ON THE ORDER, TYPE AND THE ZEROS OF AN ENTIRE FUNCTION (II)

By

Pawan Kumar KAMTHAN

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1. Let $f=f(z)=\sum_{n=0}^{\infty}c_n z^n$ be an entire function, then $\lim_{n\to\infty}|c_n|^{-1/n}=\infty$. The order ρ, lower order λ; type T and lower type t of $f(z)$ are well-known in terms of the l.u.b. of $|f(z)|$ when arg z ranges over the circle $|z|=r$. Their respective analogues in terms of the coefficients c_n are also well-known and so we write

$$(1) \qquad \rho=\overline{\lim_{n\to\infty}}\frac{n\log n}{\log|c_n|^{-1}}\,;\ e\rho T=\overline{\lim_{n\to\infty}}\, n|c_n|^{-1/n},\ \text{if } 0<\rho<\infty.$$

Further, if $x=x(n)=|c_n/c_{n+1}|$ is a non-decreasing function of n, at least for $n\geqslant n_0$, n_0 being some large but fixed positive integer, then

$$(2) \qquad \lambda=\underline{\lim_{n\to\infty}}\frac{n\log n}{\log|c_n|^{-1}}\,;\ e\rho t=\underline{\lim_{n\to\infty}}\, n|c_n|^{-1/n},\ \text{if } 0<\rho<\infty.$$

My main aim in this paper is to find out certain relationships between ρ, λ, T and t for two or more entire functions. For the sake of brevity I introduce the following abbreviations:

$$f_1=f_1(z)=\sum_{n=0}^{\infty}a_n z^n\,;\ f_2=f_2(z)=\sum_{n=0}^{\infty}b_n z^n,$$

$$x_1=x_1(n)=|a_n/a_{n+1}|\,;\ x_2=x_2(n)=|b_n/b_{n+1}|.$$

2. I state and prove the following theorems*:

THEOREM 1: *Let* (i) f_1 *and* f_2 *be two entire functions of orders* $\rho_1\,(0<\rho_1<\infty)$ *and* $\rho_2(0<\rho_2<\infty)$; *lower types* t_1 $(0<t_1<\infty)$, $t_2\,(0<t_2<\infty)$ *respectively, and* x_1 *and* x_2 *be non-decreasing functions of* n *for* $n>n_0$; *and that*

(ii) $\qquad \log|c_n|^{-1}\sim(\log|a_n|^{-1})^{\alpha}(\log|b_n|^{-1})^{\beta},\ 0<\alpha<1,\ 0<\beta<1,\ \alpha+\beta=1.$

Then f *is also an entire function of order* ρ *and lower type* t, *such that*

$$(2.\ 1) \qquad \rho\leqslant\rho_1^{\alpha}\rho_2^{\beta}\,;$$

$$(2.\ 2) \qquad t\geqslant t_1^{\alpha}t_2^{\beta}.$$

PROOF: First we show that f is an entire function; since f_1 and f_2 are entire functions, we have:

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* Most of the results cited and proved in this paper were prepared in 1961–62 and submitted in the form of a thesis to Raj. Uni. (1963). The author thanks Dr. S.C. Mitra for his kind encouragement and encouraging criticism.

$$\varliminf_{n\to\infty} |a_n|^{-1/n} = \infty\,; \tag{2. 3}$$

$$\varliminf_{n\to\infty} |b_n|^{-1/n} = \infty. \tag{2 .4}$$

Therefore for every $\varepsilon > 0$ and every arbitrarily large R,

$$(\log|a_n|^{-1})^{\alpha} > (n\log(R-\varepsilon))^{\alpha},\ n > n_1\,; \tag{2. 5}$$

$$(\log|b_n|^{-1})^{\beta} > (n\log(R-\varepsilon))^{\beta},\ n > n_2. \tag{2. 6}$$

Making use of (ii) of the theorem, we get for sufficiently large n:

$$\log|c_n|^{-1} > n\log(R-\varepsilon),$$

which means that f is an entire function.

Using (i) for f_1, we have for every $\varepsilon > 0$,

$$\frac{\log|a_n|^{-1}}{n\log n} > (\rho_1+\varepsilon)^{-1},\ n > n_1$$

or, we have:

$$(\log|a_n|^{-1})^{\alpha} > \{(\rho_1+\varepsilon)^{-1} n\log n\}^{\alpha},\ n > n_1.$$

Similarly for f_2, we get:

$$(\log|b_n|^{-1})^{\beta} > \{(\rho_2+\varepsilon)^{-1} n\log n\}^{\beta},\ n > n_2.$$

Therefore for $n > \max\ (n_1, n_2)$ and $\varepsilon > 0$,

$$(\log|a_n|^{-1})^{\alpha}(\log|b_n|^{-1})^{\beta} > \frac{n\log n}{(\rho_1+\varepsilon)^{\alpha}(\rho_2+\varepsilon)^{\beta}}. \tag{2. 7}$$

Hence making use of (ii) of the theorem and (2. 7), we find for sufficiently large n.

$$\frac{\log|c_n|^{-1}}{n\log n} > \frac{1}{(\rho_1+\varepsilon)^{\alpha}(\rho_2+\varepsilon)^{\beta}}$$

and so taking limit inferior of the preceding inequality, we get (2. 1).

We next prove (2. 2). Using (i) for f_1 and f_2, we have:

$$n|a_n|^{\rho_1/n} > (t_1-\varepsilon)e\rho_1,\ n > n_1,\ \varepsilon > 0\,; \tag{2. 8}$$

$$n|b_n|^{\rho_2/n} > (t_2-\varepsilon)e\rho_2,\ n > n_2,\ \varepsilon > 0. \tag{2. 9}$$

Inequalities (2. 8) and (2. 9) lead to

$$(\log|a_n|^{-1})^{\alpha} < \left[\frac{n}{\rho_1}\log\{n/(t_1-\varepsilon)e\rho_1\}\right]^{\alpha},\ n > n_2\,;$$

$$(\log|b_n|^{-1})^{\beta} < \left[\frac{n}{\rho_2}\log\{n/(t_2-\varepsilon)e\rho_2\}\right]^{\beta},\ n > n_2\,,$$

which when combined yield

$$(\log|a_n|^{-1})^{\alpha}(\log|b_n|^{-1})^{\beta} < \frac{n}{\rho_1^{\alpha}\rho_2^{\beta}}\left\{\log\left(\frac{n}{A}\right)\right\}^{\alpha}\left\{\log\left(\frac{n}{B}\right)\right\}^{\beta}, \tag{2. 10}$$

where

$$A = (t_1-\varepsilon)e\rho_1\,;\ B = (t_2-\varepsilon)e\rho_2.$$

So in accordance with (ii), we find from (2.10) for sufficiently large n,

$$\log|c_n|^{-1} < \frac{n}{\rho_1^{\alpha}\rho_2^{\beta}}\left\{\log\left(\frac{n}{A}\right)\right\}^{\alpha}\left\{\log\left(\frac{n}{B}\right)\right\}^{\beta}, \tag{2. 11}$$

and as $\rho_1^{-\alpha}\rho_2^{-\beta} \leqslant \rho^{-1}$ from (2. 1), we see that for sufficiently large n,

$$\log|c_n|^{-1} < \frac{n}{\rho}\ (\log n - \log A)^{\alpha}(\log n - \log B)^{\beta}$$

$$= \frac{n}{\rho}\left\{1 - \frac{\alpha \log A}{\log n} + O((\log n)^{-2})\right\}\left\{1 - \frac{\beta \log B}{\log n} + O((\log n)^{-2})\right\}\log n$$

$$= \frac{n}{\rho}\left\{1 - \frac{\log (A^{\alpha}B^{\beta})}{\log n} + O((\log n)^{-2})\right\}\log n.$$

Therefore for large n,

$$|c_n|^{-\rho/n} < n^{[1 - \frac{\log A^{\alpha}B^{\beta}}{\log n} + O((\log n)^{-2})]},$$

or,

$$\frac{\rho en|c_n|^{\rho/n}}{\rho e} > n^{[\log (A^{\alpha}B^{\beta})/\log n + O((\log n)^{-2})]},$$

and as the power of n on the right-hand side of the preceding inequality tends to $A_1^{\alpha}B_1^{\beta}$, $A_1 = e\rho_1 t_1$, $B_1 = e\rho_2 t_2$ we find $\rho et \geqslant e^{\alpha+\beta}t_1^{\alpha}t_2^{\beta}\rho_1^{\alpha}\rho_2^{\beta}$, and using (2.1) again we finally have (2.2).

THEOREM 2:* *If f_1 and f_2 be two entire functions having the same order $\rho\,(0<\rho<\infty)$, lower type $t_1\,(0<t_1<\infty)$; $t_2\,(0<t_2<\infty)$ respectively and if x_1 and x_2 be non-decreasing functions of n for $n>n_0$, then f, where* (i) $|c_n| \sim |a_n|^{\alpha}|b_n|^{\beta}$, *$\alpha$ and β satisfying the condition of Th.* 1, *is also an entire function of order* ρ and lower type t such that*

(2. 12) $$t \geqslant t_1^{\alpha}t_2^{\beta}$$

Further, if $T_1\,(0<T_1<\infty)$ and $T_2\,(0<T_2<\infty)$ are types of f_1 and f_2, then the type T of f is given by

(2. 13) $$T \leqslant T_1^{\alpha}T_2^{\beta}.$$

PROOF: We have: for every $\varepsilon>0$,

(2. 14) $$\left(\frac{n}{\rho e}\right)^{\alpha}|a_n|^{\alpha\rho/n} > (t_1-\varepsilon)^{\alpha},\ n>n_1;$$

(2. 15) $$\left(\frac{n}{\rho e}\right)^{\beta}|b_n|^{\beta\rho/n} > (t_2-\varepsilon)^{\beta},\ n>n_2.$$

On multiplying (2. 14) and (2. 15) and then using (i) of the theorem, we find for $n>\max\,(n_1, n_2)$

$$\frac{n}{\rho e}|c_n|^{\rho/n} > (t_1-\varepsilon)^{\alpha}(t_2-\varepsilon)^{\beta},$$

and so (2. 12) follows. Similarly, making use of (i) we can prove (2.13). The proof that f is an entire function can be obtained on making an appeal to the proof of Th. 1.

COR. If $t_1 = T_1$, $t_2 = T_2$, then $t = T = T_1^{\alpha}T_2^{\beta}$

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* We are interested in those $f(z)$'s for which the order, under the condition (i), is ρ. Even if the order of f is $\leq \rho$, (2. 12) will hold ; but for (2. 13) only such f's are to be considered for which the order is equal to ρ.

REMARK: We can obtained a result similar to Theorem 1 relating to types T_1 and T_2 and connecting them to the type T of f, provided we assume that the order of ρ under the conditions of Theorem 1 is given as: $\rho=\rho_1{}^{\alpha}\rho_2{}^{\beta}$ (which has to be considered as hypothesis in case we wish to obtain such results); in fact we have:

THEOREM 1′: *If f_1 and f_2 be two entire functions satisfying all the conditions of Th.* 1, *excluding that imposed on x_1 and x_2 then f is also an entire function of order ρ. If T_1, T_2, T are types of f_1, f_2, f respectively and if $\rho=\rho_1{}^{\alpha}\rho_2{}^{\beta}$, then $T \leqslant T_1{}^{\alpha}T_2{}^{\beta}$, the types being non-zero finite.*

PROOF: We omit the proof as it is based on the proof of Theorem 1. We also omit numerous obvious corollaries.

3. THEOREM 3: *Let* (i) *f_1 and f_2 be two entire functions of orders ρ_1 $(0<\rho_1<\infty)$, ρ_2 $(0<\rho_2<\infty)$; lower orders λ_1 $(0<\lambda_1<\infty)$, λ_2 $(0<\lambda_2<\infty)$ respectively and that x_1 and x_2 are non-decreasing functions of n for $n>n_0$.* (ii) $\log|c_n/c_{n+1}|\sim(\log|a_n/a_{n+1}|)^{1/r}(\log|b_n/b_{n+1}|)^{1/\delta}$, *where $1<\delta<\infty$; $1<r<\infty$; $\delta^{-1}+r^{-1}=1$. Then f is also an entire function of order ρ and lower order λ, such that*

(3. 1) $\rho \leqslant \rho_1{}^{1/r}\rho_2{}^{1/\delta}$, (3. 2) $\lambda \geqslant \lambda_1{}^{1/r}\lambda^{1/\delta}$.

PROOF: Shah [2] has proved if $F(z)=\sum_{n=0}^{\infty}A_nz^n$ is an entire function then

(A) $$\text{Order of } F=\overline{\lim_{n\to\infty}}\frac{\log n}{\log|A_n/A_{n+1}|}$$

and if $|A_n/A_{n+1}|$ is non-decreasing then

(B) $$\text{Lower order of } F=\underline{\lim_{n\to\infty}}\frac{\log n}{\log|A_n/A_{n+1}|}$$

Hence making use of (B) and (A), since x_1 is non-decreasing, we get:

(3. 3) $$\frac{\log|a_n/a_{n+1}|}{\log n}<\frac{1}{\lambda_1}+\varepsilon,\ n>n_1;$$

(3. 4) $$\frac{\log|a_n/a_{n+1}|}{\log n}>\frac{1}{\rho_1}-\varepsilon,\ n>n_2$$

for every $\varepsilon>0$. Similarly for f_2, we have, for $\varepsilon>0$:

(3. 5) $$\frac{\log|b_n/b_{n+1}|}{\log n}<\frac{1}{\lambda_2}+\varepsilon,\ n>n_1';$$

(3. 6) $$\frac{\log|b_n/b_{n+1}|}{\log n}>\frac{1}{\rho_2}-\varepsilon,\ n>n_2'.$$

From (3. 3) and (3. 5) we have for $n>\max(n_1, n_1')$

$$\frac{(\log|a_n/a_{n+1}|)^{1/r}(\log|b_n/b_{n+1}|)^{1/\delta}}{\log n}<\left(\frac{1}{\lambda_1}+\varepsilon\right)^{1/r}\left(\frac{1}{\lambda_2}+\varepsilon\right)^{1/\delta},$$

and as $|c_n/c_{n+1}|$ is non-decreasing, find from (i)

$$\overline{\lim_{n\to\infty}}\frac{\log|c_n/c_{n+1}|}{\log n}\leqslant\lambda_1{}^{-1/r}\lambda_2{}^{-1/\delta}.$$

which implies $\lambda^{-1} \leqslant \lambda_1^{-1/r}\lambda_2^{-1/\delta}$. Similarly from (3. 4) and (3. 6) we obtain $\rho^{-1} \geqslant \rho_1^{-1/r}\rho_2^{-1/\delta}$ and this proves the theorem.

COR. *If $f_j(z) = \sum_{n=0}^{\infty} a_n^{(j)} z^n$, $(j=1, 2, \cdots, s)$ be s entire functions of orders $\rho_j (0 \leqslant \rho_j \leqslant \infty)$, $(j=1, 2, \cdots, s)$ respectively and each of the function $|a_n^{(j)}/a_{n+1}^{(j)}|$, $(j=1, 2, \cdots, s)$ be non-decreasing for $n > n_0$, then the function $f(z) = \sum_{n=0}^{\infty} c_n z^n$ where*

$$\log|c_n/c_{n+1}| \sim (\log|a_n^{(1)}/a_{n+1}^{(1)}|)^{1/\alpha_1} \cdots (\log|a_n^{(s)}/a_{n+1}^{(s)}|)^{1/\alpha_s},$$

where $1 < \alpha_j < \infty (j=1, 2, \cdots, s)$; $\sum_{j=1}^{s} \alpha_j^{-1} = 1$, is also an entire function of order ρ, such that

$$\rho \leqslant \rho_1^{1/\alpha_1} \cdots \rho_s^{1/\alpha_s}.$$

and a similar type of result in case of lower orders.

4. Here I prove the following theorems:

THEOREM 4: *Let* (i) *f_1 and f_2 be two entire functions oorders ρ_1 and ρ_2.*
(ii) $2(\log|c_n|^{-1})^{-1} \sim (\log|a_n|^{-1})^{-1} + (\log|b_n|^{-1})^{-1}$.
Then f is also an entire function of order ρ, such that

(4. 1) $$2\rho \leqslant \rho_1 + \rho_2.$$

(iii) *Further, let λ_1 and λ_2 be the lower orders of f_1 and f_2, and x_1 and x_2 be non-decreasing functions of n for $n > n_0$ and that* (ii) *holds. Then f is of lower order λ, such that*

(4. 2) $$2\lambda \geqslant \lambda_1 + \lambda_2.$$

COR. *If f_1 and f_2 are of regular growths, then f is also of regular growth and*

(4. 3) $$2\rho = \rho_1 + \rho_2.$$

REMARK: The result (4. 3) can also be obtained even if f_1 and f_2 are not of regular growths. But in that case we will have to make some other supposition as the following theorem shows:

THEOREM 5: *Let* (i) *f_1 and f_2 be two entire functions of orders $\rho_1 (0<\rho_1<\infty)$, $\rho_2 (0<\rho_2<\infty)$; types T_1 $(0 < T_1 < \infty)$, $T_2 (0 < T_2 < \infty)$; lower types $t_1 (0 < t_1 < \infty)$, $t_2 (0 < t_2 < \infty)$, and x_1 and x_2 are non-decreasing functions of n for $n > n_0$.*
(ii) $2(\log|c_n|^{-1})^{-1} \sim (\log|a_n|^{-1})^{-1} + (\log|b_n|^{-1})^{-1}$.
Then f is also an entire function of order ρ, such that

(4. 4) $$2\rho = \rho_1 + \rho_2$$

PROOF OF THEOREM 4: First we show that f is an entire function. Since f_2 is an entire function, hence for every $\varepsilon > 0$ and arbitrily large R,

$$(\log|b_n|^{-1/n})^{-1} < \{\log(R-\varepsilon)\}^{-1}, \ n > n_1.$$

Similarly for f_1, we have:

$$(\log|a_n|^{-1/n})^{-1} < \{\log(R-\varepsilon)\}^{-1},\ n > n_2.$$

Hence using (ii), we have for sufficiently large n,

$$2(\log|c_n|^{-1/n})^{-1} < 2\{\log(R-\varepsilon)\}^{-1},$$

and so f is an entire function.

Further, as f_1 and f_2 are of orders ρ_1 and ρ_2, we find for every $\varepsilon>0$ and large n,

$$\frac{n\log n}{\log|a_n|^{-1}} + \frac{n\log n}{\log|b_n|^{-1}} < \rho_1+\rho_2+2\varepsilon,$$

and (4. 1) follows. The proof of (4. 2) is similar and so omitted.

PROOF OF THEOREM 5: Using (i) for f_1, we have for every $\varepsilon > 0$,

(4. 5) $$\frac{n\log n}{\log|a_n|^{-1}} < \frac{n}{\log|a_n|^{-1}} \log\{e\rho_1(T_1+\varepsilon)\} + \rho_1 = \rho_1 + o(1).$$

Similarly, for f_2, we have for $n > n_2$,

(4. 6) $$\frac{n\log n}{\log|a_n|^{-1}} < \rho_2 + o(1).$$

Inequalities (4. 5) and (4. 6) yield on making use of (ii), for sufficiently large n,

$$\frac{2n\log n}{\log|c_n|^{-1}} < \rho_1+\rho_2+o(1).$$

Therefore

(4. 7) $$2\rho \leqslant \rho_1+\rho_2.$$

Again, using (2) for f_1, we have for every $\varepsilon > 0$,

$$\frac{n\log n}{\log|a_n|^{-1}} > \frac{n}{\log|a_n|^{-1}} \log\{e\rho_1(t_1-\varepsilon)\} + \rho_1 = \rho_1 + o(1),\ n > n_1,$$

and a similar expression for f_2. Therefore for sufficiently large n,

(4. 8) $$2\lambda = \lim_{n\to\infty} \frac{2n\log n}{\log|c_n|^{-1}} \geqslant \rho_1+\rho_2.$$

But as $\lambda \leqslant \rho$ always, we find from (4. 8)

(4. 9) $$2\rho \geqslant \rho_1+\rho_2$$

Therefore (4. 4) follows from (4. 7) and (4. 9).

5. Let $n(x)$ denote the number of zeros of an entire function $f(z)$ in $|z| \leqslant x$. I prove:

THEOREM 6: *If $f(z)$ is an entire function of order zero and not a constant, then*

(5. 1) $$\lim_{r\to\infty} \frac{N_\alpha(r)}{Q_\alpha(r)} = \infty,$$

where

$$N_\alpha(r) = r^\alpha \int_0^r \frac{n(x)}{x^{\alpha+1}} dx;\ Q_\alpha(r) = r^{\alpha+1} \int_r^\infty \frac{n(x)}{x^{\alpha+2}} dx \neq 0,\ \alpha \geqslant 0.$$

PROOF: Suppose (5. 1) does not hold good. Then

$$\overline{\lim_{r\to\infty}} \frac{N_\alpha(r)}{Q_\alpha(r)} = \eta_1,\ \eta_1 < \infty.$$

Hence for $r > R$,

$$N_\alpha(r) \leqslant \eta Q_\alpha(r),$$

Let $0 \leqslant \alpha < \xi < \alpha + (1+\eta)^{-1}$. Then $\int^\infty n(x) x^{-1-\xi} dx$ is convergent and so

$$\begin{aligned}\int_R^\infty Q_\alpha(t) t^{-1-\xi} dt &= \int_R^\infty t^{\alpha-\xi} dt \int_t^\infty \frac{n(x)}{x^{\alpha+2}} dx \\ &= \int_R^\infty \frac{n(x)}{x^{\alpha+2}} dx \int_R^x t^{\alpha-\xi} dt \\ &\leqslant \frac{1}{\alpha-\xi+1} \int_R^\infty \frac{n(x)}{x^{\xi+1}} dx,\end{aligned}$$

and so $\int^\infty Q_\alpha(x) x^{-1-\xi} dx$ is convergent. Now

$$\begin{aligned}\int_R^\infty \frac{Q_\alpha(t)}{t^{1+\xi}} dt &\leqslant \frac{1}{\alpha-\xi+1} \int_R^\infty x^{\alpha-\xi} d\left[\frac{N_\alpha(x)}{x^\alpha}\right] \\ &\leqslant \frac{\eta(\xi-\alpha)}{\alpha-\xi+1} \int_R^\infty \frac{Q_\alpha(x)}{x^{1+\xi}} dx.\end{aligned}$$

But $\eta(\xi-\alpha)/(\alpha-\xi+1) < 1$, and so a contradiction.

Note: The author has obtained this result alternatively [1, p. 11].

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Post-Graduate Studies (Eve.)
Delhi University,
Delhi-6, India