarly for f_2, we have, for $\varepsilon>0$:

(3. 5) $$\frac{\log|b_n/b_{n+1}|}{\log n}<\frac{1}{\lambda_2}+\varepsilon,\ n>n_1';$$

(3. 6) $$\frac{\log|b_n/b_{n+1}|}{\log n}>\frac{1}{\rho_2}-\varepsilon,\ n>n_2'.$$

From (3. 3) and (3. 5) we have for $n>\max(n_1, n_1')$

$$\frac{(\log|a_n/a_{n+1}|)^{1/r}(\log|b_n/b_{n+1}|)^{1/\delta}}{\log n}<\left(\frac{1}{\lambda_1}+\varepsilon\right)^{1/r}\left(\frac{1}{\lambda_2}+\varepsilon\right)^{1/\delta},$$

and as $|c_n/c_{n+1}|$ is non-decreasing, find from (i)

$$\overline{\lim_{n\to\infty}}\frac{\log|c_n/c_{n+1}|}{\log n}\leqslant\lambda_1^{-1/r}\lambda_2^{-1/\delta}.$$

which implies $\lambda^{-1} \leqslant \lambda_1^{-1/r}\lambda_2^{-1/\delta}$. Similarly from (3. 4) and (3. 6) we obtain $\rho^{-1} \geqslant \rho_1^{-1/r}\rho_2^{-1/\delta}$ and this proves the theorem.

COR. *If* $f_j(z) = \sum_{n=0}^{\infty} a_n^{(j)} z^n$, $(j=1, 2, \cdots, s)$ *be* s *entire functions of orders* $\rho_j\,(0 \leqslant \rho_j \leqslant \infty)$, $(j=1, 2, \cdots, s)$ *respectively and each of the function* $|a_n^{(j)}/a_{n+1}^{(j)}|$, $(j=1, 2, \cdots, s)$ *be non-decreasing for* $n > n_0$, *then the function* $f(z) = \sum_{n=0}^{\infty} c_n z^n$ *where*

$$\log|c_n/c_{n+1}| \sim (\log|a_n^{(1)}/a_{n+1}^{(1)}|)^{1/\alpha_1} \cdots (\log|a_n^{(s)}/a_{n+1}^{(s)}|)^{1/\alpha_s},$$

where $1 < \alpha_j < \infty\,(j=1, 2, \cdots, s)$; $\sum_{j=1}^{s} \alpha_j^{-1} = 1$, *is also an entire function of order* ρ, *such that*

$$\rho \leqslant \rho_1^{1/\alpha_1} \cdots \rho_s^{1/\alpha_s}.$$

and a similar type of result in case of lower orders.

4. Here I prove the following theorems:

THEOREM 4: *Let* (i) f_1 *and* f_2 *be two entire functions oorders* ρ_1 *and* ρ_2.
(ii) $2(\log|c_n|^{-1})^{-1} \sim (\log|a_n|^{-1})^{-1} + (\log|b_n|^{-1})^{-1}$.
Then f *is also an entire function of order* ρ, *such that*

$$2\rho \leqslant \rho_1 + \rho_2. \tag{4. 1}$$

(iii) *Further, let* λ_1 *and* λ_2 *be the lower orders of* f_1 *and* f_2, *and* x_1 *and* x_2 *be non-decreasing functions of* n *for* $n > n_0$ *and that* (ii) *holds. Then* f *is of lower order* λ, *such that*

$$2\lambda \geqslant \lambda_1 + \lambda_2. \tag{4. 2}$$

COR. *If* f_1 *and* f_2 *are of regular growths, then* f *is also of regular growth and*

$$2\rho = \rho_1 + \rho_2. \tag{4. 3}$$

REMARK: The result (4. 3) can also be obtained even if f_1 and f_2 are not of regular growths. But in that case we will have to make some other supposition as the following theorem shows:

THEOREM 5: *Let* (i) f_1 *and* f_2 *be two entire functions of orders* $\rho_1\,(0<\rho_1<\infty)$, $\rho_2\,(0<\rho_2<\infty)$; *types* $T_1\ (0 < T_1 < \infty)$, $T_2\,(0 < T_2 < \infty)$; *lower types* $t_1\,(0 < t_1 < \infty)$, $t_2\,(0 < t_2 < \infty)$, *and* x_1 *and* x_2 *are non-decreasing functions of* n *for* $n > n_0$.
(ii) $2(\log|c_n|^{-1})^{-1} \sim (\log|a_n|^{-1})^{-1} + (\log|b_n|^{-1})^{-1}$.
Then f *is also an entire function of order* ρ, *such that*

$$2\rho = \rho_1 + \rho_2 \tag{4. 4}$$

PROOF OF THEOREM 4: First we show that f is an entire function. Since f_2 is an entire function, hence for every $\varepsilon > 0$ and arbitrily large R,

$$(\log|b_n|^{-1/n})^{-1} < \{\log(R-\varepsilon)\}^{-1},\ n > n_1.$$

Similarly for f_1, we have:

$$(\log|a_n|^{-1/n})^{-1} < \{\log(R-\varepsilon)\}^{-1},\ n > n_2.$$

Hence using (ii), we have for sufficiently large n,

$$2(\log|c_n|^{-1/n})^{-1} < 2\{\log(R-\varepsilon)\}^{-1},$$

and so f is an entire function.

Further, as f_1 and f_2 are of orders ρ_1 and ρ_2, we find for every $\varepsilon>0$ and large n,

$$\frac{n\log n}{\log|a_n|^{-1}}+\frac{n\log n}{\log|b_n|^{-1}}<\rho_1+\rho_2+2\varepsilon,$$

and (4. 1) follows. The proof of (4. 2) is similar and so omitted.

PROOF OF THEOREM 5: Using (i) for f_1, we have for every $\varepsilon>0$,

(4. 5) $$\frac{n\log n}{\log|a_n|^{-1}}<\frac{n}{\log|a_n|^{-1}}\log\{e\rho_1(T_1+\varepsilon)\}+\rho_1=\rho_1+o(1).$$

Similarly, for f_2, we have for $n>n_2$,

(4. 6) $$\frac{n\log n}{\log|a_n|^{-1}}<\rho_2+o(1).$$

Inequalities (4. 5) and (4. 6) yield on making use of (ii), for sufficiently large n,

$$\frac{2n\log n}{\log|c_n|^{-1}}<\rho_1+\rho_2+o(1).$$

Therefore

(4. 7) $$2\rho\leqslant\rho_1+\rho_2.$$

Again, using (2) for f_1, we have for every $\varepsilon>0$,

$$\frac{n\log n}{\log|a_n|^{-1}}>\frac{n}{\log|a_n|^{-1}}\log\{e\rho_1(t_1-\varepsilon)\}+\rho_1=\rho_1+o(1),\ n>n_1,$$

and a similar expression for f_2. Therefore for sufficiently large n,

(4. 8) $$2\lambda=\underline{\lim}_{n\to\infty}\frac{2n\log n}{\log|c_n|^{-1}}\geqslant\rho_1+\rho_2.$$

But as $\lambda\leqslant\rho$ always, we find from (4. 8)

(4. 9) $$2\rho\geqslant\rho_1+\rho_2$$

Therefore (4. 4) follows from (4. 7) and (4. 9).

5. Let $n(x)$ denote the number of zeros of an entire function $f(z)$ in $|z|\leqslant x$. I prove:

THEOREM 6: *If $f(z)$ is an entire function of order zero and not a constant, then*

(5. 1) $$\underline{\lim}_{r\to\infty}\frac{N_\alpha(r)}{Q_\alpha(r)}=\infty,$$

where

$$N_\alpha(r)=r^\alpha\int_0^r\frac{n(x)}{x^{\alpha+1}}dx;\ Q_\alpha(r)=r^{\alpha+1}\int_r^\infty\frac{n(x)}{x^{\alpha+2}}dx\neq0,\ \alpha\geqslant0.$$

PROOF: Suppose (5. 1) does not hold good. Then

$$\overline{\lim_{r\to\infty}} \frac{N_\alpha(r)}{Q_\alpha(r)} = \eta_1, \ \eta_1 < \infty.$$

Hence for $r > R$,

$$N_\alpha(r) \leqslant \eta Q_\alpha(r),$$

Let $0 \leqslant \alpha < \xi < \alpha + (1+\eta)^{-1}$. Then $\int^\infty n(x) x^{-1-\xi} dx$ is convergent and so

$$\int_R^\infty Q_\alpha(t) t^{-1-\xi} dt = \int_R^\infty t^{\alpha-\xi} dt \int_t^\infty \frac{n(x)}{x^{\alpha+2}} dx$$

$$= \int_R^\infty \frac{n(x)}{x^{\alpha+2}} dx \int_R^x t^{\alpha-\xi} dt$$

$$\leqslant \frac{1}{\alpha-\xi+1} \int_R^\infty \frac{n(x)}{x^{\xi+1}} dx,$$

and so $\int^\infty Q_\alpha(x) x^{-1-\xi} dx$ is convergent. Now

$$\int_R^\infty \frac{Q_\alpha(t)}{t^{1+\xi}} dt \leqslant \frac{1}{\alpha-\xi+1} \int_R^\infty x^{\alpha-\xi} d\left[\frac{N_\alpha(x)}{x^\alpha}\right]$$

$$\leqslant \frac{\eta(\xi-\alpha)}{\alpha-\xi+1} \int_R^\infty \frac{Q_\alpha(x)}{x^{1+\xi}} dx.$$

But $\eta(\xi-\alpha)/(\alpha-\xi+1) < 1$, and so a contradiction.

Note: The author has obtained this result alternatively [1, p. 11].

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