Journal of Gakugei, Tokushima University
Vol. XIV. 1964 59-63

ON A STEP FUNCTION

By

Pawan Kumar KAMTHAN

(Received September 30, 1963)

1. In a recent paper [2], I have defined a step function and found its appica-tions. Here I make use of the same function and find its certain more applications in the theory of entire functions defined by Dirichlet series.

2. To revise, let us define two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ to satisfy the following conditions:

(i) $0=\alpha_0<\alpha_1<\cdots<\alpha_n\to\infty$;

(ii) $\underline{\lim}_{n\to\infty}(\alpha_n-\alpha_{n-1})=h>0$;

(iii) $\overline{\lim}_{n\to\infty} n/\lambda_n=D<\infty$,

and

(iv) $0\leqslant\beta_1\leqslant\beta_2\leqslant\cdots\leqslant\beta_n\to\infty$.

Suppose now that $f(x)$ ($\geqslant 0$) is a step function having β_n as jump points. Further, let $(\alpha_n-\alpha_{n-1})$ be the jump at the point β_n $(n=1, 2, \cdots)$; so that define $f(x)$ as follows:

$$f(x)=\sum_{\beta_n\leqslant x}(\alpha_n-\alpha_{n-1}),\quad (\alpha_0=\alpha_{-1}). \tag{2.1}$$

Also define:

$$\overline{\underline{\lim}}_{x\to\infty}\frac{\log f(x)}{x}=\frac{B}{A};\quad (0<A\leqslant B<\infty).$$

LEMMA: *Let $\psi'(x)$ be a integrable for $x>0$; then*

$$\sum_{\beta_n\leqslant x}(\alpha_n-\alpha_{n-1})\psi(\beta_n)=f(x)\psi(x)-\int_0^x f(t)\psi'(t)\,dt.$$

For, we have

$$\begin{aligned}\int_0^x f(t)\psi'(t)\,dt &=(\alpha_1-\alpha_0)\big(\psi(\beta_1)-\psi(\beta_0)\big)+(\alpha_2-\alpha_0)\big(\psi(\beta_2)-\psi(\beta_1)\big)\\ &\qquad+\cdots+(\alpha_n-\alpha_0)\big(\psi(x)-\psi(\beta_n)\big)\\ &=(\alpha_1-\alpha_0)\big(\psi(x)-\psi(\beta_0)\big)+(\alpha_2-\alpha_1)\big(\psi(x)-\psi(\beta_1)\big)\\ &\qquad+\cdots+(\alpha_n-\alpha_{n-1})\big(\psi(x)-\psi(\beta_n)\big)\\ &=\sum_{\beta_n\leqslant x}(\alpha_n-\alpha_{n-1})\big(\psi(x)-\psi(\beta_n)\big),\end{aligned}$$

and the lemma follows.

3. THEOREM 1: *Let*

$$f(x)\sim L(x)e^{Bx},$$

where $L(x)$ *is a 'slowly increasing', monotonically tending to infinity, such that*

$$L(\eta x)\sim L(x);\quad 0<\eta<\infty,$$

then, if $\alpha>0$,

$$\lim_{x\to\infty}\frac{1}{f(x)e^{(\alpha-B)x}}\sum_{\beta_n\leqslant x}(\alpha_n-\alpha_{n-1})e^{(\alpha-B)\beta_n}=\frac{B}{\alpha};\tag{3.1}$$

$$\lim_{x\to\infty}\frac{1}{f(x)e^{-(\alpha+B)x}}\sum_{\beta_n>x}(\alpha_n-\alpha_{n-1})e^{-(\alpha+B)\beta_n}=\frac{B}{\alpha}.\tag{3.2}$$

PROOF: From the above lemma, we have

$$\sum_{\beta_n\leqslant x}(\alpha_n-\alpha_{n-1})e^{(\alpha-B)\beta_n}=f(x)e^{(\alpha-B)x}-(\alpha-B)\int_0^x f(t)e^{(\alpha-B)t}dt.\tag{3.3}$$

Hence

$$\frac{1}{f(x)e^{(\alpha-B)x}}\sum_{\beta_n\leqslant x}(\alpha_n-\alpha_{n-1})e^{(\alpha-B)\beta_n}\sim 1+\frac{B-\alpha}{L(x)e^{\alpha x}}\int_0^x L(t)e^{\alpha t}dt.$$

But

$$\int_0^x L(t)e^{\alpha t}\,dt<\frac{1}{\alpha}L(x)e^{\alpha x};$$

$$\int_0^x L(t)e^{\alpha t}dt>\int_{x-cx}^x L(t)e^{\alpha t}\,dt>L(\theta x)\int_{\theta x}^x e^{\alpha t}\,dt\qquad (O<\theta=1-c<1)$$

$$\sim\frac{1}{\alpha}L(x)e^{\alpha x}.$$

Therefore (3.1) follows.

Now

$$\sum_{x<\beta_n\leqslant X}(\alpha_n-\alpha_{n-1})e^{-(\alpha+B)\beta_n}=f(X)e^{-(\alpha+B)X}-f(x)e^{-(\alpha+B)x}+(B+\alpha)\int_x^X f(t)e^{-(B+\alpha)t}\,dt.$$

Hence

$$\frac{1}{f(x)e^{-(\alpha+B)x}}\sum_{\beta_n>x}(\alpha_n-\alpha_{n-1})e^{-(\alpha+B)\beta_n}\sim-1+\frac{B+\alpha}{L(x)e^{-\alpha x}}\int_x^\infty L(t)e^{-\alpha t}dt.$$

But

$$\int_x^\infty L(t)e^{-\alpha t}\,dt>\frac{L(x)e^{-\alpha x}}{\alpha}.$$

$$\int_x^\infty L(t)e^{-\alpha t}\,dt=\sum_{\nu=1}^\infty\int_{(\nu-1)\log 2+x}^{\nu\log 2+x}L(t)e^{-\alpha t}\,dt<\sum_{\nu=1}^\infty L(\nu\log 2+x)\int_{(\nu-1)\log 2+x}^{\nu\log 2+x}e^{-\alpha t}\,dt$$

$$<L(x)\sum_{\nu=1}^\infty(1+\varepsilon)^\nu\left[e^{-\alpha((\nu-1)\log 2+x)}-e^{-\alpha(\nu\log 2+x)}\right]$$

$$\to\frac{L(x)e^{-\alpha x}}{\alpha},\quad\text{on letting, after summation, }\varepsilon\to 0.$$

This proves (3. 2).

THEOREM 2: *Let $f(x)$ be a step function defined by* (2. 1), *and*

$$\varphi(x)=\int_0^x f(t)\,dt,$$

then, if A and B are defined as in (2. 2), *we have*

$$\varliminf_{x\to\infty}\frac{\varphi(x)}{f(x)}\leqslant\frac{1}{B}\leqslant\frac{1}{A}\leqslant\varlimsup_{x\to\infty}\frac{\varphi(x)}{f(x)}$$

PROOF: We have

$$\varlimsup_{n\to\infty}\frac{\log\alpha_n}{\beta_n}=B,$$

and so if $H<B$, we have $\varlimsup_{n\to\infty}\alpha_n\exp(-H\beta_n)\to\infty$. Theorefore ([3], p. 20)

$$e^{H(\beta_\mu-\beta_n)}\geqslant\frac{\alpha_\mu}{\alpha_n},\quad\mu=1,2,\cdots n\quad\text{(equality holds only if }\mu=n).$$

One can choose a number x, $\beta_n\leqslant x<\beta_{n+1}$, then

$$e^{H(\beta_\mu-x)}\geqslant\frac{\alpha_\mu}{\alpha_n},\quad\mu=1,2,\cdots,n.$$

Let $f(x)=\alpha_n$. Then

$$\frac{\varphi(x)}{f(x)}=\frac{1}{f(x)}\sum_{\beta_\mu\leqslant x}(\alpha_\mu-\alpha_{\mu-1})(x-\beta_\mu)=\frac{1}{\alpha_n}\sum_{\mu=1}^n(\alpha_\mu-\alpha_{\mu-1})(x-\beta_\mu)$$

$$\leqslant\frac{1}{H\alpha_n}\sum_{\mu=2}^n(\alpha_\mu-\alpha_{\mu-1})\log\left(\frac{\alpha_n}{\alpha_\mu}\right)+O\left(\frac{\log\alpha_n}{\alpha_n}\right)$$

$$<\frac{1}{H}\sum_{\mu=2}^n\int_{\frac{\alpha_{\mu-1}}{\alpha_n}}^{\frac{\alpha_\mu}{\alpha_n}}\log\left(\frac{1}{x}\right)dx+O\left(\frac{\log\alpha_n}{\alpha_n}\right)$$

$$=\frac{1}{H}\int_{\frac{\alpha_1}{\alpha_n}}^1\log\left(\frac{1}{x}\right)dx+O\left(\frac{\log\alpha_n}{\alpha_n}\right)$$

$$\to 1/H\quad\text{as }n\to\infty.$$

Hence, since $(B-H)$ can be made arbitrarily small, we find that

$$\underline{\lim}_{x\to\infty}\frac{\varphi(x)}{f(x)}\leqslant\frac{1}{B}.$$

Similarly it can be shown that

$$\overline{\lim}_{x\to\infty}\frac{\varphi(x)}{f(x)}\geqslant\frac{1}{A},$$

and the result follows.

4. Applications. Let $f(s)=\sum_{n=1}^{\infty}a_n e^{s\lambda_n}(s=\sigma+it)$ be an entire function represented by Dirichlet series. Further, let $\mu(\sigma)$ and $\lambda_{\nu(\sigma)}$ be respectively the maximum term of $f(s)$ and its rank. Let $\varkappa_n=\log|a_{n-1}/a_n|/(\lambda_n-\lambda_{n-1})$ $(\varkappa_0=0)$; then ([1], p. 717) $\varkappa_n$ $(n=1, 2,\cdots)$ are the points of the left-hand discontinuities of $\lambda_{\nu(\sigma)}$, where

$$\lambda_{\nu(\sigma)}=\sum_{\varkappa_n\leqslant\sigma}(\lambda_n-\lambda_{n-1}),\quad (\lambda_0=\lambda_{-1}).$$

Further $\varkappa_n$ ([1], p. 718) is a non-decreasing function of n tending to ∞ with n. It is also well-known that $(\mu(0)=1)$

$$\log\mu(\sigma)=\int_0^{\sigma}\lambda_{\nu(x)}dx,$$

and the order $(R)\rho$ and lower order λ are given by [5]:

$$\overline{\underline{\lim}}_{\sigma\to\infty}\frac{\log\lambda_{\nu(\sigma)}}{\sigma}=\frac{\rho}{\lambda};\quad (O\leqslant\lambda\leqslant\rho\leqslant\infty).$$

So replacing $f(x)$ by $\lambda_{\nu(x)}$ and $\varphi(x)$ by $\log\mu(x)$, we have from Theo. 2

$$\underline{\lim}_{\sigma\to\infty}\frac{\log\mu(\sigma)}{\lambda_{\nu(\sigma)}}\leqslant\frac{1}{\rho}\leqslant\frac{1}{\lambda}\leqslant\overline{\lim}_{\sigma\to\infty}\frac{\log\mu(\sigma)}{\lambda_{\nu(\sigma)}},\quad (0<\lambda\leqslant\rho<\infty). \tag{4.1}$$

K. N. Srivastava [6] has proved (4. 1) by an alternative mothod. Again, let $\lambda_{\nu(\sigma)}=L(\sigma)e^{\rho\sigma}$, then following (3. 1), we obtain

$$\lim_{\sigma\to\infty}\frac{\log\mu(\sigma)}{\lambda_{\nu(\sigma)}}=\frac{1}{\rho}. \tag{4.2}$$

Q. I. Rahman [4] has obtained this alternatively.

References

[1] Azpeitia, A. G., *On the maximum modulus and the maximum term of an entire Dirichlet series, Proc.* Amer. Math. Soc., **12** (1961), 717-21.

[2] Kamthan, P. K., *A theorem on step function,* J. Gakugei, Tokushima Univ., **xiii** (1962), 43-47.

[3] Pólya, G. & Szegö, G., *Aufgaben und Lehrsatze aus der Analysis,* Vol. I, Springer-Verlag, Berlin, (1954).

[4] Rahman, Q. I., *A note on entire functions (defined by Dirichlet series) of perfectly regular growth,* Quart. J. Math., **23** (1955), 173-75.

[5] ————, *On the maximum modulus and the coefficients of an entire Dirichlet series,* Tôh. Math. J., **8** (1956), 108-13.

[6] Srivastava, K. N., *On the maximum term of an entire Dirichlet Series,* Proc. Nat. Aca. Scs., (India), Allahbad, 27, Sect. A, (1958), 134-46.

University of Delhi,
Delhi-6, India.