JOURNAL OF GAKUGEI
TOKUSHIMA UNIVERSITY

MATHEMATICS

DECEMBER, 1963

Volume XIV

PUBLISHED BY

THE GAKUGEI FACULTY, TOKUSHIMA UNIVERSITY
TOKUSHIMA JAPAN

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TOKUSHIMA, JAPAN

Journal of Gakugei, Tokushima University
Vol. XIV. 1964 1-53

THE STUDENT'S DISTRIBUTION FOR A UNIVERSE BOUNDED AT ONE OR BOTH SIDES (Continued)

By

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(Received September 30, 1963)

In the present note the author treats the most general form for the volume-element of the first half, i.e. one sided case of the proposed theme, and has gained its general information, at least theoretically. Naturally to get actual solutions for concrete cases of several sizes $n=6, 7, \cdots$, it requires a vast bulk of computations which could be accomplished only by making constant use of electronic computers, &c. Also the results reported in the preceding notes[1)] are now supplemented with possible improvement, sometimes rebuilt to get a better insight, or to make more general and intelligible. Lastly to reveal the general feature of its application, the T.N.D. is examplified by the special case $n=4$.

17. *Volume-element in General, Preliminaries.* We have to find the n-dimensional volume element

$$dV_n = F_{n-2}(\bar{x}, s)\, d(\sqrt{n}\bar{x})\, d(\sqrt{n}s), \tag{17.0}$$

or, its main factor F_{n-2}, which denotes the $(n-2)$-dimensional measure of the product area $\sigma=(S \cap \bar{K})$, where $S=S_{n-1}(\bar{x})$ is the $(n-1)$-dimensional simplex, i.e. the points- aggregate

$$\{P(x_1, x_2, \cdots, x_n) \mid x_i \geqq 0, \textstyle\sum x_i = n\bar{x} = \text{determinate}\}$$

in a n-dimensional space with centroid $G(\bar{x}, \cdots, \bar{x})$, while $\bar{K}=\bar{K}_{n-1}(s)$ is a $(n-1)$-dimensional spherical surface with center G and radius $\sqrt{n}\,s$. Meanwhile we are considering the $(n-1)$-dimensional whole space R_{n-1}, where $\sum x_i = n\bar{x}$ with all x_i irrespective to signs holds at all, so that S_{n-1}, $\bar{K}_{n-1}$, F_{n-2}, as well as the prolonged S_{n-1}, all $\subset R_{n-1}$. However, the product surface F_{n-2} is not a mere compound function of $\bar{x}$ and s, but more precisely defined by the product of $(\sqrt{n}s)^{n-2}$ and a certain function of $\tau=s/\bar{x}$ with $(n-1)$ steps, what were described in [I] but also become clear by (8) below: Namely, I: $0<\tau<1/\sqrt{n-1}$, II: $1/\sqrt{n-1}<\tau<\sqrt{2/(n-2)}\cdots$, the $(n-1)$-th: $\sqrt{(n-2)/2}<\tau<\sqrt{n-1}$. However the n-th case $\sqrt{n-1}<\tau<\infty$ never takes place. In fact, given any universe

1) Although the author endeavored possiblly to make the present note understandable without referring to his previous papers, yet these are, to be available when required, cited below as [I]: Some exceptional examples to Student's distribution, this Jour. Vol. X (1959), p. 11-; [II]: the same topic as the present, Sect. 1-7, ibid. Vol. XI (1960), p. 11-; [III] and [IV]: its continuations, Sect. 8-12, ibid. Vol. XII (1961), p. 5- and Sect. 13-16, ibid. Vol. XIII (1962), p. 1-.

with a non-negative argument, and its sample $\{x_i\}$, such that $\sum x_i = n\bar{x}$, $\sum(x_i - \bar{x})^2 = ns^2$, we should have

$$0 \leq s^2 \leq (n-1)\bar{x}^2, \quad \text{i.e.} \quad 0 \leq \tau = s/\bar{x} \leq \sqrt{n-1}.$$

For, no x_i can be greater than $n\bar{x}$, because, if so, some other x_j must become negative against the presumption. If at most one $x_i = n\bar{x}$ and all remaining $x_j = 0$, then $ns^2 = (n\bar{x} - \bar{x})^2 + (n-1)\bar{x}^2 = n(n-1)\bar{x}^2$, which yields $s = \sqrt{n-1}\,\bar{x}$, while, if all $x_i = \bar{x}$ or 0, then $s = 0$. Excepting these extreme cases, let $x_i = n\bar{x}\varepsilon_i$, $0 \leq \varepsilon_i < 1$, so that $\sum \varepsilon_i = 1$ but $\sum \varepsilon_i^2 < 1$ and there are at least two $\varepsilon_j > 0$, $\varepsilon_k > 0$. Also by the known relation $\sum x_i^2 = n\bar{x}^2 + ns^2$, we have $n\bar{x}^2 \sum \varepsilon_i^2 = \bar{x}^2 + s^2$, and whence follows that $0 < s^2 = \bar{x}^2(n\sum \varepsilon_i^2 - 1) < (n-1)\bar{x}^2$, Q.E.D.

We may therefore subdivide the whole interval $0 \leq \tau \leq \sqrt{n-1}$, whose ν-th subinterval is

$$\text{(17.1)} \qquad \tau_{\nu-1} = \sqrt{(\nu-1)/(n-\nu+1)} < \tau < \tau_\nu = \sqrt{\nu/(n-\nu)}, \quad \nu = 1, 2, \cdots, n-1,$$

where the sequence $\{\tau_\nu\}$ is monotonic increasing, but the interval length

$$\text{(17.2)} \quad I_\nu = \tau_\nu - \tau_{\nu-1} \ (\nu = 1, \cdots, n-1)$$

becomes minimal at the midway and $I_1 < I_{n-1}$ (Fig. 1.). Really, under assumption that $n > 3$, $1 \leq \nu < n-1$, inequalities $I_\nu \lesseqgtr I_{\nu+1}$ afford $0 \gtreqless (n-\nu)^2(n-4\nu) + 4\nu$, which shows that for $\nu \leq n/4$ the lower sign holds, but when $\nu = n-2$, the upper. In the remaining portion $\mathfrak{T}$: $n/4 < \nu < n-2$, the upper or lower sign holds according as $\varphi(\nu) \equiv 4\nu^3 - 9n\nu^2 + (6n^2-4)\nu - n^3 \gtreqless 0$, where $\varphi(n/4) < 0$, $\varphi(n-2) > 0$. The roots of $\varphi' = 0$ are α, $\beta = (3n \pm \sqrt{n^2 + 16/3})/4$ and $\varphi''(\alpha) > 0$, $\varphi''(\beta) < 0$. Thus $\varphi(x)$ becomes minimum (< 0) at $x = \alpha > n-2$, which however lies outside $\mathfrak{T}$ and maximum (>0) at $x = \beta$ inside $\mathfrak{T}$. Therefore $\varphi(x)$ vanishes at a point γ between $n/4$ and β and becomes negative in $1 \leq \nu < \gamma$ but positive in $\gamma < \nu \leq n-1$. So that I_ν decreases up to $x = \gamma$ or thereabout, but increases ever afterwards up to $x = n-1$.

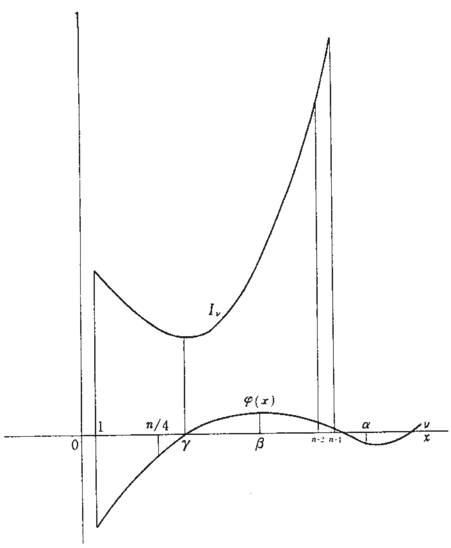

Fig. 1

The purpose of the present note is to determine several forms of the product surface $F_{n-2}(\bar{x}, s)$ for the successive subcases in general. But before going to those details, some concerned notions and magnitudes would be preluded in order to shorten and easify somewhat of the subsequent statement.

First let the successive typical subsimplexes of the main simplex S be written as

$$(17.3)\qquad S=S_{n-1}=A_1A_2\cdots A_n,\ S'=S^{n}_{n-2}=A_1A_2\cdots A_{n-1},\ S''_{n-3}=S^{n,n-1}_{n-3}=A_1A_2\cdots A_{n-2},$$
$$\cdots\cdots,\ S^{(\nu)}=S^{n,\,n-1,\,\cdots,\,n-\nu+1}_{n-\nu-1}=A_1A_2\cdots A_{n-\nu},\ \cdots\cdots,$$

where the number of dashes or bracketed ruby-figure put upperly, denotes the suborder, that is, the number of vertices typically rejected from S, which are detailed by the upperly written names of vertices and at the same time the dimension lowerly. So, at length $S^{(n-1)}=S^{n,\,n-1,\,\cdots,\,2}_{0}$ reduces to a vertex A_1. Clearly each of (3) being the base of the precedent, it holds

$$(17.4)\qquad S\supset S'\supset S''\supset\cdots\supset S^{(\nu)}\supset\cdots\supset S^{(n-1)}\,(=A_1).$$

Next, let their centroids (typical subcentroids) be

$$(17.5)\qquad G,\ G',\ G'',\ \cdots,\ G^{(\nu)},\ \cdots,\ G^{(n-1)}\,(=A_1);$$

or, more in detail, suffixed upperly and lowerly as in (3), but sometimes abridged simply as $G, G_1, G_2, \cdots, G_\nu, \cdots G_{n-1}$, where $G_\mu \in S_\nu$ if $\mu>\nu$ in (4). Their original O-$x_1x_2\cdots x_n$ coordinates were $G(\bar{x}, \cdots, \bar{x})$, $G'(\bar{x}', \cdots, \bar{x}', 0)$, $\cdots$, $G^{(\nu)}(\bar{x}^{(\nu)}, \cdots, \bar{x}^{(\nu)}, 0, \cdots 0)$, where

$$(17.6)\qquad \bar{x}^{(\nu)}=n\bar{x}/(n-\nu),\quad \text{i.e.}\quad (n-\nu)\bar{x}^{(\nu)}=n\bar{x},\quad (\nu=1, 2, \cdots, n-1,)$$

what follows from the fundamental relation $\sum x_i=n\bar{x}$. Besides, the distance of the subcentroid G_ν is given by

$$(17.7)\quad GG_\nu=g_\nu=\sqrt{(n-\nu)(\bar{x}^{(\nu)}-\bar{x})^2+\nu\bar{x}^2}=\bar{x}\sqrt{n\nu/(n-\nu)}=\sqrt{n}\,\bar{x}\tau_\nu\ (\nu=1, \cdots, n-1).$$

Hence, the ν-th stage in regard to the radius r of the sphere $\bar{K}$

$$(17.8)\qquad g_{\nu-1}<r=\sqrt{n}\,s<g_\nu\quad (g_0=0,\ \nu=1, \cdots, n-1)$$

just corresponds to the ν-th subcase $\tau_{\nu-1}<\tau=s/\bar{x}<\tau_\nu$. In other words, the subcentroid $G_{\nu-1}$ lies inside $\bar{K}$ but G_ν outside $\bar{K}$ at the ν-th subcase. Moreover

$$(17.9)\qquad \text{the distance } GG_\nu \text{ is } \perp \text{ to } S^{(\nu)}\quad (\nu=1, 2, \cdots),$$

which is evident by symmetry; or else, if any point $P(x_1, \cdots, x_{n-\nu}, 0, \cdots, 0)\in S^{(\nu)}$, we see that $\sum(x_i-\bar{x}^{(\nu)})(\bar{x}-\bar{x}^{(\nu)})=(\bar{x}-\bar{x}^{(\nu)})\sum(x_i-\bar{x}^{(\nu)})=0$ by (6).

Here we may observe that the ordinary 3 $\perp$ rs theorem can be readily generalized to the $(n-1)$-dimensional space: Let PQ be normal to $S^{(\nu)}(\supset S^{(\mu)}, \mu>\nu)$ and besides if $QR\perp S^{(\mu)}$, i.e. $QR\perp$ to any straight line on $S^{(\mu)}$, then the join PR is also $\perp$ to $S^{(\mu)}$ and inversely if $PR\perp S^{(\mu)}$, then the join QR is $\perp$ to $S^{(\mu)}$. Further the 3 $\perp$ rs theorem holds for any 3 subcentroids G_ν, G_μ, G_λ $(\nu<\mu<\lambda)$.

When $\nu<\mu$, $G_\mu\in S^{(\nu)}$ and by (9) $GG_\nu\perp G_\mu G_\nu$, so that we obtain the distance between the two subcentroids

(17.10) $$d_{\mu\nu}=G_\mu G_\nu=\sqrt{g_\mu^2-g_\nu^2}=\bar{x}\sqrt{n(\tau_\mu^2-\tau_\nu^2)}=n\bar{x}\sqrt{(\mu-\nu)/(n-\mu)(n-\nu)}\ (\mu>\nu).$$

In particular, putting $\mu=\nu+1$, we get the distance between successive subcentroids

(17.11) $$h_\nu=G_\nu G_{\nu+1}=\sqrt{g_{\nu+1}^2-g_\nu^2}=n\bar{x}/\sqrt{(n-\nu)(n-\nu-1)}\ (\nu=0,\ 1,\ \cdots,\ n-2).$$

Besides the subcentral distance is obtained from (10) and (6)

(17.12) $$k_\nu=G'G_\nu=\bar{x}'\sqrt{(n-1)(\nu-1)/(n-\nu)},\ (k_1=0,\ k_2=h_1,\ \nu=1,\ \cdots,\ n-1).$$

This may be more radically deduced from (11): The original $O-x_1\cdots x_n$ co-ordinate system was transformed into the new $G-\zeta(=\eta_0)\eta_1\cdots\eta_{n-2}$ coordinates system, which was so chosen that G is the new origin and the succeeding G_ν lies at $h_{\nu-1}$ on the $\eta_{\nu-1}$ parallel through $G_{\nu-1}$, where $h_{\nu-1}$ were taken positively, except the first one $h_0=-\bar{x}\sqrt{n/(n-1)}$. Really letting $G(\bar{x},\ \cdots,\ \bar{x})\to G(0,\ \cdots,\ 0)$ and $G'(\bar{x}',\ \cdots,\ \bar{x}')\to G'(h_0,\ 0,\ \cdots,\ 0)$, it yields the distance $|GG'|=|h_0|=\sqrt{(n-1)(\bar{x}-\bar{x}')^2+\bar{x}^2}=\bar{x}\sqrt{n/(n-1)}$ by (6). But we have taken the ζ-axis along the negative direction GG', so that

(17.13) $$\zeta=G'G=h_0=-\bar{x}\sqrt{n/(n-1)}\ (=-g_1)$$

which denotes also the equation to the hyperplane S', because, if P be any point on S', $GG'\perp G'P$ after (9) and the ζ-ordinate of P satisfies (13) always. Consequently the new coordinates of G_ν are

(17.14) $$G_\nu(h_0,\ h_1,\ \cdots,\ h_{\nu-1},\ 0,\ \cdots,\ 0)\qquad \nu=1,\ 2,\ \cdots,\ n-1.$$

Or, if the ζ-ordinate $(=\eta_0)$ be ignored, i.e. if the whole $G-\eta_1\eta_2\cdots\eta_{n-2}$ axes be projected on the hyperplane (13), we obtain the $G'-\eta_1\eta_2\cdots\eta_{n-2}$ coordinates system, in which

(17.15) $$G_\nu\equiv G_\nu(h_1,\ h_2,\ \cdots,\ h_{\nu-1},\ 0,\ \cdots,\ 0)\qquad \nu=2,\ 3,\ \cdots,\ n-1.$$

Hence the radius vector $G'G_\nu=k_\nu$ is again obtained by

$$k_\nu^2=\sum_1^{\nu-1}h_\mu^2=\sum_1^{\nu-1}(g_{\mu+1}^2-g_\mu^2)=g_\nu^2-g_1^2=n\bar{x}^2\left(\frac{\nu}{n-\nu}-\frac{1}{n-1}\right)=\frac{(n-1)(\nu-1)}{n-\nu}\bar{x}'^2$$

in agreement with (12). The broken line $G_1G_2\cdots G_{n-2}$ may be called G_ν-zigzag when stopped at G_ν, and $G_1,\ G_2,\ \cdots$ their vertices; they are all $\perp$ two by two. They may be also expressed by polar coordinates of $G_\nu(\nu=2,\ \cdots,\ n-1,\ \theta_0=1)$:

(17.16) $$\left\{\begin{array}{l} h_1=k_\nu\cos\theta_1\cos\theta_2\cdots\cos\theta_{\nu-2}\\ h_2=k_\nu\sin\theta_1\cos\theta_2\cdots\cos\theta_{\nu-2}\\ \cdots\cdots\\ h_\mu=k_\nu\sin\theta_{\mu-1}\cos\theta_\mu\cdots\cos\theta_{\nu-2}\\ \cdots\cdots\\ h_{\nu-1}=k_\nu\sin\theta_{\nu-2}\end{array}\right.\qquad \left\{\begin{array}{l} k_1=0,\ \ k_2=h_1\\ k_3=\sqrt{h_1^2+h_2^2}=k_\nu\cos\theta_2\cdots\cos\theta_{\nu-2}\\ \cdots\cdots\\ k_{\mu+1}=(\sum_1^k h_i^2)^{1/2}=k_\nu\cos\theta_\mu\cdots\cos\theta_{\nu-2}\\ \cdots\cdots\\ k_{\nu-1}=k_\nu\cos\theta_{\nu-2}.\end{array}\right.$$

Further

$$(17.17)\qquad h_\mu/k_{\mu+1}=\sin\theta_{\mu-1},\ \ k_\mu/k_{\mu+1}=\cos\theta_{\mu-1},\ \ h_\mu/k_\mu=\tan\theta_{\mu-1},$$
$$h_\mu^2+k_\mu^2=k_{\mu+1}^2\quad(\mu=1,\ \cdots,\ n-2,\ \theta_0=\pi/2).$$

More generally for the polar coordinates of any point $P(\rho,\ \eta_1,\ \cdots,\ \eta_{n-2})$, G_1 as origin,

$$(17.18)\qquad P(\eta_i=\rho\sin\theta_{i-1}\cos\theta_i\cdots\cos\theta_{n-3}\mid i=1,\ 2,\ \cdots,\ n-2,\ \theta_0=\pi/2),$$

we have the following relations

$$(17.19)\qquad \eta_2/\eta_1=\tan\theta_1,\ \ \eta_{i+1}/\eta_i=\tan\theta_i/\sin\theta_{i-1}\quad(i=1,\ \cdots,\ n-2)$$

and particularly for $P=G_\nu$

$$(17.20)\qquad \tan\theta_{\nu-2}/\sin\theta_{\nu-3}=h_{\nu-1}/h_{\nu-2}=\sqrt{(n-\nu+2)/(n-\nu)}\ \text{by (11)}.$$

The subsimplexes' centroids (5) become at the same time the centers of typical subspheres, if exist, which are their intersections with the main sphere $\bar{K}_{n-1}$

$$(17.21)\qquad \bar{K}'_{n-2},\ \bar{K}''_{n-3},\ \cdots,\ K^{(\nu)}_{n-\nu-1},\ \cdots,\ \bar{K}^{(n-2)}_1=\text{linear circle},\ \bar{K}^{(n-1)}_0=\text{point sphere}.$$

Since GG_1 is $\perp$ to S', if $r=\sqrt{n}\,s>g_1=|GG_1|=\sqrt{n/(n-1)}\,\bar{x}$, i.e. when $\tau>\tau_1$ (the second subcase and thereafter), $\bar{K}_{n-1}$ intersects S' and we get $\bar{K}'_{n-2}$ of radius $r_1=\sqrt{ns^2-n\bar{x}^2/(n-1)}=\sqrt{n-1}\ s'$, on writing similarly as $r=\sqrt{n}\ s$. Similarly when $r=\sqrt{n}\,s>g_\nu=\sqrt{n}\ \bar{x}\tau_\nu$, i.e. in the $(\nu+1)$-th subcase and thereafter there yields the intersection of $\bar{K}_{n-1}$ and $S^{(\nu)}_{n-\nu-1}$, the subsphere $\bar{K}^{(\nu)}_{n-\nu-1}$ with center G_ν. Its radius becomes, in consequence that $GG_\nu\perp S^{(\nu)}$ after (9)

$$(17.22)\qquad r_\nu=\sqrt{n-\nu}\,s^{(\nu)}=\sqrt{ns^2-g_\nu^2}=\sqrt{n}\,s\sqrt{1-\tau_\nu^2/\tau^2}=\sqrt{n(s^2-\nu\bar{x}^2)/(n-\nu)}$$
$$=\sqrt{r_{\nu-1}^2-h_{\nu-1}^2}=\sqrt{r_\nu^2-\sum_1^{\nu-1}h_i^2}=\sqrt{r_\nu^2-k_\nu^2}=\sqrt{r_\mu^2\mp d_{\mu\nu}^2}\,(\mu\lesseqgtr\nu).$$

Particularly for $\nu=n-1$, we get $r_{n-1}=s^{(n-1)}=\sqrt{n}\,s\sqrt{1-(n-1)/\tau^2}$, which is real only for the unavailable $\tau>\sqrt{n-1}$, but vanishing at the end of available $\tau\leqq\sqrt{n-1}$.

To make the matter easily understandable, it may be metaphorized as follows: Miss $\bar{K}_{n-1}$ and Mr. S_{n-1} made marriage. In the first age I: $0<r<g_1$ there was no child (no subsphere), but when Mrs. $\bar{K}$'s body measure r reached to g_1 at $\tau=\tau_1$ (the end of I or the epoch of II), she has borne the first daughter $\bar{K}'$; when r grew further to g_2 at $\tau=\tau_2$ (epoch of III), the second child $\bar{K}''$ was born, $\cdots$, however in the last age, when r became g_{n-1} at $\tau=\sqrt{n-1}$ (the last time), the last child $\bar{K}^{(n-1)}$, but alass still born!

In consequence of (4) at the ν-th subcase there exist spheres

(17.23) $\bar{K}_{n-1} \supset \bar{K}'_{n-2} \supset \bar{K}''_{n-3} \supset \cdots \supset \bar{K}^{(\nu-1)}_{n-\nu}$ or $\supset \bar{K}^{(\nu)}_{n-\nu-1}$ but no more,

according as the ν-th subinterval is open or closed at right.

We write likewise to $\tau = s/\bar{x}$, the successive ratios $\tau^{(\nu)} = s^{(\nu)}/\bar{x}^{(\nu)}$, where $\bar{x}^{(\nu)} = n\bar{x}/(n-\nu)$ denotes the non-vanishing equal original coordinates of G_ν, which however is of less use, except the first one

(17.24) $\tau' = s'/\bar{x}' = \sqrt{\dfrac{(n-1)\tau^2 - 1}{n}}$ and conversely $\tau = \sqrt{\dfrac{n\tau'^2 + 1}{n-1}}$.

Hence the ν-th stage (1) is also designated just by

(17.25) $\tau'_{\nu-2} = \sqrt{\dfrac{\nu-2}{n-\nu+1}} < \tau' < \tau'_{\nu-1} = \sqrt{\dfrac{\nu-1}{n-\nu}}$ $(\nu = 2, 3, \cdots, n-1)$.

The comparison of two systems is tabulated as follows:

Subinterval	I	II	…	the ν-th	…	the $(n-1)$ th
$\tau = s/\bar{x}$	$0 \sim 1/\sqrt{n-1}$	$1/\sqrt{n-1} \sim \sqrt{2/(n-2)}$	…	$\sqrt{(\nu-1)/(n-\nu+1)} \sim \sqrt{\nu/(n-\nu)}$	…	$\sqrt{(n-2)/2} \sim \sqrt{n-1}$
$\tau' = s'/\bar{x}'$	imaginary	$0 \sim 1/\sqrt{n-2}$	…	$\sqrt{(\nu-2)/(n-\nu+1)} \sim \sqrt{(\nu-1)/(n-\nu)}$	…	$\sqrt{(n-3/)2} \sim \sqrt{n-2}$

When $\nu > 1$ writing $\tau' = s'/\bar{x}'$ in (25) and multiplying by $\sqrt{n-1}\,\bar{x}'$, we get after (12)

(17.26) $k_{\nu-1} < \sqrt{n-1}\, s' = r_1 < k_\nu$ $(\nu = 2, 3, \cdots, n-1)$

as the ν-th substage: $\tau'_{\nu-2} < \tau' < \tau'_{\nu-1}$. Thus the radius of $\bar{K}'$ being in length between $G'G_{\nu-1}$ and $G'G_\nu$, the subcentroid $G_{\nu-1}$ lies inside $\bar{K}'$, but G_ν outside $\bar{K}'$. To continue the before made metaphor about Mrs. $\bar{K}$, now it is concerned with her daughter $\bar{K}'$, who got married Mr. S', a son of S, and the latter's body measure $r_1 = \sqrt{n-1}\, s'$. Untill the end of I, $\bar{K}'$ was not yet born. But at the end of II, $\tau = \tau_2$, when r_1 became k_2, the bride $\bar{K}'$ has borne her child $\bar{K}''$ and hereafter quite the same birth of children as described about the grandmother $\bar{K}$'s.

Lastly to write the equations to $S^{(\nu)}_{n-\nu-1}$ and $\bar{K}^{(\nu)}_{n-\nu-1}$, we remind that the former are given as a hyperplane, linearly by the original coordinates $x_n = x_{n-1} = \cdots = x_{n-\nu+1} = 0$ with $\sum_1^{n-\nu} x_\mu = n\bar{x}$ in which all summands are non-negative in the proper $S^{(\nu)}$, but some of them become negative in the prolonged $S^{(\nu)}$. Hence, in view of (13.6) in [IV], the new equations to $S^{(\nu)}_{n-\nu-1}$ are rendered as the intersection of hyperplanes,

(17.27) $S^{(\nu)}$: $\eta_\mu = h_\mu$ $(\mu = 0, 1, \cdots, \nu-1$, all positive except $\eta_0 = \zeta = h_0)$,

but the remaining $(n-\nu-1)$ ordinates η_μ's remain variable, what hold for the prolonged $S^{(\nu)}$ also. Therefore the equation to $K^{(\nu)}_{n-\nu-1} = (\bar{K} \cap S^{(\nu)})$ are

$$(17.28)\qquad \bar{K}^{(\nu)}_{n-\nu-1}:\ \eta_\mu = h_\mu\ \ (\mu = 0, 1, \cdots, \nu-1)\ \ \text{and}\ \ \sum_{\nu}^{n-2} \eta_\mu^2 = r_\nu^2,$$

where the radius $r_\nu = s\sqrt{n(1-\tau_\nu^2/\tau^2)}$ by (22) and the center lies at distance k_ν from G'. In particularr, when $\nu = 1$, we have $\zeta = -\sqrt{n/(n-1)}\,\bar{x}$ for S', and in addition

$$(17.29)\qquad \bar{K}':\ \sum_{1}^{n-2} \eta_i^2 = r_1^2 = (n-1)s'^2\ \&\text{c}.$$

Remark (17.30). If the first subsphere $\bar{K}'$ be cut by the base of (27), i.e. $\bigcup_{1}^{\nu-1}(\eta_\mu = h_\mu)$, there yields the intersection $\sum_{1}^{n-2}\eta_i^2 - \sum_{1}^{\nu-1}\eta_j^2 = r_1^2 - k_\nu^2$, i.e. $\sum_{\nu}^{n-2}\eta_i^2 = r_\nu^2$ after (22). Therefore $\bar{K}^{(\nu)}$ may be simply referred to that on $\bar{K}'$ instead to consider on $\bar{K}$ itself.

Summary. There are the following several monotonic sequences:

(17.31) Central distances $\{GG_\nu = g_\nu = \bar{x}\sqrt{n\nu/(n-\nu)} = \sqrt{n}\,\bar{x}\tau_\nu\}$,

namely $g_1 = \bar{x}\sqrt{n/(n-1)}$, $g_2 = \bar{x}\sqrt{2n/(n-2)}$, $\cdots$, $g_{n-2} = \bar{x}\sqrt{n(n-1)} = GA_1$.

(17.32) Successive subcentroids' distances $\{G_\nu G_{\nu+1} = h_\nu = n\bar{x}/\sqrt{(n-\nu)(n-\nu-1)}\}$,

i.e. $h_0 = g_1$, $h_1 = n\bar{x}/\sqrt{(n-1)(n-2)}$, $h_2 = n\bar{x}/\sqrt{(n-2)(n-3)}$, $\cdots$,

$h_{n-2} = n\bar{x}/\sqrt{2} = a/2$ (half side).

(17.33) Subcentral distances $\{G'G_\nu = k_\nu = n\bar{x}\sqrt{(\nu-1)/(n-1)(n-\nu)}$

$= \bar{x}'\sqrt{(n-1)(\nu-1)/(n-\nu)}$, i.e. $k_1 = 0$, $k_2 = h_1$, $k_3 = n\bar{x}\sqrt{2/(n-1)(n-2)}$, $\cdots$,

$k_{n-1} = n\bar{x}\sqrt{(n-2)/(n-1)} = G'A_1\}$.

(17.34) Terminals of subintervals $\{\tau_\nu = \sqrt{\nu/(n-\nu)}\}$, $\tau_0 = 0$, $\tau_1 = 1/\sqrt{n-1}$,

$\tau_2 = \sqrt{2/(n-2)}$, $\cdots$, $\tau_{n-1} = \sqrt{n-1}$.

However the following two are not monotonic:

(17.35) Subintervals' sequence $\{\tau_\nu - \tau_{\nu-1}\}$ has a minimum, and

(17.36) the sequence of number of the ν-th subsimplexes: $\{N_\nu\}$ has a maximum; namely, $N_0=1$, $N_1=n$, $\cdots$, $N_\nu = {}_nC_\nu$, $\cdots$, $N_{n-1}=n$, while their permutations are again monotonic increasing:

$$(17.37)\qquad {}_nP_1 = n,\ {}_nP_2 = n(n-1),\ \cdots,\ {}_nP_\nu = n(n-1)\cdots(n-\nu+1),\ \cdots,\ {}_nP_n = n!$$

Naturally sizes of successive subsimplexes as well as subspheres form both monotonic decreasing sequences. However, their measures are scarcely directly treated for computations of the product surface F_{n-2}. Yet the subspheres play the following important roles: firstly, whenever they make their first appearance, the stage changes, and secondly, in our determination of overlappingly calculated spaces, they constitute themselves wholly or partly together with the foregoing of them, a gate or barrier, through which further spreading vectors enter into the newly overlapping domain, as will be seen in the subsequent section.

18. *Continued, Establishment of the Product-Surfaces* $F_{n-2,\nu}$ ($\nu=1, 2, \cdots, n-1$). The product-surface $F_{n-2}=(S_{n-1}\frown \bar{K}_{n-1})$ being really of different form at every stage, let its expression corresponding to the ν-th stage be denoted by $F_{n-2,\nu}\equiv F_\nu$ and the correction necessary to be added to $F_{\nu-1}$ to obtain F_ν by $(-1)^{\nu-1}F_1\mathfrak{h}_{\nu-1}$. Thus

$$F_\nu = F_{\nu-1}+(-1)^{\nu-1}\mathfrak{h}_{\nu-1}F_1 = F_{\nu-2}+(-1)^{\nu-2}\mathfrak{h}_{\nu-2}+(-1)^{\nu-1}\mathfrak{h}_{\nu-1}F_1=\cdots, \quad \text{i.e.}$$

$$F_\nu = F_1\sum_0^{\nu-1}(-1)^i\mathfrak{h}_i \quad (\mathfrak{h}_0=1), \tag{18.1}$$

where $\mathfrak{h}_i$ is truly defined by $\tau=s/\bar{x}$. It is clear that for $\tau=\tau_{n-1}=\sqrt{n-1}$ (the final value) after (17.31) $g_{n-1}\equiv GA_1$ and $\bar{K}$ passes through all the vertices, so that $S\frown\bar{K}$ reduces to null measure. Consequently we should have

$$F_{n-1}=F_1\sum_0^{n-2}(-1)^i\mathfrak{h}_i=0, \text{ so that } \sum_0^{n-2}(-1)^i\mathfrak{h}_i(\sqrt{n-1})=0. \tag{18.2}$$

The ν-th volume element being simply

$$dV_\nu = F_\nu d(\sqrt{n}\bar{x})\,d(\sqrt{n}\,s), \tag{18.3}$$

we have only to determine F_ν successively.

I ($n\geq 2, \nu=1$): $0<\tau=s/\bar{x}<\tau_1=1/\sqrt{n-1}$, i.e. $0<\sqrt{n}\,s<\sqrt{n}\,\bar{x}/\sqrt{n-1}=|GG'|$. Here $|GG'|$ being the shortest distance from G to S', the s-sphere $\bar{K}$ lies wholly inside the simplex $S=S_{n-1}$, so that the s-sphere's surface-points all bioccupy s and $\bar{x}$ and $\epsilon\,\sigma=(S\frown\bar{K})$. Hence, we have only to find the $(n-2)$-dimensional surface area F_{n-2} of $\bar{K}_{n-1}$. Now the $\eta_1\eta_2\cdots\eta_{n-2}$ rectangular coordinates system being transformed into $\rho\theta_1\cdots\theta_{n-3}$ polar system, we have the $(n-2)$-dimensional space element

$$d\sigma = d\eta_1\cdots d\eta_{n-2}=\rho^{n-3}\cos\theta_2\cos^2\theta_3\cdots\cos^{n-4}\theta_{n-3}\,d\rho\, d\theta_1\cdots d\theta_{n-3}.$$

This being projected inversely on the upper half of the sphere $\bar{K}$: $\zeta^2+\rho^2=r^2=ns^2$, and integrated, we obtain as the required spherical surface

$$\tfrac{1}{2}F_{n-2,I}=\int_0^{2\pi}d\theta_1\int_{-\pi/2}^{\pi/2}\cos\theta_2\,d\theta_2\cdots\int_{-\pi/2}^{\pi/2}\cos^{n-4}\theta_{n-3}\,d\theta_{n-3}\int_0^r\sec\gamma\cdot\rho^{n-3}d\rho. \tag{18.4}$$

The $\eta_1\eta_2\cdots\eta_{n-2}$ coordinates axes form 2^{n-2} multiplants, a generalized name of quadrant ($2^2=4$), octant ($2^3=8$), the first multiplant being that in which all η_i's≥ 0. As there are $4\times 2^{n-4}=2^{n-2}$ quadrants, we may only consider the first multiplant by symmetry. Yet in (4), multiplying by 2 due to $\zeta\gtreqless 0$, the power becomes 2^{n-1} and we have

$$F_{n-2,I}=2^{n-1}(\sqrt{n}\,s)^{n-2}\int_0^{\pi/2}d\theta_1\int_0^{\pi/2}\cos\theta_2\,d\theta_2\cdots\int_0^{\pi/2}\cos^{n-4}\theta_{n-3}\,d\theta_{n-3}\int_0^{\pi/2}\sin^{n-3}\psi\, d\psi. \tag{18.4.1}$$

Or, applying a known formula

$$2\int_0^{\pi/2}\cos^m\theta d\theta=\sqrt{\pi}\ \Gamma\left(\frac{m+1}{2}\right)/\Gamma\left(\frac{m}{2}+1\right),\quad m=0,1,\cdots,n-2,$$

we obtain

$$(18.5)\qquad F_{n-2,I}(\sqrt{n}\,s)=2\sqrt{\pi}^{\,n-1}(\sqrt{n}\,s)^{n-2}/\Gamma\left(\frac{n-1}{2}\right),$$

which is the well known Fisher's formula. Hereafter we shall call this expression as Fisher's areal function and denote it by F_I or F_1, while F_ν in (1) as the generalized Fisher's areal function. We get for F_1, e.g.

n	2	3	4	5	6	7	8
F_1	2	$2\sqrt{3}\,\pi s$	$16\pi s^2$	$10\sqrt{5}\,\pi^2 s^3$	$96\pi^2 s^4$	$49\sqrt{7}\,\pi^3 s^5$	$8192\pi^3 s^6/15$ &c.

The volume-element are obtained on multiplying these by $nd\bar{x}ds$, respectively. Naturally the above Fisher's formula holds good equally for all $\tau=s/\bar{x}>1/\sqrt{n-1}$ also, if the simplex S be prolonged, $\bar{S}$ say, i.e. if under condition $\sum x_i=n\bar{x}$, the restriction of x_i's non-negativity be released. But then the s-sphere's radius exceeds $|GG'|$ and $\bar{K}$ protrudes outside the proper S, so that (5) should be corrected to get the successive F_ν correctly.

First consider

II $(n\geq 3,\ \nu=2)$: $1/\sqrt{n-1}<\tau=s/\bar{x}<\sqrt{2/(n-2)}$. Here comes the first subsphere $\bar{K}'$ through which the radius-vectors of $\bar{K}$ go over outside the proper S. In fact, after (17.7,8) $g_1=\sqrt{n/(n-1)}\,\bar{x}<r=\sqrt{n}\,s<g_2=\sqrt{2n/(n-2)}\,\bar{x}$ and $\bar{K}$ contains G' in its inside but those points on the produced GG' are outside S. Thus $\bar{K}$ protrudes outward S over S'. But by (17.12,26) $k_1=0<r_1=\sqrt{n-1}\ s'$ $<k_2=\sqrt{(n-1)/(n-2)}\ \bar{x}$ (Fig. 2). So that $\bar{K}'$ lies entirely inside the proper S' and accordingly the protruding portion over every first subsimplex has no common point with each other. The annexed Fig. 2a shows the configuration schematically but rather generally. In fact, the hyperplane S': $\zeta=\sqrt{n/(n-1)}$ $\bar{x}$ is a $(n-2)$-dimensional space of $\eta_1,\cdots,\eta_{n-2}$ but it is denoted by a single thick line in Fig 2a.

Now the solid sphere K'_{n-2} being divided into concentric $(n-2)$-dimensional spherical shells of radius ρ, where $0<\rho<r_1$, which is therefore $\in S'_{n-2}\subset S$, we get the elementary volume $F_{n-3}(\rho)d\rho$, where F_{n-3} is obtained by replacing n by $n-1$ and $\sqrt{n}\,s$ by ρ in (5). This being projected inversely on $\bar{K}_{n-1}$: $\zeta^2+\rho^2=ns^2$, there yields the areal element dC_{n-2} of the calotte protruding over S'. Hence, this integrated, affords the required area of the protruding calotte

$$C_{n-2}=\int_0^{r_1}\sec\gamma\cdot 2\sqrt{\pi}^{\,n-2}\rho^{n-3}d\rho/\Gamma\left(\frac{n}{2}-1\right),\quad\text{where}\quad \sec\gamma=\sqrt{n}\,s/\sqrt{ns^2-\rho^2}.$$

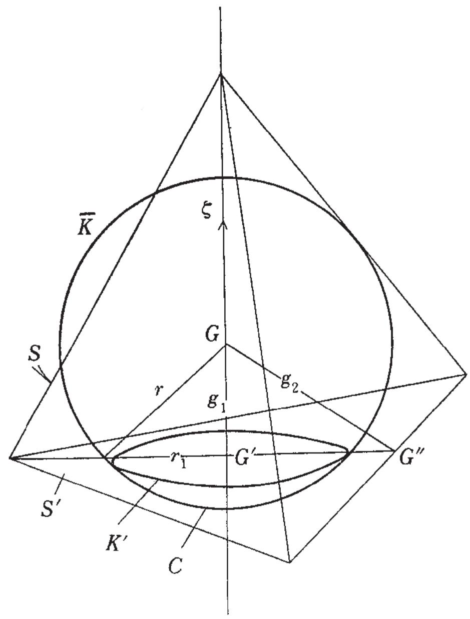

Fig. 2

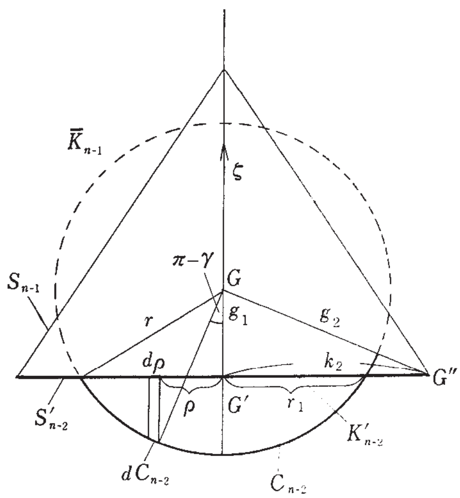

Fig. 2a

That is

$$(18.6) \qquad C_{n-2}=\frac{2(\sqrt{\pi n}s)^{n-2}}{\Gamma(n/2-1)}\int_0^{\psi_1}\sin^{n-3}\psi d\psi, \quad \text{where} \quad \psi_1=\sec^{-1}\sqrt{n-1}\,\tau.$$

Otherwise: If from the whole sphere F_{n-2} the upper hemisphere as well as the lower calotte be cut off (Fig. 2*b*), there remains a $(n-2)$-dimensional spherical frustum with lateral surface $\mathfrak{v}=\frac{1}{2}F_{n-2}-C_{n-2}$.

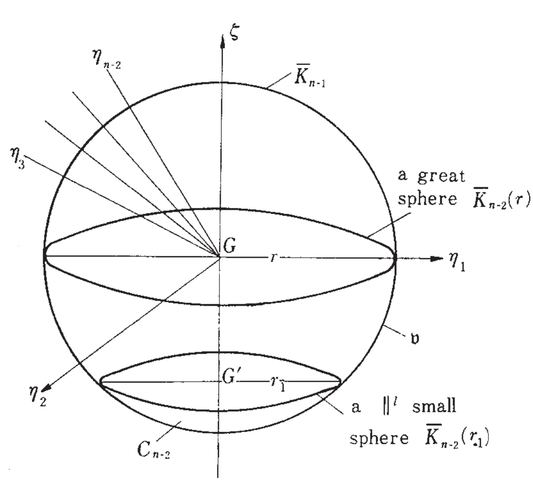

Fig. 2*b*

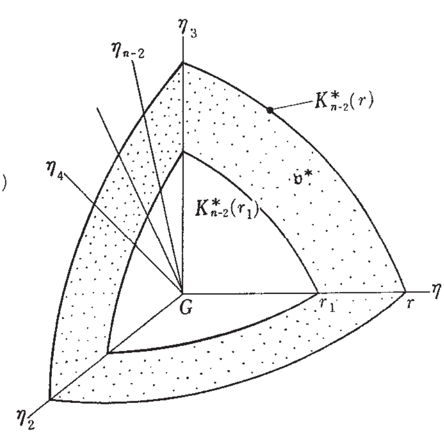

Fig. 2*c*

The lateral surface $\mathfrak{v}$ is bounded upperly by a sphere $\bar{K}_{n-3}(r=\sqrt{n}\,s)$ on $\zeta=0$ and lowerly by a parallel small sphere $\bar{K}'_{n-3}(r_1=\sqrt{n-1}\,s')$ on $\zeta=-\sqrt{n/(n-1)}\,\bar{x}$. Consequently, if this be projected on $\zeta=0$, the projected $\mathfrak{v}^*$ shall be bounded by two concentric spheres $\bar{K}^*_{n-3}(r)$ and $K^*_{n-3}(r_1)$ (Fig. 2c). Hence, if the latter be projected on $\bar{K}$ inversely, yields just as in (4)

$$(18.7)\qquad \mathfrak{v}=2^{n-2}\int_0^{\pi/2}d\theta_1\int_0^{\pi/2}\cos\theta_2 d\theta_2\cdots\int_0^{\pi/2}\cos^{n-4}\theta_{n-3}d\theta_{n-3}\int_{r_1}^{r}\frac{\sqrt{n}\,s}{\sqrt{ns^2-\rho^2}}\rho^{n-3}d\rho,$$

whose innermost integral, when $\rho=\sqrt{n}\,s\sin\psi$, $\psi_1=\sec^{-1}\sqrt{n-1}\,\tau$, reduces to

$$\int_{r_1}^{r}=\int_{\psi_1}^{\pi/2}(\sqrt{n}\,s)^{n-2}\sin^{n-3}\psi d\psi=\int_0^{\pi/2}-\int_0^{\psi_1}.$$

Therefore

$$\mathfrak{v}=\tfrac{1}{2}F_{n-2,1}-\frac{2(\sqrt{n\pi}\,s)^{n-2}}{\Gamma(n/2-1)}\int_0^{\psi_1}\sin^{n-3}\psi d\psi=\tfrac{1}{2}F_{n-2,1}-C_{n-2},$$

and whence (6) is delivered once more again.

Since there are n congruent calottes to be subtracted, we obtain

$$(18.8)\qquad F_{n-2,II}=F_1-nC_{n-2}=F_1(1-\mathfrak{h}_1),\quad\text{where}$$

$$(18.9)\qquad \mathfrak{h}_1=\frac{n\Gamma((n-1)/2)}{\sqrt{\pi}\,\Gamma(n/2-1)}\int_0^{\psi_1}\sin^{n-3}\psi d\psi\quad(\psi_1=\sec^{-1}\sqrt{n-1}\;\tau),$$

which may be called Archimedes' formula. E.g. for $n=6$ we get

$\mathfrak{h}_1(\tau)=3-3(15\tau^2-1)/10\sqrt{5}\,\tau^3$, $\mathfrak{h}_1'(\tau)=9(5\tau^2-1)/10\sqrt{5}\,\tau^4$, $\mathfrak{h}_1''(\tau)=9(2-5\tau^2)/5\sqrt{5}\,\tau^5$,

so that $\quad\mathfrak{h}_1(\sqrt{5})=2.112$, $\mathfrak{h}_1'(1/\sqrt{5})=0$, but $\mathfrak{h}_1''(1/\sqrt{5})=9/5$.

In general, at the epoch of II: $\tau_1=1/\sqrt{n-1}$, $\mathfrak{h}_1$ reduces to naught. Moreover, since $\mathfrak{h}_1'(\tau)=\text{const}\cdot\tau^{-2}\sin^{n-4}\psi_1$, besides holds for large n,

$$\mathfrak{h}_1^{(p)}(\tau)=\text{const}\cdot\sum_{q=0}^{p}\binom{p}{q}(-1)^{p-q}\frac{|p-q+1}{\tau^{p-q-2}}\frac{d^q}{d\tau^q}(\sin^{n-4}\psi_1(\tau)),$$

where the summands reduce to 0 as $\tau=\tau_1$, $\psi_1=0$, if $n-4>p$. Hence the p-th derivative $\mathfrak{h}_1^{(p)}(\tau_1)$ becomes zero for $p=1, 2, \cdots$, up to $n-5$.

Also, since $C_{n-2}=\frac{1}{2}F_{n-2}-\mathfrak{v}$, wherein (5) and (7) be substituted, yields

$$(18.10)\qquad \begin{aligned} nC_{n-2}&=n2^{n-2}(\sqrt{n}\,s)^{n-2}\int_0^{\pi/2}d\theta_1\cdots\int_0^{\pi/2}\cos^{n-4}\theta_{n-3}d\theta_{n-3}\int_0^{\psi_1}\sin^{n-3}\psi d\psi\\ &=n2^{n-2}J(\psi_1=\sec^{-1}\sqrt{n-1}\,\tau)=F_1\mathfrak{h}_1,\end{aligned}$$

where the power 2^{n-1} in (4.1) is halved, now that ζ is confined to be one sided, while on the otherhand ${}_nP_1=n$ should be multiplied. And (10) denotes the amount of correction to F_I to obtain F_{II} but with factor -1, since it must be subtracted. This form is quite of the same type as those for the subsequent F_{III}, F_{IV}, $\cdots$ below.

III $(n \geq 4, \nu = 3)$: $\sqrt{2/(n-2)} < \tau < \sqrt{3/(n-3)}$, $\sqrt{1/(n-2)} < \tau' < \sqrt{2/(n-3)}$. By (17.8, 26) we have now $g_2 < r < g_3$, $k_2 < r_1 < k_3$, so that G'' inside $\bar{K}$, $\bar{K}'$, but G''' outside. Thus here the second subsphere $\bar{K}''$ makes its appearance on S'' and plays its role as a new gate ($\bar{K}'$ was the first gate in II). If Archimedes' formula applied here, those calottes over neighbouring S'_{n-2} mutually overlap, which needs further correction. Thus after III we ought to consider not only the hitherto treated typical subsimplex $S' = S^n_{n-2}$ but also another, say S^{n-1}_{n-2}, both of which have the same subsimplex $S^{n,n-1}_{n-3} = S'' = B_{n-3}$ $(x_n = x_{n-1} = 0)$ as a common base, that is again schematically denoted by thick lines (Fig. 3).

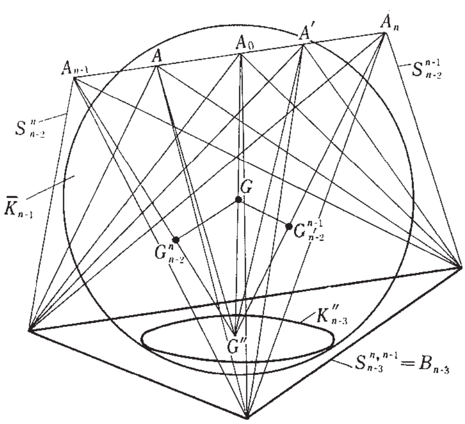

Fig. 3

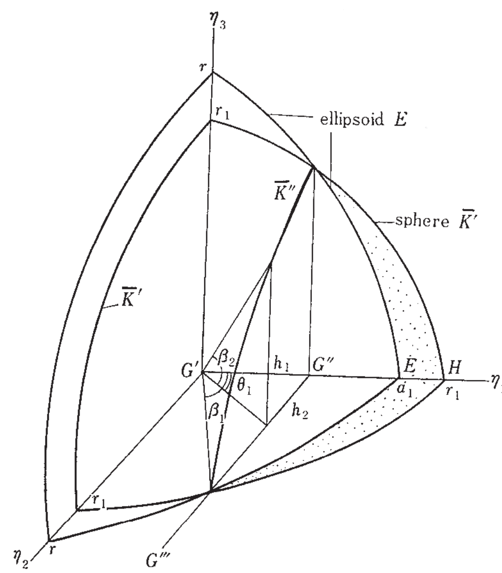

Fig. 3a

The join $A_{n-1}G''$, A_nG'' being the respective axis of S^n_{n-2}, S^{n-1}_{n-2}, they are both $\perp$ to B_{n-3} and accordingly the plane $A_nG''A_{n-1}$ is $\perp$ to B_{n-3}, as any straight line drawn through G'' on B_{n-3} becomes $\perp$ to this plane. Hence, if A be any point on A_nA_{n-1}, then AG'' is also $\perp$ to B_{n-3}. Therefore when A moves from A_{n-1} to A_n, the space $\langle\!\langle A \cdot B_{n-3} \rangle\!\rangle \equiv H_{n-2}(A)$ rotates about the common axial base B_{n-3}, the amount of rotation being measured by $\alpha = \angle AG''A_{n-1}$. Moreover, $\cup H(A)$ is clearly convex and contains all faces of S, so that $\cup H(A) \supseteq S$. But, if the point $A' \neq A$, then $H(A)$, $H(A')$ cannot have any common point except on B_{n-3} (i.e. G''). For, putting $A_{n-1}A : AA_n = \lambda$, we get $A(0, \cdots, 0, n\bar{x}/(1+\lambda), \lambda n\bar{x}/(1+\lambda))$ and similarly if $A_{n-1}A' : A'A_n = \lambda'$, $A'(0, \cdots, 0, n\bar{x}/(1+\lambda'), \lambda' n\bar{x}/(1+\lambda'))$ with $\lambda \neq \lambda'$. Assuming that $H(A)$, $H(A')$ have a common point $C(c_1, \cdots, c_{n-1}, c_n)$ with non-vanishing c_{n-1}, c_n. Produce 2 joins CA, CA' to meet B_{n-3} at B, B' both with coordinates $x_{n-1}=x_n=0$. Writing $AC: CB=\mu$ and $A'C: CB'=\mu'$, we have $c_{n-1}=n\bar{x}/(1+\lambda)(1+\mu)=n\bar{x}/(1+\lambda')(1+\mu')$ and $c_n=\lambda n\bar{x}/(1+\lambda)(1+\mu)=\lambda' n\bar{x}/(1+\lambda')(1+\mu')$, and whence $\lambda=\lambda'$ follows, which however contradicts the presumption. Hence, every point of S is once and only once swept out by the rotating $H(A)$, so that $\cup H(A)=S$.

The equation to $H(\lambda)$ or $H(A)$ can be deduced from those of its axis AG'', which in original coordinates are $x_1=\cdots=x_{n-2}=[n\bar{x}-(1+\lambda)x_{n-1}]/(n-2)$, $x_n=\lambda x_{n-1}$. When these be transformed into new coordinates, cf. (13.6) in [IV], we obtain, besides trivial $\eta_2=\cdots=\eta_{n-2}=0$,

$$(n-1+\lambda)\zeta=\sqrt{n(n-1)}\,(\lambda-1)\bar{x}-\lambda\sqrt{n(n-2)}\,\eta_1 \tag{18.11}$$

which affords the required equation for $H(\lambda)$. In particular, we get

(18.12) if $\lambda=0$, $\zeta=-\bar{x}\sqrt{n/(n-1)}$ for $H(0)=S^n_{n-2}=S'$,

if $\lambda=\infty$, $\zeta=\sqrt{n(n-1)}\,\bar{x}-\sqrt{n(n-2)}\,\eta_1$ for $H(\infty)=S^{n-1}_{n-2}$,

if $\lambda=1$, $\zeta=-\sqrt{(n-2)/n}\,\eta_1$ for $H(1)=\langle\!\langle A_0\cdot B_{n-3}\rangle\!\rangle$, the equator E_{n-2}.

The intersection $\bar{K}_{n-1}\cap H(A)$ yields a sphere $\bar{K}_{n-2}(A)$. Every summand of $\cup\bar{K}_{n-2}(A)=\sigma$ intersects all with S'' along K'' on B_{n-3} in common, but they have no common point outward K''. Especially the portion of σ extending outward beyond the gate $\bar{K}''$, say v, belongs to Archimedes' surface in II over S^n_{n-2} as well as S^{n-1}_{n-2} (rather their prolongations $\bar{S}^n_{n-2}$, $\bar{S}^{n-1}_{n-2}$ toward $x_{n-1}<0$ as well as $x_n<0$), that must have been subtracted twice, when F_{II} applied here. Hence we ought to add this v to correct F_{II}. By symmetry we have only to treat typically the portion extending outward over S'', and after remark (17.30) to consider the portion beyond $\bar{K}''$ in $\bar{K}'$, so that $\eta_1\geq h_1$ (Fig. 3*a*).

Since the equator E_{n-2} divides S into 2 congruent parts, we have only to compute the volume of canopy $v_{HE}(=v/2)$ bounded by $\bar{K}'_{n-2}(=\bar{K}\cap S')$ and $\bar{K}^E_{n-2}$ $(=\bar{K}\cap E_{n-2})$, whose latter has its equations $\rho^2+\zeta^2=r^2=ns^2$ and $\zeta=-\sqrt{(n-2)/n}$ η_1 after (12), a great sphere, while the former's are $\rho^2+\zeta^2=r^2$ and $\zeta=-\sqrt{n/(n-1)}\,\bar{x}$, a small sphese. Hence, their projections on $\zeta=0$ become an oblate ellipsoid whose η_ν-semiaxes a_ν $(\nu=2,\cdots,n-3)$ are all $=r$ except the η_1-semiaxis $=a_1=r\sqrt{n/2(n-1)}<r$:

$$\rho_0=\sqrt{n}\,s/\sqrt{1+((n-2)/n)\cos^2\theta_1\cdots\cos^2\theta_{n-3}}, \tag{18.13}$$

and $\rho_1=\sqrt{n-1}\,s'=r_1$, a small sphere, respectively.

The canopy bounded by these 2 surfaces being projected inversely on $\bar{K}$: $\zeta^2=ns^2-\rho^2$, as they $\subset v\subset S$, by virtue of S's compactness, it yields the required volume

$$v_{HE}=v_{III}=2^{n-3}(\sqrt{n}\,s)^{n-2}\int_0^{\beta_1}d\theta_1\int_0^{\beta_2}\cos\theta_2\,d\theta_2\cdots\int_0^{\beta_{n-3}}\cos^{n-4}\theta_{n-3}\,d\theta_{n-3}\int_{\psi_0}^{\psi_1}\sin^{n-3}\psi\,d\psi\quad(n\geq 4) \tag{18.14}$$

that may be called the canopied formula. Its numerical coefficient denotes the number of multiplant 2^{n-2} divided by 2, because we are to take the positive η_1 only. Therefore $(\sqrt{n}\,s)^{n-2}\times$ [the $(n-2)$-ple integral in (14)] expresses the partial volume in the first multiplant. The limits of integrations are, as will be seen from Fig. 3a,

(18.15) $\beta_1=\cos^{-1}h_1/r_1=\sec^{-1}\sqrt{n-2}\,\tau'$ and generally

$$\beta_i=\cos^{-1}(\cos\beta_1\sec\theta_1\cdots\sec\theta_{i-1}),\ i=2,3,\cdots,n-3 \text{ and}$$

$$\phi_1=\sin^{-1}(\rho_1/\sqrt{n}\,s)=\sin^{-1}\sqrt{1-1/(n-1)\tau^2}=\cos^{-1}1/\sqrt{n-1}\,\tau=\sec^{-1}\sqrt{n-1}\,\tau,$$

$$\phi_0=\sin^{-1}(1/\sqrt{1+((n-2)/n)\cos^2\theta_1\cdots\cos^2\theta_{n-3}}=\tan^{-1}(\sqrt{n/(n-2)}\sec\theta_1\cdots\sec\theta_{n-3}).$$

We have found the above v_{III} by conceiving A_n and A_{n-1} typically, but there are ${}_nP_2$ of such v_{III}. Hence we get the correction to F_{II}, or the second overlapping space's measure

(18.16) $$O_{III}=n(n-1)v_{III}=n(n-2)2^{n-3}J_{n-2,III}=\mathfrak{h}_2F_1$$

and the product surface in the subcase III is

(18.17) $$F_{III}=F_{II}+\mathfrak{h}_2F_1=F_1(1-\mathfrak{h}_1+\mathfrak{h}_2).$$

Below it will be employed the abbreviation:

(18.18) $$\prod_{i=j+1}^{n-3}\left[\int_0^{\beta_i}\cos^{i-1}\theta_i d\theta_i\right]\cdot\int_{\phi_0}^{\phi_1}\sin^{n-3}\phi d\phi\equiv\Theta_{j+1}$$

So that

(18.19) $$v_{III}=2^{n-3}(\sqrt{n}\,s)^{n-2}\int_0^{\beta_1}\Theta_2(\theta_1,\tau)d\theta_1.$$

At the epoch $\tau=\tau_2$, $\tau'=\tau_1'$, the upper and lower limits of θ_1 coincide, so that $\mathfrak{h}_2(\tau_2)=0$, $\mathfrak{h}_2'=0$.

Also, to avoid somewhat mingled name $\theta_1,\theta_2,\cdots$, we adopt the geographical or astronomical nomenclature. We call $\theta_1,\theta_2,\theta_3,\cdots,\theta_{i+1},\cdots$ in succession, the longitude, latitude, bilatitude, the i-th latitude. Thus, the free entrance K'' in v_{III} is confined between the lower meridian $\theta_1=0$ and the upper meridian $\theta_1=\beta_1=\sec^{-1}\sqrt{n-2}\tau'$ and the latitude between $\theta_2=0$ and $\theta_2=\beta_2=\cos^{-1}(\cos\beta_1\sec\theta_1)=\sec(\sqrt{n-2}\,\tau'\cos\theta_1)$, respectively. The name θ_2-meridians &c. are also of use below in V &c., but not in III, IV.

IV ($n\geq5$, $\nu=4$): $\sqrt{3/(n-3)}<\tau<2/\sqrt{n-4}$, $\sqrt{2/(n-3)}<\tau'<\sqrt{3/(n-4)}$, $k_3<r_1<k_4$ and G_3 inside $\bar{K}'$ but G_4 outside. Hence, K''' makes its new appearance, as shown again schematically by a thick line in Fig. 4.

We have reckoned $v_{III}=$ the prolonged S-space over $K''(=K_{n-3}^{n,n-1})$ as overlapping Archimedes' subspace, because $v_{III}\subset\bar{S}_{n-2}^n\cap\bar{S}_{n-2}^{n-1}=S_{n-3}^{n,n-1}$. Similarly $v_{IV}=$ the prolonged S-space over $K'''=K_{n-4}^{n,n-1,n-2}$ forms overlapping canopied subspaces, because $v_{IV}\subset\bar{S}_{n-3}^{n,n-1}\cap\bar{S}_{n-3}^{n,n-2}$. Thus the entrance K'' in III is now

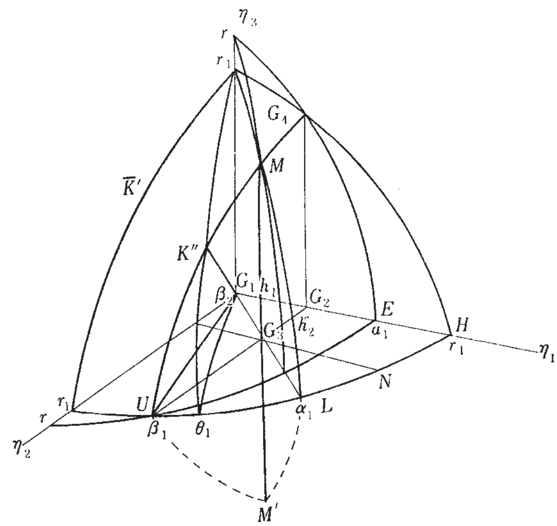

Fig. 4

barricaded by meridian plane MLM' containing G''' and the free passage K'' in III is now narrowed to MUG''', where M denotes the point at which the η_3-parallel through G''' cuts K'. So that the lower meridian becomes instead of 0 now

(18.20) $$\alpha_1 = \tan^{-1}(h_2/h_1) = \tan^{-1}\sqrt{(n-1)/(n-3)} = \sec^{-1}\sqrt{2(n-2)/(n-3)},$$

while the upper meridian β_1 and all other limits remain unchanged. Hence, we get

(18.21) $$v_{IV} = 2^{n-4}(\sqrt{n}\, s)^{n-2}\int_{\alpha_1}^{\beta_1}\Theta_2(\theta_1, \tau)\, d\theta_1 (= J_{n-2,\, IV}),$$

which may be called the semi-lunette formula, since the typical v_{IV} consists of a semi-lunette MUL. The power of 2 in v_{IV} is lowered by one than that in v_{III}, because now for θ_1 the negative quadrant should be abandoned. Besides, as v_{IV} is obtained from A_n, A_{n-1}, A_{n-2} taken in this order, there is the total number ${}_nP_3 = n(n-1)(n-2)$ of congruent v_{IV} by symmetry. Therefore the whole overlapping space is now

(18.22) $$O_{IV} = n(n-1)(n-2)v_{IV} = F_1\mathfrak{h}_3,$$

that is produced by the overaddition in III and ought to be subtracted from F_{III}:

(18.23) $$F_{IV} = F_{III} - F_1\mathfrak{h}_3 = F_1(1 - \mathfrak{h}_1 + \mathfrak{h}_2 - \mathfrak{h}_3).$$

We call the space $G' - MLU$ in K' which contains v_{IV}, to be its framework, and the surface portion MLU on K' to be its map. Further it would be conveniently cited as the *principle of diminishing domain*, that the domain of integration about any variable θ_i diminishes or at least remains unaltered but never augments, when the number of suborder ν increases. This is truly a matter of course, since the succeeding subcase concerns with the correction of the precedent, i.e. the partial space of precedent. Thus it was in III: $0<\theta_1<\beta_1$, but in IV $0<\alpha_1<\theta_1<\beta_1$, the remaining θ_i's limits being unchanged. Here we ought to write $\theta_{i|3}$ or $\theta_{i|4}$ in detail to denote that the i-th polar angle belongs to the 3rd subcase: $\sqrt{2/(n-2)}<\tau<\sqrt{3/(n-3)}$ or the 4-th subcase: $\sqrt{3/(n-3)}<\tau<\sqrt{4/(n-4)}$. But, when the domain of τ is enlarged e.g. so as the 3rd to $\sqrt{2/(n-2)}<\tau<\sqrt{n-1}$ and the 4-th to $\sqrt{3/(n-3)}<\tau<\sqrt{n-1}$, two angles $\theta_{i|3}$ and $\theta_{i|4}$ can take the same value of θ_i yet $\{\theta_{i|3}\}\supset\{\theta_{i|4}\}$, so that for their lower and upper bounds hold $\underline{\theta}_{i|3}\leq\underline{\theta}_{i|4}$ and $\bar{\theta}_{i|3}\geq\bar{\theta}_{i|4}$ in accordance with the principle of diminishing domain.

It might seem doubtful to decide how the domain of v_{IV} should be outlined: Since, for the points outside the gate K'' narrowed by K''', it appears superficially to take simply $\eta_1>h_1$, $\eta_2>h_2$ and the space v_{IV} may also comprise the portion MLN (Fig. 4). However we are concerned with the radius vector issuing from G' and which entering the gate UMM' cannot reach the domain MLN without diffraction, say. Thus the initial meridian $\theta_1=\alpha_1$ is no more retrograded—the *non-retrograde rule*, which is also a corollary of the principle of diminishing domain.

V $(n\geq 6,\ \nu=5)$: $\sqrt{4/(n-4)}<\tau<\sqrt{5/(n-5)}$, $\sqrt{3/(n-4)}<\tau'<\sqrt{4(n-5)}$. Hereafter subcentroids $G', G'', \cdots$ shall be denoted exclusively by $G_1, G_2, \cdots$, since G' used below has another meaning. In this subcase $k_4<r_1<k_5$ hold and

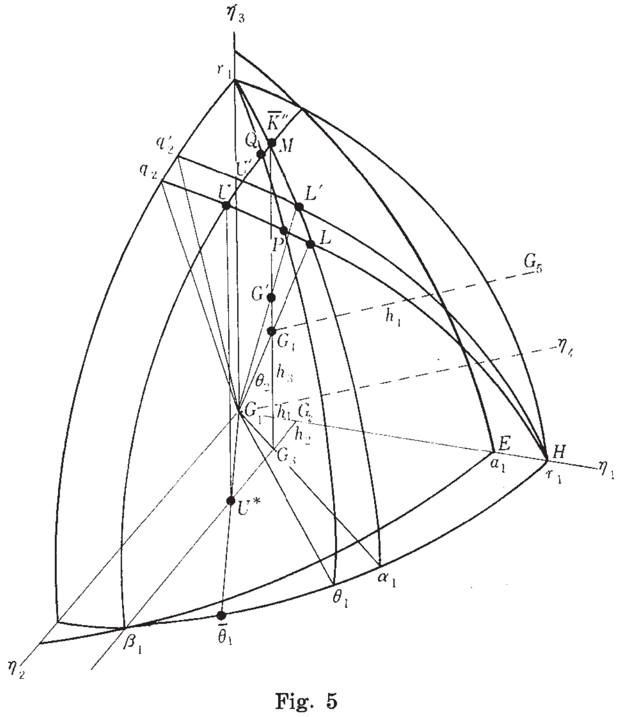

Fig. 5

$G_4(\eta_i = h_i,\ i=1, 2, 3)$ lies inside but G_5 outside $\bar{K}'$ (Fig. 5), so that the sphere $\bar{K}^{IV}$ appears and constitutes the fourth gate. Now $v_V =$ the prolonged $\bar{S}$ over $\bar{K}^{IV}$ $(=K^{n,n-1,n-2,n-3}_{n-5})$ forms further overlapping semi-lunette subspaces, because $v_V \subset (\bar{S}^{n,n-1,n-2}_{n-4} \cap \bar{S}^{n,n-1,n-3}_{n-4}) \equiv \bar{S}^{n,n-1,n-2,n-3}_{n-5}$. Hence the α_1-meridian and $\bar{K}''$ remain partly as arc ML and MU (Fig. 5). But, now that only the portion outside G_4 should be considered in v_V, we draw the confining plane $HG_1G_4 = H(G_4) = ⟪G_4 \cdot \eta_1\text{-axis}⟫$, which cuts out from $\bar{K}'$ a quadrant Hq_2 (Fig. 5). This initial plane $H(G_4)$ intersects the α_1-meridian at L and K''-arc at U, so that there yields a spherical triangle MLU, which furnishes the map of the required typical v_V.

Proof. The equation to the confining plane $H(G_4)$ is evidently

$$\begin{vmatrix} \eta_1 & \eta_2 & \eta_3 & 1 \\ 0 & 0 & 0 & 1 \\ r_1 & 0 & 0 & 1 \\ h_1 & h_2 & h_3 & 1 \end{vmatrix} = -\begin{vmatrix} \eta_2 & \eta_3 \\ h_2 & h_3 \end{vmatrix} = 0,$$

i.e. $\eta_3/\eta_2 = h_3/h_2$.
Or, in polar equation

$$\tan\theta_2/\sin\theta_1 = \sqrt{(n-2)/(n-4)};$$

that is

$$\theta_2 = \tan^{-1}(\sqrt{(n-2)/(n-4)}\sin\theta_1) = \tan^{-1}((h_3/h_2)\sin\theta_1), \tag{18.24}$$

which taken together with $\rho = r_1$ affords the equations to θ_2-quadrant, Hq_2, a great circle of $\bar{K}'$ cut out by $H(G_4)$. In particular, when $\theta_1 = \alpha_1$, we get after (24) and (20)

$$\theta_2 = \tan^{-1}(\sqrt{h_3/h_2}\sin\alpha_1) = \tan^{-1}\sqrt{(n-1)/2(n-4)}$$

as the latitude for G_4 and also for all points on G_1G_4L. But, along the arc LU the latitude θ_2 increases because of increasing θ_1. Further the η_1-axis and a moving point G' on G_3G_4 produced, such that $G_3G' = h' > h_3 = G_3G_4$, determine a plane $H(G') = ⟪G' \cdot \eta_1\text{-axis}⟫$, whose equation is similarly to (24)

$$\theta_2' = \tan^{-1}(\sqrt{h'/h_2}\sin\theta_1) \quad (h_3 < h' < h = G_3M). \tag{18.24$'$}$$

Thus when G' moves from G_4 to M, the plane $H(G')$ revolves about the η_1-axis and sweeps all the points lying outward the gate but inside $\bar{K}'$. Let $H(G')$ intersect with the α_1-meridian and K'' at L' and U' respectively. Along the arc $L'U'$ the latitude θ_2 again increases, because of increasing θ_1. And along LM also θ_2 increases, now that h' increases. Consequently the latitude θ_2 of every point in the triangle MLU except L being greater than $\theta_2(G_4)$, this triangle lies outside the gate $\bar{K}^{IV}$. It can be readily seen that if $h' < h_3$ the latitude becomes less than $\theta_2(G_4)$ and that portion remains inside the gate. Hence the required map is certainly MLU, Q.E.D.

Consequently, if any θ_1-meridian meets with UL and UM at P and Q, the limits of integration about θ_2 are given by

$$\alpha_2 = \tan^{-1}(\sqrt{(n-2)/(n-4)}\sin\theta_1) = \sec^{-1}\sqrt{1+(n-2/n-4)\sin^2\theta_1}. \tag{18.25}$$

But $\beta_2 = \cos^{-1}(h_1\sec\theta/r_1) = \sec^{-1}(\sqrt{n-2}\,\tau'\cos\theta_1)$ remains the same as in III, IV.

Also $\alpha_1 = \cos^{-1}\sqrt{(n-3)/2(n-2)}$ as in IV, but the upper limit of longitude θ_1 is now reformed to $\bar{\theta}_1 < \beta_1$ (Fig. 5). For, at the point $U(\theta_1 = \bar{\theta}_1)$, it becomes $P = Q = U$, so that we must have

$$\sec^{-1}\sqrt{1+((n-2)/(n-4))\sin^2\bar{\theta}_1} = \sec^{-1}(\sqrt{n-2}\,\tau'\cos\bar{\theta}_1)$$

and whence follows

$$\bar{\theta}_1 = \sec^{-1}\sqrt{(n-2)[1+(n-4)\tau'^2]/2(n-3)}, \tag{18.26}$$

which is $< \beta_1$, what is evident geometrically, or else analytically because of $\tau' > 1/\sqrt{n-2}$ eversince III. The other limits however remain the same as before. But, now that the negative quadrant of θ_2 must be rejected, the index of power of 2 shall be again reduced by one. Thus we obtain

$$v_V = 2^{n-5}(\sqrt{n}\,s)^{n-2}\int_{\alpha_1}^{\bar{\theta}_1} d\theta_1 \int_{\alpha_2}^{\beta_2} \Theta_3(\theta_2, \tau)\cos\theta_2 d\theta_2, \tag{18.27}$$

which may be called $\theta_1\theta_2$-triangled formula, a particular one of the general $\theta_{\nu-3}\theta_{\nu-4}$-triangled formula described later on. Since there are ${}_nP_4$ of the congruent v_V, the whole overlapping at the stage V is

$$O_V = n(n-1)(n-2)(n-3)v_V = F_1\mathfrak{h}_4. \tag{18.28}$$

Accordingly the fifth product surface is

$$F_V = F_{IV} + F_1\mathfrak{h}_4 = F_1\sum_0^4(-1)^i\mathfrak{h}_i. \tag{18.29}$$

At the epoch of V ($\tau' = \sqrt{3/(n-4)}$), $\bar{\theta}_1$ of (26) reduces to $\cos^{-1}\sqrt{(n-3)/2(n-2)} = \alpha_1$, so that $v_V = 0$, $\mathfrak{h}_4 = 0$ there.

As a check for the preceding results, let us ascertain the identity (2)

$$\sum_0^{n-2}(-1)^i\mathfrak{h}_i(\tau) = 0 \text{ for the case } n = 6,\ \tau = \sqrt{5},\ \tau' = 2. \tag{18.30}$$

It is readily obtained by the foregoing that $\mathfrak{h}_0 = 1$, $\mathfrak{h}_1 = 2.112$, $\mathfrak{h}_2 = 90J_{III}/\pi^2$, $\mathfrak{h}_3 = 180J_{IV}/\pi^2$, $\mathfrak{h}_4 = 270J_V/\pi^2$, where

$$J_N = \int_{\varphi_0}^{\varphi_1} d\varphi \int_{\theta_0}^{\theta_1}\cos\theta d\theta \int_0^{\chi_1}\cos^2\chi d\chi \int_{\psi_0}^{\psi_1}\sin^3\psi d\psi, \quad (N = III, IV, V),$$

where the limits of integrations are $\psi_1 = \sec^{-1}5$, $\psi_0 = \tan^{-1}(\sqrt{3/2}\sec\varphi\sec\theta\sec\chi) = \cos^{-1}\sqrt{1-\xi^2}/\sqrt{b^2-\xi^2}$ with $\xi = \sin\chi$, $b^2 = 1 + \frac{3}{2}a^2$, $a = \sec\varphi\sec\theta = 4c/\sqrt{1-u^2}$, $u = \sin\theta$, $c = (\sec\varphi)/4$ and $\chi_1 = \cos^{-1}a/4$ for all N, while the remaining limits differ as follows:

	φ_0	φ_1	θ_0 or	$u_0 = \sin\theta_0$	θ_1 or	$u_1 = \sin\theta_1$
J_{III}	0	$\cos^{-1}1/4$	0	0	$\cos^{-1}c$	$\sqrt{1-c^2}$
J_{IV}	$\cos^{-1}\sqrt{3/8}$	”	0	0	”	”
J_V	” $= \alpha$	$\cos^{-1}1/\sqrt{6} = \beta$	$\tan^{-1}(\sqrt{2}\sin\varphi)$	$\sqrt{(16c^2-1)/(24c^2-1)} = \gamma$	”	” $= \delta$

After having performed two inner integrations, common to all cases, we get

$$J=\int_{\varphi_0}^{\varphi_1}d\varphi\int_{u_0}^{u_1}\Big\{\frac{1}{3}\tan^{-1}\frac{\sqrt{1-c^2-u^2}}{5c}-\frac{37}{375}\tan^{-1}\frac{\sqrt{1-c^2-u^2}}{c}$$
$$+\Big(\frac{588c}{375}-\frac{192c^3}{5(1+24c^2-u^2)}\Big)\frac{\sqrt{1-c^2-u^2}}{1-u^2}\Big\}du. \tag{18.31}$$

Executing further integrations about $u=\sin\theta$ and $c=(\sec\varphi)/4$, we obtain

$$\mathfrak{h}_2=15\pi^{-1}(\tan^{-1}\sqrt{15}/5-0.296\tan^{-1}\sqrt{5}+0.0006\sqrt{15})=1.394841,$$

$$\mathfrak{h}_3=2\mathfrak{h}_2+30\pi^{-1}(0.296\tan^{-1}\sqrt{15}/3-0.002\sqrt{15})-5=0.292825.$$

Consequently, in view of (30), it should hold exactly

$$\mathfrak{h}_4=-1+\mathfrak{h}_1-\mathfrak{h}_2+\mathfrak{h}_3=0.009984.$$

Now to compute $\mathfrak{h}_4$ directly, we must evaluate J_V. First to integrate about u: the first half of the inner integral of (31) integrated by parts, reduces to

$$\Big[\frac{u}{3}\tan^{-1}\frac{\sqrt{1-c^2-u^2}}{5c}-\frac{37}{375}u\tan^{-1}\frac{\sqrt{1-c^2-u^2}}{c}\Big]_\gamma^\delta$$
$$+\int_\gamma^\delta\Big(\frac{5}{3(1+24c^2-u^2)}-\frac{37}{375(1-u^2)}\Big)\frac{cu^2du}{\sqrt{1-c^2-u^2}}$$
$$=-\frac{\gamma}{3}\tan^{-1}\frac{\sqrt{1-c^2-\gamma^2}}{5c}+\frac{37\gamma}{375}\tan^{-1}\frac{\sqrt{1-c^2-\gamma^2}}{c}+\int_\gamma^\delta\text{ ,, }\qquad\text{ ,, }$$

whose last integral considered together with the before remaining last half in (31) on putting $v=\sqrt{1-c^2-u^2}/u$, $v_1=c\sqrt{9-24c^2}/\sqrt{16c^2-1}$, yields

$$\frac{1}{3}\int_0^{v_1}\Big[\frac{5c}{(1+24c^2)v^2+25c^2}-\frac{37c-12c^3}{125(v^2+c^2)}\Big]dv$$
$$=\frac{1}{3\sqrt{1+24c^2}}\tan^{-1}\frac{\sqrt{1+24c^2}}{5c}v_1-\frac{37-12c^2}{375}\tan^{-1}\frac{v_1}{c}.$$

Hence we get

$$J_V=\frac{1}{3}\int_\alpha^\beta\Big\{\frac{1}{\sqrt{1+24c^2}}\tan^{-1}\frac{1}{5}\sqrt{\frac{(1+24c^2)(9-24c^2)}{16c^2-1}}$$
$$-(0.296-0.096c^2)\tan^{-1}\sqrt{\frac{5(9-24c^2)}{16c^2-1}}$$
$$-\sqrt{\frac{16c^2-1}{24c^2-1}}\Big(\tan^{-1}\frac{1}{5}\sqrt{\frac{9-24c^2}{24c^2-1}}-0.296\tan^{-1}\sqrt{\frac{9-24c^2}{24c^2-1}}\Big)\Big\}d\varphi$$
$$=\int_{\sqrt{6}/16}^{\sqrt{6}/4}(\text{ ,, })\frac{dc}{c\sqrt{16c^2-1}},\quad\text{when }\varphi\text{ transformed into }\sec^{-1}4c.$$

Finally writing $1+24c^2=x$, we attain at length

$$\mathfrak{h}_4=\frac{27v}{\pi^2}J_V=\frac{45}{\pi^2}\int_5^{10}\left\{\frac{1}{\sqrt{x-2}}\left(0.296\tan^{-1}\sqrt{\frac{10-x}{x-2}}-\tan^{-1}\frac{1}{5}\sqrt{\frac{10-x}{x-2}}\right)\right.$$

$$\left.-\sqrt{\frac{3}{2x-5}}\left[(0.3-0.004\,x)\tan\sqrt{\frac{3(10-x)}{2x-5}}-\frac{1}{\sqrt{x}}\tan^{-1}\frac{1}{5}\sqrt{\frac{3(10-x)}{2x-5}}\right]\right\}\frac{dx}{x-1}$$

which is enough complicate to integrate directly. However, on applying Gauss' method of numerical integration by 5 ordinates, the author has obtained its value 0.009986, which almost coincides with the expected value 0.009984, the error being only 0.02%, that is probably due to having rounded figures at the last place in the computing ways.

Superficially it seems that the map of v_V shall be bounded by the α_1-meridian and some arc of $\bar{K}''$ and thirdly a parallel small circular arc of $\bar{K}'$ cut by the plane $\eta_3=h_3$ through G_4. However, the map thus obtained makes the value (30) far much deviate from zero.

We can proceed to the succeeding subcases quite similarly as in V.. But to save the repetition, we would rather treat generally and partly inductively.

VI The general ν-th subcase $(n\geq 7,\ 6\leq\nu\leq n-1)$: $\sqrt{\frac{\nu-1}{n-\nu+1}}<\tau<\sqrt{\frac{\nu}{n-\nu}}$, $\sqrt{\frac{\nu-2}{n-\nu+1}}<\tau'<\sqrt{\frac{\nu-1}{n-\nu}}$, where $k_{\nu-1}<r_1<k_\nu$ and $G_1, G_2, \cdots, G_{\nu-1}$ all lie inside $\bar{K}'$ but $G_\nu, G_{\nu+1}, \cdots$ outside $\bar{K}'$. The $(\nu-1)$-th subsphere $K_{n-\nu}^{(\nu-1)}$ makes its appearance as the $(\nu-1)$-th gate with center $G_{\nu-1}(h_1, \cdots, h_{\nu-2}, 0, \cdots, 0)$ and radius $r_{\nu-1}=\sqrt{n}\,s\sqrt{1-\tau_{\nu-1}^2/\tau^2}$ after (17.24), which accordingly disappears at the epoch $\tau_{\nu-1}$. Here we contrive the $\xi_{\nu-4}\eta_{\nu-3}\eta_{\nu-2}$ octant, where the $\xi_{\nu-4}$-axis being the

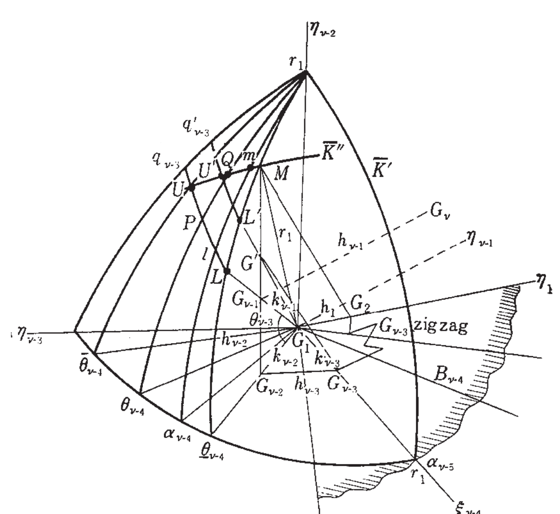

Fig. 6

join $G_1G_{\nu-3}$ produced $(=G_1\alpha_{\nu-5}$, Fig. 6). Now the $\theta_{\nu-4}$ meridian is constructed by the intersection of $\bar{K}'$ with the plane determined by a point in the octant and the $\eta_{\nu-2}$ axis, the values of $\theta_{\nu-4}$ being entered on the $\xi_{\nu-4}\eta_{\nu-3}$-quadrant, e.g. $\theta_{\nu-4}=\cos^{-1}k_{\nu-3}/k_{\nu-2}$ for $G_{\nu-1}$. Let the $\eta_{\nu-2}$ ordinate of $G_{\nu-1}(=G_{\nu-2}G_{\nu-1})$ be produced to cut $\bar{K}'$ at M, where $G_{\nu-2}M=h=\sqrt{r_1^2-k_{\nu-2}^2}$. Also $\bar{K}''$ passes through M as will be seen later on. Take further G' on $G_{\nu-2}M$, so as $G_{\nu-2}G'=h'$, where $h_{\nu-2}\leq h'\leq h$. Since $G'G_{\nu-2}$ is $\perp$ to the space $R_{\nu-3}$ determined by $\eta_1\cdots\eta_{\nu-3}$ axes and $G_{\nu-2}G_{\nu-3}$ $\perp$ to $R_{\nu-4}$ determined by $\eta_1\cdots\eta_{\nu-4}$ axes $\equiv B_{\nu-4}$ by reason of the 3 $\perp$ rs theorem, $G'G_{\nu-3}$ is normal to $B_{\nu-4}$. Accordingly, when G' moves on $G_{\nu-2}M$, the space $R_{\nu-3}(G'\cdot B_{\nu-4})=$ hyperplane $H_{\nu-3}(G')$ revolves about the base $B_{\nu-4}$ as axis, and in particular $H_{\nu-3}(G_{\nu-1})$ becomes the confining space of $v_{VI}=v_\nu$, the ν-th typical correction volume. The equation to this hyperplane $H_{\nu-3}(G_{\nu-1})\equiv R_{\nu-3}$ $(G_{\nu-1}\cdot B_{\nu-4})$ is

$$\begin{vmatrix} \eta_1 & \eta_2 & \cdots & \eta_{\nu-4} & \eta_{\nu-3} & \eta_{\nu-2} & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ r_1 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & r_1 & \cdots & 0 & 0 & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & r_1 & 0 & 0 & 1 \\ h_1 & h_2 & \cdots & h_{\nu-4} & h_{\nu-3} & h_{\nu-2} & 1 \end{vmatrix} = -\begin{vmatrix} \eta_{\nu-3} & \eta_{\nu-2} \\ h_{\nu-3} & h_{\nu-2} \end{vmatrix} = 0,$$

that is $\eta_{\nu-2}/\eta_{\nu-3}=h_{\nu-2}/h_{\nu-3}$.

Or, in polar coordinates

$$\frac{\tan\theta_{\nu-3}}{\sin\theta_{\nu-4}}=\sqrt{\frac{n-\nu+3}{n-\nu+1}}, \quad \text{i.e.}$$

$$\text{(18.32)} \qquad \theta_{\nu-3}=\tan^{-1}\left(\frac{h_{\nu-2}}{h_{\nu-3}}\sin\theta_{\nu-4}\right).$$

Quite similarly for the hyperplane $H_{\nu-3}(G')$

$$\text{(18.32)}' \qquad \theta'_{\nu-3}=\tan^{-1}((h'/h_{\nu-3})\sin\theta_{\nu-4}), \qquad \text{where } h_{\nu-2}\leq h'\leq h.$$

The planes $(G_{\nu-1}\cdot\xi_{\nu-4}$ axis$)\subset H_{\nu-3}(G_{\nu-1})$ and $(G'\cdot\xi_{\nu-4}$-axis$)\subset H_{\nu-3}(G')$ intersect $\bar{K}'$ along quadrants $\alpha_{\nu-5}q_{\nu-3}$ and $\alpha_{\nu-5}q'_{\nu-3}$ which by turns cut $\theta_{\nu-4}$-meridian at L and L' as well as $\bar{K}''$-arc at U and U', respectively. But $\bar{K}''$ also passes through M, because it holds for $M(r_1, \theta_1, \cdots, \theta_{\nu-3}, 0, \cdots, 0)$ that $\cos\theta_{\nu-3}=k_{\nu-2}/r_1$ (Fig. 6) and $k_{\nu-2}=h_1/\cos\theta_1\cdots\cos\theta_{\nu-4}$ after (17.16), so that M lies on $\bar{K}''$: $\eta_1=r_1\cos\theta_1\cdots$ $\cos\theta_{\nu-3}=h_1$. Thus we get the apparent map LMU for v_ν, since every point in LMU has $\theta_{\nu-3}\geq\theta_{\nu-3}(G_{\nu-1})$ after (32), (32)'. Hence, if a $\theta_{\nu-4}$-meridian cut LU and UM at P and Q, the $\theta_{\nu-3}$-limits of integrations are given by

$$\text{(18.33)} \quad \theta_{\nu-3}(P)=\alpha_{\nu-3|\nu}=\tan^{-1}\sqrt{\frac{n-\nu+3}{n-\nu+1}}\sin\theta_{\nu-4}=\sec^{-1}\sqrt{1+\frac{n-\nu+3}{n-\nu+1}\sin^2\theta_{\nu-4}},$$

$$\text{(18.34)} \qquad \theta_{\nu-3}(Q)=\beta_{\nu-3|\nu}=\sec^{-1}\sqrt{n-2}\,\tau'\cos\theta_1\cdots\cos\theta_{\nu-4}.$$

Particularly when P, Q coincide at $U(\bar{\theta}_{\nu-4})$, on equating the expressions (33), (34), we have

(18.35) $\bar{\theta}_{\nu-4|\nu}=\sec^{-1}\sqrt{[n-\nu+3+(n-\nu+1)(n-2)\tau'^2\cos^2\theta_1\cdots\cos^2\theta_{\nu-5}]/2(n-\nu+2)}.$

Furthermore, this value (35) in conformity with the principle of diminishing domain lies between the old limits $\alpha_{\nu-4|\nu-1}$ and $\beta_{\nu-4|\nu-1}$, both of which are obtainable by replacing ν by $\nu-1$ in (33) (34) as follows:

$$\alpha_{\nu-4|\nu-1}=\sec^{-1}\sqrt{1+\frac{n-\nu+4}{n-\nu+2}\sin^2\theta_{\nu-5}}, \tag{18.36}$$

$$\beta_{\nu-4|\nu-1}=\sec^{-1}(\sqrt{n-2}\,\tau'\cos\theta_1\cdots\cos\theta_{\nu-5}). \tag{18.37}$$

For, on writing those inequalities (36) < (35) < (37) in details, they reduce to consistent inequalities:

$$\theta_{\nu-5}<\sec^{-1}(\sqrt{n-1}\,\tau'\cos\theta_1\cdots\cos\theta_{\nu-6}) \text{ and } \theta_{\nu-6}<\sec^{-1}(\sqrt{n-2}\,\tau'\cos\theta_1\cdots\cos\theta_{\nu-7}).$$

On the otherhand the lower bound $\underline{\theta}_{\nu-4|\nu}$ is found from Fig. 6 to be

$$\underline{\theta}_{\nu-4|\nu}=\sec^{-1}k_{\nu-2}/k_{\nu-3}==\sec^{-1}\sqrt{(\nu-3)(n-\nu+3)/(\nu-4)(n-\nu+2)}, \tag{18.38}$$

and which is really $\leq\alpha_{\nu-4|\nu-1}$. For, writing this inequality in detail by using (38) (36), we obtain $\theta_{\nu-5|\nu-1}'\geq\sec^{-1}\sqrt{(\nu-4)(n-\nu+4)/(\nu-5)(n-\nu+3)}$. But, since this last expression just denotes $\underline{\theta}_{\nu-5|\nu-1}$, the inequality hold quite correctly. However by reason of the non-retrograde rule the new lower limit for $\theta_{\nu-4|\nu}$ should still remain to be $\alpha_{\nu-4|\nu-1}$ as it stands, and the new interval of the integration about $\theta_{\nu-4}$ ought to be

$$\alpha_{\nu-4|\nu-1}<\theta_{\nu-4|\nu}<\bar{\theta}_{\nu-4|\nu}. \tag{18.39}$$

Thus the before found *LMU* being merely a supermap, it reduces to the true map *lmU* bounded lowerly by the $\alpha_{\nu-4}$-meridian, as shown in Fig. 6.

Just similarly as for $\bar{\theta}_{\nu-4}$ in (35) we can find $\bar{\theta}_{\nu-5|\nu}$ on equating the 2 members bounding inequalities (39) by use of (35) (36), as

(18.40)

$$\bar{\theta}_{\nu-5|\nu}=\sec^{-1}\sqrt{[2(n-\nu+4)+(n-\nu+1)(n-2)\tau'^2\cos^2\theta_1\cdots\cos^2\theta_{\nu-6}]/3(n-\nu+3)}.$$

More generally with reference to (35) (40), we get inductively

(18.41) $\bar{\theta}_{\nu-\mu|\nu}=\sec^{-1}$

$$\sqrt{[(\mu-3)(n-\nu+\mu-1)+(n-\nu+1)(n-2)\tau'^2\cos^2\theta_1\cdots\cos^2\theta_{\nu-\mu-1}]/(\mu-2)(n-\nu+\mu-2)}$$

for $\mu=4, 5, \cdots, \nu-1$. Or, writing $\nu-\mu=m(\mu=\nu-m)$, we get

(18.42) $\bar{\theta}_{m|\nu}=\sec^{-1}$

$$\sqrt{[(n-m-1)(\nu-m-3)+(n-\nu+1)(n-2)\tau'^2\cos^2\theta_1\cdots\cos^2\theta_{m-1}]/(n-m-2)(\nu-m-2)}$$

for $m=1, 2, \cdots, \nu-4$. Thus in the ν-th subcase, the upper limits of integration about $\bar{\theta}_{m|\nu}$ $(m=1, 2, \cdots)$ are found to be

(18.43)

$$\bar{\theta}_{1|\nu} = \sec^{-1}\sqrt{[(n-2)(\nu-4)+(n-\nu+1)(n-2)\tau'^2]/(n-3)(\nu-3)},$$

(cf. $\bar{\theta}_{1|5}$ in (26))

$$\bar{\theta}_{2|\nu} \equiv \sec^{-1}\sqrt{[(n-3)(\nu-5)+(n-\nu+1)(n-2)\tau'^2\cos^2\theta_1]/(n-4)(\nu-4)},$$

$$\bar{\theta}_{3|\nu} = \sec^{-1}\sqrt{[(n-4)(\nu-6)+(n-\nu+1)(n-2)\tau'^2\cos^2\theta_1\cos^2\theta_2]/(n-5)(\nu-5)},$$

.... up to (40) and (35).

The remaining upper limits are generally (but $\beta_1 = \sec^{-1}\sqrt{n-2}\,\tau'$ as in III, IV)

(18.44) $$\beta_m = \sec^{-1}(\sqrt{n-2}\,\tau' \prod_1^{m-1}\cos\theta_i) \qquad \text{for } m=\nu-3, \nu-2, \cdots, n-3.$$

Lastly, the lower limits are

(18.45)
$$\begin{aligned} \alpha_m &= \tan^{-1}\sqrt{n-1/(n-3)} && \text{for } m=1, \\ &= \tan^{-1}(\sqrt{(n-m)/(n-m-2)}\sin\theta_{m-1}) && \text{for } m=2, \cdots, \nu-3, \\ &= 0 && \text{for } m=\nu-2, \cdots, n-3, \end{aligned}$$

which contain neither τ nor ν explicitly, so that independent of the suborder.

It is well to be remarked that all upper limits depend on τ (τ' being also the funcition of τ), while the lower limits are all free from τ, τ'.

Accordingly we obtain the $\theta_{\nu-3}\theta_{\nu-4}$-triangled formula

(18.46) $$v_\nu = 2^{n-\nu}\sqrt{n}\,s^{n-2}\left[\prod_1^{\nu-4}\int_{\alpha_i}^{\bar{\theta}_i}\cos^{i-1}\theta_i d\theta_i\right]\int_{\alpha_{\nu-3}}^{\beta_{\nu-3}}\Theta_{\nu-2}(\theta_{\nu-3}, \tau)\cos^{\nu-4}\theta_{\nu-3}d\theta_{\nu-3} (=J_\nu)$$

where

(18.47) $$\Theta_{\nu-2} = \left(\prod_{\nu-2}^{n-3}\int_0^{\beta_i}\cos^{i-1}\theta_i d\theta_i\right)\int_{\psi_0}^{\psi_1}\sin^{n-3}\psi d\psi \quad \text{with}$$

(18.48) $$\psi_1 = \sin^{-1}(\rho_\nu/\sqrt{n}\,s) = \sec^{-1}\sqrt{n-1}\,\tau \quad \text{and}$$

$$\psi_0 = \sin^{-1}(\rho_0/\sqrt{n}\,s) = \tan^{-1}(\sqrt{n/(n-2)}\sec\theta_1\cdots\sec\theta_{n-3}).$$

In particular, for the final subcase $\nu = n-1$, (46) reduces to

(18.49) $$v_{n-1} = 2(\sqrt{n}\,s)^{n-2}\left[\prod_1^{n-5}\int_{\alpha_i}^{\bar{\theta}_i}\cos^{i-1}\theta_i d\theta_i\right]\int_{\alpha_{n-5}}^{\beta_{n-5}}\cos^{n-5}\theta_{n-4}d\theta_{n-4}$$
$$\int_0^{\beta_{n-4}}\cos^{n-4}\theta_{n-3}d\theta_{n-3}\int_{\psi_0}^{\psi_1}\sin^{n-3}\psi d\psi.$$

Although there was the factor 2^{n-1} as the number of quadrants in (4), now that up to the ν-th subcases, there are $(\nu-1)$ gates which confine their one sided portion only to be adopted and thus their negative quadrants entirely rejected, the power 2^{n-1} is to be divided by $2^{\nu-1}$ and it reduces to $2^{n-\nu}$. The volume v_ν has been computed about the typical one, i.e. that portion lying outward $\bar{K}'$, $\bar{K}''$, $\cdots$, $K^{(\nu-1)}$ typically chosen, and forms $(\nu-1)$-ple overlapping.

Really there being ${}_nP_{\nu-1}$ of such portions, the whole overlapping volume O_ν shall be obtained by multiplying this number to v_ν:

$$(18.50)\qquad O_\nu = {}_nP_{\nu-1}v_\nu = F_1\mathfrak{h}_{\nu-1} \text{ with } F_1 = 2\sqrt{\pi}^{n-1}(\sqrt{n}\,s)^{n-2}/\Gamma((n-1)/2).$$

So that

$$(18.51)\qquad \mathfrak{h}_{\nu-1} = 2^{n-\nu-1}\sqrt{\pi}^{1-n}{}_nP_{\nu-1}\Gamma\left(\frac{n-1}{2}\right)J_\nu,$$

where the $(n-2)$-ple integrl J_ν in (46) is really a function of τ alone, because, among its limits of integrations the upper limits solely contain τ though the lower as well as integrands are all free from τ. Consequently the product surface as well as the volume element at the ν-th stage become

$$(18.52)\qquad F_\nu = F_{\nu-1} + (-1)^{\nu-1}\mathfrak{h}_{\nu-1}(\tau) = F_1(\sqrt{n}\,s)\sum_0^{\nu-1}(-1)^i\mathfrak{h}_i(\tau),$$

$$(18.53)\qquad dV_\nu = F_\nu d(\sqrt{n}\,\bar{x})\,d(\sqrt{n}\,s), \qquad \nu = 1, 2, \cdots, n-1.$$

Thus our problem is completely resolved, at least theoretically, so far concerned with the volume element, although there remains still to contemplate thoroughly how to manipulate the integral with so enormously large multiplicity and also to investigate its behaviour toward the central limit theorem.

N. B. The above method of consideration VI may also be applied to V or IV, by making the axial space $B_{\nu-4}$ degenerate into η_1-or η_3-axis, and taking the moving point G' on the η_3-or η_2-parallel through G_4 or G_3, respectively, although the conjecture may be naturally done in the order of IV, V, VI.

The ν-th correction factor $\mathfrak{h}_{\nu-1}$ is essentially defined by the repeated integral J_ν, where its integrands as well as limits of integrations behave regular about τ, in $0 \leq \tau \leq \sqrt{n-1} < \infty$, $0 \leq \tau' \leq \sqrt{n-2} < \infty$. Besides at the epoch $\tau \equiv \tau_{\nu-1}$, J_ν and $\mathfrak{h}_{\nu-1}$ do vanish, because the domain of integration reduces to naught as $\tau \to \tau_{\nu-1}$. This fact might be seen as a matter of course, since the correction at any stage vanishes at its epoch. But we can say a little more[1]):

$$(18.54)\qquad \frac{d^p}{d\tau^p}\mathfrak{h}_{\nu-1}(\tau_{\nu-1}) = 0, \text{ so far } p = 0, 1, \cdots, \nu-4.$$

For, at the epoch $\tau = \tau_{\nu-1}$, $\tau' = \tau'_{\nu-2} = \sqrt{\dfrac{\nu-2}{n-\nu+1}}$, $n \geq 7$, $\nu \geq 6$, every upper limit of the first $(\nu-4)$ factor-integrals in (46) coincides with the respective lower: typically the m-th become both $\sec^{-1}\sqrt{(m+1)(n-m-1)/m(n-m-2)} \neq \infty$, while every of them as well as the partial integrals of J_ν all possess successive derivatives that remain finite at $\tau = \tau_{\nu-1}$.

1) These facts are similar to those phenomena illustrated about the fr. fs. $f_1, f_2, \cdots$ in Cramér's Mathematical Methods of Statistics, p. 245-.

We may write the whole correction factor to be multiplied to F_1 as

$$\mathfrak{h}(\tau)=\sum_1^{\nu}(-1)^{i-1}\mathfrak{h}_{i-1} \quad \text{in} \quad \tau_{\nu-1}\leq\tau\leq\tau_\nu, \quad \nu=1, 2, \cdots, n-1, \tag{18.55}$$

that has at least the first derivative continuous throughout the whole interval $0\leq\tau\leq n-1$. The partial sum $\sum_1^{\nu-1}$ is thus well continued without any jump not only in its value but also in its direction by the following $\sum_1^{\nu}$, since even at the point of continuation $\tau_{\nu-1}$ not only these two sums themselves, but also their first few derivatives just coincide.

Summarizing all the above, the general Fisher's function (product surface) can be defined as a certain continuation of the proper Fisher's function F_1 by

$$F_\nu=F_1(\sqrt{n}s)\,\mathfrak{h}(\tau) \tag{18.56}$$

which is derivable continuously through the whole interval $0\leq\tau\leq\sqrt{n-1}$, although $\mathfrak{h}_0$, $\mathfrak{h}_1$, $\cdots$ are apparently stepwise defined. Whence we can furthermore conclude that the Student's function with any truncated originally regular parent fr. f. is likewise stepwise smooth in its whole intrval throughout. However a more general treatment for $\mathfrak{h}(\tau)$ shall be postponed for a future.

19. *Student's Function for the T.N.D. as Universe.* In general, let any universe truncated negatively[1] be $f(x)\,(x>0)$ and from which a $n(>2)$-sized sample $\{x_1, \cdots, x_n\}$ be drawn with sample mean $\bar{x}$ and S.D. s. So that the elementary probabilities are

$$dp=f(x_1)\cdots f(x_n)dx_1\cdots dx_n=g(\bar{x}, s)dv, \quad dP=g(\bar{x}, s)dV,$$

where the volume element $dV(=\int dv$ taken as $\bar{x}$, s determinate) is given by (17.0). Hence, the joint probability for $\bar{x}$, s is

$$dP=f(\bar{x}, s)nd\bar{x}ds=g(\bar{x}, s)F_{n-2}(\bar{x}, s)nd\bar{x}ds.$$

Or, denoted by Student's ratio $t=\sqrt{n-1}\,(\bar{x}-m)/s$,

$$\begin{aligned}dP&=g\big(\bar{x}, \sqrt{n-1}\,(\bar{x}-m)/t\big)\,F_{n-2}\big(\bar{x}, \sqrt{n-1}\,(\bar{x}-m)/t\big)\,n\sqrt{n-1}\,|\bar{x}-m|\,dxdt/t^2\\&=f(\bar{x}, t)d\bar{x}dt.\end{aligned}$$

Therefore the Student's fr. $f.$ $s(t)$ is obtained as

$$s(t)=\int f(\bar{x}, t)d\bar{x},$$

where the integration is taken in the $\bar{x}s$-space so far $\sqrt{n-1}\,(\bar{x}-m)/s=t$ (assigned value) holds. Whenever the universe $f(x)$ is known, $g(\bar{x}, s)$ can be calculated, while F_{n-2} is decided most generally after the foregoing section.

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1) The author used before to call it as a 'positively' truncated one, but it seems rather suitable to say 'negatively'.

As an example to adequate application of the foregoing theory, we would discuss the said theme in outline. If the original complete normal distribution

$$\varphi(x-a)=\frac{1}{\sqrt{2\pi}}\exp[-\tfrac{1}{2}(x-a)^2], \quad a\gtreqless 0, \quad -\infty<x<\infty,$$

assumed the variance $\sigma^2=1$ for simplicity, be truncated negatively, i.e. its negative side be rejected out, there remains the truncated normal distribution[1)]

(19.1) $$f(x)=\frac{\varphi(x-a)}{D}=\frac{1}{D\sqrt{2\pi}}\exp[-\tfrac{1}{2}(x-a)^2], \quad x>0,$$

where

(19.2) $$D=\int_0^\infty \varphi(x-a)dx=\Phi(a)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-z^2/2}dz>0$$

with the parent mean

(19.3) $$m=a+\lambda>0,\ \lambda=\varphi(a)/D>0,\ \text{besides} >-a,\ \text{if}\ a<0.$$

If a sample $\{x_1, \cdots, x_n\}$ be drawn from (1) with the sample mean $\bar{x}$ and S.D. s, we get the partial $\bar{x}s$-joint probability due to the ν-th subinterval I_ν:

(19.4) $$dP_\nu=f(x_1)\cdots f(x_n)dV_\nu=g(\bar{x}, s)F_1(\sqrt{n}\,s)\mathfrak{h}(\tau)d(\sqrt{n}\,\bar{x})d(\sqrt{n}\,s), \qquad \text{where}$$

(19.5) $$g(\bar{x}, s)=\exp\left\{-\frac{n}{2}(s^2+(\bar{x}-a)^2)\right\}/(\sqrt{2\pi}\,D)^n,$$

$$F_1(\sqrt{n}\,s)=2\sqrt{\pi}^{\,n-1}(\sqrt{n}\,s)^{n-2}/\Gamma((n-1)/2),$$

$$\mathfrak{h}(\tau)=\sum_1^{\nu}(-1)^{i-1}\mathfrak{h}_{i-1}(\tau),\ \ \tau=s/\bar{x}\in I_\nu:\ \tau_{\nu-1}<\tau<\tau_\nu=\sqrt{\nu/(n-\nu)}.$$

Or, transforming s into Student's ratio $t=\sqrt{n-1}\,(\bar{x}-m)/s$ which is $\lesseqgtr 0$ according as $\bar{x}\lesseqgtr m$, we obtain the ν-th partial $\bar{x}t$-joint probability

(19.6) $$dP_\nu=g(\bar{x}, s=\sqrt{n-1}\,(\bar{x}-m)/t)F_1(\sqrt{n(n-1)}(\bar{x}-m)/t)\mathfrak{h}(\tau=\sqrt{n-1}\,(\bar{x}-m)/\bar{x}t)$$
$$\times\, n\sqrt{n-1}\,|\bar{x}-m|\,d\bar{x}dt/t^2$$

$$=c|\bar{x}-m|^{n-1}e^{-Q(\bar{x},t)}\mathfrak{h}\,d\bar{x}dt/|t|^n, \qquad \text{where}$$

(19.7) $$Q(x,t)=\tfrac{1}{2}n(n-1)(\bar{x}-m)^2/t^2+\tfrac{1}{2}n(x-a)^2=\tfrac{1}{2}n\{R(x-m)^2+2\lambda(x-m)+\lambda^2\},$$

$$R=1+\frac{n-1}{t^2},\quad c=\frac{1}{D^n}\sqrt{\frac{n(n-1)}{2}}^{\,n}\bigg/\sqrt{\frac{\pi}{n-1}}\,\Gamma\left(\frac{n+1}{2}\right),\quad \tau=\frac{\sqrt{n-1}\,(\bar{x}-m)}{\bar{x}t}.$$

Hence, the ν-th partial Student fr. f. is defined by

(19.8) $$s_\nu(t)=\frac{c}{|t|^n}\int_{\bar{x}_0}^{\bar{x}_1}|\bar{x}-m|^{n-1}e^{-Q}\mathfrak{h}\,d\bar{x},$$

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1) H. Cramér, loc. cit., pp. 247-8, 381-.

the integration being taken so far τ belongs to the ν-th subinterval for the given t. We write, for the sake of convenience, when $t(\neq 0)$ is assigned,

(19.9.1) $$G(\bar{x}, t) = \int_0^{\bar{x}} (m-x)^{n-1} e^{-Q} dx (>0) \qquad \text{for } 0 < \bar{x} < m,\ t<0 \text{ and}$$

(19.9.2) $$= \int_\infty^{\bar{x}} (x-m)^{n-1} e^{-Q} dx (<0) \qquad \text{for } m < \bar{x} < \infty,\ t<0,$$

whose $\bar{x}$-derivatives yield both

(19.10) $G'(\bar{x}, t) = |\bar{x}-m|^{n-1} e^{-Q}$ (Fig. 7).

Making use of this G' we may express the partial fr. f.

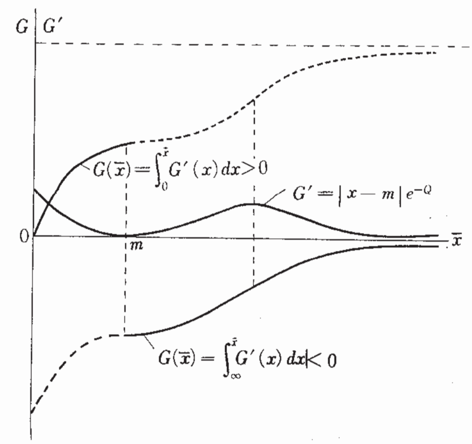

Fig. 7

(19.11) $$s_\nu(t) = \frac{c}{|t|^n} \int_{\bar{x}_0}^{\bar{x}_1} G'(\bar{x}, t)\, \mathfrak{h}(\tau) d\bar{x}$$

$$= \int_{\bar{x}_0}^{\bar{x}_1} f_\nu(\bar{x}, t) d\bar{x}$$

$$= \frac{c}{|t|^n} \sum_1^\nu (-1)^{\mu-1} \int_{\bar{x}_0}^{\bar{x}_1} G'(\bar{x}, t) \mathfrak{h}_{\mu-1}(\tau) d\bar{x}$$

$$= \sum_1^\nu (-1)^{\mu-1} \int_{\bar{x}_0}^{\bar{x}_1} f_1 \mathfrak{h}_{\mu-1} d\bar{x}, \quad \text{where } f_1 = c\,G'/|t|^n.$$

It remains to decide limits of integrations. Since in the ν-th subcase

$$\tau_{\nu-1} = \sqrt{(\nu-1)/(n-\nu+1)} < \tau = \sqrt{n-1}\,(\bar{x}-m)/\bar{x}t < \tau_\nu = \sqrt{\nu/(n-\nu)}$$

hold, we have

(19.12) $$\bar{x}(1-\tau_\nu t/\sqrt{n-1}) \gtrless m \gtrless \bar{x}(1-\tau_{\nu-1} t/\sqrt{n-1}),$$

where the double signs take the upper or lower according as $t \lessgtr 0$. Besides for $t<0$ the bracketed expressions become both positive and we get

(19.13) $$m/(1-\tau_\nu t/\sqrt{n-1}) < \bar{x} < m/(1-\tau_{\nu-1} t/\sqrt{n-1}),$$

while, for $t>0$ if the two expressions be both positive, alike inequalities but with the reversed sense. We take further the positive constants

(19.14) $$\sqrt{n-1}/\tau_\nu = \sqrt{(n-1)(n-\nu)/\nu} = t_{n-\nu},$$

i.e. $(t_0 = 0)$, $t_1 = 1$, $t_2 = \sqrt{2(n-1)/(n-2)}, \cdots, t_\nu = \sqrt{(n-1)\nu/(n-\nu)}, \cdots, t_{n-1} = n-1$, $t_n = \infty$, which correspond to $(\tau = \infty)$, $\tau_\nu = \sqrt{n-1}$, $\sqrt{(n-2)/2}, \cdots, \sqrt{\nu/(n-\nu)}, \cdots$, $1/\sqrt{n-1}$, 0, respectively. The whole t-interval $(-\infty < t < \infty)$ is thus divided into $n+1$ subintervals: 1° $-\infty < t < 0$, 2° $0 < t < 1$, 3° $1 < t < t_2, \cdots, (n-\nu+1)°$ $t_{n-\nu-1} < t < t_{n-\nu}, \cdots, (n+1)°\ n-1 < t < \infty$. Further putting

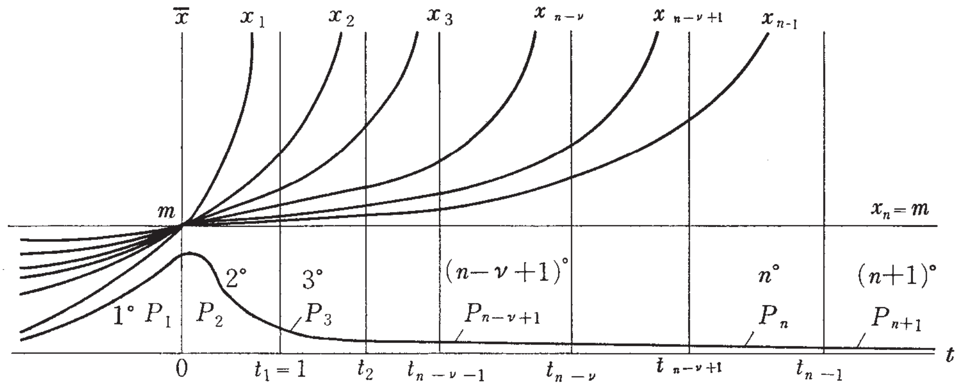

Fig. 8

(19.15) $$x_\nu = m/(1-t/t_\nu) = m/(1-t\tau_{n-\nu}/\sqrt{n-1}),$$

$$x_{n-\nu} = m/(1-t/t_{n-\nu}) = m/(1-t\tau_\nu/\sqrt{n-1}),$$

that is $x_1 = m/(1-t)$, $x_2 = m/(1-t/t_2)$, $\cdots$, $x_{n-1} = m/(1-t/t_{n-1}) = m/(1-t/(n-1))$, $x_n = m$, we obtain a bundle of hyperbolas (however their upper positive branches only considered, Fig. 8), all passing through $(t=0, \bar{x}=m)$. Since on the hyperbola $x_{n-\nu} = m/(1-t\tau_\nu/\sqrt{n-1})$ the τ-ratio $(= s/\bar{x} = \sqrt{n-1}\,(\bar{x}-m)/\bar{x}t)$ becomes τ_ν, every hyperhola $x_{n-\nu}$ is characterized by τ_ν, that is

(19.16) $$\bar{x} = x_{n-\nu} \text{ means } \tau = \tau_\nu \text{ and vice versa.}$$

Now returning to inequalities (12) (13) relating to the ν-th subcase, we rewrite them

(19.17) $$x_{n-\nu} < \bar{x} < x_{n-\nu+1}, \text{ so that } \bar{x}_0 = x_{n-\nu} \text{ and } \bar{x}_1 = x_{n-\nu+1} \text{ for } t<0,$$

while, on the contrary,

(19.18) $$x_{n-\nu+1} < \bar{x} < x_{n-\nu} \text{ and } \bar{x}_0 = x_{n-\nu+1},\ \bar{x}_1 = x_{n-\nu} \text{ for } t>0.$$

However, the latter is only permissible when $0 < x_{n-\nu+1} < x_{n-\nu}$, i.e. so far as $0<t<t_{n-\nu}$ and thus t lies within $2^\circ \cdots$ up to $(n-\nu+1)^\circ$. If $x_{n-\nu+1} > 0$, but $x_{n-\nu} < 0$, namely if $t_{n-\nu} < t < t_{n-\nu+1}$ or $t \in (n-\nu+2)^\circ$, then we should take

(19.19) $$\bar{x}_0 = x_{n-\nu+1} \text{ but } \bar{x}_1 = \infty.$$

Lastly, if $x_{n-\nu} < 0$ also, i.e. $t > t_{n-\nu+1}$, then the ν-th subcase must be abandoned.

Accordingly the full Student's fr. f. is obtained by combining (11) as

(19.20) $$s(t) = \sum_1^{n-1} s_\nu(t) = \sum_1^{n-1} \int_{\bar{x}_0(\nu)}^{\bar{x}_1(\nu)} f_\nu(\bar{x}, t)\,d\bar{x},$$

where the limits of integrations are determined after the foregoing as follows:

1° $-\infty < t < 0$: In view of (11) and (17) we get

$$s(t) = \sum_{1}^{n-1} \int_{x_{n-\nu}}^{x_{n-\nu+1}} f_\nu(\bar{x}, t) d\bar{x} = \sum_{\nu=1}^{n-1} \sum_{\mu=1}^{\nu} (-1)^{\mu-1} \int_{x_{n-\mu}}^{x_{n-\mu+1}} f_1 \mathfrak{h}_{\mu-1} d\bar{x}.$$

But, since the double summations become $\sum_{\mu=1}^{n-1} \sum_{\nu=\mu}^{n-1}$, we obtain

$$s(t) = \frac{c}{|t|^n} \sum_{1}^{n-1} (-1)^{\mu-1} \int_{x_1}^{x_{n-\mu+1}} G' \mathfrak{h}_{\mu-1} d\bar{x}. \tag{19.21}$$

2° $0 < t < t_1 = 1$: Quite similarly as in 1° but now after (18), we get

$$s(t) = \sum_{1}^{n-1} \int_{x_{n-\nu+1}}^{x_{n-\nu}} f_\nu(\bar{x}, t) d\bar{x} = \frac{c}{t^n} \sum_{1}^{n-1} (-1)^{\mu-1} \int_{x_{n-\mu+1}}^{x_1} G' \mathfrak{h}_{\mu-1} d\bar{x}. \tag{19.22}$$

3° $1 < t < t_2$: Now that the upper limit x_1 becomes negative, we must replace it by ∞ after (19)

$$s(t) = \sum_{1}^{n-1} \int_{x_{n-\nu+1}}^{\infty} f_\nu(\bar{x}, t) d\bar{x} = \frac{c}{t^n} \sum_{1}^{n-1} (-1)^{\mu-1} \int_{x_{n-\mu+1}}^{\infty} G' \mathfrak{h}_{\mu-1} d\bar{x}. \tag{19.23}$$

4° $t_2 < t < t_3$: Here $x_2 = m/(1-t/t_2)$ becomes also negative, so that the summand for $\nu = n-1$ is to be rejected and

$$s(t) = \sum_{1}^{n-2} \int_{x_{n-\nu+1}}^{\infty} f_\nu(\bar{x}, t) d\bar{x} = \frac{c}{t^n} \sum_{1}^{n-2} (-1)^{\mu-1} \int_{x_{n-\mu+1}}^{\infty} G' \mathfrak{h}_{\mu-1} d\bar{x}. \tag{19.24}$$

In general $(i+1)°$ $t_{i-1} < t < t_i$: Because here $x_1, x_2, \cdots$ up to x_{i-1} become negative, so the summand ends at $\nu = n-i+1$:

$$s(t) = \sum_{1}^{n-i+1} \int_{x_{n-\nu+1}}^{\infty} f_\nu(\bar{x}, t) d\bar{x} = \frac{c}{t^n} \sum_{1}^{n-i+1} (-1)^{\mu-1} \int_{x_{n-\mu+1}}^{\infty} G' \mathfrak{h}_{\mu-1} d\bar{x}. \tag{19.25}$$

The uttermost case $(n+1)°$: $t_{n-1} = n-1 < t < \infty$. Here all x_j's except $x_n = m$ become negative and only $\nu = 1$ is alone to be adopted. Hence, putting $i = n$ in (25)

$$s(t) = \int_m^\infty f_1(\bar{x}, t) d\bar{x} = \frac{c}{t^n} \int_m^\infty G' dx = \frac{c}{t^n} [G(\infty) - G(m)] = -\frac{c}{t^n} G(m, t>0). \tag{19.26}$$

Since this form (26) not only gives the full Student's fr. f. in $(n+1)°$, but also comes in the other positive t-interval, so we specify it by denoting by $s_0(t)$ and calling the first (principal) constituent in the positive t-interval. Thus

$$s_0(t) = \frac{c}{t^n} [G(\infty) - G(m)] = -\frac{c}{t^n} G(m, t) = \frac{c}{t^n} \int_m^\infty (\bar{x} - m)^{n-1} e^{-Q} dx > 0. \tag{19.27}$$

The before last $n°$: $t_{n-2} = \sqrt{(n-1)(n-2)/2} < t < n-1$. We have by (25)

(19.28) $$s(t)=\frac{c}{t^n}\left[\int_m^\infty G'd\bar{x}-\int_{x_{n-1}}^\infty G'\mathfrak{h}_1 d\bar{x}\right]$$

$$=\frac{c}{t^n}\left[-G(m,t>0)+\int_{x_{n-1}}^\infty G(\bar{x},t)\mathfrak{h}_1'\frac{\partial\tau}{\partial\bar{x}}d\bar{x}(<0)\right]$$

since, when the last integral is integrated by parts, the integrated parts vanish because of $G(\infty)=0$ and $\mathfrak{h}_1(\tau_1)=0$. Or, if the integration variable $\bar{x}$ be transformed into τ by the relation $\bar{x}=m/(1-t\tau/\sqrt{n-1})$, we obtain

(19.29) $$s(t)=s_0(t)+\frac{c}{t^n}\int_\tau^{\sqrt{n-1}/t}G(\bar{x}=m/(1-t\tau/\sqrt{n-1}),t)\,\mathfrak{h}_1'(\tau)\frac{\partial\tau}{\partial\bar{x}}\frac{\partial\bar{x}}{\partial\tau}d\tau$$

$$=s_0(t)-\frac{c}{t^n}\int_{\tau_1}^{\sqrt{n-1}/t}\mathfrak{h}_1'\,d\tau\int_{\bar{x}}^\infty(y-m)^{n-1}e^{-Q(y,t)}dy \text{ by (19.9.2).}$$

Similarly for the case once more before last: $(n-1)^\circ\ t_{n-3}=\sqrt{(n-1)(n-3)/3}<t<t_{n-2}=\sqrt{(n-1)(n-2)/2}$,

(19.30) $$s(t)=s_0(t)+\frac{c}{t^n}\int_{x_{n-1}}^\infty G(\bar{x},t)\,\mathfrak{h}_1'\frac{\partial\tau}{\partial\bar{x}}d\bar{x}-\frac{c}{t^n}\int_{x_{n-2}}^\infty G(\bar{x},t)\,\mathfrak{h}_2'\frac{\partial\tau}{\partial\bar{x}}d\bar{x}$$

(19.30.1) $$=s_0(t)-\frac{c}{t^n}\int_{\tau_1}^{\sqrt{n-1}/t}\mathfrak{h}_1'\,d\tau\int_{\bar{x}}^\infty(y-m)^{n-1}e^{-Q}dy+\frac{c}{t^n}\int_{\tau_2}^{\sqrt{n-1}/t}\mathfrak{h}_2'\,d\tau\int_{\bar{x}}^\infty ''$$

and so on. Similar consideration may still be applied to the negative interval:
$1^\circ\ -\infty<t<0$: Integrating (21) by parts we obtain

(19.31) $$s(t)=\frac{c}{|t|^n}\sum_1^{n-1}(-1)^{\nu-1}\left[G(\bar{x},t)\mathfrak{h}_{\nu-1}(\tau)\Big|_{x_1}^{x_{n-\nu+1}}-\int_{x_1}^{x_{n-\nu+1}}G(\bar{x},t)\mathfrak{h}_{\nu-1}'(\tau)\frac{\partial\tau}{\partial\bar{x}}d\bar{x}\right].$$

Since the integrated parts together reduce to $G(m,t)$, because the upper limit $x_{n-\nu+1}$ means $\tau=\tau_{\nu-1}$ by (16) and $\mathfrak{h}_{\nu-1}(\tau_{\nu-1})=0$ except $\mathfrak{h}_0=1$, while, the lower limit $\bar{x}=x_1$ yields $\tau=\tau_{n-1}$ and $\sum_1^{n-1}(-1)^{\nu-1}\mathfrak{h}_{\nu-1}(\tau_{n-1})=0$. Therefore

(19.32) $$s(t)=\frac{c}{|t|^n}\left[G(m,t)+\sum_2^{n-1}(-1)^\nu\int_{x_1}^{x_{n-\nu+1}}G(\bar{x},t)\,\mathfrak{h}_{\nu-1}'\frac{\partial\tau}{\partial\bar{x}}d\bar{x}\right].$$

Here the first term may be again denoted by $s_0(t)$ which has a similar shape as (27)

(19.33) $$s_0(t)=\frac{c}{|t|^n}G(m,t)=\frac{c}{|t|^n}\int_0^m(m-\bar{x})^{n-1}\exp(-Q(\bar{x},t))d\bar{x}.$$

Or else, if the integration variable $\bar{x}$ be replaced by $\tau=\sqrt{n-1}(\bar{x}-m)/\bar{x}t$, we obtain

(19.34) $$s(t)=s_0(t)-\frac{c}{|t|^n}\sum_2^{n-1}(-1)^\nu\int_{\tau_{\nu-1}}^{\tau_{n-1}}G\left(\bar{x}=\frac{m}{1-t\tau/\sqrt{n-1}},\ t\right)\mathfrak{h}_{\nu-1}'d\tau.$$

So also for 2° $0<t<1$

(19.35.1) $$s(t)=s_0(t)-\frac{c}{t^n}\sum_{x}^{n-1}(-1)^\nu\int_{x_{n-\nu+1}}^{x_1}G(\bar{x},t)\mathfrak{h}'_{\nu-1}\frac{\partial\tau}{\partial\bar{x}}d\bar{x}$$

(19.35.2) $$=s_0(t)-\frac{c}{t^n}\sum_{1}^{n-1}(-1)^\nu\int_{\tau_{\nu-1}}^{\tau_{n-1}}G\left(\bar{x}=\frac{m}{1-t\tau/\sqrt{n-1}},\ t\right)\mathfrak{h}'_{\nu-1}d\tau,\ \&c.$$

We have described so far Student's fr. f. $s(t)$. Accordingly Student's d. f. is given by

(19.36) $$S(t_\alpha)=\int_{-\infty}^{t_\alpha}s(t)dt$$

which in case $t_\alpha<0$ by (34) becomes on putting $\bar{x}=m/(1-t\tau/\sqrt{n-1})$

(19.37) $$S(t_\alpha)=c\int_{-\infty}^{t_\alpha}G(m,t)\frac{dt}{|t|^n}$$
$$+c\sum_{2}^{n-1}(-1)^{\nu-1}\int_{-\infty}^{t_\alpha}\frac{dt}{|t|^n}\int_{\tau_{\nu-1}}^{\tau_{n-1}}\mathfrak{h}'_{\nu-1}d\tau\int_0^{\bar{x}}(m-y)^{n-1}e^{-Q(y,t)}dy$$
$$=S_0(t_\alpha)+\sum_{\nu=2}^{n-1}(-1)^{\nu-1}S_{\nu-1}(t_\alpha),$$

whose first term is again called the main value, while the following terms being the first, second, $\cdots$ corrections.

In case $t_\alpha>0$, we may otherwise conceive the complementary d. f.

(19.38) $$\bar{S}(t_\alpha)=\int_{t_\alpha}^{\infty}s(t)dt(=1-S(t_\alpha)).$$

First, if $t_{n-1}=n-1\leq t_\alpha<t_n=\infty$, we may evaluate after (26) (27)

(19.39) $$\bar{S}(t_\alpha)=c\int_{t_\alpha}^{\infty}\frac{-G(m,t)}{t^n}dt\equiv\bar{S}_0(t_\alpha),\quad \bar{S}(t_{n-1})=-c\int_{n-1}^{\infty}G(m,t)\frac{dt}{t^n}.$$

Next, if $t_{n-2}<t_\alpha<t_{n-1}$, availing (29)

(19.40) $$\bar{S}(t_\alpha)=\bar{S}_0(t_\alpha)+c\int_{t_\alpha}^{n-1}\frac{dt}{t^n}\int_{\tau_1}^{\sqrt{n-1}/t}G\left(\bar{x}=\frac{m}{1-t\tau/\sqrt{n-1}},\ t\right)\mathfrak{h}'_1d\tau$$
$$=\text{''}-c\int_{t_\alpha}^{n-1}\frac{dt}{t^n}\int_{\tau_1}^{\sqrt{n-1}/t}\mathfrak{h}'_1d\tau\int_{\bar{x}}^{\infty}(y-m)^{n-1}e^{-Q(y,t)}dy\qquad\left(\bar{x}=\frac{m}{1-t\tau/\sqrt{n-1}}\right).$$

Further if $t_{n-3}\leq t_\alpha<t_{n-2}$, we have after (30)

(19.41) $$\bar{S}(t_\alpha)=\bar{S}(t_{n-2})+c\int_{t_\alpha}^{t_{n-2}}\frac{-G(m,t)}{t^n}dt+c\int_{t_\alpha}^{t_{n-2}}\frac{dt}{t^n}\int_{\tau_1}^{\sqrt{n-1}/t}G(\bar{x},t)\mathfrak{h}'_1d\tau$$
$$-c\int_{t_\alpha}^{t_{n-2}}\frac{dt}{t^n}\int_{\tau_2}^{\sqrt{n-1}/t}G(x,t)\mathfrak{h}'_2d\tau$$
$$=\bar{S}_0(t_\alpha)+c\int_{t_\alpha}^{t_{n-1}}\frac{dt}{t^n}\int_{\tau_1}^{\sqrt{n-1}/t}G(\bar{x},t)\mathfrak{h}'_1d\tau-c\int_{t_\alpha}^{t_{n-2}}\frac{dt}{t^n}\int_{\tau_2}^{\sqrt{n-1}/t}G(\bar{x},t)\mathfrak{h}'_2d\tau,$$

where $G(\bar{x}, t)$ may be expressed by $\int_{\bar{x}}^{\infty}(y-m)^{n-1}e^{-Q}dy$ and the first term being the main value and the second, third integrals are the first, second correction, and so on.

The d. f. thus obtained being equated to $\alpha/2$ (the level $\alpha=0.1, 0.05, 0.01$ &c.) and solved for t_α, we can seek the significant lower or upper critical point by interpolation. Or, when α is small so that $|t_\alpha|$ is pretty great, we may expand $S(t_\alpha)$ or $\bar{S}(t_\alpha)$ in power series of $1/t_\alpha$ to certain term, neglecting the remainder. Thus obtained equations solved by Horner, we can find the approximate critical values t_α. The detailes would be described by numerical example in the following section.

20. *Continued, Numerical Examples for the T.N.D.* We investigate this special case in order to examplify the foregoing general results, since the general feature can be seen therefrom. We take only 3 typical T.N.D., i.e. those truncated at (i) the centroid ($a=0$), (ii) the left quartile and (iii) the right quartile ($a=0.674489750$[1]) and for simplicity confining the sample size to be $n=4$ (the cases $n=2, 3$ had been already outlined in [II] Sect. 3). The below requisite constants are calculated after (19.2, 3, 7) as follows:

species	D	c	a	λ	$m=\lambda+a$	$e^{-2\lambda^2}$	$ce^{-2\lambda^2}=c'$
(i)	1/2	423.4206	0	.797885	$=\sqrt{2/\pi}$	.279923	118.525
(ii)	3/4	83.6386	+.674490	.423702	1.098192	.698342	58.4083
(iii)	1/4	6774.73	−.674490	1.271106	0.596616	.0395013	267.611

Let us begin with the negative subinterval $1°\ -\infty<t<0$. Employing (19.34) we have the fr. f.

$$(20.1)\qquad s(t)=\frac{c}{t^4}\left[G(m,t)-\int_{\tau_1}^{\tau_3}G(\bar{x},t)\mathfrak{h}_1'd\tau+\int_{\tau_2}^{\tau_3}G(\bar{x},t)\mathfrak{h}_2'd\tau\right]$$

$$=s_0(t)+s_1(t)+s_2(t),\quad \text{where}\quad \bar{x}=m/(1-t\tau/\sqrt{3}).$$

First consider the main term in (19.33)

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1) It used to take 0.6745 roughly but roundly, as the so-called probable error in the classical theory of errors. However, after E. Czuber, Wahrsheinlichkeitsrechnung, I, S. 124, for the quartile of the probability curve ρ_0, such that $\frac{1}{\sqrt{\pi}}\int_0^{\rho_0}e^{-t^2}dt=1/4$, the detailed value $\rho_0=$ 0.47693 62762 is computed by H. Opitz. Hence, the quartile for the normal d. f. $\Phi(x_0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_0}e^{-x^2/2}dx$ is $x_0=\sqrt{2}\rho_0=1.414213562\times0.4769362762=0.674489750$. Also this classical book contains a table for the probability integral $\frac{2}{\sqrt{\pi}}\int_0^{\xi}e^{-t^2}dt=\Phi^*(\xi)$, say, with seven-eleven decimal figures. The recent books treat the normal integral $\Phi(x)$, but only with five or six decimal figures, which is theoretically somewhat lacking. The author has below availed to convert $\Phi^*(\xi)$ into $\Phi(x)$ in the following way: Given any $x\gtrless0$, we put $|x|/\sqrt{2}=\xi$ and find $\Phi^*(\xi)$ from Czuber, half of which being added to or subtracted from 0.5 according as $x\gtrless0$, yields just the seven or eleven decimal placed values of $\Phi(x)$.

$$(20.2) \qquad s_0(t) = \frac{c}{t^4} G(m, t) = \frac{c}{t^4} \int_0^m (m-x)^3 e^{-Q(x)} dx,$$

where $Q(x) = Q(x, t)$

$$= 6(x-m)^2/t^2 + 2(x-a)^2 = 2R(x-m)^2 + 4\lambda(x-m) + 2\lambda^2, \quad R = 1 + 3/t^2.$$

To make the integral explicit, we prepare an easy

Lemma A. Let

$$I_\nu \equiv \int_{a(t)}^{b(t)} (m-x)^\nu e^{-Q(x)} dx \text{ and } K_\nu \equiv (m-b)^\nu e^{-Q(b)} - (m-a)^\nu e^{-Q(a)},$$

where limits of integration are constant or some function of t, and naturally the lower limit does not mean the constant $a = m - \lambda$, although m, λ, Q, R are those used before. By integrating by parts, we obtain a recurrence formula

$$4RI_{\nu+1} = 4\lambda I_\nu + \nu I_{\nu-1} + K_\nu, \quad \nu = 0, 1, 2, \cdots$$

and whence

$$(20.3) \qquad I_1 = (4\lambda I_0 + K_0)/4R, \quad I_2 = [(1+\mu^2)RI_0 + \lambda K_0 + RK_1]/4R^2,$$

$$I_3 = \{(3+\mu^2)4\lambda I_0 + (2+\mu^2)K_0 + 4\lambda K_1 + 4RK_2\}/16R^2, \text{ \&c.},$$

where

$$\mu = 2\lambda/\sqrt{R} \text{ and } 4\lambda I_0 = e^{-2\lambda^2}\mu[\Phi(\mu - 2(m-b)\sqrt{R}) - \Phi(\mu - 2(m-a)\sqrt{R})]/\varphi(\mu).$$

First, on putting $a=0$, $b=m$, $\nu=3$ in (3), we obtain an explicit form for (2)

$$(20.4) \qquad s_0(t) = (ce^{-2\lambda^2}/16R^2t^4)\cdot\{(3+\mu^2)\mu[\Phi(\mu) - \Phi(\mu - 2m\sqrt{R})]/\varphi(\mu)$$

$$+ 2 + \mu^2 - (2 + \mu^2 + 4\lambda m + 4Rm^2)\exp(4\lambda m - 2Rm^2)\}.$$

Thus, e.g. when $t = 0$, $R = \infty$ but $Rt^2 \to 3$, and we have for the 3 species

$$(20.4.0) \qquad s_0(-0) = ce^{-2\lambda^2}/72 = c'/72 = c_0 \text{ say}$$

$$= 1.64618,\ 0.81123,\ 3.71682, \text{ respectively.}$$

Next, the first corrections $s_1(t)$ is given by (19.32) or (19.34). But here because of $\mathfrak{h}_1' = \dfrac{2}{\sqrt{3\tau^2}}$, $\tau = \dfrac{\sqrt{3}(\bar{x}-m)}{t\bar{x}}$,

$$\frac{\partial\tau}{\partial\bar{x}} = \frac{\sqrt{3}m}{t\bar{x}^2}, \text{ it holds } \mathfrak{h}_1'\frac{\partial\tau}{\partial\bar{x}} = \frac{2mt}{3(m-\bar{x})^2},$$

so that (19.32) is more advantageous. Also, in view of (19.9.1) we obtain

$$s_1(t) = \frac{c}{t^4}\int_{x_1}^{x_3} \frac{2mt\,dx}{3(x-m)^2} \int_0^x (m-y)^3 e^{-Q(y)} dy.$$

Or, interchanging the order of integration (Fig. 9),

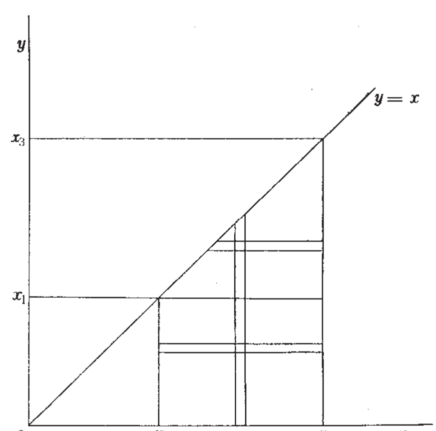

Fig. 9

$$s_1(t) = \frac{2cm}{3t^3}\int_0^{x_1}(m-y)^3 e^{-Q}dy\int_{x_1}^{x_3}\frac{dx}{(x-m)^2} + \int_{x_1}^{x_3} " \int_y^{x_3} "$$

$$= -\frac{2c}{3t^4}[2J + (1-t)I_3 + mtI_2],$$

in which $J = \int_0^{x_3}(m-y)^3 e^{-Q}dy$, and $I_\nu = \int_{x_1}^{x_3}(m-y)^\nu e^{-Q}dy$.

Hence, upon employing Lemma A, we get

$$(20.5)\quad s_1(t) = \frac{c'}{24R^2t^4}\Big\{2(3+\mu^2)\Phi_0 + \Big[(1-t)(3+\mu^2) + \frac{mtR}{\lambda}(1+\mu^2)\Big]\Phi_1$$

$$-\Big[(3-t)(3+\mu^2) + \frac{mtR}{\lambda}(1+\mu^2)\Big]\Phi_3 + 2(2+\mu^2+4\lambda m+4Rm^2)E_0$$

$$+(1-t)(2+\mu^2)E_1 - (3-t)(2+\mu^2)E_3\Big\},$$

where $\Phi_\nu = \mu\Phi\Big(\mu + \frac{2mt\sqrt{R}}{\nu - t}\Big)\Big/\varphi(\mu)$, $E_\nu = \exp\Big[-\frac{4\lambda mt}{\nu-t} - \frac{2Rm^2t^2}{(\nu-t)^2}\Big]$,

for $\nu = 0, 1, 3$. And consequently

(20.5.0)

$$s_1(-0) = c'\sqrt{\frac{2\pi}{108}}\Big\{\varphi(2m\sqrt{3}) - 3\varphi\Big(\frac{2m}{\sqrt{3}}\Big) - m\sqrt{3}\Big[\Phi(2m\sqrt{3}) - \Phi\Big(\frac{2m}{\sqrt{3}}\Big)\Big]\Big\}.$$

Further, as to the second correction, adopting (19.34), we have

$$s_2(t) = \frac{2\sqrt{3}c}{\pi t^4}\int_1^{\sqrt{3}}T(\tau)\,I_3(\tau)\frac{d\tau}{\tau^2},\quad \text{where}$$

$$I_3(\tau) = \int_0^{\bar{x}}(m-y)^3 e^{-Q(y)}dy,\quad \bar{x} = \frac{m}{1-t\tau/\sqrt{3}},\quad T = \tan^{-1}\sqrt{\frac{3}{2}(\tau^2-1)},$$

which requires a numerical integration. However it need not treat as a double integral, rather it can solely be dealt with the ordinary Gauss method: For, given t and selected abscissas τ_ν, in order to compute the inner integral I_3, we may still appeal to Lemma A. Thus, we have

$$(20.6)\quad s_2(t) = \frac{(3-\sqrt{3})c'}{8\pi R^2t^4}\sum_\nu\frac{A_\nu}{\tau_\nu^2}T_\nu\Big\{(3+\mu^2)\frac{\mu}{\varphi(\mu)}\Big[\Phi\Big(\mu + \frac{2mt\tau_\nu\sqrt{R}}{\sqrt{3}-t\tau_\nu}\Big) - \Phi(\mu - 2m\sqrt{R})\Big]$$

$$+\Big[2+\mu^2 - \frac{4\lambda mt\tau_\nu}{\sqrt{3}-t\tau_\nu} + \frac{4Rm^2t^2\tau_\nu^2}{(\sqrt{3}-t\tau_\nu)^2}\Big]\exp\Big(-\frac{4\lambda mt\tau_\nu}{\sqrt{3}-t\tau_\nu} - \frac{2Rm^2t^2\tau_\nu^2}{(\sqrt{3}-t\tau_\nu)^2}\Big)$$

$$-(2+\mu^2+4\lambda m+4Rm^2)\exp(4\lambda m - 2Rm^2)\Big\},$$

where $\tau_\nu = \frac{1}{2}(\sqrt{3}+1) + \frac{1}{2}(\sqrt{3}-1)\xi_\nu$ and A_ν, ξ_ν denote Gaussian constants. Also

$$(20.6.0)\quad s_2(-0) = (\sqrt{3}-1)c'/6\sqrt{6\pi}\sum A_\nu T_\nu(2m^2 + 1/\tau_\nu^2)\varphi(2m\tau_\nu).$$

Otherwise. In order to see how the fr. f. $s(t)$ behaves in the vicinity of the origin, we conceive another expression for $s(t)$: If the integration variable x or y in $s_0(t)$ &c. be replaced by $u=(x-m)/t$ or $(y-m)/t$, we obtain

(20.7) $$s_0(t)=c\int_0^{-m/t} u^3 e^{-Q} du, \quad \text{where } Q=6u^2+2(ut+\lambda)^2,$$

which for $t\rightarrow 0$ furnishes a second proof of (4.0):

$$s_0(0)=c'\int_0^{\infty} u^3 e^{-6u^2} du = c'/72 = c_0.$$

Further, differentiating (7) about t, yields

(20.8) $$s'(t)=-\frac{cm^4}{t^5}\exp\left[-\frac{6m^2}{t^2}-2a^2\right]-c\int_0^{-m/t} u^3 e^{-Q} 4(ut+\lambda) u du.$$

When $t\rightarrow 0$, the first term $\rightarrow 0$ and the integral tends

(20.8.0) $$s_0'(0)=-4\lambda c'\int_0^{\infty} u^4 e^{-6u^2} du = -3\lambda c_0\sqrt{\pi/6} = -0.030150\lambda c'.$$

Thus $s_0(t)$ decreasing at origin. But, to see whether the same holds for the full fr. f. or not, we must still investigate the whole derivative $s'=s_0'+s_1'+s_2'$.

We have similarly for the first correction $s_1(t)$ by replacing its inner integration-variable y by $u=(y-m)/t$

(20.9) $$s_1(t)=-\frac{2c}{3}\left\{2\int_{u_3}^{-m/t} u^3 e^{-Q} du + \int_{u_3}^{u_1}[(1-t)u-m]u^2 e^{-Q} du\right\},$$

where $u_1=m/(1-t)$, $u_3=m/(3-t)$. So that

(20.9.0) $$s_1(0)=-\frac{2c'}{3}\left\{\int_{m/3}^{\infty} 2u^3 e^{-6u^2} du + \int_{m/3}^{m}(u^3-mu^2)e^{-6u^2} du\right\}$$

$$=\frac{c'}{108}\sqrt{2\pi}\,\{m\sqrt{3}\,[\Phi(v_1)-\Phi(v_0)]+\varphi(v_1)-3\varphi(v_0)\},$$

where and below $v_0=2m/\sqrt{3}$, $v_1=2m\sqrt{3}$. And the derivative is

(20.10) $$s_1'(t)=\frac{4cm^4}{3t^5}\exp\left(-\frac{6m^2}{t^2}-2a^2\right)+\frac{16c}{3}\int_{u_3}^{-m/t} u^4(ut+\lambda)e^{-Q} du$$

$$+\frac{8c}{3}\int_{u_3}^{u_1} u^3[(ut+\lambda)(\overline{1-tu}-m)+1/4]e^{-Q} du,$$

(20.10.0) $$s_1'(0)=\frac{\sqrt{2\pi}}{54}c'\left\{\frac{\sqrt{3}}{2}\lambda[2+\Phi(v_1)-3\Phi(v_0)]\right.$$

$$\left.+(\lambda m+m^2/3+1/2)\varphi(v_0)-(\lambda m+3m^2+1/2)\varphi(v_1)\right\}.$$

Lastly, for the second correction, we get

(20.11) $$s_2(t) = \frac{2\sqrt{3}\,c}{\pi}\int_1^{\sqrt{3}} T\frac{d\tau}{\tau^2}\int_{u_0}^{-m/t} u^3 e^{-Q}du, \quad \text{where } u_0 = \frac{m\tau}{\sqrt{3}-t\tau},$$

(20.11.0) $$s_2(0) = \frac{c'}{6\sqrt{6\pi}}\int_1^{\sqrt{3}}\left(2m^2+\frac{1}{\tau^2}\right)T(\tau)\varphi(2m\tau)d\tau$$

$$= \frac{(\sqrt{3}-1)c'}{6\sqrt{6\pi}}\sum A_\nu(2m^2+1/\tau_\nu^2)T(\tau_\nu)\varphi(2m\tau_\nu),$$

which agrees with (6.0). The derivative becomes

(20.12) $$s_2'(t) = -\frac{2\sqrt{3}\,c}{\pi}\left\{\int_1^{\sqrt{3}}\frac{m^4\tau^5}{(\sqrt{3}-t\tau)^5}\exp\left[-\frac{6m^2\tau^2+2(at\tau+\sqrt{3}\lambda)^2}{(\sqrt{3}-t\tau)^2}\right]\frac{Td\tau}{\tau^2}\right.$$

$$\left. + \frac{m^4}{t^5}\exp\left(-\frac{6m^2}{t^2}-2a^2\right) - \int_{u_0}^{-m/t}u^3\exp[-6u^2-2(ut+\lambda)^2]\cdot 4(ut+\lambda)udu\right\};$$

(20.12.0) $$s_2'(0) = \frac{-\lambda c'}{9\sqrt{2\pi}}\int_1^{\sqrt{3}} T\left\{\left(\frac{4m^2\tau^3}{\lambda}+4m\tau+\frac{3}{m\tau}\right)m^2\varphi(2m\tau)+\frac{3}{2\tau^2}[1-\Phi(2m\tau)]\right\}d\tau$$

$$= -\frac{\lambda c'(\sqrt{3}-1)}{9\sqrt{2\pi}}\sum A_\nu T_\nu\left\{\left(\frac{4m^2}{\lambda}\tau_\nu^3+4m\tau_\nu+\frac{3_\nu}{m\tau_\nu}\right)m^2\varphi(2m\tau_\nu)+\frac{3}{2\tau_\nu^2}[1-\Phi(2m\tau_\nu)]\right\}.$$

Evaluating all of them about $t=0$, we obtain the following

(20.13) values of $s(0)$, and $s'(0)$

species	$s_0(0)$	$s_1(0)$	$s_2(0)$	$s(0)$
(i)	+1.6462	−0.8216	+0.1987	+1.0233
(ii)	+0.8112	−0.2856	+0.0262	+0.5518
(iii)	+3.7168	−1.9053	+0.8219	+2.6352
species	$s'_0(0)$	$s'_1(0)$	$s'_2(0)$	$s'(0)$
(i)	−0.6567	+3.2617	−1.5930	+1.0120
(ii)	−0.1719	+0.9638	−0.2386	+0.5533
(iii)	−2.362	+17.926	−6.207	+9.357

From this table we may conclude that the Student's fr. f. for T.N.D. distributes skew against the ordinarv symmetrical N.D. and the mode deviates toward the positive side. Although it is desirous to find the exact values of their modes, i.e. where $s'(t)$ does vanish, we are hastening to obtain the critical points, so that these investigations are postponed.

We should discuss the Student's d.f. that with the argument t_α is

(20.14) $$S(t_\alpha) = \int_{-\infty}^{t_\alpha} s(t)dt = S_0(t_\alpha)+S_1(t_\alpha)+S_2(t_\alpha),$$

whose 3 components correspond to s_0, s_1, s_2 in (1). They can be somehow computed by integrating (4) (5) (6) or (7) (9) (11) in turns. But, most of them are not capable to be elementarily integrated; they require numerical computations by Gauss, Simpson &c. So that we may rather proceed straightforwardly to evaluate the triple integrals of (19.37). Thus

$$(20.15)\qquad S_0(t_\alpha)=c\int_{-\infty}^{t_\alpha}\frac{dt}{t^4}\int_0^m(m-x)^3\exp\left[-\frac{6(m-x)^2}{t^2}-2(m-x-\lambda)^2\right]dx,$$

$$(20.16)\qquad -S_1(t_\alpha)=c\int \text{''} \int_{1\sqrt{3}}^{\sqrt{3}}\frac{2d\tau}{\sqrt{3}\,\tau^2}\int_0^{\bar{x}} \text{''} \left(\bar{x}=\frac{m}{1-t\tau/\sqrt{3}}\right),$$

$$(20.17)\qquad S_2(t_\alpha)=c\int \text{''} \int_1^{\sqrt{3}}\frac{2\sqrt{3}}{\pi\tau^2}\tan^{-1}\sqrt{\frac{3}{2}(\tau^2-1)}\,d\tau\int \text{''} \qquad \text{''}$$

However, to facilitate the mechanical quadrature after Gauss, it needs some rewriting: To avoid ∞ as integration limits and to lighten calculations, we write $-1/t=u$, $-1/t_\alpha=u_\alpha$, $\sqrt{3}/\tau=v$, $m-x=w$, so that

$$(20.15)'\qquad S_0(t_\alpha)=c\int_0^{u_\alpha}u^2du\int_0^m w^3\exp[-6u^2w^2-2(w-\lambda)^2](\equiv W)dw,$$
$$=cmu_\alpha\sum A_iu_i^2\sum C_kW_{ik},$$

$$(20.16)'\qquad -S_1(t_\alpha)=\frac{2}{3}c\int_0^{u_\alpha}u^2du\int_1^3dv\int_{mV}^m Wdw=\frac{4}{3}cmu_\alpha\sum A_iu_i^2\sum B_j(1-V_{ij})\sum C_kW_{ijk},$$

$$(20.17)'\qquad S_2(t_\alpha)=\frac{2c}{\pi}\int \text{''} \int_1^{\sqrt{3}}\tan^{-1}\sqrt{\frac{3}{2}\left(\frac{3}{v^2}-1\right)}(\equiv T)\,dv\int_{mV}^m Wdw$$
$$=2(\sqrt{3}-1)cmu_\alpha/\pi\sum A_iu_i^2\sum B_j(1-V_{ij})T(v_j)\sum C_kW_{ijk}$$

in which $u_i=\frac{1}{2}(1+x_i)u_\alpha$, $w_k=\frac{1}{2}m(1+z_k)$ in (15)′,

$v_j=2-y_j$ or $\frac{1}{2}[\sqrt{3}+1+(\sqrt{3}-1)y_j]$ in (16)′ or (17)′,

$V_{ij}=1/(1+u_iv_j)$, $w_k=\frac{1}{2}m(1+z_k)+\frac{1}{2}m(1-z_k)V_{ij}$ in (16)′ and (17)′,

where and below x_i, y_j z_k, A_i, B_j, C_k denote Gaussian abscissas and coefficients.

In particular, when $t_\alpha\to 0$, we replace u by $\tan\theta$, u_α by $\pi/2$ and obtain

$$(20.15)''\qquad S_0(0)$$
$$=c\int_0^{\pi/2}\tan^2\theta(1+\tan^2\theta)(\equiv U)d\theta\int_0^m w^3\exp[-6w^2\tan^2\theta-2(w-\lambda)^2](\equiv W)\,dw$$
$$=\tfrac{1}{2}\pi cm\sum A_iU_i\sum C_kW_{ik},\quad U_i=\tan^2\theta_i(1+\tan^2\theta_i),\quad \theta_i=\tfrac{1}{4}\pi(1+x_i),$$

$$(20.16)''\qquad -S_1(0)=\frac{2}{3}c\int_0^{\pi/2}Ud\theta\int_1^3dv\int_{mV}^m Wdw=\frac{2\pi}{3}cm\sum A_iU_i\sum B_j(1-V_{ij})\sum C_kW_{ijk},$$
$$V_{ij}=1/(1+v_j\tan\theta_i),$$

$$(20.17)''\qquad S_2(0)=\frac{2c}{\pi}\int_0^{\pi/2}Ud\theta\int_1^{\sqrt{3}}Tdv\int_{mV}^m Wdw$$
$$=(\sqrt{3}-1)cm\sum A_iU_i\sum B_jT_j(1-V_{ij})\sum C_kW_{ijk},$$

It is true that Gauss' n determinate ordinates method is theoretically so excellent and exhaustive even for a highly ordered multiple integral by iterations, but its application already for the triple integral needs n^3 ordinates, thus 125 if $n=5$ and 1000 if $n=10$, which is almost a sheer impossibility to do with a small handy calculator only. Therefore it is practically very desirous to lessen the order of multiplicity. The above last three, i.e. S_0, S_1, S_2 being somewhat important and besides apt to have rather simple forms, let us try to lower their multiplicities of integral possibly. First, considering (15), the main part $S_0(0)$, we interchange the order of tx-integrations, in which Fubini's rule is allowable, since our integrals are absolutely convergent, and write $(x-m)/t=u$, then we get immediately

$$\text{(20.15.0)} \quad S_0(-0)=c\int_{-\infty}^{-0} dt/t^4 \int_0^m (m-x)^3 \exp[-6(m-x)^2/t^2-2(x-a)^2]dx$$
$$=c\int_0^m e^{-2(x-a)^2}dx\int_0^\infty u^2e^{-6u^2}du=\frac{c\pi}{48\sqrt{3}}[\Phi(2\lambda)-\Phi(-2a)].$$

Secondly, for the first correction S_1, we have because of $\mathfrak{h}_1'\dfrac{\partial\tau}{\partial x}=2mt/3(x-m)^2$

$$S_1(-0)=\frac{2cm}{3}\int_{-\infty}^0\frac{dt}{t^3}\int_{x_1}^{x_3}\frac{dx}{(x-m)^2}\int_0^x Y_3dy$$

where $\quad Y_\nu=(m-y)^\nu e^{-Q(y)},\ x_1=\dfrac{m}{1-t},\ x_3=\dfrac{m}{1-t/3},$

whose inner double integral becomes, when the order of xy-integrations interchanged and integrated about x,

$$-\frac{2}{mt}\int_0^{x_3}Y_3dy-\frac{1-t}{mt}\int_{x_1}^{x_3}Y_3dy-\int_{x_1}^{x_3}Y_2dy.$$

Further, changinging the order of ty-integrations and putting $(y-m)/t=v$, we obtain

$$\text{(20.16.0)} \quad S_1(0)=-\frac{2c}{3}\int_0^m e^{-2(y-a)^2}dy\,[2\int_{y/3}^\infty v^2e^{-6v^2}dv+\int_{y/3}^y(v-y)ve^{-6v^2}dv]$$
$$=-\frac{c}{18}\sqrt{\frac{\pi}{6}}\int_0^m E[2+\Phi(2y\sqrt{3})-3\Phi(2y/\sqrt{3})]dy,$$

where and below $E=e^{-2(y-a)^2}$, which however seems yet to need Gauss method. Thirdly, the second correction is

$$S_2(0)=-c\int_{-\infty}^0\frac{dt}{t^4}\int_{x_1}^{x_2}G(x,t)\mathfrak{h}_2'\frac{\partial\tau}{\partial x}dx=-\frac{2cm}{\pi}\int_{-\infty}^0\frac{dt}{t^3}\int_{x_1}^{x_2}\frac{Tdx}{(x-m)^2}\int_0^x(m-y)^3e^{-Q}dy,$$

where

$$T=\tan^{-1}\sqrt{\frac{3}{2}(\tau^2-1)}=\tan^{-1}\sqrt{\frac{9}{2}\Big(\frac{m-x}{xt}\Big)^2-\frac{3}{2}}\ \text{ and }\ x_1=\frac{m}{1-t},\quad x_2=\frac{m}{1-t/\sqrt{3}}.$$

Interchanging the order of the inner double integral about x, y, yields

$$\int_0^{x_1}(m-y)^3e^{-Q(y)}dy\int_{x_1}^{x_2}Tdx/(m-x)^2+\int_{x_1}^{x_2}\text{,,}\int_y^{x_2}\text{,,}=S_{21}+S_{22}\ \text{say.}$$

Executing the innerst integral indefinitely, we have

$$I(\xi)=\int\frac{Tdx}{(m-x)^2}=\frac{T}{m-x}-\frac{1}{mt}[(t-3)A+(t+3)B]$$

$$=\frac{1}{m}\left[\frac{x}{m-x}T+\frac{3}{t}U\right],$$

where $\xi=\sqrt{\dfrac{\sqrt{3}(m-x)+xt}{\sqrt{3}(m-x)-xt}}$, $A=\tan^{-1}\dfrac{\sqrt{3}+1}{\sqrt{2}}\xi$, $B=\tan^{-1}\dfrac{\sqrt{3}-1}{\sqrt{2}}\xi$,

$$T=\tan^{-1}\frac{\sqrt{6}\,\xi}{1-\xi^2}=A+B,\quad\text{and}\quad U=\tan^{-1}\frac{\sqrt{2}\,\xi}{1+\xi^2}=A-B.$$

Hence, for the innerst upper limit of S_{21}, $x=x_2=m/(1-t/\sqrt{3})$, it hold $\xi=0$ and consequently $A=B=T=U=0$, so that $I(0)=0$, while, the lower limit $x=x_1=m/(1-t)$ yields $\xi=(\sqrt{3}-1)/\sqrt{2}=\omega$, $A=\pi/4$, $B=\pi/12$, $T=\pi/3$, so that $I(\omega)=\pi/6mt$. Therefore the innerst integral of S_{21} reduces to $-\pi/6mt$. To perform its whole integration, we proceed likewise as in S_0, S_1, and attain

(20.17.0.1) $$S_{21}(-0)=\frac{c}{3}\int_{-\infty}^0\frac{dt}{t^4}\int_0^{x_1}(m-y)^3e^{-Q}dy=\frac{c}{3}\int_0^m Edy\int_y^\infty v^2e^{-6v^2}dv$$

$$=\frac{c}{36}\int_0^m E\left[ye^{-6y^2}+\sqrt{\frac{\pi}{6}}(1-\Phi(2y\sqrt{3}))\right]dy$$

$$=\frac{c}{576}[e^{-a^2/2}-e^{-(4m-a)^2}+a\Phi(4m-a)-a\Phi(-a)]$$

$$+\frac{c}{36}\sqrt{\frac{\pi}{6}}\int_0^m e^{-2(y-a)^2}(1-\Phi(2y\sqrt{3}))dy,$$

whose last integral S_{21}'' say, combined with that in S_1 of (16.0) yields

$$S_1+S_{21}''=-\frac{c}{12}\sqrt{\frac{\pi}{6}}\int_0^m E[1+\Phi(2y\sqrt{3})-2\Phi(2y/\sqrt{3})]\,dy.$$

Lastly, S_{22} may probably become relatively small, yet enough intricate. Treating in a similar way, we get

$$S_{22}(0)=-\frac{2cm}{\pi}\int_{-\infty}^0\frac{dt}{t^3}\int_{x_1}^{x_2}(m-y)^3e^{-Q}[-I(\eta)]dy\quad\text{with}\quad\eta=\sqrt{\frac{\sqrt{3}(m-y)+yt}{\sqrt{3}(m-y)-yt}}$$

$$=\frac{2cm}{\pi}\int_0^m Edy\int_{y/\sqrt{3}}^y(-v)e^{-6v^2}[(m-y)I(\eta)]dv,\quad\text{,,}\quad\eta=\sqrt{\frac{\sqrt{3}v-y}{\sqrt{3}v+y}}.$$

Or, writing $v=yz$, we get

$$S_{22}(0)=\frac{2cm}{\pi}\int_0^m y^2e^{-2(y-a)^2}dy\int_{1/\sqrt{3}}^1(-z)e^{-6y^2z^2}(m-y)I(\zeta)dz \text{ with } \zeta=\sqrt{\frac{\sqrt{3}z-1}{\sqrt{3}z+1}},$$

where $(m-y)I(\zeta)=\frac{y}{m}(T-3zU),\ T=\tan^{-1}\frac{\sqrt{6}\zeta}{1-\zeta^2},\ U=\tan^{-1}\frac{\sqrt{2}\zeta}{1+\zeta^2}.$

Hence, we reach at length

$$\text{(20.17.0.2)}\qquad S_{22}(0)=\frac{2c}{\pi}\int_0^m y^3e^{-2(y-a)^2}dy\int_{1/\sqrt{3}}^1 e^{-6y^2z^2}(3z^2U-zT)dz.$$

Thus S_{22} can be computed by aid of an iterated Gauss' n^2 ordinates method, e.g. taking $n=3$ and making use of the matrix $[e^{-6y_\mu{}^2z_\nu{}^2}]$ the labour is not so much heavy.

By the way, the d.f. components in 2° being alike in form as those in 1°, we can quite similarly evaluate

$$\int_0^1 s(t)dt=S(1)-S(0)=\bar{S}(1),$$

which is outlined as follows: First

$$\bar{S}_0(0)=\int_0^1 s_0(t)dt=c\int_0^1 dt/t^4\int_m^\infty (x-m)^3e^{-Q}dy.$$

Interchanging the order of integrations and replacing t by $(x-m)/t=u$ and x by $y=x-m$, we get

$$\bar{S}_0(0)=\frac{1}{192}[c'-c\lambda e^{-\frac{3}{2}a^2}\sqrt{2\pi}\,(1-\Phi(\lambda))+\frac{c}{12}\sqrt{\frac{\pi}{6}}\int_0^\infty e^{-2(y+\lambda)^2}(1-\Phi(2y\sqrt{3}))dy].$$

Next, just as in (16.0), we obtain

$$\bar{S}_1(0)=\frac{2cm}{3}\int_0^1\frac{dt}{t^3}\int_{x_2}^{x_1}\frac{dx}{(x-m)^2}\int_\infty^x(y-m)^3e^{-Q}dy$$

$$=-\frac{c}{18}\sqrt{\frac{\pi}{6}}\left\{\int_m^{\frac{3}{2}m}E[1+\Phi(2y\sqrt{3})-2\Phi(2y/\sqrt{3})]dy+\int_{3/2m}^\infty E[2+\Phi(2y\sqrt{3})\right.$$

$$-3\Phi(2(y-m)\sqrt{3})]\,dy-\frac{c'}{144}[e^{-\frac{1}{2}(3m-a)^2}-(3m-a)\sqrt{2\pi}\,(1-\Phi(3m-a))].$$

Further

$$\bar{S}_2(0)=\frac{2cm}{\pi}\int_0^1\frac{dt}{t^3}\int_{x_2}^{x_1}\frac{Tdx}{(x-m)^2}\int_{x_1}^\infty Y_3dy\left(=\int_{x_1}^\infty Y_3dy\int_{x_2}^{x_1}\frac{T}{(x-m)^2}dx+\int_x^{x_1}\text{''}\int_{x_2}^y\text{''}\right)$$

$$=\bar{S}_{21}+\bar{S}_{22},$$

whose innerst integral is as found above,

$$I(\xi)=\int\frac{Tdx}{(x-m)^2}=\frac{3}{mt}U(\xi)-\frac{x}{m(x-m)}T(\xi),\quad \xi=\sqrt{\frac{\sqrt{3}(x-m)-xt}{\sqrt{3}(x-m)+xt}},$$

which reduces to 0 and $I(0)=0$ for the lower limit $x=x_2=m/(1-t/\sqrt{3})$ but to $\xi=(\sqrt{3}-1)/\sqrt{2}=\omega$, $I(\omega)=\pi/6mt$ for $x=x_1=m/(1-t)$. So that

$$\bar{S}_{21}=2c\int_0^1\frac{dt}{t^4}\int_{x_1}^{\infty}Y_3dy=\frac{c}{3}\int_m^{\infty}Edy\int_x^{\infty}u^2e^{-6u^2}du$$

on changing the order and replacing t by $(y-m)/t=v$. Hence we obtain

$$\bar{S}_{21}=\frac{1}{576}\left[c'e^{-6m^2}+ca\sqrt{2\pi}\,(1-\Phi(4m-a))+16\sqrt{\frac{\pi}{6}}\int_m^{\infty}e^{-2(y-a)^2}(1-(2y\sqrt{3}))dy\right]$$

Lastly

$$\bar{S}_{22}=\frac{2c}{\pi}\int_0^1\frac{dt}{t^4}\int_{x_2}^{x_1}Y_3\left[3U(\eta)-\frac{yt}{y-m}T(\eta)\right]dy$$

$$=\frac{2c}{\pi}\int_m^{m_1}Edy\int_{y/\sqrt{3}}^{y}e^{-6v^2}(3vU(\eta)-yT(\eta))vdv,$$

where $\eta=\sqrt{\dfrac{\sqrt{3}(y-m)-yt}{\sqrt{3}(y-m)+yt}}=\sqrt{\dfrac{\sqrt{3}v-\eta}{\sqrt{3}v+\eta}}$, $v=\dfrac{y-m}{t}$, $m_1=\dfrac{m}{1-1/\sqrt{3}}$.

Or, writing $v=yz$, $\zeta=\sqrt{\dfrac{\sqrt{3}z-1}{\sqrt{3}z+1}}$, we get

$$\bar{S}_{22}=\frac{c}{6\pi}\left[\int_m^{m_1}y^3e^{-2(y-2)^2}dy\int_{1/\sqrt{3}}^{1}e^{-6y^2z^2}[3zU(\zeta)-T(\zeta)]zdz\right.$$
$$\left.+\int_{m_1}^{\infty}ye^{-2(y-a)^2}dy\int_{1-m/y}^{1}e^{-6y^2z^2}(3zU-T)zdz\right].$$

By virtue of all the above formulas we have evaluated[1)]

(20.18) $S(0)$ i.e. the area under the fr. f. $s(t)$ in 1°

species	$S_0(0)$	$S_1(0)+S''_{21}(0)$	$S'_{21}(0)$	$S_{22}(0)$	$S(0)$
(i)	7.1156	−7.5552	0.7285	0.2489	0.5378
(ii)	2.2532	−1.9758	0.1510	0.1070	0.5354
(iii)	2.1292	−23.3348	2.1723	0.4219	0.5508

Therefore, the medians all lie on the negative side, while the modes were oppositely on the positive side. These phenomena were really the case for the truncated Laplace distribution (cf. [II] sect. 2).

1) The author is grateful to M. Watanabe, Institute of Industrial Science, Tokyo University, for his endeavor with which many intricate integrations were carried out by electronic computer.

We now consider the ordinary representation by series in t^{-1} for $S(t)$ to get its approximate values. Although this can be done directly by expanding the foregoing several expressions, the following way may be conveniently utilized. Interchanging the order of xt-integrations in (15) and expanding the integrand in a power series of t^{-1} from the starts, we have for the main term in (14)

(20.19)

$$S_0(t_\alpha)=c\int_{-\infty}^{t_\alpha}\frac{dt}{t^4}\int_0^m(m-x)^3e^{-Q}dx=c\int_0^m(m-x)^3e^{-2(x-a)^2}dx\int_{-\infty}^{t_\alpha}\exp\Big[-\frac{6(m-x)^2}{t^2}\Big]\frac{dt}{t^4}$$

$$=c\sum(-1)^\nu 6^\nu/\nu!\int_0^m e^{-2(a-x)^2}(m-x)^{2\nu+3}dx\int_{-\infty}^{t_\alpha}dt/t^{2\nu+4}$$

$$=c\sum_0^\infty\frac{(-1)^{\nu-1}6^\nu J_{2\nu+3}}{\nu!(2\nu+3)t^{2\nu+3}}=c[-J_3/3t_\alpha^3+6J_5/5t_\alpha^5-18J_7/7t_\alpha^7+\cdots].$$

Since here

$$J_{2\nu+3}=\int_0^m(m-x)^{2\nu+3}e^{-2(a-x)^2}dx\quad(m>x)$$

are clearly positive, the series for $S_0(t_\alpha)$ becomes alternate. Further, expanding the binomial, we get

(20.20) $$J_{2\nu+3}=\sum_0^{2\nu+3}\binom{2\nu+3}{k}\lambda^{2\nu+3-k}j_k\quad\text{where}$$

(20.21) $$j_k=\int_0^m(a-x)^k\exp[-2(a-x)^2]dx.$$

By integrating by parts we obtain a recurrence formula

$$4j_k=(k-1)j_{k-2}+(-\lambda)^{k-1}e^{-2\lambda^2}-a^{k-1}e^{-2a^2}$$

and whence the following formulas, according as k is odd or even,

(20.22) $$j_{2p+1}=\frac{1}{4}\sum_0^p\frac{\lfloor p}{2^q\lfloor p-q}(\lambda^{2p-2q}e^{-2\lambda^2}-a^{2p-2q}e^{-2a^2}),$$

(20.23) $$j_{2p}=\frac{\lfloor 2p}{2^{3p}\lfloor p}\sqrt{\frac{\pi}{2}}[\Phi(2\lambda)-\Phi(-2a)]$$

$$-\frac{1}{4}\sum_0^{p-1}\frac{\lfloor 2p\lfloor p-q}{2^{3q}\lfloor 2p-2q\lfloor p}\times(\lambda^{(2p-2q-1)}e^{-2\lambda^2}+a^{2p-2q-1}e^{-2a^2})$$

For examples we get few even or odd-numbered j_k:

(20.24) $$j_0=\sqrt{\frac{\pi}{2}}[\Phi(2\lambda)-\Phi(-2a)],\qquad j_1=\frac{1}{4}(e^{-2\lambda^2}-e^{-2a^2}),$$

$$j_2=\frac{1}{4}j_0-\frac{1}{4}(\lambda e^{-2\lambda^2}+ae^{-2a^2}),\quad j_3=\frac{1}{4}\Big(\lambda^2+\frac{1}{2}\Big)e^{-2\lambda^2}-\frac{1}{4}\Big(a^2+\frac{1}{2}\Big)e^{-2a^2},$$

$$j_4 = \frac{3}{16} j_0 - \frac{\lambda}{4}\left(\lambda^2 + \frac{3}{4}\right)e^{-2\lambda^2} - \frac{a}{4}\left(a^2 + \frac{3}{4}\right)e^{-2a^2},$$

$$j_5 = \frac{1}{4}\left(\lambda^4 + \lambda^2 + \frac{1}{2}\right)e^{-2\lambda^2} - \frac{1}{4}\left(a^4 + a^2 + \frac{1}{2}\right)e^{-2a^2},$$

$$j_6 = \frac{15}{64} j_0 - \frac{\lambda}{4}\left(\lambda^4 + \frac{5}{4}\lambda^2 + \frac{15}{16}\right)e^{-2\lambda^2} - \frac{a}{4}\left(a^4 + \frac{5}{4}a^2 + \frac{15}{16}\right)e^{-2a^2},$$

$$j_7 = \frac{1}{4}\left(\lambda^6 + \frac{3}{2}\lambda^4 + \frac{3}{2}\lambda^2 + \frac{3}{4}\right)e^{-2\lambda^2} - \frac{1}{4}\left(a^6 + \frac{3}{2}a^4 + \frac{3}{2}a^2 + \frac{3}{4}\right)e^{-2a^2}$$

and so on. So that

(20.25) $$J_3 = \left(\lambda^2 + \frac{3}{4}\right)\lambda\sqrt{\frac{\pi}{2}}\left[\Phi(2\lambda) - \Phi(-2a)\right] + \frac{1}{4}\left(\lambda^2 + \frac{1}{2}\right)e^{2\lambda^2} - \frac{1}{4}\left(3\lambda^2 + 3a\lambda + a^2 + \frac{1}{2}\right)e^{-2a^2},$$

$$J_5 = \left(\lambda^4 + \frac{5}{2}\lambda^2 + \frac{15}{16}\right)\lambda\sqrt{\frac{\pi}{2}}\left[\Phi(2\lambda) - \Phi(-2a)\right] + \frac{1}{4}\left(\lambda^4 + \frac{9}{4}\lambda^2 + \frac{1}{2}\right)e^{-2\lambda^2}$$
$$- \frac{1}{4}\left[5\lambda^4 + 10a\lambda^3 + 10\left(a^2 + \frac{1}{2}\right)\lambda^2 + 5a\left(a^2 + \frac{3}{4}\right)\lambda + a^4 + a^2 + \frac{1}{2}\right]e^{-2a^2}, \text{ \&c.}$$

Thus, the first few terms of $S_0(t_\alpha)$ can be found. Next, as to the first correction

(20.26) $$S_1(t_\alpha) = c\int_{-\infty}^{t_\alpha}\frac{dt}{t^4}\int_{1/\sqrt{3}}^{\sqrt{3}} G\left(\bar{x} = \frac{m}{1 - t\tau/\sqrt{3}},\ t\right)\frac{2d\tau}{\sqrt{3}\tau^2}$$

where $$G(\bar{x}) = \int_0^{\bar{x}}(m - x)^3 \exp\left[-\frac{6(x-m)^2}{t^2} - 2(x-a)^2\right],$$

in which however t_α is not given, we should somehow estimate the integrand $G(\bar{x})$. When $|t_\alpha|$ is a pretty large, those terms in the integrand which are e.g. of higher order than t^{-2} may be neglected and it can be approximated by

(20.27) $$G(\bar{x}) \cong \int_0^{\bar{x}}(m^3 - 3m^2x + 3mx^2)(1 + 6m^2/t^2)e^{-2a^2}\{1 + 4ax + (8a^2 - 2)x^2\}\,dx$$
$$\cong m^4 e^{-2a^2}\left(1 - \frac{6}{t^2}\right)\left(\frac{-\sqrt{3}}{t\tau}\right)\left\{1 + \frac{\sqrt{3}}{t^2}(5 - 4am) + \frac{M}{t^2\tau^2}\right\},$$

where $M = 15 - 24am - (2 - 8a^2)m^2$. This being substituted in (26) and integrated, yields

(20.28) $$S_1(t_\alpha) \cong cm^4\left[2/t_\alpha^4 + 26(1 - 0.8am)/9t_\alpha^5 + (20M - 72m)/27t_\alpha^6\right].$$

Similarly treated about the second correction, we obtain

(20.29) $S_2(t_\alpha) \cong S_1(t_\alpha) - cm^4 e^{-2a^2}\{3M/4\pi t_\alpha^4 + (1-0.8am)(3+4\sqrt{3}\pi)/2t_\alpha^5$

$+[(2/\sqrt{6}\tan\sqrt{2}+2\sqrt{3})\cdot M - 72\sqrt{6}\tan^{-1}\sqrt{2}\cdot m^2]/24\pi t_\alpha^6\}.$

Consequently $S(t_\alpha)$ is approximately determined in the form

(20.30) $S(t_\alpha) = S_0 - S_1 + S_2 \cong A/t_\alpha^3 + B/t_\alpha^4 + C/t_\alpha^5 + D/t_\alpha^6,$

which equated to $\alpha/2$ (e.g. $\alpha = 0.1, 0.05, 0.01$) and putting $x = -1/t_\alpha$, we get

(20.31) $f(x) = a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 - \alpha/2 = 0.$

And really

(i) $13.2313x^3 - 95.867x^4 + 425.70x^5 - 1399.6x^6 - \alpha/2 = 0,$

(ii) $6.5224x^3 - 27.382x^4 + 34.87x^5 + 15.95x^6 - \alpha/2 = 0,$

(iii) $20.876x^3 - 193.04x^4 + 1169.2x^5 - 5811x^6 - \alpha/2 = 0.$

They solved by Horner, the lower critical values $t_\alpha = -1/x$ may be found. However, in case that $a_6 < 0$ with relatively large absolute value, $f(x)$ becomes decreasing after all, though increasing at the start because of $a_3 = cJ_3/3 > 0$. So that if $|a_6|$ be large and max $f(x) < 0$, the equation cannot have any positive root. Surely, since $a_3x^3 > 0$, if this term alone taken, the equation gives some rough root, but we wish a little more. Really among 3 cases the coefficient $a_6 > 0$ in (ii) and it provides all roots for 3 values of α. However, in cases of (i) and (iii) it is $a_6 < 0$ and their equations have no positive root for $\alpha = 0.1$ or 0.05, though they will do still for $\alpha = 0.01$. Hence, in these cases we have rejected the term a_6x^6 and began with a_5x^5 and contented with the less exact roots. Of course, if we had calculated some more terms, say up to $a_{10}x^{10}$, we should have obtained still adequate roots, instead of which, however, it might be corrected by a manner described later on about upper critical points. At any rate, the following results were obtained:

(20.32) the lower critical values for T.N.D. ($n = 4$)

species	term taken up to	level $\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
(i) truncated at the centroid	a_6	/	/	−11.23
	a_5	−5.739	−6.909	−11.77
(ii) 〃 Left quartile	a_6	−4.322	−5.422	−9.436
(iii) 〃 right quartile	a_6	/	/	−12.62
	a_5	−7.035	−8.346	−13.76
untruncated N.D.		−2.352	−3.182	−5.841

We now proceed to the case $t>0$, $\bar{x}>m$. Firstly considering the main constituent $s_0(t)$ which serves the full fr. f. in 5° $3<t<\infty$, it is after (19.27)

$$s_0(t) = -cG(m, t) = c\int_m^\infty (x-m)^3 e^{-Q(x)}dx/t^4, \tag{20.33}$$

where $Q(x)=Q(x, t)=6(x-m)^2/t^2+2(x-a)^2=2R(x-m)^2+4\lambda(x-m)^2+2\lambda^2$, $R=1+3/t^2$, as before. We retake Lemma A, however somewhat modified

Lemma B. If for $\nu=0, 1, 2, \cdots$

$$J_\nu \equiv \int_{a(t)}^\infty (x-m)^\nu e^{-Q(x)}dx \quad \text{and} \quad K_\nu \equiv (a(t)-m)^\nu e^{-Q(a)},$$

we obtain again a recurrence formula

$$4RJ_{\nu+1} = K_\nu + \nu J_{\nu-1} - 4\lambda J_\nu \tag{20.34}$$

and whence

$$J_1=(K_0-4\lambda J_0)/4R,\ J_2=[RK_1-\lambda K_0+(1+\mu^2)RJ_0]/4R^2,$$

$$J_3=[4RK_2-4\lambda K_1+(2+\mu^2)K_0-(3+\mu^2)4\lambda J_0]/16R^2, \text{\&c.},$$

where

$$4\lambda J_0 = e^{-2\lambda^2}\mu[1-\Phi(\mu+2(a-m)\sqrt{R})]/\varphi(\mu),\quad \mu=2\lambda/\sqrt{R}.$$

Here putting $a=m$, $\nu=3$, we have

$$s_0(t) = c'\{2+\mu^2-(3+\mu^2)\mu[1-\Phi(\mu)]/\varphi(\mu)\}/16R^2t^4, \tag{20.35}$$

so that $s_0(+0)=c'/72$, coincident with $s_0(-0)=c_0$ of (4.0).

Secondly, when $t<3$, there comes the first correction $s_1(t)$ for which (19.28) is fitting in the combined intervals 3°, 4°: $1<t<3$, so that, as in (5),

$$-s_1(t) = \frac{2cm}{3t^3}\int_{x_3}^\infty \frac{dx}{(x-m)^2}\int_x^\infty (y-m)^3 e^{-Q(y)}dy = \frac{2c}{3t^4}[(3-t)J_3 - mtJ_2],$$

where $J_\nu = \int_{x_3}^\infty (y-m)^\nu e^{-Q(y)}dy$, $x_3=m/(1-t/3)$. Here Lemma B twice applied, yields

$$s_1(t) = \frac{-c'}{24R^2t^4}\Big\{(3-t)(2+\mu^2)\exp\Big[-\frac{4\lambda mt}{3-t}-\frac{2Rm^2t^2}{(3-t)^2}\Big] \tag{20.36}$$

$$-[(3-t)(3+\mu^2)\mu+2mt\sqrt{R}(1+\mu^2)][1-\Phi(\mu+2mt\sqrt{R}/(3-t))]/\varphi(\mu)\Big\}.$$

However, when 2° $0<t<1$, we must after (19.35) take x_1 instead of ∞, as the upper limit:

$$-s_1(t)=\frac{2cm}{3t^3}\int_{x_3}^{x_1}\frac{dx}{(x-m)^2}\int_x^\infty(y-m)^3e^{-Q(y)}dy,\quad x_1=\frac{m}{1-t},\quad x_3=\frac{m}{1-t/3},\ \text{so that}$$

$$=\frac{2c}{3t^4}\int_{x_3}^{x_1}(y-m)^3e^{-Q}dy-mt\int_{x_3}^{x_1}(y-m)^2e^{-Q}dy+2m\int_{x_3}^{\infty}(y-m)^3e^{-Q}dy.$$

Hence, we require the third

Lemma C. When $H_\nu\equiv\int_{a(t)}^{b(t)}(x-m)^\nu e^{-Q(x)}dx,\ K_\nu\equiv(b-m)^\nu e^{-Q(b)}-(a-m)^\nu e^{-Q(a)}$,

we have $$4RH_{\nu+1}=-K_\nu+\nu H_{\nu-1}-4\lambda H_\nu$$

just alike B, except that K_ν's signs in (34) shall be here changed; naturally the expression for K_ν itself is alike that in Lemma A, however whose $m-b$, $m-a$ are now taken as $b-m$, $a-m$, but $4\lambda H_0$ is just the same with $4\lambda I_0$, as it stands, namely

$$4\lambda H_0=e^{-2\lambda^2}\mu[\Phi(\mu+2(b-m)\sqrt{R})-\Phi(\mu+2(a-m)\sqrt{R})]/\varphi(\mu).$$

However, on making use of Lemma B and C to above $-s_1(t)$, we get quite the same expression as (5), so that $s_1(+0)=s_1(-0)$ holds also.

Thirdly, in the subinterval 3° $1<t<\sqrt{3}$, there comes the second correction $s_2(t)$ that is after (19.30.1)

$$s_2(t)=\frac{2\sqrt{3}c}{\pi t^4}\int_1^{\sqrt{3}/t}\frac{1}{\tau^2}\tan^{-1}\sqrt{\frac{3}{2}(\tau^2-1)}\,J_3(\tau)d\tau,$$

where the inner factor is

$$J_3=\int_{\bar{x}}^\infty(y-m)^3e^{-Q(y)}dy\ \text{ with }\ \bar{x}=\frac{m}{1-t\tau/\sqrt{3}}.$$

Hence, applying Lemma C, B into J_3 and consulting Gauss, we obtain

(20.37)

$$s_2(t)=\frac{(3/t-\sqrt{3})c'}{8\pi R^2t^4}\sum\frac{A_\nu}{\tau_\nu^2}\tan^{-1}\sqrt{\frac{3}{2}(\tau_\nu^2-1)}\left\{\left(2+\mu^2-\frac{4\lambda mt\tau_\nu}{\sqrt{3}-t\tau_\nu}+\frac{4Rm^2t^2\tau_\nu^2}{(\sqrt{3}-t\tau_\nu)^2}\right)\times\right.$$

$$\left.\times\exp\left[-\frac{4mt\tau_\nu}{\sqrt{3}-t\tau_\nu}-\frac{2Rm^2t^2\tau_\nu^2}{(\sqrt{3}-t\tau_\nu)^2}\right]-(3+\mu^2)\mu\left[1-\Phi\left(\mu+\frac{2mt\sqrt{R}\tau_\nu}{\sqrt{3}-t\tau_\nu}\right)\right]\Big/\varphi(\mu)\right\},$$

where $\tau_\nu=\frac{1}{2}(\sqrt{3}/t+1)+\frac{1}{2}(\sqrt{3}/t-1)\xi_\nu$ and A_ν, ξ_ν denote Gaussian constants.

Finally, when 2° $0<t<1$, in view of (19.35.1), the above upper limit of τ must be replaced by $\sqrt{3}$ instead of $\sqrt{3}/t$, so that in (37) also τ_ν by $\frac{1}{2}(\sqrt{3}+1)+\frac{1}{2}(\sqrt{3}-1)\xi_\nu$ and the heading factor $(3/t-\sqrt{3})$ by $(3-\sqrt{3})$, and accordingly $s_2(t)$ is continuous at $t=1$. Besides, when $t\to+0$, (37) tends

$$s_2(+0)=\frac{(\sqrt{3}-1)c'}{6\sqrt{6\pi}}A_\nu\tan^{-1}\sqrt{\frac{3}{2}(\tau_\nu^2-1)}\left(2m^2+\frac{1}{\tau_\nu^2}\right)\varphi(2m\tau_\nu)=s_2(-0),\ \text{(cf. (6.0))}.$$

Consequently the full fr. f. $s(t)$ is continuous and truly derivable at the origin $t=0$.

Evaluating s_0, s_1, s_2 when $t=3$, $\sqrt{3}$, 1 and $s=s_0+s_1+s_2$ for 3 species, we get

(20.38) the values of $s(t)$ $(t>0)$

species	$s(3)$	$s(\sqrt{3})$	$s(1)$
(i)	0.0113	0.0644	0.2074
(ii)	0.0144	0.0695	0.2052
(iii)	0.0093	0.0596	0.2085

Finally the complementary distribution function (c.d.f.):

(20.39) $$\bar{S}(t_\alpha)=\int_{t_\alpha}^{\infty}s(t)\,dt=1-S(t_\alpha)=\int_{t_\alpha}^{\infty}s_0dt+\int_{t_\alpha}^{3}s_1dt+\int_{t_\alpha}^{\sqrt{3}}s_2dt$$
$$=\bar{S}_0(t_\alpha)+\bar{S}_1(t_\alpha)+\bar{S}_2(t_\alpha)$$

can be evaluated by integrating (35) (36) (37) in turns. However, these formulas being somewhat heavy, we try to make a new start.

The full c.d.f. $\bar{S}(t_\alpha)$ in 5° $3<t_\alpha<\infty$, which also composes the principal part $\bar{S}_0(t_\alpha)$ in 2°–4° is after (19.39) (19.9.2)

(20.40) $$\bar{S}_0(t_\alpha)=c\int_{t_\alpha}^{\infty}\frac{G(\infty)-G(m)}{t^4}dt=c\int_{t_\alpha}^{\infty}\frac{dt}{t^4}\int_m^{\infty}(x-m)^3e^{-Q(x)}dx.$$

The double integral can readily be reduced to a single one: On writing $2\lambda/\sqrt{R}=\mu$ and $2\sqrt{R}(x-m)+\mu=z$, the inner integral becomes

$$\int_m^{\infty}(x-m)^3e^{-Q}dx=\frac{e^{-2\lambda^2}}{16R^2\varphi(\mu)}\frac{1}{\sqrt{2\pi}}\int_\mu^{\infty}(z-\mu)^{\nu}e^{-z^2/2}dz.$$
$$=\frac{e^{-2\lambda^2}}{16\sqrt{2\pi}R^2}\{2+\mu^2-(3+\mu^2)\mu[1-\Phi(\mu)]/\varphi(\mu)\}.$$

Further the variable t replaced by $R=1+3/t^2$, or its reciprocal $1/R=\xi$ gives

(20.41)

$$\bar{S}_0(t_\alpha)=\frac{c'}{48\sqrt{3}}\int_{\xi_\alpha}^{1}\{1+2\lambda^2\xi-\lambda\sqrt{\xi}(3+4\lambda^2\xi)[1-\Phi(2\lambda\sqrt{\xi})]/\varphi(2\lambda\sqrt{\xi})\}\sqrt{(1-\xi)/\xi}\,d\xi$$
$=\bar{S}_{01}-\bar{S}_{02}$ say, where $\xi_\alpha=1/R_\alpha=1/(1+3/t_\alpha^2)$.

The first part $\bar{S}_{01}$ can immediately be integrated: Since

$$\int_{\xi_\alpha}^{1}(1+2\lambda^2\xi)\sqrt{\frac{1-\xi}{\xi}}d\xi=2\int_0^{u_\alpha}\left(\frac{1}{1+u^2}+\frac{2\lambda^2-1}{(1+u^2)^2}-\frac{2\lambda^2}{(1+u^2)^3}\right)du,$$

if $u=\sqrt{\dfrac{1-\xi}{\xi}}$, $u_\alpha=\dfrac{\sqrt{3}}{t_\alpha}$, when the integration performed, yields

$$(20.41.1)\qquad \bar{S}_{01}=\frac{c'}{48\sqrt{3}}\left\{\left(1+\frac{\lambda^2}{2}\right)\tan^{-1}\frac{\sqrt{3}}{t_\alpha}-\left(1+\frac{\lambda^2}{2}\frac{t_\alpha^2-3}{t_\alpha^2+3}\right)\frac{\sqrt{3}\,t_\alpha}{t_\alpha^2+3}\right\}.$$

As to the second part

$$(20.41.2)\qquad \bar{S}_{02}=\frac{c'\lambda}{48\sqrt{3}}\int_{\xi_\alpha}^{1}(3+4\lambda^2\xi)\cdot\frac{1-\Phi(2\lambda\sqrt{\xi})}{\varphi(2\lambda\sqrt{\xi})}\cdot\sqrt{1-\xi}\,d\xi,$$

we need Gauss' method of numerical computation. Thus we obtain

(20.42) the values of $\bar{S}_0(t_\alpha)$

species	$\bar{S}_0(3)$	$\bar{S}_0(\sqrt{3})$	$\bar{S}_0(1)$
(i)	0.0128	0.0498	0.1453
(ii)	0.0168	0.0602	0.1547
(iii)	0.0103	0.0432	0.1422

However, for $0<t<3$, it requires corrections: In the combined intervals 3° 4°: $1<t<3$ after (19.28) and (19.1) there comes the first correction

$$\bar{S}_1(t_\alpha)=c\int_{t_\alpha}^{3}\frac{dt}{t^4}\int_{x_3}^{\infty}G(x)\mathfrak{h}_1'\frac{\partial\tau}{\partial x}dx=-\frac{2cm}{3}\int_{t_\alpha}^{3}\frac{dt}{t^4}\int_{x_3}^{\infty}\frac{dx}{(x-m)^2}\int_{x}^{\infty}(y-m)^3e^{-Q(y)}dy.$$

To simplify the triple integral, interchange the order of integrations, first about x and y (and integrate about x) then about y and t (Fig. 10) and lastly replacing t by $z=(y-m)/t$, the triple integral becomes a double one

$$-\frac{2c}{3}\int_{y_\alpha}^{\infty}e^{-2(y-a)^2}dy\int_{z_0}^{z_1}(3z-y)ze^{-6z^2}dz,$$

where $\quad y_\alpha=3m/(3-t_\alpha),\ z_0=y/3,\ z_1=(y-m)/t_\alpha.$

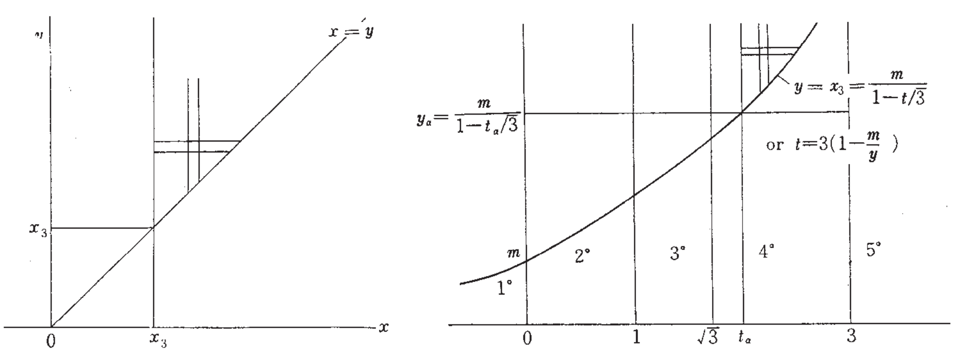

Fig. 10

But the new inner integral after integrating by parts reduces to

$$\frac{1}{12}(y-3(y-m)/t_\alpha)\exp\left[-6(y-m)^2/t_\alpha^2\right]+\frac{1}{4}\int_{z_0}^{z_1}e^{-6z^2}dz.$$

So that $\bar{S}_1(t_\alpha)$ reduces to a difference of two components as follows:

(20.43) $$\bar{S}_1(t_\alpha)=\frac{c}{18}\int_{y_\alpha}^{\infty}\left(\frac{(3-t_\alpha)(y-m)}{t_\alpha}-m\right)\exp\left[-Q(y,t_\alpha)\right]dy$$

$$-\frac{c}{6}\int_{y}^{\infty}e^{-2(y-a)^2}dy\int_{z_0}^{z_1}e^{-6^2z}dz$$

$$=\bar{S}_{11}-\bar{S}_{12},\ \text{say.}$$

Firstly, integrating $\bar{S}_{11}$ by parts, we get

(20.43.1) $$\bar{S}_{11}=\frac{c_0}{\varphi(\mu)R_\alpha t_\alpha}\{(3-t_\alpha)\varphi(w_0)-[(3-t_\alpha)\mu+2mt_\alpha\sqrt{R_\alpha}]\,(1-\Phi(w_0))\}$$

where $$w_0=\mu+\frac{2m\sqrt{R_\alpha}}{3-t_\alpha},\quad \mu=\frac{2\lambda}{\sqrt{R_\alpha}}.$$

Next, for $\bar{S}_{12}$ transforming y, z into $\eta=2(y-a)$, $\zeta=2\sqrt{3z}$ and putting $\eta_0=2(y_\alpha-a)=(6\lambda+2at_\alpha)/(3-t_\alpha)$, $\zeta_0=2y/\sqrt{3}=(\eta+2a)/\sqrt{3}$, $\zeta_1=2\sqrt{3}(y-m)/t_\alpha=\sqrt{3}(\eta-2\lambda)/t_\alpha$, we obtain

(20.43.2) $$\bar{S}_{12}=\frac{c}{24\sqrt{3}}\int_{\eta_0}^{\infty}e^{-1/2\eta^2}d\eta\int_{\zeta_0}^{\zeta_1}e^{-1/2\zeta^2}d\zeta$$

$$=\frac{c}{24\sqrt{3}}\int_{\theta_0}^{\theta_{1/2}}d\theta\int_{\rho_0}^{\infty}e^{-\rho^2/2}d\rho+\int_{\theta_{1/2}}^{\theta_1}\text{''}\int_{\rho_1}^{\infty}\text{''}$$

$$=\frac{c}{24\sqrt{3}}\left[\int_{\theta_0}^{\theta_{1/2}}\exp\left(\frac{-a^2}{1-\cos 2(\theta-\theta_0)}\right)d\theta+\int_{\theta_{1/2}}^{\theta_1}\exp\left(\frac{-b^2}{1-\cos 2(\theta_1-\theta)}\right)d\theta\right],$$

where

$$b^2=12\lambda^2/(3+t_\alpha^2),\quad \theta_0=\pi/6,\quad \theta_{1/2}=\tan^{-1}\sqrt{3}\,m/(3\lambda+at_\alpha),\quad \theta_1=\tan^{-1}\sqrt{3}/t_\alpha.$$

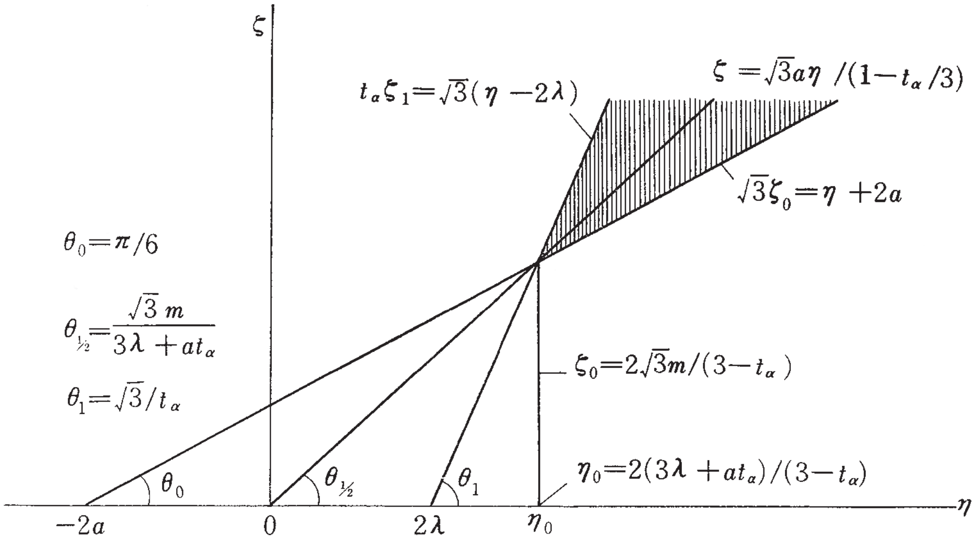

Fig. 11

Thus $\bar{S}_{12}$ being numerically integrated and subtracted from $\bar{S}_{11}$, we find $\bar{S}_1$ and then $\bar{S}$ by taking account of Table (42) and also (45) below:

(20.44) the values of the first correction $\bar{S}_1(t_\alpha)$ as well as $\bar{S}(t_\alpha)$

species	$\bar{S}_{11}(\sqrt{3})$	$\bar{S}_{12}(\sqrt{3})$	$\bar{S}_1(\sqrt{3})$	$\bar{S}(\sqrt{3})$	$\bar{S}_{11}(1)$	$\bar{S}_{12}(1)$	$\bar{S}_1(1)$	$\bar{S}_2(1)$	$\bar{S}(1)$
(i)	0.0^51	-0.0^51	-0	.0498	.0079	−.0123	−.0044	+0	.1409
(ii)	0.0^53	-0.0^64	-0	.0602	.0015	−.0026	−.0011	+0	.1536
(iii)	0.0^31	-0.0^49	-0	.0431	.0234	−.0329	−.0095	+0	.1327

Lastly for 3° $1<t<\sqrt{3}$ we must still compute the second correction, that is after (19.41)

$$\bar{S}_2(t_\alpha)=c\int_{t\alpha}^{\sqrt{3}}dt/t^4\int_1^{\sqrt{3}/t}\mathfrak{h}_2'\,d\tau\int_{y_0}^{\infty}(y-m)^3e^{-Q(y,t)}dy,$$

where $\qquad y_0=m/(1-t\tau/\sqrt{3})$, and $\mathfrak{h}_2'=\dfrac{2\sqrt{3}}{\pi\tau^2}\tan^{-1}\sqrt{\dfrac{3}{2}(\tau^2-1)}$.

Interchanging the order of integration about t, τ (Fig. 12), and replacing y and y_0 by $z=(y-m)/t$, and $z_0=m\tau/(\sqrt{3}-t\tau)$, then again interchanging the order about t, z, we obtain

$$\bar{S}_2(t_\alpha)=\frac{2\sqrt{3}\,c}{\pi}\int_1^{\sqrt{3}/t\alpha}\tan^{-1}\sqrt{\frac{3}{2}(\tau^2-1)}\,\frac{d\tau}{\tau^2}\int_{z_\alpha}^{\infty}z^3e^{-6z^2}dz\int_{t_\alpha}^{t_1}\exp\left[-2(zt+\lambda)^2\right]dt$$

where $\qquad z_\alpha=m\tau/(\sqrt{3}-t_\alpha\tau)$, $t_1=\sqrt{3}/\tau-m/z$.

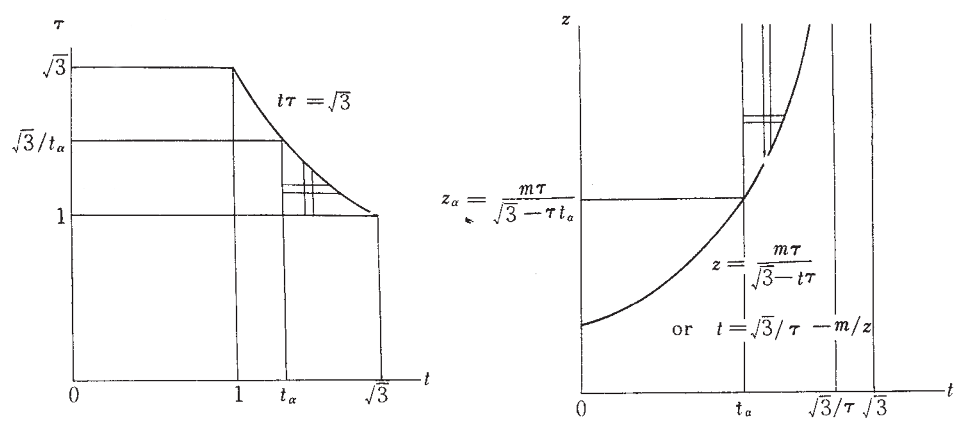

Fig. 12

Or, to facilitate Gaussian integration, making $u=\sqrt{3}/\tau$, $v=1/z$, $v_\alpha=1/z_\alpha$, we get

(20.45) $$\bar{S}_2(t_\alpha)=c\sqrt{\frac{2}{\pi}}\int_{t_\alpha}^{\sqrt{3}}\tan^{-1}\sqrt{\frac{3}{2}\left(\frac{3}{u^2}-1\right)}du$$

$$\times\int_0^{v_\alpha}e^{-6/v^2}\left[\Phi\left(\frac{2u}{v}-2a\right)-\Phi\left(\frac{2t_\alpha}{v}+2\lambda\right)\right]\frac{dv}{v^4}$$

$$=(\sqrt{3}-t_\alpha)\frac{c}{m}\sqrt{\frac{2}{\pi}}\sum A_i(u_i-t_\alpha)\tan^{-1}\sqrt{\frac{3}{2}\left(\frac{3}{u_i^2}-1\right)}$$

$$\times\sum\frac{B_j}{v_{ij}^4}\exp\left(\frac{-6}{v_{ij}^2}\right)\left[\Phi\left(\frac{2u_i}{v_{ij}}-2a\right)-\Phi\left(\frac{2t_\alpha}{v_{ij}}+2\lambda\right)\right],$$

where $u_i=\frac{1}{2}(\sqrt{3}+t_\alpha)+\frac{1}{2}(\sqrt{3}-t_\alpha)\xi_i,\quad v_{ij}=\frac{1}{2}(u_i-t_\alpha)/m[1+\eta_j]$,
and ξ_i, η_j, A_i, B_j are Gaussian constants.

In this way we may compute $S_2(t_\alpha)$ in general. However, for the present case with such small value as $n=4$, Φ's in the above expression approach to 1 sufficiently, so that the values of $S_2(1)$ become extremely small and the second corrections are really negligible Therefore, summing up all the above, we get

(20.46) the area under $s(t)$ in each subinterval

species	$P_5=\bar{S}(3)$	$P_4=\bar{S}(\sqrt{3})-P_5$	$P_3=\bar{S}(1)-P_4-P_5$	$P_2=1-P_1-P_3-P_4-P_5$	P_1
(i)	0.0128	0.0370	0.0911	0.3213	0.5378
(ii)	0.0168	0.0434	0.0934	0.3110	0.5354
(iii)	0.0103	0.0328	0·0896	0.3165	0.5508

Before closing this note we wish still to find the upper critical points t_α. These points with the significant level α (e.g. $\alpha=0.01, 0.05, 0.1$, &c.) are obtainable from the equation $\bar{S}(t_\alpha)=\alpha/2$. Thereupon, in view of Tables (42) (46), we can predict that the upper 1% point $t_{0.01}$ (or to be denoted by $t_{0.005}$?) lies certainly in the subintervals 5° $3<t<\infty$ and the 5% point $t_{0.05}$ in the vicinity of the middle of 4° $t=\frac{1}{2}(3+\sqrt{3})=2.366$ and the 10% point $t_{0.1}$ shall be about coincident with $t=\sqrt{3}$. On the otherhand, in these domains the contributions from the corrections being small enough, we have only to consider the main d.f. $\bar{S}_0(t)$ solely and to determine the critical points from the equation $\bar{S}_0(t)=\alpha/2$. But, for this purpose we must know the first few terms in the t^{-1} series of $\bar{S}_0(t)$. Just similarly to (15)—(21), we have

(20.47) $$\bar{S}_0(t)=c\int_m^\infty(x-m)^3e^{-2(x-a)^2}dx\int_t^\infty\exp\left[-6(x-m)^2/t^2\right]dt/t^4$$

$$=\quad ''\quad ''\quad\int_t^\infty\sum(-6(x-m)^2)^\nu dt/t^{2\nu+4}$$

$$=c\sum\frac{(-1)^\nu 6^\nu}{(2\nu+3)t^{2\nu+3}}\int_m^\infty(x-m)^{2\nu+3}e^{-2(x-a)^2}dx$$

$$=c\{J_3/3t^3-6J_5/5t^5+18J_7/7t^7-4J_9/t^9+\cdots\},$$

where in virtue of $m=a+\lambda$

(20.48) $$J_h=\int_m^\infty (x-a-\lambda)^h e^{-2(x-a)^2}dx=\sum_0^h \binom{h}{k}(-\lambda)^{h-k}j_k \quad \text{with}$$

(20.49) $$j_k=\int_m^\infty (x-a)^k e^{-2(x-a)^2}dx,$$

which has a recurrence formula and whence we get

(20.50) $$j_{2p+1}=\frac{1}{4}e^{-2\lambda^2}\sum_{q=0}^{p}\underline{|p}\,\lambda^{2p-2q}/2^q\,\underline{|p-q}, \quad j_1=\frac{1}{4}e^{-2\lambda^2},$$

(20.51) $$j_{2p}=\frac{1}{4}e^{-2\lambda^2}\sum_{q=0}^{p-1}\frac{\underline{|p-q}\,\underline{|2p}}{2^{3q}\,\underline{|p}\,\underline{|2p-2q}}\lambda^{2p-2q-1}+\frac{\underline{|2p}}{2^{3p}\,\underline{|p}}j_0, \quad j_0=\sqrt{\frac{\pi}{2}}[1-\Phi(2\lambda)].$$

For examples, the odd numbered j are

$$j_1=\frac{1}{4}e^{-2\lambda^2}, \quad j_3=\left(\lambda^2+\frac{1}{2}\right)j_1, \quad j_5=\left(\lambda^4+\lambda^2+\frac{1}{2}\right)j_1,$$

$$j_7=\left(\lambda^6+\frac{3}{2}\lambda^4+\frac{3}{2}\lambda^2+\frac{3}{4}\right)j_1,\cdots,$$ while the even numbered j are

$$j_0=\sqrt{\frac{\pi}{2}}[1-\Phi(2\lambda)], \quad j_2=\lambda j_1+\frac{1}{4}j_0, \quad j_4=\left(\lambda^3+\frac{3}{4}\lambda\right)j_1+\frac{3}{4}j_0,$$

$$j_6=\left(\lambda^5+\frac{5}{4}\lambda^3+\frac{15}{16}\lambda\right)j_1+\frac{15}{64}j_0,$$

$$j_8=\left(\lambda^7+\frac{7}{4}\lambda^5+\frac{35}{16}\lambda^3+\frac{105}{64}\lambda\right)j_1+\frac{105}{256}j_0, \quad \&c.$$

On substituting these in (48) (47), we obtain

(20.52) $$\bar{S}_0(t)=c'\left\{\frac{1}{12t^3}\left(\lambda^2+\frac{1}{2}\right)-\frac{3}{10t^5}\left(\lambda^4+\frac{9}{4}\lambda^2+\frac{1}{2}\right)\right.$$
$$\left.+\frac{9}{14t^7}\left(\lambda^6+5\lambda^4+\frac{87}{16}\lambda^2+\frac{3}{4}\right)-\frac{1}{t^9}\left(\lambda^8+\frac{35}{4}\lambda^6+\frac{345}{16}\lambda^4+\frac{975}{64}\lambda^2+\frac{3}{2}\right)\right\}$$
$$-c[1-\Phi(2\lambda)]\sqrt{\frac{\pi}{2}}\lambda\left\{\frac{1}{t^3}\left(\frac{\lambda^2}{3}+\frac{1}{4}\right)-\frac{1}{t^5}\left(\frac{6}{5}\lambda^4+3\lambda^2+\frac{9}{8}\right)\right.$$
$$\left.+\frac{9}{t^7}\left(\frac{2}{7}\lambda^6+\frac{3}{2}\lambda^4+\frac{15}{8}\lambda^2+\frac{15}{32}\right)-\frac{1}{t^9}\left(4\lambda^8+36\lambda^6+\frac{189}{2}\lambda^4+\frac{315}{4}\lambda^2+\frac{945}{64}\right)\right\},$$

which being equated to $\alpha/2$ and putting $1/t=x$, the resulting equation

(20.53) $$f(x)\equiv\bar{S}_0(1/x)-\alpha/2=0$$

is the required one. Thus we obtain the equations for 3 species:

$$\text{(i)}\quad 0.409316x^3 - 0.714228x^5 + 1.23649x^7 - 2.19038x^9 - \alpha/2 = 0,$$

$$\text{(ii)}\quad 0.577379x^3 - 1.40478x^5 + 2.07120x^7 - 7.58082x^9 - \alpha/2 = 0,$$

$$\text{(iii)}\quad 0.304134x^3 - 0.329924x^5 + 0.16015x^7 - 1.5879x^9 - \alpha/2 = 0.$$

Solving them and taking again the reciprocals of the roots, $1/x = t_0$, we obtain

(20.54) The upper critical pts. for $n=4$ found from $\bar{S}_0(t_0) = \alpha/2$:

species \ level	(A) $\alpha = 0.1$	(B) $\alpha = 0.05$	(C) $\alpha = 0.01$
(i) truncated at the centroid	1.725	2.295	4.208
(ii) " right quartile	1.874	2.443	4.967
(iii) " left "	1.741	2.132	3.840
untruncated normal distribution	2.353	3.182	5.841

The C-class may be adopted as it is, since they are all>3. But, for those in B lying in 4° $\sqrt{3} < t < 3$, we are to investigate how much the effect of the first correction $\bar{S}_1$ does change the figures of the decimal place. By computing $\bar{S}_1(t_0)$ after (43) as well as $s(t_0)$ after (35) (36), we obtain

(i) $\bar{S}_1(2.295) = 0.0^{14}26$ (ii) $\bar{S}_1(2.443) = 0.0^{37}15$ (iii) $\bar{S}_1(2.132) = 0.0^{7}88$,
$s_0(2.295) = 0.0276$, $s(2.443) = 0.0275$, $s(2.132) = 0.0314$.

Writing $\bar{S}_1(t_0) = -\delta$ (deviation), $s(t_0) = \eta$ (the ordinate of the fr. f. at t_0) and the true $t_{0.05} = t_0 - \varepsilon$ ($-\varepsilon$, the correction), we have approximately $\delta = \eta\varepsilon$, so that $\varepsilon = \delta/\eta$. Hence, we find the order of correction in the B-class to be

$$\varepsilon \text{ in (i)} = 0.0^{12}1, \qquad \text{(ii)} = 0.0^{36}5, \qquad \text{(iii)} = 0.0^{5}3, \text{ respectively,}$$

which show that the values found from our equations shall make shift, as it stands.

Lastly for the 10% points $\gtreqless 0$ $t_{0.1}$ we devise the following heuristic method: we calculate the deviations $\delta = \bar{S}(\sqrt{3}) - 0.5$ and the ordinates $s(\sqrt{3})$ after (44) and (38). Then the corrections to $t_0 = \sqrt{3}$ are $\varepsilon = \delta/\eta (\gtreqless 0)$. In this way we find

$$\text{the 10\% point} = \sqrt{3} + \varepsilon = 1.727 \text{ in (i)}, 1.876 \text{ in (ii)}, 1.618 \text{ in (iii)},$$

which agree pretty well with our roots, except the last one. The deviations are rather to be expected, as the result that the points of A-class being somewhat remote from $t = 3$, our equations, which were simply constructed from $\bar{S}_0(t_\alpha) = \alpha/2$, become naturally insufficient.

Journal of Gakugei, Tokushima University
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A RELATION BETWEEN HANKEL AND HARDY TRANSFORMS

By

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(Received September 30, 1963)

1. A generalization of Hankel transform is due to Hardy [3] who gave the following formula:

$$g(x)=\int_0^\infty u\cdot f(u)\cdot G_\nu(ux)\cdot dx \tag{1}$$

where

$$f(u)=\int_0^\infty x\cdot g(x)\cdot F_\nu(ux)\cdot dx, \tag{2}$$

and

$$\begin{aligned} G_\nu(x) &= \cos(a\pi)\cdot J_\nu(x)+\sin(a\pi)\cdot Y_\nu(x) \\ &= \operatorname{cosec}(\pi\nu)[\sin\{(a+\nu)\pi\}\cdot J_\nu(x)-\sin(a\pi)\cdot J_{-\nu}(x)], \end{aligned} \tag{3}$$

$$F_\nu(x)=\frac{(x/2)^{2a+\nu}}{\Gamma(a+1)\cdot\Gamma(\nu+a+1)}\,{}_1F_2[1;\ a+1,\ \nu+a+1,\ -x^2/4]. \tag{4}$$

This formula is valid under the following conditions given by Cooke [1]:

i) $a>-1,\ a+\nu>-1,\ \nu+2a<3/2,\ |\nu|<3/2,$

ii) $t^\sigma g(t)$ is integrable over $(0,\delta)$, $\sigma=\min(2a+\nu+1, 1/2)$,

iii) $t^{1/2}\cdot g(t)$ is integrable over (δ,∞), $\delta>0$.

At another place [4], we have obtained a relation between Hankel transforms of different order. The object of this note is to obtain a relation between Hankel and Hardy transforms. The result of [4] is obtained as a particular case of this result by taking $a=0$.

The result in this note is based on the following integrals which are special cases of Weber-Schafheiltin integral. The results in question, which are easily derived from the more general ones given by Watson [5], are

i) if m is zero or a positive integer, $\nu>-1-m$ and $k>0$, then

$$\int_0^\infty y^{1-k}\cdot J_\nu(xy)\cdot J_{k+2m+\nu}(uy)\cdot dy=0,\ u<x, \tag{5}$$

$$=\frac{\Gamma(m+1)}{2^{k-1}\Gamma(m+k)}u^{k-2}\left(\frac{x}{u}\right)^\nu\left(1-\frac{x^2}{u^2}\right)^{k-1}P_m^{(\nu,k-1)}\left(1-\frac{2x^2}{u^2}\right),\ u>x;$$

ii) if m is zero or a positive integer and $k>0$, then

(6)
$$\int_0^\infty y^{1-k}\cdot J_{-\nu}(xy)\cdot J_{k+2m+\nu}(uy)\cdot dy$$

$$=\frac{(-1)^\nu\Gamma(m+1)}{2^{k-2}\Gamma(m+k)}\,\frac{\sin(\pi\nu)}{\pi}\,u^{k-2}\left(\frac{x}{u}\right)^\nu\left(1-\frac{x^2}{u^2}\right)^{k-1}\left[Q_m^{(\nu,k-1)}\left(1-\frac{2x^2}{u^2}\right)\right.$$

$$\left.+\frac{\pi\cdot P_m^{(\nu,k-1)}\left(1-2\frac{x^2}{u^2}\right)}{2\sin(\pi\nu)}\right],\quad u>x,$$

$$=\frac{(-1)^\nu\Gamma(m+1)}{2^{k-2}\Gamma(m+k)}\,\frac{\sin(\pi\nu)}{\pi}\,u^{k-2}\left(\frac{x}{u}\right)^\nu\left(1-\frac{x^2}{u^2}\right)^{k-1}Q_m^{(\nu,k-1)}\left(1-\frac{2x^2}{u^2}\right),\quad u<x,$$

where $P_m^{(\alpha,\beta)}(x)$ and $Q_m^{(\alpha,\beta)}(x)$ are Jacobi polynomials and Jacobi functions of the second kind respectively.

2. We prove the following theorem:

THEOREM: *Let*

(7)
$$g(x)=\int_0^\infty y\cdot f(y)\cdot G_\nu(xy)\cdot dy,$$

and

(8)
$$h(x)=\int_0^\infty y\cdot J_{k+2m+\nu}(xy)\cdot y^k\cdot f(y)\cdot dy,$$

then

(9)
$$\frac{2^{k-1}\Gamma(k+m)}{\Gamma(m+1)}g(x)=\frac{\sin\{(a+\nu)\pi\}-(-1)^\nu\cdot\sin(a\pi)}{\sin(\pi\nu)}\times$$

$$\int_x^\infty u^{k-1}\cdot h(u)\cdot(x/u)^\nu(1-x^2/u^2)^{k-1}\cdot P_m^{(\nu,k-1)}(1-2x^2/u^2)\cdot du-$$

$$-(-1)\cdot 2\cdot\sin(a\pi)\int_0^\infty u^{k-1}\cdot h(u)\cdot(x/u)^\nu\cdot(1-x^2/u^2)^{k-1}Q_m^{(\nu,k-1)}\left(1-2\frac{x^2}{u^2}\right)du,$$

provided

i) *m is zero or a positive integer, $\nu>-1$ and $k>0$,*

ii) *$\int_0^1|t^{k+2m+\nu+1}\cdot f(t)|\cdot dt$ and $\int_1^\infty|t^{k+1/2}\cdot f(t)|dt$ are convergent,*

iii) *$\int_0^\infty|t\cdot h(t)|\cdot dt$ is convergent.*

PROOF: If the conditions under (i) and (ii) are satisfied then by Hankel inversion theorem [2], we have

(10)
$$y^k\cdot f(y)=\int_0^\infty u\cdot J_{k+2m+\nu}(uy)\cdot h(u)\cdot du.$$

Hence from (7) after substituting the value of $f(y)$, we obtain

(11)
$$
\begin{aligned}
g(x) &= \int_0^\infty y^{1-k} \cdot G_\nu(xy) \cdot dy \int_0^\infty u \cdot J_{k+2m+\nu}(uy) \cdot h(u) \cdot du \\
&= \int_0^\infty u \cdot h(u) \cdot du \int_0^\infty y^{1-k} \cdot J_{k+2m+\nu}(uy) \cdot G_\nu(xy) \cdot dy \\
&= \operatorname{cosec}(\pi\nu) \int_0^\infty u \cdot h(u) \cdot du \int_0^\infty y^{1-k} \cdot J_{k+2m+\nu}(uy) \times \\
&\quad \times [\sin\{(a+\nu)\pi\} \cdot J_\nu(xy) - \sin(a\pi) \cdot J_{-\nu}(xy)] \cdot dy.
\end{aligned}
$$

The change of the order of integration is justified under the conditions mentioned in the theorem. The final result is obtained by using (5) and (6).

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Journal of Gakugei, Tokushima University
Vol. XIV. 1964 59-63

ON A STEP FUNCTION

By

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(Received September 30, 1963)

1. In a recent paper [2], I have defined a step function and found its appica-tions. Here I make use of the same function and find its certain more appli-cations in the theory of entire functions defined by Dirichlet series.

2. To revise, let us define two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ to satisfy the following conditions:

(i) $0=\alpha_0<\alpha_1<\cdots<\alpha_n\to\infty$;

(ii) $\varliminf_{n\to\infty}(\alpha_n-\alpha_{n-1})=h>0$;

(iii) $\varlimsup_{n\to\infty} n/\lambda_n=D<\infty$,

and

(iv) $0\leqslant\beta_1\leqslant\beta_2\leqslant\cdots\leqslant\beta_n\to\infty$.

Suppose now that $f(x)$ $(\geqslant 0)$ is a step function having β_n as jump points. Further, let $(\alpha_n-\alpha_{n-1})$ be the jump at the point β_n $(n=1, 2, \cdots)$; so that define $f(x)$ as follows:

(2.1) $$f(x)=\sum_{\beta_n\leqslant x}(\alpha_n-\alpha_{n-1}),\quad (\alpha_0=\alpha_{-1}).$$

Also define:

$$\varlimsup_{x\to\infty}\frac{\log f(x)}{x}=\frac{B}{A};\quad (0<A\leqslant B<\infty).$$

LEMMA: *Let $\psi'(x)$ be a integrable for $x>0$; then*

$$\sum_{\beta_n\leqslant x}(\alpha_n-\alpha_{n-1})\psi(\beta_n)=f(x)\psi(x)-\int_0^x f(t)\psi'(t)\,dt.$$

For, we have

$$\begin{aligned}\int_0^x f(t)\psi'(t)\,dt&=(\alpha_1-\alpha_0)(\psi(\beta_1)-\psi(\beta_0))+(\alpha_2-\alpha_0)(\psi(\beta_2)-\psi(\beta_1))\\&\quad+\cdots+(\alpha_n-\alpha_0)(\psi(x)-\psi(\beta_n))\\&=(\alpha_1-\alpha_0)(\psi(x)-\psi(\beta_0))+(\alpha_2-\alpha_1)(\psi(x)-\psi(\beta_1))\\&\quad+\cdots+(\alpha_n-\alpha_{n-1})(\psi(x)-\psi(\beta_n))\\&=\sum_{\beta_n\leqslant x}(\alpha_n-\alpha_{n-1})(\psi(x)-\psi(\beta_n)),\end{aligned}$$

and the lemma follows.

3. THEOREM 1: *Let*

$$f(x) \sim L(x)e^{Bx},$$

where $L(x)$ *is a 'slowly increasing', monotonically tending to infinity, such that*

$$L(\eta x) \sim L(x); \quad 0 < \eta < \infty,$$

then, if $\alpha > 0$,

$$\lim_{x\to\infty} \frac{1}{f(x)e^{(\alpha-B)x}} \sum_{\beta_n \leqslant x} (\alpha_n - \alpha_{n-1})e^{(\alpha-B)\beta_n} = \frac{B}{\alpha}; \tag{3.1}$$

$$\lim_{x\to\infty} \frac{1}{f(x)e^{-(\alpha+B)x}} \sum_{\beta_n > x} (\alpha_n - \alpha_{n-1})e^{-(\alpha+B)\beta_n} = \frac{B}{\alpha}. \tag{3.2}$$

PROOF: From the above lemma, we have

$$\sum_{\beta_n \leqslant x} (\alpha_n - \alpha_{n-1})e^{(\alpha-B)\beta_n} = f(x)e^{(\alpha-B)x} - (\alpha - B)\int_0^x f(t)e^{(\alpha-B)t}dt. \tag{3.3}$$

Hence

$$\frac{1}{f(x)e^{(\alpha-B)x}} \sum_{\beta_n \leqslant x} (\alpha_n - \alpha_{n-1})e^{(\alpha-B)\beta_n} \sim 1 + \frac{B-\alpha}{L(x)e^{\alpha x}} \int_0^x L(t)e^{\alpha t}dt.$$

But

$$\int_0^x L(t)e^{\alpha t}\,dt < \frac{1}{\alpha}L(x)e^{\alpha x};$$

$$\int_0^x L(t)e^{\alpha t}dt > \int_{x-cx}^x L(t)e^{\alpha t}\,dt > L(\theta x)\int_{\theta x}^x e^{\alpha t}\,dt \qquad (O < \theta = 1 - c < 1)$$

$$\sim \frac{1}{\alpha}L(x)e^{\alpha x}.$$

Therefore (3.1) follows.

Now

$$\sum_{x<\beta_n \leqslant X} (\alpha_n - \alpha_{n-1})e^{-(\alpha+B)\beta_n} = f(X)e^{-(\alpha+B)X} - f(x)e^{-(\alpha+B)x} + (B+\alpha)\int_x^X f(t)e^{-(B+\alpha)t}\,dt.$$

Hence

$$\frac{1}{f(x)e^{-(\alpha+B)x}} \sum_{\beta_n > x} (\alpha_n - \alpha_{n-1})e^{-(\alpha+B)\beta_n} \sim -1 + \frac{B+\alpha}{L(x)e^{-\alpha x}} \int_x^\infty L(t)e^{-\alpha t}dt.$$

But

$$\int_x^\infty L(t)e^{-\alpha t}\,dt > \frac{L(x)e^{-\alpha x}}{\alpha}.$$

$$\int_x^\infty L(t)e^{-\alpha t}\,dt = \sum_{\nu=1}^{\infty}\int_{(\nu-1)\log 2+x}^{\nu\log 2+x} L(t)e^{-\alpha t}\,dt < \sum_{\nu=1}^{\infty} L(\nu\log 2 + x)\int_{(\nu-1)\log 2+x}^{\nu\log 2+x} e^{-\alpha t}\,dt$$

$$< L(x)\sum_{\nu=1}^{\infty}(1+\varepsilon)^\nu\left[e^{-\alpha((\nu-1)\log 2+x)} - e^{-\alpha(\nu\log 2+x)}\right]$$

$$\to \frac{L(x)e^{-\alpha x}}{\alpha}, \quad \text{on letting, after summation, } \varepsilon\to 0.$$

This proves (3. 2).

THEOREM 2: *Let $f(x)$ be a step function defined by* (2. 1), *and*

$$\varphi(x) = \int_0^x f(t)\,dt,$$

then, if A and B are defined as in (2. 2), *we have*

$$\varliminf_{x\to\infty}\frac{\varphi(x)}{f(x)} \leqslant \frac{1}{B} \leqslant \frac{1}{A} \leqslant \varlimsup_{x\to\infty}\frac{\varphi(x)}{f(x)}$$

PROOF: We have

$$\varlimsup_{n\to\infty}\frac{\log\alpha_n}{\beta_n} = B,$$

and so if $H<B$, we have $\varlimsup_{n\to\infty}\alpha_n\exp(-H\beta_n)\to\infty$. Theorefore ([3], p. 20)

$$e^{H(\beta_\mu-\beta_n)} \geqslant \frac{\alpha_\mu}{\alpha_n}, \quad \mu = 1, 2, \cdots n \quad \text{(equality holds only if } \mu = n).$$

One can choose a number x, $\beta_n \leqslant x < \beta_{n+1}$, then

$$e^{H(\beta_\mu - x)} \geqslant \frac{\alpha_\mu}{\alpha_n}, \quad \mu = 1, 2, \cdots, n.$$

Let $f(x)=\alpha_n$. Then

$$\frac{\varphi(x)}{f(x)} = \frac{1}{f(x)}\sum_{\beta_\mu\leqslant x}(\alpha_\mu-\alpha_{\mu-1})(x-\beta_\mu) = \frac{1}{\alpha_n}\sum_{\mu=1}^{n}(\alpha_\mu-\alpha_{\mu-1})(x-\beta_\mu)$$

$$\leqslant \frac{1}{H\alpha_n}\sum_{\mu=2}^{n}(\alpha_\mu-\alpha_{\mu-1})\log\left(\frac{\alpha_n}{\alpha_\mu}\right) + O\left(\frac{\log\alpha_n}{\alpha_n}\right)$$

$$< \frac{1}{H}\sum_{\mu=2}^{n}\int_{\frac{\alpha_{\mu-1}}{\alpha_n}}^{\frac{\alpha_\mu}{\alpha_n}}\log\left(\frac{1}{x}\right)dx + O\left(\frac{\log\alpha_n}{\alpha_n}\right)$$

$$= \frac{1}{H}\int_{\frac{\alpha_1}{\alpha_n}}^{1}\log\left(\frac{1}{x}\right)dx + O\left(\frac{\log\alpha_n}{\alpha_n}\right)$$

$$\to 1/H \quad \text{as } n\to\infty.$$

Hence, since $(B-H)$ can be made arbitrarily small, we find that

$$\lim_{x\to\infty} \frac{\varphi(x)}{f(x)} \leqslant \frac{1}{B}.$$

Similarly it can be shown that

$$\overline{\lim_{x\to\infty}} \frac{\varphi(x)}{f(x)} \geqslant \frac{1}{A},$$

and the result follows.

4. Applications. Let $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} (s = \sigma + it)$ be an entire function represented by Dirichlet series. Further, let $\mu(\sigma)$ and $\lambda_{\nu(\sigma)}$ be respectively the maximum term of $f(s)$ and its rank. Let $\varkappa_n = \log|a_{n-1}/a_n|/(\lambda_n - \lambda_{n-1})$ $(\varkappa_0 = 0)$; then ([1], p. 717) $\varkappa_n$ $(n = 1, 2, \cdots)$ are the points of the left-hand discontinuities of $\lambda_{\nu(\sigma)}$, where

$$\lambda_{\nu(\sigma)} = \sum_{\varkappa_n \leqslant \sigma} (\lambda_n - \lambda_{n-1}), \quad (\lambda_0 = \lambda_{-1}).$$

Further $\varkappa_n$ ([1], p. 718) is a non-decreasing function of n tending to ∞ with n. It is also well-known that $(\mu(0) = 1)$

$$\log \mu(\sigma) = \int_0^{\sigma} \lambda_{\nu(x)} dx,$$

and the order $(R)\rho$ and lower order λ are given by [5]:

$$\overline{\lim_{\sigma\to\infty}} \frac{\log \lambda_{\nu(\sigma)}}{\sigma} = \frac{\rho}{\lambda}; \quad (O \leqslant \lambda \leqslant \rho \leqslant \infty).$$

So replacing $f(x)$ by $\lambda_{\nu(x)}$ and $\varphi(x)$ by $\log \mu(x)$, we have from Theo. 2

(4.1) $$\underline{\lim_{\sigma\to\infty}} \frac{\log \mu(\sigma)}{\lambda_{\nu(\sigma)}} \leqslant \frac{1}{\rho} \leqslant \frac{1}{\lambda} \leqslant \overline{\lim_{\sigma\to\infty}} \frac{\log \mu(\sigma)}{\lambda_{\nu(\sigma)}}, \quad (0 < \lambda \leqslant \rho < \infty).$$

K. N. Srivastava [6] has proved (4. 1) by an alternative mothod. Again, let $\lambda_{\nu(\sigma)} = L(\sigma) e^{\rho\sigma}$, then following (3. 1), we obtain

(4.2) $$\lim_{\sigma\to\infty} \frac{\log \mu(\sigma)}{\lambda_{\nu(\sigma)}} = \frac{1}{\rho}.$$

Q. I. Rahman [4] has obtained this alternatively.

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昭和39年 3 月20日印刷
昭和39年 3 月30日発行

発 行 者　徳島大学学芸学部数学教室

編集責任者　霜　田　伊　左　衛

広島市空鞘町 111 番地
印 刷 所　大 学 印 刷 株 式 会 社

印 刷 者　増　田　訓　清
電話 ㉛ 4231・4232 番