NOTE ON RING EXTENSIONS OF REGULAR LOCAL RINGS

By

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1. STATEMENT.

Let R be a regular local ring with maximal ideal $\mathfrak{m}$ and S be an over ring of R such that (i) S is a finite R-module and (ii) no non-zero element in R is a zero divisor in S. Such a pair of rings sometimes occurs in the theory of rings so that it might be interesting, under this situation, to find a condition that the ideal $\mathfrak{m}S$ coincides with the J-radical[1] of S.

Our theorem will be stated as:

The ideal $\mathfrak{m}S$ coincides with the J-radical of S if and only if the equality

(a) $$[S:R]=\sum[S/\mathfrak{P}:R/\mathfrak{m}]$$

holds, where $[S:R]$ is the maximum number of linearly independent elements of S over R[2] and the sum runs over all maximal ideals $\mathfrak{P}$ of S.

And as a corollary we have :

If (a) *holds, then the quotient ring $S_{\mathfrak{P}}$ is a regular local ring for any maximal ideal $\mathfrak{P}$ of S.*

2. PROOF OF THE THEOREM.

Let R be a (commutative Noetherian) semi-local ring and S be an over ring of R. And assume the pair of rings (R, S) satisfies the conditions (i) and (ii). Then it is well known that S is a semi-local ring and $\dim S=\dim R$. Moreover, if R is a normal local ring (i.e. integrally closed in its quotient field), then $\dim S_{\mathfrak{P}}=\dim S$ for any maximal ideal $\mathfrak{P}$ of S (cf. [1] and [2]).

We say that a semi-local ring R is unmixed if $\dim \hat{R}/\mathfrak{p}=\dim R$ for any prime divisor $\mathfrak{p}$ of the zero ideal in $\hat{R}$ where $\hat{R}$ is the completion of R (cf. [1]).

PROPOSITION. *If R is unmixed, then S is also unmixed.*

PROOF. Let $\hat{R}$ and $\hat{S}$ be completions of R and S respectively. Then $\hat{S}$ is an over ring of $\hat{R}$ and is a finite $\hat{R}$-module. Let $\mathfrak{P}$ be any prime divisor of the zero ideal in $\hat{S}$. Putting $\mathfrak{p}=\hat{R}\cap\mathfrak{P}$, then $\mathfrak{p}$ is a prime ideal of $\hat{R}$ and any element of $\mathfrak{p}$ is a zero divisor in $\hat{S}$. Therefore any element of $\mathfrak{p}$ is a zero divisor in $\hat{R}$ (cf. [2]). Hence $\mathfrak{p}$ is contained in some prime divisor of the zero ideal in $\hat{R}$. Since R is unmixed, the zero

1) We mean by the J-radical the intersection of all maximal ideals of S.

2) It is equal to the dimension of the total quotient ring of S considered as a vector space over the quotient field of R.

ideal of $\hat{R}$ has no imbeded prime divisor. Whence $\mathfrak{p}$ is a prime divisor of the zero ideal in $\hat{R}$, and we have $dim\, \hat{R}/\mathfrak{p}=dim\, R$. On the otherhand, $\hat{S}/\mathfrak{P}$ is an over ring of $\hat{R}/\mathfrak{p}$ and is a finite $\hat{R}/\mathfrak{p}$-module. This shows that $dim\, \hat{S}/\mathfrak{P}=dim\, \hat{R}/\mathfrak{p}$. Therefore we have $dim\, \hat{S}/\mathfrak{P}=dim\, S$, and the proof is complete.

LEMMA. *If R is a regular local ring with maximal ideal $\mathfrak{m}$ and if $\mathfrak{m}S$ coincides with the J-radical of S, then $S_{\mathfrak{P}}$ is a regular local ring for any maximal ideal $\mathfrak{P}$ of S.*

The proof is easy and we omit it.

Now, assume that R is a regular local ring with maximal ideal $\mathfrak{m}$, and we prove the theorem stated in the paragraph 1. Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_t$ be the set of all maximal ideals of S and put $\mathfrak{M}=\mathfrak{P}_1\cap\ldots\cap\mathfrak{P}_t$. Let $\mathfrak{m}S=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_t$ be the irredundant primary decomposition of $\mathfrak{m}S$. Then we have

$$[S:R]=\sum_{i=1}^{t}[S/\mathfrak{P}_i:R/\mathfrak{m}]e(\mathfrak{q}_i), \tag{b}$$

where $e(\mathfrak{q}_i)$ is the multiplicity of the primary ideal $\mathfrak{q}_iS_{\mathfrak{P}_i}$ (cf. [2]). Now we assume that (a) is satisfied. Then, from (b) we have $e(\mathfrak{q}_i)=1$, and hence $e(\mathfrak{P}_i)=1$. Therefore we have $e(\mathfrak{M})=t$, that is, the multiplicity of the semi-local ring S is equal to the number of maximal ideals of S. On the otherhand, since a regular local ring is unmixed (cf. [1]), S is unmixed by virtue of the preceding proposition. Therefore $S_{\mathfrak{P}_i}$ is a regular local ring (cf. [3]). Recall that in a regular local ring the maximal ideal is the only primary ideal (belonging to the maximal ideal) whose multiplicity is equal to 1. This shows that $\mathfrak{q}_i=\mathfrak{P}_i$ and consequently $\mathfrak{m}S=\mathfrak{M}$. The only if part is an easy consequence of the preceding lemma and the equality (b).

References

[1] M. Nagata, Local Rings, Interscience Tracts, New York (1962).

[2] O. Zariski and P. Samuel, Commutative Algebra Vol. **II**, Van Nostrand, Princeton (1960).

[3] T. Matsuoka, Note on semi-local rings, J. of Gakugei, Tokushima Univ. Vol. **XII** (1962) pp. 51–55.