Journal of Gakugei, Tokushima University
Vol. XIII. 1962 43–47

A THEOREM ON STEP FUNCTION

By

Pawan Kumar KAMTHAN

(Received September 30, 1962)

1. In this paper we prove a theorem on step function and apply it to prove certain results in the theory of entire functions represented by Dirichlet series.

2. Let us define two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying the following conditions:

(i) $0=\alpha_0<\alpha_1<\alpha_2<\cdots<\alpha_n\to\infty$ with n.

(ii) $\underline{\lim}_{n\to\infty}(\alpha_n-\alpha_{n-1})=h>0$.

(iii) $\overline{\lim}_{n\to\infty}\frac{n}{\alpha_n}=D<\infty,\ Dh\leqslant 1.$

and

(iv) $0\leqslant\beta_1\leqslant\beta_2\leqslant\cdots\leqslant\beta_n\to\infty$ with n.

Suppose now that $f(x)$ is a step function having β_n as jump points. Further, let $(\alpha_n-\alpha_{n-1})$ be the jump at the point $\beta_n(n=1, 2, 3,\cdots)$, so that define $f(x)$ as

$$f(x)=\sum_{x\geqslant\beta_n}(\alpha_n-\alpha_{n-1}) \quad\cdots\cdots(1)$$

We prove:

THEOREM: *Let $f(x)$ be a step function defined by* (1) *and let*

$$\varphi(x)=\int_1^x f(t)\,dt$$

$$\overline{\lim}_{x\to\infty}\frac{\log f(x)}{x}=\frac{B}{A};\quad 0<B\leqslant\infty,\quad 0\leqslant A<\infty.$$

Then

$$I(f)=\overline{\lim}_{x\to\infty}\frac{\varphi(x)}{xf(x)}\leqslant 1-\frac{A}{B}. \quad\cdots\cdots(2)$$

PROOF: We have

$$\varphi(x)=\sum_{\beta_n\leqslant x}(\alpha_n-\alpha_{n-1})(x-\beta_n)$$

$$=xf(x)-\sum_{\beta_n\leqslant x}\beta_n(\alpha_n-\alpha_{n-1})$$

Further

$$\overline{\lim_{x\to\infty}} \frac{\log f(x)}{x} = \overline{\lim_{n\to\infty}} \frac{\log \alpha_n}{\beta_n} = B,$$

so that

$$\frac{\log \alpha_n}{\beta_n} < B + \epsilon ,$$

for all $n > n_0$, and hence

$$\sum_{\beta_n \leqslant x} (\alpha_n - \alpha_{n-1})\beta_n > \sum_{\beta_n \leqslant x,\, n > n_0} (\alpha_n - \alpha_{n-1}) \frac{\log \alpha_n}{(B + \epsilon)}$$

Let N be the largest integer such that $\beta_n \leqslant x$, then we get

$$\sum_{\beta_n \leqslant x} (\alpha_n - \alpha_{n-1})\beta_n > \frac{1}{B+\epsilon} \{\alpha_N \log \alpha_N + O(\alpha_N)\}$$

$$= \frac{1}{B+\epsilon} [f(x) \log f(x)] + O\{f(x)\},$$

and therefore

$$\varphi(x) \leqslant x f(x) - \frac{1}{B+\epsilon} f(x) \log f(x) + O\{f(x)\}, \quad \ldots\ldots\ldots\ldots(3)$$

and so

$$I(f) \leqslant 1 - \frac{1}{B+\epsilon} \underline{\lim_{x\to\infty}} \frac{\log f(x)}{x} = 1 - \frac{A}{B+\epsilon},$$

and since ϵ is arbitrary, we get (2).

COROLLARY (i) *Let* $\psi(x)$ *be integrable in any interval* (1, X). (ii) $\psi(x) \sim f(x)$, *where* $f(x)$ *is a step function as defined in the theorem, then*

$$\overline{\lim_{x\to\infty}} \left(\int_1^x \psi(t)\,dt \right) \Big/ x\psi(x) \leqslant 1 - \frac{A}{B}.$$

For let

$$\psi(x) = f(x) + \theta(x).$$

Then $\theta(x)$ is integrable in any interval (1, X) and obviously

$$\theta(x) = o\{f(x)\},$$

and so

$$\int_1^x \psi(t)\,dt = \int_1^x f(t)\,dt + o\{x\psi(x)\}.$$

But from (i), we also have

$$\overline{\lim_{x\to\infty}} \frac{\log \psi(x)}{x} = \frac{B}{A};$$

therefore

$$\overline{\lim_{x\to\infty}} \frac{\int_1^x \psi(t)\,dt}{x\psi(x)} \leqslant 1 - \frac{A}{B}.$$

3. *Applications*: Let

$$f(s) = \sum_{x=1}^{\infty} a_n e^{s\lambda n} \quad (s = \sigma + it)$$

be an entire function repredented by Dirichlet series of order (R) ρ and lower order λ $(0 < \rho < \infty)$ and let $\mu(\sigma)$ and $\lambda_{\nu(\sigma)}$ be respectively its maximum term and the rank of the maximum term. Then, since [1]

$$\overline{\lim_{\sigma\to\infty}} \frac{\log \lambda_{\nu(\sigma)}}{\sigma} = \frac{\rho}{\lambda};$$

and

$$\log \mu(\sigma) = A_1 + \int_1^\sigma \lambda_{\nu(t)}\,dt,$$

we have from the above theorem $(f(x) = \lambda_{\nu(x)})$

$$\overline{\lim_{\sigma\to\infty}} \frac{\log \mu(\sigma)}{\sigma\lambda_{\nu(\sigma)}} \leqslant 1 - \frac{\lambda}{\rho}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Also from (3) we have

$$\overline{\lim_{\sigma\to\infty}} \frac{\log \mu(\sigma)}{\lambda_{\nu(\sigma)} \log \lambda_{\nu(\sigma)}} \leqslant \overline{\lim_{\sigma\to\infty}} \frac{\sigma}{\log \lambda_{\nu(\sigma)}} - \frac{1}{\rho + \epsilon} = \frac{1}{\lambda} - \frac{1}{\rho}, \quad \ldots\ldots(5)$$

since ϵ is arbitrary. The results (4) and (5) have also been obtained by R. P. Srivastav [2] by a different method.

4. In this article we below give an example to show that (2) is the best possible result. We also show that for one value of α the equality sign in (2) holds and for the other value of α the inequality sign in (2) holds. It is also shown here that when $A = B = \infty$, $I(f)$ may have any assigned value α such that $0 \leqslant \alpha \leqslant 1$.

If $0 < A = B < \infty$, it follows from (2) that $I(f) = 0$. We give an example to demonstrate that the converse, viz., may be zero but $A \neq B$, is not necessarily true.

(a) Let $0 < A < B < \infty$, $1 < \alpha \leqslant B/A$ and let*

* In what follows, we suppose that $\{x_n\}$ is a sequence of positive numbers increasing sufficiently rapidly.

$$f(x, \alpha)=f(x)=\begin{cases}[e^{Bx_n}] & \text{when } x \in I_1(x_n \leqslant x < \alpha x_n)\\ [e^{Ax}e^{(B-\alpha A)x_n}] & \text{when } x \in I_2(\alpha x_n \leqslant x < e^{x_n})\\ [e^{Bx}] & \text{when } x \in I_3(e^{x_n} \leqslant x < x_{n+1})\end{cases}$$

Then

$$\overline{\lim_{x\to\infty}} \frac{\log f(x)}{x} = B; \qquad \underline{\lim_{x\to\infty}} \frac{\log f(x)}{x} = A.$$

Further for x in I_1

$$\frac{\varphi(x)}{xf(x)} = \frac{e^{Bx_n}(x - x_n) + O(e^{Bx_n})}{xe^{Bx_n}} \sim 1 - \frac{x_n}{x}.$$

Hence for x in I_1

$$I(f, x) \leqslant 1 - \frac{1}{\alpha} + o(1)$$

$$I(f, \alpha x_n) = 1 - \frac{1}{\alpha} + o(1).$$

Similarly for x in I_2

$$\varphi(x) = \int_{\alpha x_n}^{x} f(t)\,dt + \int_1^{x_n} f(t)\,dt + \int_{x_n}^{\alpha x_n} f(t)\,dt$$

$$= \frac{e^{(B-\alpha A)x_n}(e^{Ax} - e^{\alpha A x_n})}{A} + O(e^{Bx_n}) + (\alpha - 1)x_n e^{Bx_n},$$

and so for x in I_2

$$I(f) = \frac{(\alpha - 1)x_n e^{Bx_n}}{xe^{Ax}e^{(B-\alpha A)x_n}} + o(1)$$

$$\leqslant 1 - \frac{1}{\alpha} + o(1).$$

And for x in I_3

$$I(f) = \left\{\frac{e^{Bx} - e^{Be^{x_n}}}{B} + O(e^{(B-\alpha A)x_n} e^{Ae^{x_n}})\right\} \Big/ xe^{Bx}$$

$$\to 0.$$

Therefore

$$I(f) = \overline{\lim_{x\to\infty}} \frac{\varphi(x)}{xf(x)} = 1 - \frac{1}{\alpha}.$$

(b). Let

$$f(x,\alpha,\beta)=f(x)=\begin{cases}2xe^{x^2} & \text{when } x_n\leqslant x<X_n=\alpha x_{n+1}+\beta\log x_{n+1}\\ 2X_n e^{X^2 n} & \text{when } X_n\leqslant x<x_{n+1},\ n=1,2,\cdots\end{cases}$$

where either $0<\alpha<1$ or $\alpha=0$, $\beta>1$ or $\alpha=1$, $\beta<0$.
Clearly $A=B=\infty$. When x lies in the first interval $I(f)=0$. Consider x in the second interval and also we see that

$$\frac{\int_{X_n}^{x} f(t)\,dt}{2x\,X_n\,e^{X_n^2}}=\frac{2X_n e^{X_n^2}(x-X_n)}{2x\,X_n\,e^{X_n^2}}=1-\frac{X_n}{x}$$

$$=1-\frac{\alpha x_{n+1}+\beta\log x_{n+1}}{x},$$

and so when $x=x_{n+1}$ we see that

$$I(f)=1-\alpha.$$

(c). Let

$$f(x)=x^{\Delta}\log x,\quad \Delta\geqslant 0.$$

Then

$$A=B=0 \quad\text{and}\quad I(f)=\frac{1}{\Delta+1}.$$

(d). Let $0\leqslant A<B<\infty$,

$$X_n=x_n\left(1+\frac{1}{\log x_n}\right)$$

$$C=\left(\frac{Bx_n}{X_n}-A\right),$$

$$f(x)=\begin{cases}[e^{Bx_n}] & (x_n\leqslant x<X_n)\\ [e^{Ax}e^{CX_n}] & (X_n\leqslant n<e^{x_n})\\ [e^{Bx}] & (e^{x_n}\leqslant x<x_{n+1})\end{cases}$$

Then it is easily verified that

$$\overline{\lim_{x\to\infty}}\frac{\log f(x)}{x}=\frac{B}{A};\ I(f)=0.$$

References

[1.] Rahman, Q. I.—On the maximum modulus and the coefficients of an entire Dirichlet series, Tohoku Math. Jour., No. 1, 8, (1956), 108–113.

[2.] Srivastav, R. P.—On the entire functions and their derivatives represented by Dirighlet series, Ganita (Lucknow), No. 2, 9. (1958), 83–93.

Department of Mathematics;
Birla College, Pilani, India.