ESTIMATION OF COEFFICIENTS OF ENTIRE FUNCTIONS REPRESENTED BY DIRICHLET SERIES

By

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(Received September 30, 1964)

1. Let $f(s)=\sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ be an entire function represented by Dirichlet series, where $0=\lambda_0<\lambda_1<\cdots<\lambda_n\to\infty$ as $n\to\infty$, and

(1.1) $\overline{\lim_{n\to\infty}}\frac{n}{\lambda_n}=D$; (1.2) $\underline{\lim_{n\to\infty}}\frac{n}{\lambda_n}=d,\ 0<d<D<\infty$;

(1.3) $\underline{\lim_{n\to\infty}}(\lambda_{n+1}-\lambda_n)=h$; (1.4) $\overline{\lim_{n\to\infty}}(\lambda_{n+1}-\lambda_n)=m,\ 0<h\leqslant m<\infty$;

where $h\leqslant D^{-1}$, $m\leqslant d^{-1}$.

I wish to prove certain results involving the coefficients a_n of $f(s)$. Throughout, it is supposed that λ_n's satisfy the above relations, unless specified.

2. Define:

$$\theta(n)=\frac{\log|a_n|^{-1}}{\lambda_n\log\lambda_n};\quad \varphi(n)=\frac{\log|a_n/a_{n+1}|}{\log\lambda_n},$$

and let

$$\overline{\underline{\lim_{n\to\infty}}}\,\varphi(n)=\frac{\beta}{\alpha};\qquad \overline{\underline{\lim_{n\to\infty}}}[\varphi(n)]^{-1}=\frac{\delta}{\gamma};$$

$$\overline{\underline{\lim_{n\to\infty}}}\,\theta(n)=\frac{B}{A};\qquad \overline{\underline{\lim_{n\to\infty}}}[\theta(n)]^{-1}=\frac{\Delta}{C};$$

Then we have the following

THEOREM 1: *The following relations will hold:*

(i) $\alpha d\leqslant A=\frac{1}{\Delta}$; (ii) $\frac{1}{C}=B\leqslant D\beta$,

(iii) $\frac{\gamma}{D}\leqslant C\leqslant\frac{\gamma}{d}$; (iv) $\alpha=\frac{1}{\delta}$.

Further, if $x_n=\log|a_{n-1}/a_n|/(\lambda_n-\lambda_{n-1})$ *is a non-decreasing function of* n *for* $n>n_0$, *then*

(v) $B\geqslant\beta/m$; (vi) $\Delta\geqslant h\delta$.

PROOF: (i) We have:

$$\log\left|\frac{a_n}{a_{n+1}}\right|>(\alpha-\varepsilon)\log\lambda_n,\ n\geqslant n_0.$$

But

(2. 1) $$\log|a_n|^{-1}=\log\left\{\left|\frac{a_{n_0}}{a_{n_0+1}}\cdots\frac{a_{N-1}}{a_N}\frac{a_N}{a_{N+1}}\cdots\frac{a_{n-1}}{a_n}\right|\right\}+O(1)$$
$$>(\alpha-\varepsilon)[(N-n_0)\log\lambda_{n_0}+\{(d-\varepsilon)-(D+\varepsilon)\lambda_N\}\log\lambda_N]+O(1),$$
where N is larger than n_0. Let
$$\lambda_n=[\lambda_N(\log\lambda_N)^2]+1,$$
then $\log\lambda_n\sim\log\lambda_N$ as $N\to\infty$. Hence for sufficiently large n,
$$\log|a_n|^{-1}>(\alpha-\varepsilon)(d-\varepsilon)(1+o(1))\lambda_n\log\lambda_n,$$
or,
$$A\geqslant\alpha d.$$

The proofs of (ii) and (iii) are similar and so omitted; the proof of (iv) being straight forward.

PROOF (v): Let $0<\beta<\infty$. Then
$$x_n=\frac{\log|a_{n-1}/a_n|}{\lambda_n-\lambda_{n-1}}>\frac{(\beta-\varepsilon)\log\lambda_{n-1}}{(\lambda_n-\lambda_{n-1})},$$
for a sequence of n, say $n=N_p+1$ $(p=1, 2,\cdots)$, $N_1>n_0$, $N_p\to\infty$ as $p\to\infty$. Then as in (2.1),
$$\log|a_n|^{-1}=\log\left\{\left|\frac{a_{N_1-1}}{a_{N_1}}\cdots\frac{a_{N_p-1}}{a_{N_p}}\cdots\frac{a_{n-1}}{a_n}\right|\right\}+O(1)$$
$$=(\lambda_{N_1}-\lambda_{N_1-1})x_{N_1}+\cdots+(\lambda_{N_p}-\lambda_{N_p-1})x_{N_p}+\cdots+(\lambda_n-\lambda_{n-1})x_n+O(1)$$
$$\geqslant(N_p-N_1)x_{N_1}+(\lambda_n-\lambda_{N_p-1})x_{N_p}+O(1).$$

Let
$$\lambda_{n-1}=[\lambda_{N_p}(\log\lambda_{N_p})^{\eta}]+1,\ \eta>0.$$
Then $\log\lambda_{n-1}\sim\log\lambda_{N_p}$ as n and $p\to\infty$. Then
$$\log|a_n|^{-1}>(N_p-N_1)x_{N_1}+\frac{(\lambda_n-\lambda_{N_p})(\beta-\varepsilon)\log\lambda_{N_p}}{(\lambda_{N_p}-\lambda_{N_p-1})}+O(1)$$
$$>\frac{(1+o(1))(\beta-\varepsilon)\lambda_n\log\lambda_{N_p}}{(m+\varepsilon)}.$$
Therefore $B\geqslant\beta/m$. The proof of (vi) being on the same lines, is therefore omitted.

REMARK: It is to be noted that under some less strictly followed conditions on λ_n's, Δ is equal to order (R) of $f(s)$ (see for example [8], p. 217) and C is equal to the lower order (R) of $f(s)$ (see [6]).

3. Let $1/h=D=d$. Then, if ρ denotes the order $(R)\rho$ of $f(s)$, we have from (i), (iv) and (vi) of the previous theorem:

(3. 1) $$D\rho=D\overline{\lim_{n\to\infty}}\frac{\lambda_n\log\lambda_n}{\log|a_n|^{-1}}=\overline{\lim_{n\to\infty}}\frac{\log\lambda_n}{\log|a_n/a_{n+1}|}.$$

In the following theorem we suppose $\lambda_n=n$. Then

THEOREM 2: *If $a_n>0$ for all n and*

(3. 2) $$\lim_{n\to\infty}n\left(\frac{a_n^2}{a_{n-1}\,a_{n+1}}-1\right)=\frac{1}{\rho},$$

then (i) $f(s)=\sum_{n=1}^{\infty}a_ne^{ns}$ *is an entire function of order* $(R)\rho$ *and* (ii) $\log\mu(\sigma)\sim\frac{\lambda_{\nu(\sigma)}}{\rho}$.

PROOF: The proof follows from (3. 1) and from the method adopted by Pólya and Szegö ([9], p. 13).

The converse of the theorem is not necessarily true. Consider

$$f(s)=\exp(e^{2s})+\exp(e^{s})=\sum_{n=0}^{\infty}e^{2ns}\left\{\frac{1}{n!}+\frac{1}{(2n)!}\right\}+\sum_{n\in E}\frac{e^{ns}}{n!},$$

where E is the set of all positive odd integeres except 1. This is an entire function of order (R) equal to 2. Here the limit in (3. 2) does not exist (see for example [10]). Also

$$\log\mu(\sigma)\sim e^{2\sigma};\ \lambda_{\nu(\sigma)}\sim 2e^{2\sigma}.$$

Thus (ii) follows.

We can construct examples to show that the above theorem holds good, in its converse sense; for example we have ([1], p. 27-28).

$$f(s)=\sum_{n=1}^{\infty}\frac{e^{ns}}{\exp(x_1+\cdots+x_n)};$$

$$x_n=\frac{1}{\rho}\log n+S_n,\ n\geqslant n_0;\ x_n=0,\ n<n_0,\ \varliminf_{n\to\infty}S_n=-\frac{1}{\rho}\log\gamma,$$

$$\varlimsup_{n\to\infty}S_n=-\frac{1}{\rho}\log\delta,\ \ (0<\delta\leqslant\gamma<\infty);\ (S_{n+1}-S_n)=O(1/\log n);$$

$$\left(S_n-\frac{S_1+\cdots+S_n}{n}\right)=O(1/\log n).$$

Then we have ([5], p. 76-77) omitting the details:

$$\lim_{\sigma\to\infty}\frac{\log\mu(\sigma)}{\lambda_{\nu(\sigma)}}=\frac{1}{\rho},$$

and clearly (3. 2) holds.

4. Here I prove a theorem furnishing a systematic study of the results obtained in the preceding two theorems. λ_n's satisfy (1. 1)-(1. 4). Let us introduce the following notation:

$$k(n+1,\ n)=(\lambda_{n+1}-\lambda_n)x_{n+1}-(\lambda_n-\lambda_{n-1})x_n$$

THEOREN 3: *Let $f(s)$ in the usual series form be an entire function of order $(R)\,\rho<\infty$ and if*

$$\varlimsup_{n\to\infty}\lambda_n\left(\frac{|a_n|^2}{|a_{n-1}a_{n+1}|}-1\right)=\frac{L}{l};$$

then

(4. 1) $$ld\leqslant\varliminf_{n\to\infty}\frac{\log|a_n/a_{n+1}|}{\log\lambda_n}\leqslant LD;$$

and if $\lambda_n=n$, then

(4. 2) $$\varliminf_{n\to\infty}n\ k(n+1,\ n)=\max(\text{o},\ l);$$

and further

(4. 3) $$\overline{\lim_{n\to\infty}}\, \lambda_n\, k(n+1,n) \leqslant L.$$

PROOF: It is quite clear that if

$$\overline{\lim_{n\to\infty}}\, \lambda_n \log\left\{\frac{|a_n|^2}{|a_{n-1}\, a_{n+1}|}\right\} = \frac{L_1}{l_1};$$

then $L=L_1$; $l=l_1$. We now prove the results. Let $-\infty < l < \infty$, then

$$\left|\frac{a_n^2}{a_{n+1}\, a_{n-1}}\right| > \exp\left\{\frac{l-\varepsilon}{\lambda_n}\right\},\ n \geqslant N.$$

Now

$$\left|\frac{a_n}{a_{n+1}}\right| = |k| \left|\frac{a_{N+1}^2}{a_{N+2}\, a_N} \cdots \frac{a_n^2}{a_{n+1}\, a_{n-1}}\right|.$$

Therefore

$$\log\left|\frac{a_n}{a_{n+1}}\right| > \log|k| + (l-\varepsilon) \sum_{p=N+1}^{n} \lambda_p^{-1}$$
$$\sim (l-\varepsilon)\,(d-\varepsilon) \log \lambda_n,$$

and (4. 1) follows.

To prove (4. 2), we take $\lambda_n = n$. Either $\log|a_n/a_{n+1}| > \log|a_{n-1}/a_n|$ for all $n > n_0$, in which case for all large n,

$$n\ k(n+1,\ n) = n \log\left|\frac{a_n^2}{a_{n+1}\, a_{n-1}}\right|,$$

hence

(4. 4) $$l \geqslant \underline{\lim}_{n\to\infty}\, n\ k(n+1,\ n) \geqslant l,\ l \geqslant 0,$$

or, we shall have $\log\left|\frac{a_n}{a_{n+1}}\right| \leqslant \log\left|\frac{a_{n-1}}{a_n}\right|$ for an infinity of n say $n=m_1,\ m_2, \cdots$. Then $|a_n| e^{\sigma\lambda_n}$ $(n=m_1,\ m_2, \cdots)$ does not become the maximum term for any σ, and $x_{n+1} < x_n$ infinitely often, which is contrary to our hypothesis. Therefore

(4. 5) $$\underline{\lim}_{n\to\infty}\, n\ k(n+1,\ n) = 0 \geqslant l,$$

and so (4. 2) follows.

We now prove (4. 3). We may suppose that $L < \infty$. Let $|a_m| e^{\sigma\lambda_m}$, $|a_n| e^{\sigma\lambda_n}$ and $|a_p| e^{\sigma\lambda_p}$ be three consecutive maximum terms $(m \leqslant n-1 \leqslant p-2)$. Now if $|a_m| e^{\sigma\lambda_m}$ is to become the maximum term, then $x_m \leqslant \sigma < x_{m+1}$ (for exhaustive arguments see [5], Ch. I, Part I). Similarly $|a_{m+2}| e^{\sigma\lambda_{m+2}}$, $|a_{m+3}| e^{\sigma\lambda_{m+3}}$, $\cdots$, $|a_{n-1}| e^{\sigma\lambda_{n-1}}$ are all maximum terms if

$$x_{m+2} \leqslant \sigma < x_{m+3},$$
$$\cdots \quad \cdots \quad \cdots$$
$$x_{n-1} \leqslant \sigma < x_n.$$

Thus $x_m < x_{m+1} = x_{m+2} = \cdots = x_{n-1} = \sigma < x_n$. But by hypothesis $|a_n| e^{\sigma\lambda_n}$ is a maximum term, hence $x_n \leqslant \sigma < x_{n+1}$. Therefore

(4. 6) $$x_{m+1} = x_{m+2} = \cdots = x_n.$$

Similarly we have:

(4. 7) $$x_{n+1}=x_{n+2}=\cdots=x_p.$$

We require a

LEMMA: *Let m be a positive integer such that* $\lambda_m=\lambda_{\nu(\sigma)}$, $\lambda_m>\lambda_{\nu(0)}$, *then*

$$x_m=\max\left(\frac{\log\left|\frac{a_0}{a_m}\right|}{\lambda_m-\lambda_0}, \frac{\log\left|\frac{a_1}{a_m}\right|}{\lambda_m-\lambda_1}, \cdots, \frac{\log\left|\frac{a_{m-1}}{a_m}\right|}{\lambda_m-\lambda_{m-1}}\right).$$

PROOF OF THE LEMMA: With the smallest value of σ if all the following inequalities hold, viz.,

$$|a_0|\leqslant|a_m|e^{\sigma\lambda_m},\ |a_1|e^{\sigma\lambda_1}\leqslant|a_m|e^{\sigma\lambda_m},\cdots,\ |a_{m-1}|e^{\sigma\lambda_{m-1}}\leqslant|a_m|e^{\sigma\lambda_m},$$

then $|a_m|e^{\sigma\lambda_m}$ becomes the maximum term. This value of σ is, therefore, equal to x_m. Then if we refer to the convex polygon construction for the maximum term (see for instance [5], fig. 1), it is clear that

$$\frac{-\log|a_m|+\log|a_0|}{\lambda_m-\lambda_0}, \frac{-\log|a_m|+\log|a_1|}{\lambda_m-\lambda_1}, \cdots, \frac{-\log|a_m|+\log|a_{m-1}|}{\lambda_m-\lambda_{m-1}}$$

are all $\leqslant$ the slope x_m and therefore the lemma follows.

From the lemma, it follows that

$$x_n=\max\left\{\frac{\log\left|\frac{a_{n-1}}{a_n}\right|}{\lambda_n-\lambda_{n-1}}, \cdots, \frac{\log\left|\frac{a_m}{a_n}\right|}{\lambda_n-\lambda_m}, \cdots\right\}$$

or,

(4. 8) $$x_n=\frac{\log\left|\frac{a_m}{a_n}\right|}{\lambda_n-\lambda_m},\ n>m;$$

since the maximum term goes from $|a_m|e^{\sigma\lambda_m}$ to $|a_n|e^{\sigma\lambda_n}$. Similarly

$$x_p=\max\left\{\frac{\log\left|\frac{a_{p-1}}{a_p}\right|}{\lambda_p-\lambda_{p-1}}, \cdots, \frac{\log\left|\frac{a_n}{a_p}\right|}{\lambda_p-\lambda_n}, \cdots\right\}$$

(4. 9) $$=\frac{\log\left|\frac{a_n}{a_p}\right|}{\lambda_p-\lambda_n},\ p>n.$$

Hence from (4. 6) and (4. 7), we have:

$$\lambda_n k(n+1,\ n)=\lambda_n\{(\lambda_{n+1}-\lambda_n)x_p-(\lambda_n-\lambda_{n-1})x_{m+1}\}$$

(4. 10) $$=\lambda_n\left\{\left(\frac{\lambda_{n+1}-\lambda_n}{\lambda_p-\lambda_n}\right)\log\left|\frac{a_n}{a_p}\right|-\left(\frac{\lambda_n-\lambda_{n-1}}{\lambda_n-\lambda_m}\right)\log\left|\frac{a_m}{a_n}\right|\right\}.$$

From (4. 8), we have:

(4. 11) $$\log\left|\frac{a_m}{a_n}\right|\geqslant\left(\frac{\lambda_n-\lambda_m}{\lambda_n-\lambda_{n-1}}\right)\log\left|\frac{a_{n-1}}{a_n}\right|,$$

and from (4. 7) and (4. 9) we have:

(4. 12) $$\log\left|\frac{a_n}{a_p}\right|=\left(\frac{\lambda_p-\lambda_n}{\lambda_{n+1}-\lambda_n}\right)\log\left|\frac{a_n}{a_{n+1}}\right|.$$

Combining (4. 10), (4. 11) and (4. 12) we obtain

$$\lambda_n k(n+1,\ n) \leqslant \lambda_n\left\{\log\left|\frac{a_n}{a_{n+1}}\right| - \log\left|\frac{a_{n-1}}{a_n}\right|\right\}$$
$$= \lambda_n \log\left|\frac{a_n{}^2}{a_{n+1}a_{n-1}}\right|.$$
$$\therefore\quad \lambda_n k(n+1,\ n) < (L+\varepsilon),\ n > n_0,$$

and so (4. 3) follows.

5. If $f(s)$ is an entire function of order $(R)\rho$ and lower order $(R)\lambda$, $0 < \lambda \leqslant \rho < \infty$, then I have shown ([3], p. 45; equa. (5))

$$\overline{\lim_{\sigma\to\infty}} \frac{\log \mu(\sigma)}{\lambda_{\nu(\sigma)} \log \lambda_{\nu(\sigma)}} \leqslant \frac{1}{\lambda} - \frac{1}{\rho}.$$

I wish to prove now a result of the above type in terms of the coefficient a_n. We have then the following.

THEOREM 4: *If* $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ *is an entire function of finite order* $(R)\rho < \infty$, $\rho > 0$ *finite lower order* $(R)\lambda$, $\lambda > 0$ *and finite type* T $(T > 0)$, *where* (i) $\lim_{n\to\infty} (\lambda_{n+1} - \lambda_n) = h$ *and* (ii) $\log|a_{n-1}/a_n|/(\lambda_n - \lambda_{n-1})$ *is a non-decreasing function of* n *for* $n > n_0$, *then*

$$\lim_{n\to\infty} \frac{\log|a_n/a_{n+1}|}{\lambda_n \log \lambda_n} \leqslant \frac{1}{\lambda} - \frac{1}{\rho}.$$

PROOF: If order $(R)\rho$ of $f(s)$ is finite and λ_n's satisfy (i), then we have from a result on type ([4], p. 276), (taking $\rho(\sigma) = \rho$ in particular):

(5. 1) $$|a_n| < \left\{\frac{(T+\varepsilon)\rho e}{\lambda_n}\right\}^{\lambda_n/\rho},\ n > n_0;\ \varepsilon > 0.$$

Again, as ρ is finite and (i) holds good, we have ([2], Th. 2):

$$\log m(\sigma) \sim \log \mu(\sigma),$$

and so ([6], following after (1.6)), for all $n > n_0$, and $\varepsilon > 0$,

(5. 2) $$|a_{n+1}|^{-1} < \{\lambda_{n+1}\}^{\lambda_{n+1}/(\lambda-\varepsilon)}.$$

Combining (5. 1) and (5. 2)

$$\log\left|\frac{a_n}{a_{n+1}}\right| < \frac{\lambda_n}{\rho}\log((T+\varepsilon)e\rho) - \frac{\lambda_n}{\rho}\log\lambda_n + \frac{\lambda_{n+1}\log\lambda_{n+1}}{\lambda-\varepsilon}$$

(5. 3) $$< \frac{\lambda_n}{\rho}\log((T+\varepsilon)e\rho) - \frac{\lambda_n}{\rho}\log\lambda_n + \frac{\lambda_n+h+\varepsilon}{\lambda-\varepsilon}\log(\lambda_n+h+\varepsilon),$$

for arbitrarily large n and the result follows.

REMARK: In case we restrict λ_n's further, that is if in addition to (i), λ_n's also satisfy: $\overline{\lim_{n\to\infty}}(\lambda_{n+1} - \lambda_n) = m$, we get a stronger result:

$$\overline{\lim_{n\to\infty}} \frac{\log|a_n/a_{n+1}|}{\lambda_n \log \lambda_n} \leqslant \frac{1}{\lambda} - \frac{1}{\rho},$$

but it is weker in the sense that λ_n's are subject to more stringent conditions. The proof is almost the same, except in (5. 3) we get on the right-hand side $m+\varepsilon$ in place of $h+\varepsilon$ and this being true for all $n > n_0$.

NOTE: Theorem 1 can be applied to provide numerous results involving orders (R) and lower orders (R) for two and more entire functions, for instance results of the type of Theorems 3, 4 & 5 in [7].

The author thanks Dr. S.C. Mitra for his encouraging criticism and interest in this work.

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