Journal of Gakugei, Tokushima University
Vol. XIV. 1964 55-57

A RELATION BETWEEN HANKEL AND HARDY TRANSFORMS

By

K. N. Srivastava

(Received September 30, 1963)

1. A generalization of Hankel transform is due to Hardy [3] who gave the following formula:

$$g(x) = \int_0^\infty u \cdot f(u) \cdot G_\nu(ux) \cdot dx \tag{1}$$

where

$$f(u) = \int_0^\infty x \cdot g(x) \cdot F_\nu(ux) \cdot dx, \tag{2}$$

and

$$\begin{aligned} G_\nu(x) &= \cos(a\pi) \cdot J_\nu(x) + \sin(a\pi) \cdot Y_\nu(x) \\ &= \operatorname{cosec}(\pi\nu)[\sin\{(a+\nu)\pi\} \cdot J_\nu(x) - \sin(a\pi) \cdot J_{-\nu}(x)], \end{aligned} \tag{3}$$

$$F_\nu(x) = \frac{(x/2)^{2a+\nu}}{\Gamma(a+1) \cdot \Gamma(\nu+a+1)} \, {}_1F_2[1;\ a+1,\ \nu+a+1,\ -x^2/4]. \tag{4}$$

This formula is valid under the following conditions given by Cooke [1]:

i) $a > -1,\ a+\nu > -1,\ \nu + 2a < 3/2,\ |\nu| < 3/2,$

ii) $t^\sigma g(t)$ is integrable over $(0, \delta)$, $\sigma = \min(2a+\nu+1, 1/2)$,

iii) $t^{1/2} \cdot g(t)$ is integrable over (δ, ∞), $\delta > 0$.

At another place [4], we have obtained a relation between Hankel transforms of different order. The object of this note is to obtain a relation between Hankel and Hardy transforms. The result of [4] is obtained as a particular case of this result by taking $a = 0$.

The result in this note is based on the following integrals which are special cases of Weber-Schafheiltin integral. The results in question, which are easily derived from the more general ones given by Watson [5], are

i) if m is zero or a positive integer, $\nu > -1-m$ and $k > 0$, then

$$\begin{aligned} &\int_0^\infty y^{1-k} \cdot J_\nu(xy) \cdot J_{k+2m+\nu}(uy) \cdot dy = 0,\ u < x, \\ &= \frac{\Gamma(m+1)}{2^{k-1}\Gamma(m+k)} u^{k-2} \left(\frac{x}{u}\right)^\nu \left(1 - \frac{x^2}{u^2}\right)^{k-1} P_m^{(\nu, k-1)}\left(1 - \frac{2x^2}{u^2}\right),\ u > x; \end{aligned} \tag{5}$$

ii) if m is zero or a positive integer and $k > 0$, then

$$\int_0^\infty y^{1-k} \cdot J_{-\nu}(xy) \cdot J_{k+2m+\nu}(uy) \cdot dy \tag{6}$$

$$= \frac{(-1)^\nu \Gamma(m+1)}{2^{k-2}\Gamma(m+k)} \frac{\sin(\pi\nu)}{\pi} u^{k-2}\left(\frac{x}{u}\right)^\nu \left(1-\frac{x^2}{u^2}\right)^{k-1} \left[Q_m^{(\nu,k-1)}\left(1-\frac{2x^2}{u^2}\right) + \frac{\pi \cdot P_m^{(\nu,k-1)}\left(1-2\frac{x^2}{u^2}\right)}{2\sin(\pi\nu)}\right], \quad u > x,$$

$$= \frac{(-1)^\nu \Gamma(m+1)}{2^{k-2}\Gamma(m+k)} \frac{\sin(\pi\nu)}{\pi} u^{k-2}\left(\frac{x}{u}\right)^\nu \left(1-\frac{x^2}{u^2}\right)^{k-1} Q_m^{(\nu,k-1)}\left(1-\frac{2x^2}{u^2}\right), \quad u < x,$$

where $P_m^{(\alpha,\beta)}(x)$ and $Q_m^{(\alpha,\beta)}(x)$ are Jacobi polynomials and Jacobi functions of the second kind respectively.

2. We prove the following theorem:

THEOREM: *Let*

$$g(x) = \int_0^\infty y \cdot f(y) \cdot G_\nu(xy) \cdot dy, \tag{7}$$

and

$$h(x) = \int_0^\infty y \cdot J_{k+2m+\nu}(xy) \cdot y^k \cdot f(y) \cdot dy, \tag{8}$$

then

$$\frac{2^{k-1}\Gamma(k+m)}{\Gamma(m+1)} g(x) = \frac{\sin\{(a+\nu)\pi\} - (-1)^\nu \cdot \sin(a\pi)}{\sin(\pi\nu)} \times \tag{9}$$

$$\int_x^\infty u^{k-1} \cdot h(u) \cdot (x/u)^\nu (1 - x^2/u^2)^{k-1} \cdot P_m^{(\nu,k-1)}(1 - 2x^2/u^2) \cdot du -$$

$$-(-1) \cdot 2 \cdot \sin(a\pi) \int_0^\infty u^{k-1} \cdot h(u) \cdot (x/u)^\nu \cdot (1 - x^2/u^2)^{k-1} Q_m^{\nu,k-1)}\left(1 - 2\frac{x^2}{u^2}\right) du,$$

provided

i) *m is zero or a positive integer, $\nu > -1$ and $k > 0$,*

ii) *$\int_0^1 |t^{k+2m+\nu+1} \cdot f(t)| \cdot dt$ and $\int_1^\infty |t^{k+1/2} \cdot f(t)| dt$ are convergent,*

iii) *$\int_0^\infty |t \cdot h(t)| \cdot dt$ is convergent.*

PROOF: If the conditions under (i) and (ii) are satisfied then by Hankel inversion theorem [2], we have

$$y^k \cdot f(y) = \int_0^\infty u \cdot J_{k+2m+\nu}(uy) \cdot h(u) \cdot du. \tag{10}$$

Hence from (7) after substituting the value of $f(y)$, we obtain

(11)
$$g(x)=\int_0^\infty y^{1-k}\cdot G_\nu(xy)\cdot dy\int_0^\infty u\cdot J_{k+2m+\nu}(uy)\cdot h(u)\cdot du$$
$$=\int_0^\infty u\cdot h(u)\cdot du\int_0^\infty y^{1-k}\cdot J_{k+2m+\nu}(uy)\cdot G_\nu(xy)\cdot dy$$
$$=\operatorname{cosec}(\pi\nu)\int_0^\infty u\cdot h(u)\cdot du\int_0^\infty y^{1-k}\cdot J_{k+2m+\nu}(uy)\times$$
$$\times[\sin\{(a+\nu)\pi\}\cdot J_\nu(xy)-\sin(a\pi)\cdot J_{-\nu}(xy)]\cdot dy.$$

The change of the order of integration is justified under the conditions mentioned in the theorem. The final result is obtained by using (5) and (6).

References

1. R. G. Cooke, Proc. Lond. Math. Soc., **24** (1925), 381-420.
2. A. Erdelyi (Editor), *Higher transcendental functions*, (McGraw-Hill 1953) vol **2**, 73.
3. G. H. Hardy, Proc. Lond. Math. Soc., **24** (ix) (1925).
4. K. N. Srivastava and B. R. Bhonsle, *A relation between Hankel transforms of different orders*, (under communication).
5. G. N. Watson, *Theory of Bessel functions*, (Cambridge 1944), 401, 404.

M. A. College of Technology
Bhopal (M. P) India.