ON A CRITERION FOR ANALYTICALLY UN-RAMIFICATION OF A LOCAL RING

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1. Let $\mathfrak{o}$ be a local ring with maximal ideal $\mathfrak{m}$. $\mathfrak{o}$ is called analytically unramified in case the completion of $\mathfrak{o}$ has no nilpotent element. A criterion that $\mathfrak{o}$ is analytically unramified, obtained by D. Rees in [5], is stated in terms of the integral closures of ideals: $\mathfrak{o}$ is analytically unramified in case there is an integer k such that $\mathfrak{q}_n \subset \mathfrak{q}^{n-k}$ for all $n \geqq k$ where $\mathfrak{q}_n$ is the integral closure of the n-th power of a zero dimensional ideal $\mathfrak{q}$. Moreover, if this condition is satisfied $\mathfrak{a}_n \subset \mathfrak{a}^{n-k}$ for all $n \geqq k$, where $\mathfrak{a}$ is any ideal and k is an integer depending on $\mathfrak{a}$.

In this note we shall show that this result of Rees can be translated in the finiteness condition of the integral closure of the Rees' ring associated with $\mathfrak{q}$. In the course of proof we shall also show $l(\mathfrak{q}_n)$, the length of $\mathfrak{q}_n$, can be expressed as the Hilbert function $\mu(n)$ if n is large. This is a theorem due to Muhly. He discussed it under some additional restrictions placing on $\mathfrak{o}$ [1].

2. Let $\mathfrak{a}$ be an ideal in a commutative Noetherian ring $\mathfrak{o}$. An element $x \in \mathfrak{o}$ is called integral over $\mathfrak{a}$ in case x satisfies the equation of the form, $x^{\rho}+c_1x^{\rho-1}+\dots+c_{\rho}=0$, where $c_i \in \mathfrak{a}^i$. The set of elements which are integral over $\mathfrak{a}$ forms an ideal [4]. We call it the integral closure of $\mathfrak{a}$, and is denoted by $\mathfrak{a}_a$. We write $\mathfrak{a}_n$ in place of $(\mathfrak{a}^n)_a$. If the set $a_1, \dots, a_r$ is a basis of $\mathfrak{a}$ we associate with $\mathfrak{a}$ the Ree's ring $\mathfrak{o}(\mathfrak{a})$ defined by $\mathfrak{o}(\mathfrak{a})=\mathfrak{o}[a_1t, \dots, a_rt, t^{-1}]$ where t is an indeterminate over $\mathfrak{o}$. Obviously, $\mathfrak{o}(\mathfrak{a})$ is a graded subring of $\mathfrak{o}[t, t^{-1}]$. Consider the integral closure $\mathfrak{o}^*(\mathfrak{a})$ of $\mathfrak{o}(\mathfrak{a})$ in $\mathfrak{o}[t, t^{-1}]$. Then, it is immediately seen that $\mathfrak{o}^*(\mathfrak{a})$ is a graded ring. Moreover, if $x \in \mathfrak{o}$, then $xt^n \in \mathfrak{o}^*(\mathfrak{a})$ if and only if $x \in \mathfrak{a}_n$.

LEMMA 1. *Suppose there is an integer k such that $\mathfrak{a}_n \subset \mathfrak{a}^{n-k}$ for all $n \geqq k$. Then $\mathfrak{o}^*(\mathfrak{a})$ is a finite $\mathfrak{o}(\mathfrak{a})$-module* (Rees [6]).

PROOF. If $xt^n \in \mathfrak{o}^*(\mathfrak{a})$ and if $n \geqq k$, then $xt^n \in \mathfrak{a}_nt^n \subset \mathfrak{a}^{n-k}t^n = (\mathfrak{a}^{n-k}t^{n-k})t^k \subset t^k\mathfrak{o}(\mathfrak{a})$. If $n < k$, we also have $xt^n = x(t^{-1})^{k-n}t^k \in t^k\mathfrak{o}(\mathfrak{a})$. Hence in either case $xt^n \in t^k\mathfrak{o}^*(\mathfrak{a})$.

Since $\mathfrak{o}^*(\mathfrak{a})$ is graded we can consider homogeneous ideals in $\mathfrak{o}^*(\mathfrak{a})$. For such ideal A, we associate ideals A_n in $\mathfrak{o}$ defined by $A_n = \{x \in \mathfrak{o};\ xt^n \in A\}$. Then, as the converse of lemma 1, we get

LEMMA 2. *If A is a homogeneous ideal in $\mathfrak{o}^*(\mathfrak{a})$ and if $\mathfrak{o}^*(\mathfrak{a})$ is finite over $\mathfrak{o}(\mathfrak{a})$, then there is an integer k such that $A_n = \mathfrak{a}^{n-k}A_k$ for all integer $n \geqq k$. In particular, we have $A_n \subset \mathfrak{a}^{n-k}$.*

PROOF. We show first $\mathfrak{a}^p A_q \subset A_{p+q}$. In fact, if $a \in \mathfrak{a}^p$ and $b \in A_q$, then $at^p \in \mathfrak{o}^*(\mathfrak{a})$ and $bt^q \in A$. Hence $abt^{p+q} \in A$. Therefore $ab \in A_{p+q}$. Now, since an $\mathfrak{o}(\mathfrak{a})$-module $\mathfrak{o}^*(\mathfrak{a})$ is generated by homogeneous elements, A is also generated by homogeneous elements as an $\mathfrak{o}(\mathfrak{a})$-module. Let

$$A = \mathfrak{o}(\mathfrak{a})\omega_1 + \ \dots\ + \mathfrak{o}(\mathfrak{a})\omega_m,$$

with $\omega_i = x_i t^{\lambda_i}$ and $x_i \in A_{\lambda_i}$ $(i=1, \dots, m)$. If k is an integer such that $k \geqq \operatorname{Max} \lambda_i$ and if $x \in A_n$ $(n \geqq k)$, then $xt^n \in A$ and can be written as

$$xt^n = (y_1 t^{n-\lambda_1})(x_1 t^{\lambda_1}) + \ \dots\ + (y_m t^{n-\lambda_m})(x_m t^{\lambda_m})$$

with $y_i \in \mathfrak{a}^{n-\lambda_i}$. Therefore we have

$$x \in \mathfrak{a}^{n-\lambda_1}A_{\lambda_1} + \ \dots\ + \mathfrak{a}^{n-\lambda_m}A_{\lambda_m} = \mathfrak{a}^{n-k}\mathfrak{a}^{k-\lambda_1}A_{\lambda_1} + \ \dots\ + \mathfrak{a}^{n-k}\mathfrak{a}^{k-\lambda_m}A_{\lambda_m}$$
$$\subset \mathfrak{a}^{n-k}A_{(k-\lambda_1)+\lambda_1} + \ \dots\ + \mathfrak{a}^{n-k}A_{(k-\lambda_m)+\lambda_m} = \mathfrak{a}^{n-k}A_k.$$

In case $A = \mathfrak{o}^*(\mathfrak{a})$, we have $A_n = \mathfrak{a}_n$. Hence

COROLLARY. *The converse of lemma 1 is true. Moreover we have $\mathfrak{a}_n = \mathfrak{a}^{n-k}\mathfrak{a}_k$ for all $n \geqq k$.*

Now, recall that an ideal $\mathfrak{b}$ in $\mathfrak{o}$ is called normal in case $\mathfrak{b}^n = \mathfrak{b}_n$ for all integers n [2]. Then, summarizing lemma 1 and 2, we have the following

THEOREM 1. *$\mathfrak{o}^*(\mathfrak{a})$ is a finite module over $\mathfrak{o}(\mathfrak{a})$ if and only if there exists an integer k such that $\mathfrak{a}_n \subset \mathfrak{a}^{n-k}$ for all $n \geqq k$ and when this is so $\mathfrak{a}_k$ is a normal ideal.*

PROOF. Put $\mathfrak{b} = \mathfrak{a}_k$. Then the last part of the theorem follows from the relation; $\mathfrak{b}_n = \mathfrak{a}_{nk} = \mathfrak{a}^{nk-k}\mathfrak{a}_k = (\mathfrak{a}^k)^{n-1}\mathfrak{a}_k \subset \mathfrak{b}^{n-1}\mathfrak{b} = \mathfrak{b}^n$.

3. In this section we put the restriction on $\mathfrak{o}$ and $\mathfrak{o}$ is assumed to be a semi-local ring with maximal ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$. Then the theorem of Rees can be stated as follows:

LEMMA 3. *Let $\mathfrak{v}$ be a defining ideal in $\mathfrak{o}$. Then $\mathfrak{o}$ is analytically unramified if and only if we can find an integer k such that $\mathfrak{v}_n \subset \mathfrak{v}^{n-k}$ for $n \geqq k$. If this condition is satisfied we have $\mathfrak{a}_n \subset \mathfrak{a}^{n-k}$ for any ideal $\mathfrak{a}$ where k is an integer depending on $\mathfrak{a}$.*

PROOF. Let $\hat{\mathfrak{o}}$ be the completion of $\mathfrak{o}$. Then $\hat{\mathfrak{o}}$ is a direct sum of complete local rings $\mathfrak{o}_i$ $(i=1, \ldots, r)$ and $\mathfrak{o}_i$ is isomorphic to the completion $\mathfrak{o}_{\mathfrak{p}_i}$ [3]. Hence if $\hat{\mathfrak{o}}$ has no nilpotent element, then each $\mathfrak{o}_{\mathfrak{p}_i}$ is analytically unramified and we can apply the Rees' result to the pair of rings $\mathfrak{o}_{\mathfrak{p}_i}$ and $\mathfrak{o}_i$. Whence we can find an integer k such that $(\mathfrak{a}\mathfrak{o}_{\mathfrak{p}_i})_n \subset (\mathfrak{a}\mathfrak{o}_{\mathfrak{p}_i})^{n-k}$ for $n \geqq k$ $(i=1, \ldots, r)$. Since $\mathfrak{b}_a\mathfrak{o}_s=(\mathfrak{b}\mathfrak{o}_s)_a$ for any multiplicatively closed set S, $0 \notin S$, and since $\mathfrak{b}=\cap_{i=1}^{r}(\mathfrak{b}\mathfrak{o}_{\mathfrak{p}_i}\cap\mathfrak{o})$ holds for any ideal $\mathfrak{b}$ [3], we get $\mathfrak{a}_n \subset \mathfrak{a}^{n-k}$ if we contract the above relation back to $\mathfrak{o}$. As for the converse, it is enough to mention that the proof of lemma 1 of [5] is still true without any change.

Now, from theorem 1 and lemma 3, we obtain immediately our main theorem:

THEOREM 2. *In a semi-local ring the following three conditions are equivalent.*

(1) $\mathfrak{o}$ *is analytically unramified.*

(2) $\mathfrak{o}^*(\mathfrak{v})$ *is finite over* $\mathfrak{o}(\mathfrak{v})$ *for some defining ideal* $\mathfrak{v}$.

(3) *Existence of a normal defining ideal.*

Moreover, when this is so, for any ideal $\mathfrak{a}$, (2) *is still true and* $\mathfrak{a}_k$ *is normal for some* k.

If E is a finite module over $\mathfrak{o}$ and $\mathfrak{v}$ is a defining ideal of $\mathfrak{o}$, then it is well known that the length of $E/\mathfrak{v}^nE$ is expressed as a polynomial if n is large [3]. Therefore from corollary of lemma 2, jointly with theorem 2, we have the following.

COROLLARY *If* $\mathfrak{v}$ *is a defining ideal of an analytically unramified semi-local ring, then* $l(\mathfrak{v}_n)$, *the length of the integral closure of* $\mathfrak{v}^n$, *is represented as a Hilbert function* $\mu(n)$ *if* n *is sufficiently large.*

References

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