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THE STUDENT'S DISTRIBUTION FOR A UNIVERSE BOUNDED AT ONE OR BOTH SIDES (Continued)

By

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In the foregoing notes[1] the author had dealt with some special Student's ratio for sizes $n=2, 3, 4$, but laying less stress on the general treatment. In the present note he intends to investigate the matter more generally. If the sample mean and S. D. be $\bar{x}$ and s, there are the following $(n-1)$ subcases: I: $0<\tau=s/\bar{x}<1/\sqrt{n-1}$, II: $1/\sqrt{n-1}<\tau<\sqrt{2/(n-2)}$, III: $\sqrt{2/(n-2)}<\tau<\sqrt{3/(n-3)}$, ……, the final $(n-1)$-th: $\sqrt{(n-2)/2}<\tau<\sqrt{n-1}$. The partial volume-element dV_I, dV_{II},…… can be formulated without much difficulty. Nevertheless its actual determinations for several concrete values of $n\geq 5$ are so seriously intricate that below the special case $n=5$ only could be newly examplified.

13. *The Volume-Element for the Sample Mean and S. D. taken from a Universe $f(x)$ with $x>0$.* Let the sample point drawn from the universe be $P(x_1,\cdots, x_n)$. The elementary probabilities are

$$dp=f(x_1)\cdots f(x_n)dx_1\cdots dx_n=g(\bar{x}, s)dv, \quad \text{and} \tag{13.1}$$

$$dP=f(\bar{x}, s)d\bar{x}ds=g(\bar{x}, s)dV \quad \left(=\int dv \text{ taken as } \bar{x}, s \text{ determinate}\right), \tag{13.2}$$

where the product $f(x_1)\cdots f(x_n)$ is assumed to reduce to a certain function of the fundamental symmetrical sums $\sum x_i=n\bar{x}$ and $\sum x_i^2=n(s^2+\bar{x}^2)$.

Now after Cramér[2] we transform $x_1,\cdots, x_n$ orthogonally into $y_1,\cdots, y_n$, e.g. as

$$y_1=(-x_1+x_2)/\sqrt{2}=(-\xi_1+\xi_2)/\sqrt{2}, \text{ where and below } \xi_i=x_i-\bar{x},$$

$$y_2=(-x_1-x_2+2x_3)/\sqrt{6}=(-\xi_1-\xi_2+2\xi_3)/\sqrt{6}, \tag{13.3}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y_i=(-x_1-\cdots-x_i+ix_{i+1})/\sqrt{i(i+1)}=(-\xi_1-\cdots-\xi_i+i\xi_{i+1})/\sqrt{i(i+1)},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

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1) Y. Watanabe, Some exceptional examples to Student's distribution, Jour. of Gakugei, Tokushima Univ., vol. X (1959); The Student's distribution for a universe bounded at one or both sides, ibid., vol. XI (1960) and its continuation, ibid., vol. XII (1962). They are below referred as [I], [II] and [III], respectively.

2) H. Cramér, Mathematical Methods of Statistics, p. 383—, where Prof. Cramér treats in two different ways, one statistically, one geometrico-analytically, after which latter the present author has followed, who wishes here to express his heartily gratitude to Prof. Cramér, whose treatise indeed so much helped the author to plan and perform his work.

$$y_{n-1} = (-x_1 - \cdots - x_{n-1} + (n-1)x_n)/\sqrt{(n-1)n}$$
$$= (-\xi_1 - \cdots - \xi_{n-1} + (n-1)\xi_n)/\sqrt{(n-1)n},$$
$$y_n = (x_1 + x_2 + \cdots + x_n)/\sqrt{n} = \sqrt{n}\,\bar{x}, \text{ so that } \sum_1^n (x_i - \bar{x}) = \sum_1^n \xi_i = 0.$$

Or, solved for $x_i - \bar{x} = \xi_i$, rewriting coefficients in columns to rows,

$$\xi_1 = x_1 - \bar{x} = -y_1/\sqrt{2} - y_2/\sqrt{6} - y_3/\sqrt{12} - \cdots$$
$$- y_{n-2}/\sqrt{(n-1)(n-2)} - y_{n-1}/\sqrt{n(n-1)},$$

(13.4) $$\xi_2 = x_2 - \bar{x} = +y_1/\sqrt{2} - y_2/\sqrt{6} - y_3/\sqrt{12} - \cdots$$
$$- y_{n-2}/\sqrt{(n-1)(n-2)} - y_{n-1}/\sqrt{n(n-1)},$$

$$\xi_3 = x_3 - \bar{x} = \quad + 2y_2/\sqrt{6} - y_3/\sqrt{12} - \cdots$$
$$- y_{n-2}/\sqrt{(n-1)(n-2)} - y_{n-1}/\sqrt{n(n-1)},$$

....

$$\xi_{n-1} = x_{n-1} - \bar{x} = \quad + (n-2)y_{n-2}/\sqrt{(n-1)(n-2)} - y_{n-1}/\sqrt{n(n-1)},$$
$$\xi_n = x_n - \bar{x} = \quad + (n-1)y_{n-1}/\sqrt{n(n-1)}.$$

Whence directly or by the known orthogonal property, we obtain

(13.5) $$\sum_1^n (x_i - \bar{x})^2 = \sum_1^{n-1} y_i^2 = ns^2,$$

as the equation to the $(n-1)$-dimensional[1)] spherical surface $\bar{K}_{n-1}$ of radius $\sqrt{n}\,s$ with center $G(\bar{x}, \cdots, \bar{x})$. Also y_i's equations (3) represent n hyperplanes, which two by two are perpendicular to each other. In particular, the last one expresses the $\bar{x}$-hyperplane, whose normal from the origin O inclines equally to all x_i-axis and along which the length $OG = \sqrt{n}\,\bar{x}$ being measured, the end point denotes the centroid $G(\bar{x}, \bar{x}, \cdots, \bar{x})$. Or, else, y_i denotes the i-th component of the vector OP with orthogonal components $x_1, x_2, \cdots, x_n$, which has the direction cosines $l_1 = l_2 = \cdots = l_i = -1/\sqrt{i(i+1)}$, $l_{i+1} = \sqrt{i/(i+1)}$, $l_{i+2} = \cdots = l_n = 0$. So that y_i's direction is precisely that of OY_i drawn through O with direction cosines $l_1, \cdots, l_n$. This is however by the description of the simplex S_{n-1} inconvenient, since we ignore here the origin and instead of which, the centroid G is predominated. Hence, if the centroid's translation $y_n = \sqrt{n}\,\bar{x}$ be for a while kept aside and considered only the rotation around it, we may imagine to have given $-\bar{x}$ to every x_i, so that G coincides with O and now the i-th component y_i may be regarded as the projection of GP on y_i's direction, as really described by ξ_i's in (3). We interpret these directions in the following ways: First, considered a vertex A_n of the simplex $S_{n-1} = A_1 A_2 \cdots A_n$, its height is the per-

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1) It is true that $\sum_1^n (x_i - \bar{x})^2 = ns^2$ denotes a n-dimensional sphere. However, since the sample point $(x_1, \cdots, x_n)$ is confined to satisfy not only this equation but also the equation $\sum_1^n x_i = n\bar{x}$ necessarily, the dimension is lowered to $n-1$.

pendicular from A_n on the base simplex $B_{n-2}=S_{n-2}^{(n)}=A_1A_2\cdots A_{n-1}$, or what is the same thing as the join of the vertex $A_n(0,\cdots, 0, n\bar{x})$ and $G_{n-2}^{(n)}(n\bar{x}/(n-1),\cdots, n\bar{x}/(n-1), 0)$, the centroid of the base $S_{n-2}^{(n)}$. In fact, its direction ratios being -1: -1:$\cdots$ -1: $(n-1)$, which are the same as y_{n-1}'s, we take this direction as $\zeta(=y_{n-1})$. Next, conceiving the base simplex $S_{n-2}^{(n)}=A_1A_2\cdots A_{n-1}$, its height against the further base $B_{n-3}=S_{n-3}^{(n,n-1)}=A_1A_2\cdots A_{n-2}$ is the join of $A_{n-1}(0,\cdots, n\bar{x}, 0)$ to the base center $G_{n-3}^{(n,n-1)}=(n\bar{x}/(n-2),\cdots, n\bar{x}/(n-2), 0, 0)$, but now converting the direction for a later convenience' sake, its direction ratios now become 1: 1:$\cdots$: $-(n-2)$: 0, which are those of y_{n-2}'s with all signs changed, and we shall denote it by $\eta_1(=-y_{n-2})$. Similarly, the following subsimplex $S_{n-4}^{(n,n-1,n-2)}=A_1A_2\cdots A_{n-3}$ has its height $A_{n-3}G_{n-4}$ with direction ratios when the sense converted: 1: 1:$\cdots$: $-(n-3)$: 0: 0, which are those of y_{n-3}'s with the signs changed, and we shall denote it by $\eta_2(=-y_{n-3})$, and so on. Lastly the subsimplex $S_1^{(n,n-1,\cdots,3)}=A_1A_2$ has its height A_2A_1 with direction cosines $1/\sqrt{2}$, $-1/\sqrt{2}$ and we put $\eta_{n-2}=-y_1$. In the accompanying Fig. 1, it might become a left-handed system by some choice of numbering vertices. Thus

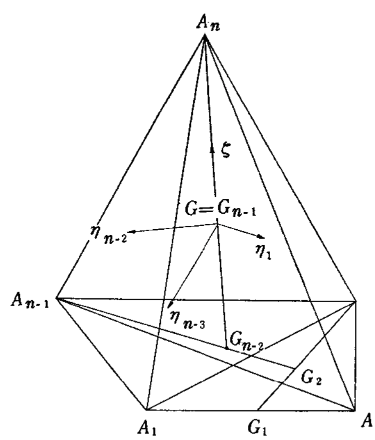

Fig. 1.

$$
\begin{aligned}
\zeta &= y_{n-1}=[-x_1-x_2-\cdots-x_{n-1}+(n-1)x_n]/\sqrt{n(n-1)},\\
\eta_1 &= -y_{n-2}=[x_1+x_2+\cdots+x_{n-2}-(n-2)x_{n-1}]/\sqrt{(n-1)(n-2)},\\
\eta_2 &= -y_{n-3}=[x_1+\cdots+x_{n-3}-(n-3)x_{n-2}]/\sqrt{(n-2)(n-3)},\\
\eta_3 &= -y_{n-4}=[x_1+\cdots+x_{n-4}-(n-4)x_{n-3}]/\sqrt{(n-3)(n-4)}, \qquad (13.6)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\eta_{n-3} &= -y_2=(x_1+x_2-2x_3)/\sqrt{6},\\
\eta_{n-2} &= -y_1=(x_1-x_2)/\sqrt{2}.
\end{aligned}
$$

However, in certain circumstances, ζ shall be conveniently written as η_0 or

η_{n-1}. Naturally all these directions form a $(n-1)$ rectangular co-ordinates axes, G as origin and they suffice to describe S_{n-1} and $\bar{K}_{n-1}$. We have also in view of (6) and (4)

$$\zeta^2 + \sum_1^{n-2} \eta_i^2 = \sum_1^{n-1} y_i^2 = ns^2 \tag{13.7}$$

as the equation to the s-spherical surface $\bar{K}_{n-1}$. If its half, upper or lower according as $\zeta \gtrless 0$, be projected orthogonally on $\zeta=0$, it becomes a $(n-2)$ dimensional spherical solid

$$\sum_1^{n-2} \eta_i^2 = \rho^2 = ns^2 - \zeta^2 \qquad (0 < \zeta < \sqrt{n}s). \tag{13.8}$$

We write further

$$\begin{aligned}
\eta_1 &= \rho \cos\theta_1 \cos\theta_2 \cos\theta_3 \cdots\cdots \cos\theta_{n-4} \cos\theta_{n-3} \\
\eta_2 &= \rho \sin\theta_1 \cos\theta_2 \cos\theta_3 \cdots\cdots \cos\theta_{n-4} \cos\theta_{n-3} \\
\eta_3 &= \qquad \sin\theta_2 \cos\theta_3 \cdots\cdots \cos\theta_{n-4} \cos\theta_{n-3} \\
&\cdots\cdots\cdots\cdots \\
\eta_{i+1} &= \qquad \rho \sin\theta_i \cos\theta_{i+1} \cdots\cdots \cos\theta_{n-4} \cos\theta_{n-3} \\
&\cdots\cdots\cdots\cdots \\
\eta_{n-3} &= \qquad \rho \sin\theta_{n-4} \cos\theta_{n-3} \\
\eta_{n-2} &= \qquad \rho \sin\theta_{n-3},
\end{aligned} \tag{13.9}$$

where $\rho = GQ$ is the projection of GP on $\zeta=0$ and θ_i denotes the angle which the projection of GQ on the $(i+1)$-dimensional subspace formed by $\eta_1, \cdots, \eta_{i+1}$ makes with the i-dimensional subspace formed by $\eta_1, \cdots, \eta_i$, $i=1, 2, \cdots, n-3$; and $0<\theta_1<2\pi$ or $-\pi<\theta_1<\pi$, but all other θ_i's are between $\pm\pi/2$.

In the foregoing successive transformations, the Jacobians being

$$\frac{\partial(x_1, \cdots, x_n)}{\partial(y_1, \cdots, y_n)} = (-1)^{n-1}, \quad \frac{\partial(y_1, \cdots, y_{n-1}=\zeta, y_n=\sqrt{n}\,\bar{x})}{\partial(\eta_{n-2}, \cdots, \eta_1, \zeta, \bar{x})} = (-1)^n\sqrt{n}, \tag{13.10}$$

we have

$$dv = dx_1 \cdots dx_n = dy_1 \cdots dy_n = \sqrt{n}\, d\bar{x}\, d\zeta\, d\eta_1 \cdots d\eta_{n-2}.$$

Furthermore

$$\frac{\partial(\eta_1, \eta_2, \eta_3, \cdots, \eta_{n-2}, \zeta, \bar{x})}{\partial(\rho, \theta_1, \theta_2, \cdots, \theta_{n-3}, s, \bar{x})} = \frac{ns}{\zeta} \rho^{n-3} \cos\theta_2 \cos^2\theta_3 \cdots \cos^{n-4}\theta_{n-3}, \tag{13.11}$$

where $\zeta = \pm\sqrt{ns^2-\rho^2}$, whose signs can be both taken, and thus obtained two systems yield the same value of the probability element dP, so that the Jacobian should be multiplied by 2[1]. Consequently we have

1) Cramér. loc. cit., p. 385.

(13.12) $dv = 2\sqrt{n}\,d\bar{x}\sqrt{n}\,ds\sqrt{\dfrac{ns^2}{ns^2-\rho^2}}\,\rho^{n-3}\cos\theta_2\cdots\cos^{n-4}\theta_{n-3}\,d\rho\,d\theta_1\,d\theta_2\cdots d\theta_{n-3}$

$= n\,d\bar{x}\,ds\,dF_{n-2},$

where dF_{n-2} denotes the surface-element of the s-sphere $\bar{K}_{n-1}$, which lies in S_{n-1} and whose integral extended over all possible arguments yields $F_{n-2}(= S_{n-1}\cap\bar{K}_{n-1})$. Hence the required volume-element is

(13.13) $$dV = F_{n-2}\,n\,d\bar{x}\,ds \quad \text{with}$$

(13.14) $$F_{n-2}(\bar{x}, s) = 2\int_{\alpha_1}^{\beta_1} d\theta_1 \int_{\alpha_2}^{\beta_2}\cos\theta_2\,d\theta_2\cdots\int_{\alpha_{n-3}}^{\beta_{n-3}}\cos^{n-4}\theta_{n-3}\,d\theta_{n-3}\int_0^{\rho_1}\sqrt{\frac{ns^2}{ns^2-\rho^2}}\,\rho^{n-3}\,d\rho,$$

whose innermost integral may be replaced also by $(\sqrt{n}\,s)^{n-2}\int_0^{\psi_1}\sin^{n-3}\psi\,d\psi$ on putting $\rho = \sqrt{n}\,s\sin\psi$, $\psi_1 = \sin^{-1}\rho_1/\sqrt{n}\,s$.

It remains only to determine the limits of integrations $\alpha_i\ \beta_i$ in conformity with the various subcases, cf. [I]: I: $0<\tau=s/\bar{x}<1/\sqrt{n-1}$, II: $1/\sqrt{n-1}<\tau<\sqrt{2/(n-2)}$, ..., the finale: $\sqrt{(n-2)/2}<\tau<\sqrt{n-1}$.

First, the opening scene is

I: $0<\sqrt{n}\,s<\sqrt{n/(n-1)}\,\bar{x}=GG_{n-2}$, so that the whole s-sphere lies inside S_{n-1}. But the whole area of the $(n-1)$-dimensional sphere of radius $\rho_1=\sqrt{n}\,s$ being as well known,

(13.15) $$F_{n-2,\,I} = 2\sqrt{\pi}^{\,n-1}(\sqrt{n}\,s)^{n-2}/\Gamma((n-1)/2),$$

the required volume element is given by

(13.16) $$dV_{n,\,I} = nF_{n-2,\,I}\,d\bar{x}\,ds = 2\sqrt{n}^{\,n}\sqrt{\pi}^{\,n-1}s^{n-2}\,d\bar{x}\,ds/\Gamma((n-1)/2),$$

which is the well-known Fisher's formula. Otherwise, we obtain (15) immediately on writing the evident limits of integrations in (14)

(13.17) $$F_{n-2,\,I} = 2\int_0^{2\pi} d\theta_1\int_{-\pi/2}^{\pi/2}\cos\theta_2\,d\theta_2\cdots\int_{-\pi/2}^{\pi/2}\cos^{n-4}\theta_{n-3}\,d\theta_{n-3}\int_0^{\pi/2}(\sqrt{n}\,s)^{n-2}\sin^{n-3}\psi\,d\psi,$$

and employing the formula $2\int_0^{\pi/2}\cos^m\theta\,d\theta = \sqrt{\pi}\,\Gamma\left(\frac{m+1}{2}\right)\Big/\Gamma\left(\frac{m+2}{2}\right)$, for $m=0, 1,\cdots, n-3$. For example, if $n=4$, 5, we have

(13.18) $$F_{2,I} = 16\pi s^2,\quad dV_{4,\,I} = F_{2,\,I}\,d(2\bar{x})\,d(2s) = 64\pi s^2\,d\bar{x}\,ds,$$

(13.19) $$F_{3,\,I} = 10\pi^2\sqrt{5}\,s^3,\quad dV_{5,\,I} = F_{3,\,I}\,d(\sqrt{5}\,\bar{x})\,d(\sqrt{5}\,s) = 50\sqrt{5}\,\pi^2 s^3\,d\bar{x}\,ds.$$

For the following several subcases, however, it is too intricate to determine actually every α_i, β_i in (14). More preferably we would consider adequate corrections of F_I to be made for successive subcases, as will be seen in the subsequent section. However the final subcase

(13.20) $$\sqrt{(n-2)/2} < s/\bar{x} = \tau < \sqrt{n-1}$$

would especially be treated, since here the determination of limits of integrations can be found with ease. Observing that there are n vertices in the simplex $S_{n-1} = A_1 A_2 \cdots A_n$ and at every of them $(n-1)$ side-simplexes S_{n-2} meet symmetrically, we may only consider that surface σ_{n-2} bounded by a said S_{n-2} at a said vertex A_n, or its half by symmetry. Consequently the whole surface is

(13.21) $$F_{n-2,\ \text{finale}} = 2n(n-1)\sigma_{n-2},$$

so that

(13.22) $$dV_{n,\ \text{finale}} = 2n^2(n-1)\sigma_{n-2}\,d\bar{x}\,ds.$$

But, since $A'_1, \cdots, A'_{n-1}$, the projections of $A_1, \cdots, A_{n-1}$ on the $\eta_1\eta_2$-plane, which is perpendicular to $GA_n = \zeta$-axis, situate symmetrically about G, e.g. $\angle A'_1 G A'_2 = 2\pi/(n-1)$ and its half is $\pi/(n-1)$. Hence, we have $\beta_1 = \pi/(n-1)$, but α_1 as well as all α_i are 0. First, considering the innermost integral of (14)

(13.23) $$I_1 = \int_0^{\rho_1} \sqrt{\frac{ns^2}{ns^2-\rho^2}}\,\rho^{n-3}\,d\rho,$$

the upper limit is clearly the radius vector of the boundary of s-sphere $\bar{K}_{n-1}$: $\rho^2 : +\zeta^2 = ns^2$ cut out by a side face $S_{n-2}^{(n-1)} = A_1 \cdots A_{n-2} A_n (x_{n-1} = 0)$, whose equation will be obtained as follows: The axis or height of $S_{n-2}^{(n-1)}$ is $h_{n-2} = A_n G_{n-3}^{(n,n-1)}$, which is the join of the vertex A_n to the centroid of its base $B_{n-3} = S_{n-3}^{(n,n-1)} = A_1 \cdots A_{n-2}$, where $A_n = (0, 0, \cdots, 0, n\bar{x})$ and $G_{n-3}^{(n,n-1)} = (n\bar{x}/(n-2), \cdots, n\bar{x}/(n-2), 0, 0)$. Therefore,, the equations to h_{n-2} become $x_1 = x_2 = \cdots = x_{n-2} = (n\bar{x} - x_n)/(n-2)$ and $x_{n-1} = 0$. Or, in view of (6) these reduce to

(13.24) $$\zeta = \sqrt{n(n-1)}\,\bar{x} - \sqrt{n(n-2)}\,\eta_1, \quad \eta_2 = \cdots = \eta_{n-2} = 0.$$

But this height h_{n-2} being perpendicular to the base B_{n-3}, which is parallel to the subspace formed by η_2-, η_3-, $\cdots$, η_{n-2}-axis, the side simplex $S_{n-2}^{(n-1)}$ is generated by the parallels to h_{n-2} drawn through every point of B_{n-3}, so that the coordinates of any point P in $S_{n-2}^{(n-1)}$ satisfy the first one of (24), but with variable $\eta_2, \cdots, \eta_{n-2}$. Thus the first equation, i.e.

(13.25) $$\zeta = \sqrt{n(n-1)}\,\bar{x} - \sqrt{n(n-2)}\,\rho\cos\theta_1 \cdots \cos\theta_{n-3}$$

may be regarded as that of $S_{n-2}^{(n-1)}$, rather with its prolonged subspace together. Hence, its intersection with $\bar{K}_{n-1} : \zeta^2 + \rho^2 = ns^2$ is given by

(13.26) $$\rho^2 + n(\sqrt{n-1}\,\bar{x} - \sqrt{n-2}\,\rho\cos\theta_1 \cdots \cos\theta_{n-3})^2 = ns^2;$$

or, if $\cos\theta_i = u_i$,

$$[1 + n(n-2)u_1^2 \cdots u_{n-3}^2]\rho^2 - 2n\sqrt{(n-1)(n-2)}\,\bar{x}\rho u_1 \cdots u_{n-3} + n[(n-1)\bar{x}^2 - s^2] = 0.$$

This being solved for ρ and its smaller root selected, because the greater one corresponds to those about other remote vertices at farther side, we have, as the upper limit of I_1,

(13.27) $$\rho_1 = (n\sqrt{(n-1)(n-2)}u_1\cdots u_{n-3}\bar{x} - \sqrt{E})/F,$$

where

(13.28) $$E = n[Fs^2 - (n-1)\bar{x}^2], \quad F = 1 + n(n-2)u_1^2\cdots u_{n-3}^2.$$

For its integration, we write $\rho = \sqrt{n}\,s \sin\psi$ and obtain

(13.29) $$I_1 = (\sqrt{n}\,s)^{n-2}\int_0^{\psi_1} \sin^{n-3}\psi\, d\psi$$

where $\psi_1 = \sin^{-1}\rho_1/\sqrt{n}\,s$ and

(13.30) $$\sin\psi_1 = [\sqrt{n(n-1)(n-2)}u_1\cdots u_{n-3}\bar{x} - \sqrt{E/n}]/Fs,$$

(13.31) $$\cos\psi_1 = [\sqrt{n-1}\,\bar{x} + \sqrt{n-2}\,u_1\cdots u_{n-3}\sqrt{E}]/Fs.$$

When all u_i's$=1$, the radical expression $\sqrt{E}$ reduces to $\sqrt{n(n-1)[(n-1)s^2 - \bar{x}^2]}$, which is real positive after (20) and $\sin\psi_1$, $\cos\psi_1$ become non-negative fractions and accordingly ψ_1 becomes an acute angle. Thus every upper limit of u_i is all unity, i.e. $\alpha_i = 0$. However, if one of u_i be $=0$ ($\beta_i = \pi/2$), the radical expression $\sqrt{E}$ reduces to $\sqrt{n(s^2-(n-1)\bar{x}^2)}$, which is imaginary. Therefore, in the existing continuous domain, each u_i should be between 1 and a certain positive fraction δ_i. In general, to make ρ real, we must have $E \geq 0$, i.e.

(13.32) $$n(n-2)u_1^2\cdots u_{n-3}^2 s^2 \geq (n-1)\bar{x}^2 - s^2 (\geq 0).$$

Hence

(13.33) $$1 \geq u_1 \geq u_1u_2 \geq \cdots \geq u_1u_2\cdots u_{n-3} \geq \sqrt{\frac{(n-1)\bar{x}^2 - s^2}{n(n-2)s^2}} = \sqrt{\frac{n-1-\tau^2}{n(n-2)\tau^2}} = \delta,$$

where $0 \leq \delta \leq 1/(n-2) < 1$, because of (20). Therefore, for δ_i, the lower limit of u_i, we should have

(13.34) $$1 \geq u_i \geq \frac{\sqrt{(n-1)\bar{x}^2 - s^2}}{\sqrt{n(n-2)}\,u_1\cdots u_{i-1}s} = \frac{\delta}{u_1\cdots u_{i-1}} = \delta_i \quad (i = 2, 3, \cdots, n-3),$$

and $u_1 \geq \delta_1 = \cos\pi/(n-1)$. Consequently we get the main factor of F_{n-2}

(13.35) $$\sigma_{n-2} = (\sqrt{n}\,s)^{n-2}\int_{\delta_1}^1 \frac{du_1}{\sqrt{1-u_1^2}}\int_{\delta_2}^1 \frac{u_2 du_2}{\sqrt{1-u_2^2}}\cdots\int_{\delta_{n-3}}^1 \frac{u_{n-3}^{n-4}du_{n-3}}{\sqrt{1-u_{n-3}^2}}\int_0^{\psi_1}\sin^{n-3}\psi\, d\psi.$$

In particular, for the ending value $\tau = \sqrt{n-1}$, ρ_1 as well as ψ_1 reduces to naught and we have $\sigma_{n-2} = 0$, as a matter of course. Thus the general formulation is rather plainly accomplished. Notwithstanding, the actual computation for the $(n-2)$-ple integral with a given $n \geq 5$ is enough intricate. For examples, we pick up only few cases:

Ex. 1. Case $n=3$, II: $1/\sqrt{2}<\tau<\sqrt{2}$, we have $\zeta^2+\eta^2=3s^2$, $\eta=\rho$ from (7) (8) but none for (9), so that (35) reduces to

$$\sigma_1=\int_0^{\rho_1}\sqrt{\frac{3s^2}{3s^2-\rho^2}}\,d\rho=\sqrt{3}s\int_0^{\psi_1}d\psi=\sqrt{3}s\psi_1,\quad \text{on putting}\quad \rho=\sqrt{3}s\sin\psi.$$

Here $\rho_1(=\eta_1)$ is found by eliminating ζ between (24) $\zeta=\sqrt{6}\bar{x}-\sqrt{3}\eta$ and $\zeta^2+\eta^2=3s^2$ to be

$$\rho_1=\eta_1=[3\bar{x}-\sqrt{3(2s^2-\bar{x}^2)}]/2\sqrt{2}$$

Hence

$$\sin\psi_1=[\sqrt{3}-\sqrt{2\tau^2-1}]/2\sqrt{2}\tau,\quad \cos\psi_1=[1+\sqrt{3(2\tau^2-1)}]/2\sqrt{2}\tau.$$

And whence $\cos(\pi/3-\psi_1)=1/\sqrt{2}\tau$, i.e. $\psi_1=\pi/3-\cos^{-1}1/\sqrt{2}\tau$. Thus obtained σ_1 being multiplied by 6, because there are 3 vertices and we have halved at vertex. Hence

$$(13.36)\quad F_{1,II}=6\sqrt{3}s(\pi/3-\cos 1/\sqrt{2}\tau),\quad dV_{3,II}=F_{1,II}\,d(\sqrt{3}\bar{x})\,d(\sqrt{3}s),$$

which coincides with (3.9) in [1].

Ex. 2. For $n=4$, III: $1<\tau<\sqrt{3}$, (35) delivers

$$\sigma_2=4s^2\int_{\delta_1}^1\frac{du}{\sqrt{1-u^2}}\int_0^{\psi_1}\sin\psi\,d\psi,$$

where $u=\cos\theta$, $\delta_1=\cos\pi/3$, $\cos\psi_1=(\sqrt{3}\bar{x}+\sqrt{2}\cos\theta\cdot\sqrt{E})/Fs$, where $E=4(1+8\cos^2\theta)s^2-12\bar{x}^2$, $F=1+8\cos^2\theta$ after (28) (30) (31). Executing the integration, we get

$$\sigma_2=4s^2\left[\pi\left(\frac{\bar{x}}{3\sqrt{3}s}-\frac{1}{6}\right)-\frac{\bar{x}}{\sqrt{3}s}\tan^{-1}\sqrt{\frac{3(s^2-x^2)}{2\bar{x}^2}}+\tan^{-1}\sqrt{\frac{s^2-x^2}{2s^2}}\right],$$

which multiplied by $2n(n-1)=24$ yields

$$(13.37)\quad F_{2,III}=96s^2\left[\frac{\pi}{6}\left(\frac{2}{\sqrt{3}\tau}-1\right)-\frac{1}{\sqrt{3}\tau}\tan^{-1}\sqrt{\frac{3}{2}(\tau^2-1)}+\tan^{-1}\sqrt{\frac{1}{2}\left(1-\frac{1}{\tau^2}\right)}\right].$$

This multiplied by $4d\bar{x}ds$ becomes (3.12) in [I].

Ex. 3. In case $n=5$, IV: $\sqrt{3/2}<\tau<2$, a much more intricate results are outlined as follows: We have after (35)

(13.38) $$\sigma_3 = 5\sqrt{5}s^3\int_{1/\sqrt{2}}^{1}\frac{du}{\sqrt{1-u^2}}\int_{v_0}^{1}\frac{vdv}{\sqrt{1-v^2}}\int_0^{\psi_1}\sin^2\psi d\psi (\equiv J \text{ say}),$$

where

$$v_0 = \sqrt{\frac{4-\tau^2}{15u^2\tau^2}},\quad \psi_1 = \sin^{-1}\frac{\rho_1}{\sqrt{5}\,s},$$

$$\rho_1 = \frac{10\sqrt{3}uv\bar{x} - \sqrt{5[(1+15u^2v^2)s^2 - 4\bar{x}^2]}}{1+15u^2v^2}.$$

This triple integral on putting $p = uv$ becomes

(13.39) $$J = \int_{1/\sqrt{2}}^{1}\frac{du}{u\sqrt{1-u^2}}\int_{p_0}^{u}(\psi_1 - \sin\psi_1\cos\psi_1)\frac{pdp}{2\sqrt{u^2-p^2}},$$

where

(13.40) $$p_0 = \sqrt{\frac{4-\tau^2}{15\tau^2}},\quad \rho_1 = \frac{10\sqrt{3}p\bar{x} - \sqrt{5X}}{1+15p^2},\quad X = (1+15p^2)s^2 - 4\bar{x}^2.$$

The inner integral of J integrated by parts yields

(13.41) $$\frac{1}{2}\sqrt{u^2-p_0^2}\left(\cos^{-1}\frac{\tau}{2} - \frac{\tau}{2}\sqrt{1-\frac{\tau^2}{4}}\right) + \int_{p_0}^{u}\sqrt{u^2-p^2}\,R\,dp.$$

Because X vanishes for $p=p_0$, we get $\rho_{10} = \rho_1(p=p_0) = s\sqrt{5(4-\tau^2)}/2$ and $\psi_1 = \cos^{-1}\tau/2$. Hence, the integrated parts, when further integrated about u, yield, as the first component of J,

(13.42) $$J_1 = U(\tau)\left[\frac{\pi}{2}(1-p_0) - T(p_0)\right],$$

where

(13.43) $$U(\tau) = \frac{1}{2}\left(\cos^{-1}\frac{\tau}{2} - \frac{\tau}{2}\sqrt{1-\frac{\tau^2}{4}}\right), \text{ and}$$

$$T(p) = \tan^{-1}\sqrt{1-2p^2} - p\tan^{-1}\sqrt{1-2p^2}/p.$$

The not-yet integrated parts are

(13.44) $$J_2 = \int_{1/\sqrt{2}}^{1}\frac{du}{u\sqrt{1-u^2}}\int_{p_0}^{u}\sqrt{u^2-p^2}\,R\,dp,$$

where

$$R = \frac{1}{2}\frac{d}{d\psi_1}(\psi_1 - \sin\psi_1\cos\psi_1)\frac{\partial\psi_1}{\partial\rho_1}\frac{\partial\rho_1}{\partial p} = \frac{\rho_1^2}{5s^2\sqrt{15s^2-\rho_1^2}}\frac{\partial\rho_1}{\partial\rho},$$

which reduces after some calculations to

(13.45) $$R=\frac{P_0}{Q}+\frac{P_1}{Q\sqrt{X}},$$

where

(13.46) $$Q=(1+15p^2)^3,\quad \sqrt{X}=\bar{x}\tau\sqrt{15(p^2-p_0^2)},$$

(13.47) $$P_0=\sqrt{15}\left[1-\frac{4}{\tau^2}+\left(1+\frac{12}{\tau^2}\right)15p^2\right],$$

(13.48) $$P_1=-90\left[1-\frac{4}{\tau^2}+\left(1+\frac{4}{3\tau^2}\right)15p^2\right]p\bar{x}.$$

Interchanging the order of double integration in J_2, we have

(13.49) $$J_2=\frac{\pi}{2}\int_{p_0}^{1}(1-p)R\,dp-\int_{p_0}^{1/\sqrt{2}}T(p)R\,dp=V-W$$
$$=(V_0+V_1)-(W_0+W_1),$$

suffixes 0, 1 corresponding to P_0 and P_1 respectively. They become after all

(13.50) $$V(\tau)=\frac{\pi}{4}\left[\frac{1}{\sqrt{15}}\left(\frac{\tau^2}{4}+\frac{1}{4\tau^2}-2\right)+\cos^{-1}\frac{1}{4}-2U(\tau)-2U\left(\frac{1}{\tau}\right)\right],$$

(13.51) $$W_0(\tau)=T(p_0)\left(\frac{\tau^2}{2}p_0-\frac{1}{2}\sqrt{15}p_0\right)+T'(p_0)\left(\frac{1}{2\sqrt{15}}+\tan^{-1}\sqrt{15}p_0\right)$$
$$+\frac{1}{8\tau^2}\left(\frac{1}{\sqrt{15}}\tan^{-1}\frac{\sqrt{1-2p_0^2}}{p_0}+\sqrt{\frac{15}{17}}\tan^{-1}\sqrt{\frac{1-2p_0^2}{17p_0^2}}\right)$$
$$+\int_0^{\zeta_1}\tan^{-1}\sqrt{\frac{15}{2}(1-\zeta^2)}\cdot\frac{d\zeta}{1+\zeta^2},$$

(13.52) $$W_1(\tau)=\left(\frac{0.0951}{\tau}-\frac{0.3802}{\tau^3}\right)+\left(\frac{0.7746}{\tau}-\frac{2.4099}{\tau^3}\right)\int_0^{\zeta_1}\frac{\tan^{-1}\sqrt{1-2p_0^2-\zeta^2}/p_0}{1+15p_0^2-\zeta^2}d\zeta$$
$$+\left(\frac{0.0004}{\tau^8}+\frac{0.0012}{\tau^{10}}\right)\int_{p_0}^{1/\sqrt{2}}\tan^{-1}\left(\frac{\tau}{2}\sqrt{15(p^2-p_0^2)}\right)\tan^{-1}\frac{\sqrt{1-2p^2}}{p}dp,$$

where $p_0=\sqrt{(4-\tau^2)/15\tau^2}$, $\zeta_1=\sqrt{17\tau^2-8/15\tau^2}$. Thus obtained J multiplied by $200\sqrt{5}s^3$ would yield $F_{3,IV}$. However, to complete the calculations, the last integral in (51) and two integrals in (52) are to be still computed e.g. by means of Taylor-Laurent series expansions, which are extremely troublesome. So that, unless some ingenious device about the method of literal integration corresponding to Gauss' method of numerical integration might be furnished or we could tabulate the values of $J(\tau)$ e.g. by use of the electronic calculator, the above obtained result should be less worthy. We would rather contrive to manage otherwise as in the following section.

14. *The Principle of Projection, the Correction to F_I for Subcases II, III,$\cdots$.*

Below we shall rewrite the preceding ζ by η_{n-1} for a while in order to facilitate the notation of summation, and re-establish the notion of projection. Let

$$(14.1)\qquad \sum_1^{n-1} a_\nu \eta_\nu = a_0 \quad \text{with} \quad \sum_1^{n-1} a_\nu^2 = a^2 > 0$$

be a hyperplane $H = H_{n-1}(\eta)$ in the $(n-1)$-dimensional space R_{n-1}. First, let us obtain the minimal distance from a fixed point $P = P(\xi_1, \cdots, \xi_{n-1})$ outside H to the surface, which is to find after Lagrange the absolute minimum of the squared distance

$$y = \sum_1^{n-1} (\eta_\nu - \xi_\nu)^2 - 2\lambda \Big(\sum_1^{n-1} a_\nu \eta_\nu - a_0\Big),$$

so that $\dfrac{\partial y}{\partial \eta_\nu} = 2(\eta_\nu - \xi_\nu) - 2\lambda a_\nu = 0$. Consequently

$$\eta_\nu - \xi_\nu = \lambda a_\nu, \quad \text{i.e.} \quad \eta_\nu = \xi_\nu + \lambda a_\nu, \quad \nu = 1, 2, \cdots, n-1.$$

Substituting these values in (1), we get $\lambda a^2 = a_0 - \sum_1^{n-1} a_\nu \xi_\nu$. Hence we have

$$(14.2)\qquad \eta_\nu = \xi_\nu + a_\nu \Big(a_0 - \sum_1^{n-1} a_\nu \xi_\nu\Big)/a^2 = \xi'_\nu \quad \text{say,}$$

where $a_0 - \sum a_\nu \xi_\nu \neq 0$ and $a^2 \neq 0$, so that $P'(\xi')$ on H is uniquely determined.

Furthermore, if $Q(\eta)$ be any point on H, we have

$$\begin{aligned} PQ^2 &= \sum(\xi_\nu - \eta_\nu)^2 = \sum(\xi_\nu - \xi'_\nu)^2 + \sum(\eta_\nu - \xi'_\nu)^2 - 2\sum(\xi_\nu - \xi'_\nu)(\eta_\nu - \xi'_\nu) \\ &= P'P^2 + P'Q^2 + 2\sum \lambda a_\nu (\eta - \xi'_\nu) \\ &= \quad '' \qquad '' \quad + 2\lambda\Big(\sum a_\nu \eta_\nu - \sum a_\nu \xi'_\nu\Big) \ (= a_0 - a_0 = 0 \text{ by } (1)). \end{aligned}$$

Therefore

$$(14.3)\qquad PQ^2 = P'P^2 + P'Q^2 \quad \text{and} \quad \angle PP'Q = \text{a right angle.}$$

Thus PP' being perpendicular to all straight lines drawn on H, we may call PP' the perpendicular or normal to H from P. Or, we say P is projected to P' on H. Its uniqueness should be especially remarked.

Naturally the converse i.e. that for a determinate P' on H there is only one point P, is not true, as a matter of course. However, to abbreviate words, we say frequently, P is the inverse projection of P'.

Moreover, any straight line which passes through P and meets H, intersects H only once, just likely as any straight line in R_3 behaves against the ordinary plane. For, if it meets H at two points Q_1 and Q_2, then $\angle PP'Q_1 = \angle PP'Q_2 =$ a right angle, what is absurd in Euclidian space unless $Q_1 = Q_2$. This explains why H is called a hyperplane in R_{n-1}.

As was stated at (13.24–25) as to $S_{n-2}^{(n-1)}$, the first subsimplex of S_{n-1} is nothing but a hyperplane H_{n-1}, what can be generally recognized about all other first subsimplexes, since they are symmetric in S_{n-1}, all things in regard to one special S_{n-2} must also hold to other S_{n-2}. The sum $\cup S_{n-1}^{(\nu)}$ surrounds S_{n-1} and serves as the boundary surface, since any ray issuing from the centroid G meets $\cup S_{n-2}^{(\nu)}$ once and only once, just as in the sphere K_{n-1} its surface $\bar{K}_{n-1}$ relates with its radius vector.

Next, let $P(\xi_\nu) \in B_{n-1} = B$, a bounded second hyperplane (the projecting hyperplane) with an equation

$$(14.4) \qquad \sum_1^{n-1} b_\nu \xi_\nu = b_0, \quad \text{but} \quad \sum_1^{n-1} \xi_\nu^2 < b^2, \quad \text{say.}$$

By projecting B on H, we obtain a third hyperplane (the projected hyperplane)

$$(14.5) \qquad \sum_1^{n-1} c_\nu \xi_\nu = c_0,$$

which yields by eliminating $\xi_1, \xi_2, \cdots, \xi_{n-1}$ between (2) and (4), or else (2) solved for ξ_ν and these being substituted in (4), generally. However, if $a_\mu = 0$ in (1), i.e. ξ_μ be lacking in (1), the corresponding $\xi'_\mu = \xi_\mu$ and $c_\mu = b_\mu$ follows. E.g. if (1) be merely

$$(14.6) \qquad \eta_{n-1} = -\bar{x}\sqrt{n/(n-1)} = a_0 \quad \text{(a first subsimplex } S_{n-2}^{(n)})$$

then (4) becomes $\sum_1^{n-2} b_\nu \xi'_\nu$ (lacking ξ'_{n-1})$= b_0 - a_0 b_{n-1} (= b'_0)$, that moreover somehow bounded. By the same reasoning, even if the projecting space $\mathfrak{B}$ is any surface of dimension $n-1$ at most with the equation $\beta(\xi_1, \cdots, \xi_{n-1}) = 0$, the projected space $\mathfrak{B}'$ becomes $\beta(\xi'_1, \cdots, \xi'_{n-1}, a_0) = 0$. In particular, if the projecting space be an inferior spherical segment of the $(n-1)$-dimensional sphere $\bar{K}_{n-1}$

$$(14.7) \qquad \sum_1^{n-1} \xi_\nu^2 = ns^2$$

cut out by the hyperplane (6), its projection on (6) becomes $\sum_1^{n-2} \xi_\nu'^2 = ns^2 - \xi_{n-1}^2$ with $\sqrt{\frac{n}{n-1}}\bar{x} \leq \xi_{n-1} \leq \sqrt{n}s$, that is, a $(n-2)$-dimensional spherical solid

$$(14.8) \qquad \sum_1^{n-2} \eta_\nu^2 = ns^2 - \frac{n}{n-1}\bar{x}^2 (= (n-1)s'^2).$$

Here the last dashing mark relates merely to the first subsimplex $S' = S_{n-2}^{(\nu)}$, but by no means to projection. Similarly, coordinates of the centroid of S' being $(n\bar{x}/(n-1), \cdots, n\bar{x}/(n-1), 0)$, it may briefly be written as $G'(\bar{x}', \cdots, \bar{x}', 0)$ in accordance with

$$(14.9) \qquad n\bar{x} = (n-1)\bar{x}'$$

and further by (8) and (9) it follows that

(14.10) $\tau' = s'/\bar{x}' = \sqrt{((n-1)\tau^2-1)/n}$ e.g. $\tau' = \sqrt{(4\tau^2-1)/5}$ for $n=5$.

Now we return to our main problem: Supposing a surface element $d\sigma$ at a point $P(\xi_\nu)$, e.g. that of the s-sphere $\bar{K}_{n-1}: \zeta^2+\rho^2=\sum_1^{n-1}\eta_\nu^2=ns^2$ is projected on H_{n-1}, hyperplane (1), its projected elementary area $d\sigma'$ at $P'(\xi'_\nu)$ is given by

(14.11) $$d\sigma' = d\sigma \cos\gamma,$$

where γ denotes the angle the sphere's outward normal GPN makes with that of H_{n-1}. The equation to H_{n-1}, (1) may be written after Hesse $\sum_1^{n-1} l_\nu\eta_\nu = p$, where $p=GG'$ expresses the normal drawn from the origin G on H_{n-1} and l_ν denotes the ν-th direction cosine of $GG': l_\nu = \cos\varphi_\nu = a_\nu/\sqrt{\sum a_\nu^2}$, $\nu=1,\cdots,n-1$, while those for $\bar{K}_{n-1}$ are $\lambda_\nu=\xi_\nu/\sqrt{n}s$. Hence

(14.12) $$\cos\gamma = \sum_1^{n-1}\lambda_\nu l_\nu = \sum_1^{n-1} a_\nu\xi_\nu/s\sqrt{n\sum a_\nu^2}.$$

In particular, if H_{n-1} be $\zeta=a_0$, i.e. $\eta_{n-1}=a_0$, but all other η_ν arbitrary, then H_{n-1} reduces to the whole $(\eta_1,\cdots,\eta_{n-2})$ coordinates system itself, i.e. an entire $(n-2)$-dimensional space. In this particular case $a_1=\cdots=a_{n-2}=0$, $a_{n-1}=1$, $\sum a_\nu^2=1$ and the normal GG' coincides with the ζ-direction. Hence, for the s-sphere $\bar{K}_{n-1}: \zeta^2=ns^2-\rho^2$ we get simply

(14.12.1) $$\cos\gamma = \xi_{n-1}/\sqrt{n}s = \zeta/\sqrt{n}s = \sqrt{ns^2-\rho^2}/\sqrt{n}s$$

and γ is the angle between the radius vector GP and ζ-axis.

The truthfulness of (11) can be shown as follows: generally H_{n-1} contains $(n-1)$ unknowns and one equation, so that it is a $(n-2)$-dimensional space, and its surface element shall be given by $d\sigma'=dx'_1\cdots dx'_{n-2}$ at P', in whatsoever way we may take the $(n-2)$ rectangular P'-$(x'_1,\cdots,x'_{n-2})$ co-ordinates axes. On the other hand $\bar{K}_{n-1}$ being a $(n-1)$-dimensional surface, its surface element can be replaced approximately by that of the tangential hyperplane T_{n-1}: $\sum_1^{n-1} y_\nu\eta_\nu=ns^2$. Hence, the corresponding surface element at P shall be represented as $d\sigma=dx_1\cdots dx_{n-2}$ also. Now it is in our power to select these x-, x'-systems suitably. For a moment we rotate the G-$(\eta_1,\cdots,\eta_{n-2},\eta_{n-1}=\zeta)$ co-ordinates system to a new G-$(x_1,\cdots,x_{n-2},x_{n-1})$ system and further translate to P-$(x_1,\cdots,x_{n-1})$ system, so as the x_{n-1}-axis to coincide with the radius vector GPN and the x_1-axis to lie in the plane PGG' and perpendicular to PN, while, correspondingly G'-$(x'_1,\cdots,x'_{n-1})$ system (or translated to P'-$(x'_1,\cdots,x'_{n-1})$), naturally so as the x'_{n-1}-axis to coincide with $G'G$ in direction and x'_1-axis to lie in the plane PGG' and perpendicular to $G'G$ so that $\angle(Gx_1, G'x'_1)=\angle(Gx_{n-1}, G'x'_{n-1})=\gamma$. All the remaining x_2-$,\cdots, x_{n-2}$-axes as well as x'_2-$,\cdots, x'_{n-2}$-axes being all perpendicular to the plane PGG', independent of γ, and their directions before and after projection remain the same. Therefore $dx_1 = dx'_1 \sec\gamma$, but $dx_\nu = dx'_\nu$ $(\nu = 2, \cdots, n-2)$. Hence we have $dx_1 dx_2 \cdots dx_{n-2} = dx'_1 \sec\gamma \cdot dx'_2 \cdots dx'_{n-2}$, that is

$$(14.11.1) \qquad d\sigma = d\sigma' \sec\gamma \qquad \text{Q.E.D.}$$

All the above being a prologue, we are now able to produce the correction formula for F_I necessary to obtain $F_{II}, \cdots$. In all subcases II, III, $\cdots$, the s-sphere $\bar{K}_{n-1}$ protrudes outside S_{n-1}, whose protruded portion accordingly should be subtracted from F_I in (13.15). First, for the subcase

II: $1/\sqrt{n-1} \lessdot \tau < \sqrt{2/(n-2)}$, since $GG_{n-2} = \sqrt{n/(n-1)}\bar{x} < \sqrt{n}s < \sqrt{2n/(n-2)}\bar{x} = GG_{n-3}$ (Fig. 2), the s-sphere protrudes partly over each of n faces of S_{n-1}, but, the protruding portions evidently have no common part. The area of the protruded calotte C_{n-2} can be found by an application of the principle of projection. Projecting the protruded part of $\bar{K}_{n-1}$, e.g. over the base simplex $S' = S^{(n)}_{n-2} = A_1 \cdots A_{n-1}$, whose centroid G' has the co-ordinates $(\eta_1 = \cdots = \eta_{n-2} = 0$, $\zeta = -\sqrt{n/(n-1)}\bar{x})$ after (13.6), and which is perpendicular to GG', produced A_nG (ζ-axis). The intersection of S' and $\bar{K}_{n-1}$ having the equations: $\zeta = -\sqrt{n/(n-1)}\bar{x}$, $\rho^2 = ns^2 - n\bar{x}^2/(n-1) = (n-1)s'^2$, it is a $(n-2)$ dimensional sphere $\bar{K}'_{n-2}$ with G' as center and radius $\rho_1 = \sqrt{n-1}s'$ (Fig. 2a), which wholly lies inside S', since $\sqrt{n-1}s' = \sqrt{ns^2 - n\bar{x}^2/(n-1)} < G_{n-2}G_{n-3} = \sqrt{GG^2_{n-3} - GG^2_{n-2}} = n\bar{x}/\sqrt{(n-1)(n-2)}$, because of II: $\tau = s/\bar{x} < \sqrt{2/(n-2)}$. Hence, the area of calotte is obtained as an inverse projection by applying (11) in aid of (13.15)

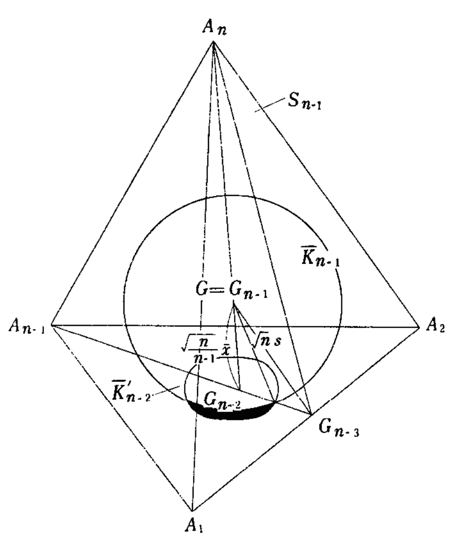

Fig. 2

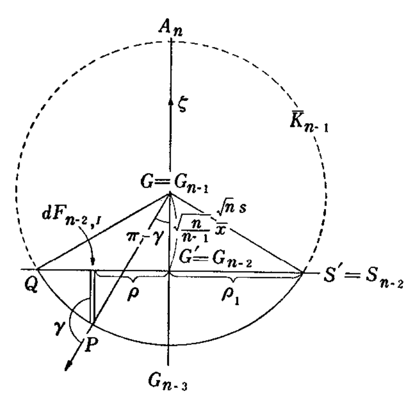

Fig. 2a

$$C_{n-2}=\int_{\text{inside } \bar{K}'}|\sec\gamma|\,dF_{n-3,I}=\int_0^{\rho_1}\frac{\sqrt{n}s}{\sqrt{ns^2-\rho^2}}\cdot 2\sqrt{\pi}^{n-2}\rho^{n-3}d\rho/\Gamma\left(\frac{n-2}{2}\right),$$

where $\rho_1=\sqrt{ns^2-n\bar{x}^2/(n-1)}$. Or, on writing $\rho=\sqrt{n}s\sin\psi$, $\psi_1=\cos^{-1}\bar{x}/\sqrt{n-1}s$, we obtain

$$(14.13)\qquad C_{n-2}=\frac{2\sqrt{n\pi s^2}^{\,n-2}}{\Gamma(n/2-1)}\int_0^{\psi_1}\sin^{n-3}\psi\,d\psi,$$

a generalized Archimedes' formula. When the integral performed indefinitely, it becomes

$$-\frac{\cos\psi\sin^{n-4}\psi}{n-3}-\frac{n-4}{(n-3)(n-5)}\cos\psi\sin^{n-6}\psi-\cdots\cdots\text{ending in}$$

$$+\frac{(n-4)(n-6)\cdots4\cdot2}{(n-3)(n-5)\cdots5\cdot3}(1-\cos\psi)\quad\text{or}\quad+\frac{(n-4)(n-6)\cdots3\cdot1}{(n-3)(n-5)\cdots4\cdot2}\psi,$$

according as n is even or odd, however, for $n=4$ or 3 the last coefficients being taken as unity. Thus, e.g. for $n=3$, 4, 5 we have

(14.13.1) $C_1=2\sqrt{3}s\psi_1=2\sqrt{3}s\cos^{-1}\bar{x}/\sqrt{2}s=2\sqrt{3}\bar{x}\tau\cos^{-1}(1/\sqrt{2}\tau)$ (cf. Ex. 1.),

(14.13.2) $C_2=8\pi s^2(1-\cos\psi_1)=2\pi(2s)(2s-2\bar{x}/\sqrt{3})$

$=$ (circumference) (height), i.e. Archimedes' proper formula[1],

(14.13.3) $C_3=10\sqrt{5}\pi s^3(\cos^{-1}1/2\tau-\sqrt{4\tau^2-1}/4\tau^2)$, and so on.

Subtracting nC_{n-2} from (13.15), we attain

$$(14.14)\quad F_{n-2,II}=\frac{2\sqrt{\pi}^{\,n-1}(\sqrt{n}s)^{n-2}}{\Gamma\left(\frac{n-1}{2}\right)}\left[1-\frac{n\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}-1\right)}\int_0^{\psi_1}\sin^{n-3}\psi\,d\psi\right]$$

$$=F_{n-2,I}(1-H_{n-2}(\tau))\quad\text{say},$$

where $\psi_1=\sec^{-1}\sqrt{n-1}\tau$. E.g. for $n=4$ and $n=5$, we have

$$(14.15)\qquad F_{2,II}=16\pi s^2[1-(2-2/\sqrt{3}\tau)]=16\pi s^2(2/\sqrt{3}\tau-1),\ \text{and}$$

$$(14.16)\qquad F_{3,II}=F_{3,I}(1-H_{3,I}(\tau))$$

$$=10\sqrt{5}\pi^2s^3\left[1-\frac{5}{\pi}\left(\cos^{-1}\frac{1}{2\tau}-\sqrt{4\tau^2-1}/4\tau^2\right)\right].$$

---

1) In Archimedes' proper formula, the height being quite arbitrary between 0 and radius $2s$, our formula seems to be less general. However, our $\tau=s/\bar{x}$ being arbitrary between $1/\sqrt{3}$ and $\sqrt{3}$, our height $=2s-2\bar{x}/\sqrt{3}$ is also arbitrary between 0 and $2s$ and thus just equally general as Archimedes'.

The volume-element dV_{II} is obtained by multiplying $F_{n-2,II}$ by $nd\bar{x}ds$.

We proceed to treat the subcase

III: $\sqrt{2/(n-2)}<\tau<\sqrt{3/(n-3)}$. Now that the calottes considered in II have some common portion, it needs to subtract the overlapping area O_{III} from nC_{n-2}, i.e. to add O_{III} into F_{II}. By its quadrature, a leading principle may be stated as follows: Since, the simplex S as well as spherical surface $\bar{K}$ are compact and convex (cf. section 1 in [I]), if some two curves or surfaces be both $\subset\mathfrak{S}=(S\frown\bar{K})$, and they are closed, then, the space bounded by the 2 boundaries ought to belong to $\mathfrak{S}$ also. For the sake of clarity, we consider preliminarily a particular

Ex. 4. Case $n=4$, III: $1<\tau<\sqrt{3}$. Here the s-sphere $\bar{K}_3:\zeta^2+\rho^2=4s^2$ intersects firstly the subsimplex S_1: e.g. A_1A_2 at 2 points P, Q, which may be understood as a linear circle K_1 (Fig. 3), secondly $S_2^{(4)}=A_1A_2A_3$ along a small circle $\bar{K}_2$ of radius $\rho_1=G_2P=G_2Q=\sqrt{4s^2-4\bar{x}^2/3}=\sqrt{3}s'$, and thirdly the plane of symmetry, an equator $A_0A_1A_2$, whose equation is $\zeta=m\eta_1$, where $m=-\tan\alpha/2=-1/\sqrt{2}$ along a great circle PQA_0, where A_0 is the middle point of A_3A_4 and α is the angle between G_1A_3, G_1A_4. Their protruding arcs, one, the small circle's arc PHQ with the central angle 2β, where $\beta=\angle PG_2G_1$ and $\cos\beta=G_1G_2/\sqrt{3}s'=\sqrt{2/(3\tau^2-1)}$, and the other, the great circle's arc PEQ together bound a protruding lunette $L_2\subset\bar{K}_3$. To find the area of L_2, we project it on the hyperplane $\zeta=0$ by which $\bar{K}_2$'s radius $\rho_1=\sqrt{3}s'$ remains the same, while the great circle PQA_0 becomes an ellipse E_2: $\eta_1^2\sec^2\alpha/2+\eta_2^2=4s^2$, or $\rho_0^2=4s^2/(1+1/2\cos^2\theta)$ with semiaxes $2s$ and $2s\sqrt{2/3}$, where $2s>\sqrt{3}s'>2s\sqrt{2/3}$, because of $\sqrt{3}>\tau>1$. The plane area σ' outside the ellipse but inside the circle, if projected on $\bar{K}_3$ inversely, yields the required area of lunette (Fig. 3a),

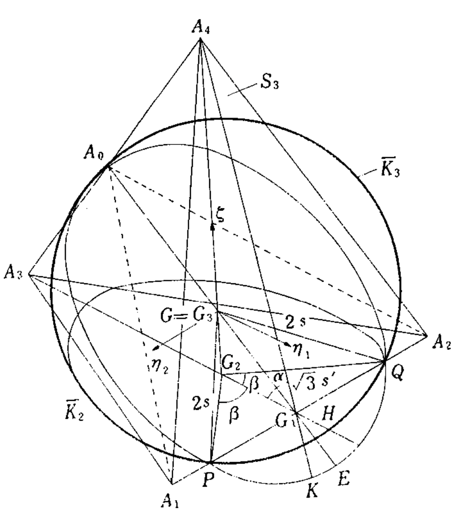

Fig. 3

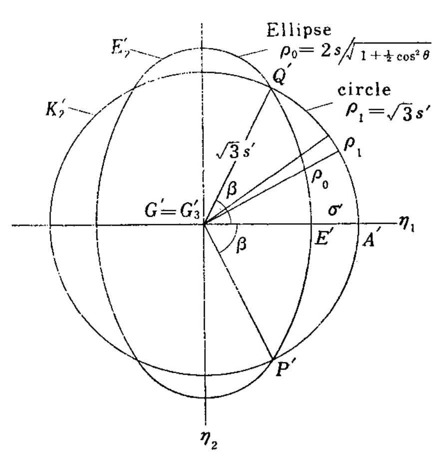

Fig. 3a

$$\sigma=2\int_0^{\beta}d\theta\int_{\rho_0}^{\rho_1}\frac{2s\rho d\rho}{\sqrt{4s^2-\rho^2}}=8s^2\left[\tan^{-1}\sqrt{\frac{1}{2}\left(1-\frac{1}{\tau^2}\right)}-\frac{1}{\tau\sqrt{3}}\tan^{-1}\sqrt{\frac{3}{2}(\tau^2-1)}\right].$$

Just similarly, the protruding arc over PQ of $\bar{K}_3$'s intersection with $S_2^{(3)}=A_1A_2A_4$ and arc PEQ again bound a congruent lunette of area σ, by reason of symmetry. These portions belong both to the protruding calottes over $S_2^{(3)}$ as well as $S_2^{(4)}$ and thus there gives rize an overlapping. Hence, if the formula F_{II} be applied to case III, we have subtracted too much. To correct this, we ought to add the whole overlapping $O_{III}=2\sigma\times$ number of sides$=12\sigma$. This being added to $F_{2,II}$ in (15), we get rightly

(14.17) $F_{2,III}=$

$$96s^2\left[\frac{\pi}{6}\left(\frac{2}{\tau\sqrt{3}}-1\right)+\tan^{-1}\sqrt{\frac{1}{2}\left(1-\frac{1}{\tau^2}\right)}-\frac{1}{\sqrt{3}\tau}\tan^{-1}\sqrt{\frac{3}{2}(\tau^2-1)}\right],$$

which agrees with (13.37).

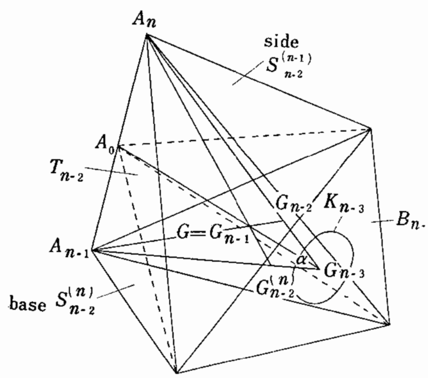

Fig. 4

Now speaking about the general simplex S_{n-1} $(n\geq 5)$ with a vertex A_n (Fig. 4), we conceive only 2 first subsimplexes, e.g. the base $S_{n-2}^{(n)}\left(\zeta=-\sqrt{\frac{n}{n-1}}\bar{x}\right)$ and a side $S_{n-2}^{(n-1)}(\zeta=\sqrt{n(n-1)}\bar{x}-\sqrt{n(n-2)}\eta_1)$, which have a common base $B_{n-3}=S_{n-3}^{(n;n-1)}$ with the equation $\eta_1=n\bar{x}/\sqrt{(n-1)(n-2)}$. Their axes $A_{n-1}G_{n-2}^{(n)}$, $A_nG_{n-2}^{(n-1)}$ both produced, meet at $G_{n-3}^{(n;n-1)}=G_{n-3}$. Besides they are both perpendicular to B_{n-3} and make an angle $\alpha=\sec^{-1}(n-1)$ with each other. We say figuratively the subsimplex $S_{n-2}^{(n-1)}$ comes on $S_{n-2}^{(n)}$ by a rotation α about the common base B_{n-3} as axis, this axis remaining in its initial situation invariablly, what is evident when $n=4$, $\alpha=\sec^{-1}3$ (Fig. 3). Also, if the base B_{n-3} be projected from A_0, the middle point of A_nA_{n-1}, thus obtained subspace is A_0-$B_{n-3}\equiv T_{n-2}\subset S_{n-1}$ and becomes an equator, a space of symmetry with respect to $S_{n-2}^{(n)}$ and $S_{n-2}^{(n-1)}$. So that its axis A_0G_{n-3} bisects the angle α and accordingly we may imagine the axis $G_{n-3}A_{n-1}$ rotates about B_{n-3} first an angle $\alpha/2$ to coincide with $G_{n-3}A_0$ and then still $\alpha/2$ to come finally on $G_{n-3}A_n$.

Under III, the s-sphere $\bar{K}_{n-1}$ having the radius $\sqrt{n}s$ which are between $GG_{n-3}=\sqrt{2n/(n-2)}\bar{x}$ and $GG_{n-4}=\sqrt{3n/(n-3)}\bar{x}$, it holes a pitfall cavity, which is edged by a circle K_{n-3} on B_{n-3}, and forms there, as the lowest part of the protruding calotte, a concave canopy $C'_{n-2}=\widehat{K_{n-3}N}$ where N denotes the nadir, the end point of the radius vector $GG_{n-3}N=\sqrt{n}s$. The s-sphere $\bar{K}_{n-1}$ intersects the spaces $S^{(n)}_{n-2}$, $S^{(n-1)}_{n-2}$ and T_{n-2} along spheres $\bar{K}^{(n)}_{n-2}$, $\bar{K}^{(n-1)}_{n-2}$ and K^{T}_{n-2} respectively, all of which pass through K_{n-3} and make their respective canopies $\widehat{K_{n-3}H}$, $\widehat{K_{n-3}K}$ and $\widehat{K_{n-3}E}$, where H, K, E are the ends of radii $G^{(n)}_{n-2}G_{n-3}H=G^{(n-1)}_{n-2}G_{n-3}K=\sqrt{ns^2-n\bar{x}^2/(n-1)}=\sqrt{n-1}s'$ and $GG_{n-2}E=\sqrt{n}s$, respectively. It should be noticed that e.g. although $\bar{K}_{n-2}\subset\mathfrak{S}_{n-2}(=\bar{K}_{n-1}\cap S_{n-1})$, the inner or outer point P of $\bar{K}_{n-2}$ is also an inner or outer point of $\bar{K}_{n-1}$ and $P\bar{\in}\mathfrak{S}_{n-2}$. The canopies $\widehat{K_{n-3}H}$, $\widehat{K_{n-3}K}$ coincides, if one receive a rotation amounting α about the circle K_{n-3} as axis, the canopy $\widehat{K_{n-3}E}$ being an intermediate position. By reason of symmetry the space v_{HE} extending between 2 surfaces $\widehat{K_{n-3}H}$ ($x_n=0$, $x_{n-1}<0$) and $\widehat{K_{n-3}E}$ is congruent to v_{KE} that extends between $\widehat{K_{n-3}K}$($x_{n-1}=0$, $x_n<0$) and $\widehat{K_{n-3}E}$. Both v_{HE} and $v_{KE}\subset C'_{n-2}\subset(\bar{K}_{n-1}\cap S_{n-1}$ extended), in consequence of compactness of $\bar{K}_{n-1}$ and S_{n-1} extended toward $x_{n-1}<0$, $x_n<0$. We are going to find $v=2v_{HE}$, the volume between two canopies $\widehat{K_{n-3}H}$, $\widehat{K_{n-3}K}$, which shall be overlappingly subtracted if formula F_{II} be applied, and the whole overlapping is

$$O_{III}=v\times{}_nC_2=n(n-1)v_{HE}, \tag{14.18}$$

where v_{HE} denotes the volume bounded between 2 canopies $\widehat{K_{n-3}H}$ and $\widehat{K_{n-3}E}$. So that we have

$$F_{n-2,III}=F_{n-2,II}+O_{III}=F_{n-2,I}(1-H(\tau)+K(\tau)). \tag{14.19}$$

The volume v_{HE} should be computed by integrations about ρ, $\theta_1,\cdots,\theta_{n-3}$, as in (13.14). The canopy $\widehat{K_{n-3}H}$ being the intersection of

$$\bar{K}^{(n)}_{n-2}:\rho^2+\left(\zeta+\sqrt{\frac{n}{n-1}}\bar{x}\right)^2=(n-1)s'^2 \quad\text{and}\quad S^{(n)}_{n-2}:\zeta=-\sqrt{\frac{n}{n-1}}\bar{x},$$

if projected on $\zeta=0$, it becomes simply

$$\rho_1=\sqrt{n-1}s', \tag{14.20}$$

which gives the upper limit of ρ. The equations to the axis of equator $T_{n-2}=A_0\text{-}S^{(n\cdot n-1)}_{n-2}$ are $\zeta=-\sqrt{\frac{n-2}{n}}\eta_1$, $\eta_2=\cdots=\eta_{n-2}=0$, whose first equation may be seen as that of T_{n-2}, so that the equation to $\bar{K}^{T}_{n-2}$ becomes $\rho^2+\frac{n-2}{n}\eta_1^2=ns^2$, i.e. $\frac{2(n-1)}{n}\eta_1^2+\eta_2^2+\cdots+\eta_{n-2}^2=ns^2$, a $(n-2)$-dimensional ellipsoid, and its projection on $\zeta=0$ is

(14.21) $$\rho_0=\sqrt{n}s\Big/\sqrt{1+\frac{n-2}{n}\cos^2\theta_1\cdots\cdots\cos^2\theta_{n-3}}.$$

The space enclosed by (20) and (21) being projected inversely on the sphere $\bar{K}_{n-1}:\zeta^2=ns^2-\rho^2$, we obtain the required measure

(14.22) $$v_{HE}=2^{n-3}\iint\cdots\int\sec\gamma\cdot\rho^{n-3}\cos\theta_2\cos^2\theta_3\cdots\cos^{n-4}\theta_{n-3}d\rho\, d\theta_1\cdots d\theta_{n-3},$$

where $\sec\gamma=\sqrt{1+\left(\frac{\partial\zeta}{\partial\rho}\right)^2}=\sqrt{\frac{ns^2}{ns^2-\rho^2}}$, and 2^{n-3} denotes the number of quadrants. For example,

Ex. 5. If $n=5$, III: $\sqrt{2/3}<s/\bar{x}<\sqrt{3/2}$, we have to compute

(14.23) $$v_{HE}=4\int_0^\beta d\varphi\int_0^\delta\cos\vartheta\, d\vartheta\int_{\rho_0}^{\rho_1}\sec\gamma\cdot\rho^2 d\rho \quad\text{with}\quad \sec\gamma=\sqrt{5}s/\sqrt{5s^2-\rho^2},$$

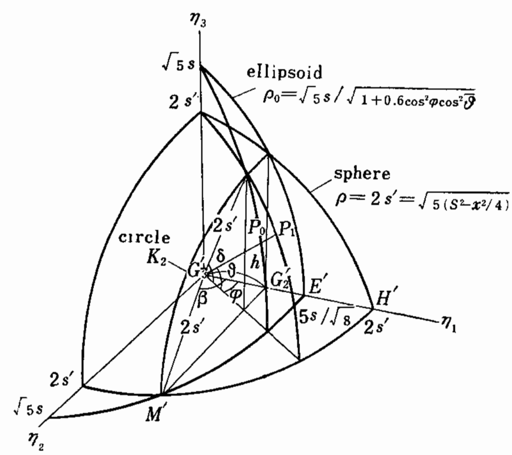

Fig. 5

where the limits of integrations are found by Fig. 5 as follows: $\rho_1=2s'=\sqrt{5(s^2-\bar{x}^2/4)}$, $\rho_0=\sqrt{5}s/\sqrt{1+0.6\cos^2\varphi\cos^2\vartheta}$, $h=G_2G_3=5\bar{x}/2\sqrt{3}$, $\cos\beta=h/2s'=\sqrt{5}\bar{x}/2\sqrt{3(s^2-\bar{x}^2/4)}=\sqrt{5/3(4\tau^2-1)}=1/\sqrt{3}\tau'$ and $\cos\delta=h\sec\varphi/2s'=\cos\beta\sec\varphi$. Executing the integrations, we obtain

(14.24) $$v_{HE}=10\sqrt{5}s^3\left[\sqrt{\frac{3\tau^2-2}{5\tau^2}}\int_0^1\frac{\tan^{-1}\sqrt{\left(\frac{3}{2}\tau^2-1\right)(1-\zeta^2)}}{1+(3\tau^2-2)\zeta^2/5\tau^2}d\zeta\right.$$
$$\left.-\frac{\pi}{2}\left(1-\frac{1}{\sqrt{3}\tau'}\right)\frac{\sqrt{4\tau^2-1}}{4\tau^2}\right].$$

Fortunately the not-yet completed integral can be expressed finitely as[1)]

1) Y. Watanabe, Zur einigen Integral-Abschätzung, the present volume, p. 49.

(14.25) $$\frac{\pi}{2}\sqrt{\frac{5\tau^2}{3\tau^2-2}}\left[\tan^{-1}\sqrt{\frac{3}{5}}-\tan^{-1}\frac{1}{\sqrt{4\tau^2-1}}\right].$$

Hence

(14.26) $$v_{HE}=5\sqrt{5}\pi s^3(\tan^{-1}\sqrt{4\tau^2-1}-\tan\sqrt{15}/3-\sqrt{4\tau^2-1}/4\tau^2+\sqrt{15}/12\tau^2),$$

and

(14.27) $$O_{III}=20v_{HE}=100\sqrt{5}\pi s^3(\quad '' \quad\quad '' \quad).$$

So that

(14.28) $$F_{III}=F_{II}+O_{III}=F_I(1-H(\tau)+K(\tau)),$$

where

(14.29) $$K(\tau)=10\pi^{-1}[\tan^{-1}\sqrt{4\tau^2-1}-\tan^{-1}\sqrt{15}/3-\sqrt{4\tau^2-1}/4\tau^2+\sqrt{15}/12\tau^2].$$

E.g. we have for $\tau=2$, $\tau'=\sqrt{3}$,

$$O_{III}=702.4815s^3(\tan^{-1}\sqrt{15}-\tan^{-1}\sqrt{15}/3-\sqrt{15}/24)=172.1107s^3.$$

But, we get after (13.19) and (14.16)

$$F_{3,I}=10\sqrt{5}\pi^2s^3=220.6911s^3,$$

$$F_{3,II}=220.6911s^3\left[1-\frac{5}{\pi}(\tan^{-1}\sqrt{15}-\sqrt{15}/16)\right]=-157.263s^3.$$

Therefore

$$F_{3,III}=F_{3,II}+O_{III}=14.8476s^3,$$

which shall be further compensated by the following O_{IV}.

To illustrate the subsequent subcase

IV: $\sqrt{3/(n-3)}<s/\bar{x}=\tau<\sqrt{4/(n-4)}$, we examplify by

Ex. 6. Case $n=5$, IV: $\sqrt{3/2}<\tau<2$, as its generalization can be made without difficulty. Now that $GG_1=\sqrt{15/2}\bar{x}<\sqrt{5}s<GA_1=2\sqrt{5}\bar{x}$ hold, the s-sphere $\bar{K}_4$: $\rho^2+\zeta^2=5s^2$ intersects the third subsimplex, e.g. $S_1^{(3,4,5)}=A_1A_2(x_3=x_4=x_5=0)$ at a linear circle $K_1:PQ$ (Fig. 6). There are the following first and second subsimplexes containing A_1A_2 as their common side: 1° $S_3^{(5)}=A_1A_2A_3A_4(x_5=0)$, 2° $S_3^{(4)}=A_1A_2A_3A_5(x_4=0)$ and further 3° $S_2^{(4,5)}=A_1A_2A_3(x_4=x_5=0)$, 4° $S_2^{(3,5)}=A_1A_2A_4(x_3=x_5=0)$, 5° $S_2^{(3,4)}=A_1A_2A_5(x_3=x_4=0)$. Their respective equations are 1° $\zeta=-\sqrt{5}\bar{x}/2$, 2° $\zeta=\sqrt{5}(2\bar{x}-\sqrt{3}\eta_1)$, 3° $\zeta=-\sqrt{5}\bar{x}/2$, $\eta_1=5\bar{x}/2\sqrt{3}$, 4° $\zeta=-\sqrt{5}\bar{x}/2$, $\eta_1=-5\bar{x}/6\sqrt{3}$, 5° $\zeta=\sqrt{5}(2\bar{x}-\sqrt{3}\eta_1)$, $\eta_2=\sqrt{2}\eta_1$, whose first two are 3-dimensional, while the remaining three 2-dimensional. To save the trouble, we may conceive again the space of symmetry (equator): 6° $T_3^{(4,5)}=A_0A_1A_2A_3$, where A_0

denotes the middle point of side A_4A_5, and whose equation is $\zeta=-\sqrt{\frac{3}{5}}\eta_1$. The intersection of $\bar{K}_4$ with 1°, 2° and 6° are subspheres $K_3^{(5)}$, $K_3^{(4)}$ and $\bar{K}_3^T$, all

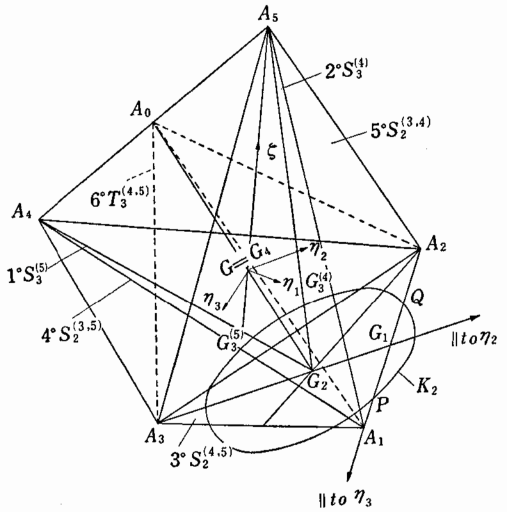

Fig. 6

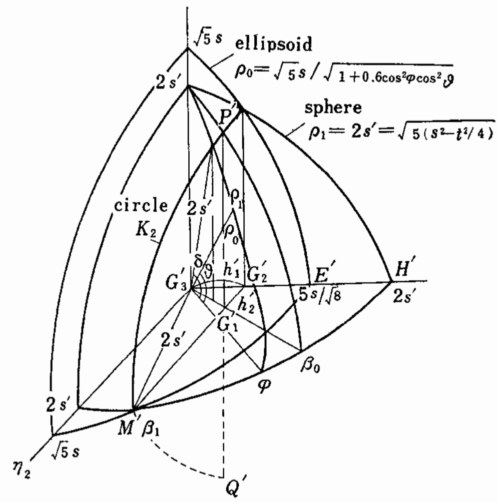

Fig. 6a

of which pass through K_1 and consequently they intersect furthermore with 3° along a same circle K_2 that contain K_1 as its chord and protrude over K_1. Hence, there takes place again some overlapping among the subspheres' portions as in Ex. 4. Since, however, the overlapping portion produced between $\bar{K}_3^{(5)}$ and $\bar{K}_3^T$ is congruent to that between $\bar{K}_3^{(4)}$ and $\bar{K}_3^T$ by reason of symmetry, we can halve the trouble of calculations by taking only the first half. First, the equation to $\bar{K}_3^{(5)}$, as intersection of $\bar{K}_4:\zeta^2+\rho^2=5s^2$ and 1° $S_3^{(5)}:\zeta=-\sqrt{5}\bar{x}/2$, yields a sphere

(14.30) $$\rho_1=\sqrt{5(s^2-\bar{x}^2/4)}=2s',$$

where $\frac{5}{\sqrt{3}}s<2s'<\sqrt{5}s$ because of IV, and which remains the same even when projected on $\zeta=0$. Next, by eliminating ζ between $\bar{K}_4:\zeta^2+\rho^2=5s^2$ and T_3: $\zeta=-\sqrt{\frac{3}{5}}\eta_1$, we get the equation to their intersection $\bar{K}_3^T$ (or its projection on $\zeta=0$): $\rho^2+\frac{3}{5}\eta_1^2=5s^2$, i.e. $\frac{8}{5}\eta_1^2+\eta_2^2+\eta_3^2=5s^2$. Thus, it yields an ellipsoid

(14.31) $$\rho_0=\sqrt{5}s/\sqrt{1+0.6\cos^2\varphi\cos^2\vartheta}.$$

Both of $\bar{K}_3^{(5)}$ and $\bar{K}_3^T$ pass through K_2 which contains the protruding arc $\widehat{PMQ}$, and they form, as in Ex. 5, their respective canopies $\widehat{K_2H}$ and $\widehat{K_2E}$. These canopies are both $\subset\mathfrak{S}_3=(\bar{K}_3^{(5)}\cap\text{prolonged } S_3)$, so the space bounded by them $\subset\mathfrak{S}_3$ also. But, as in Ex. 4, the portion of $\mathfrak{S}_3$ bounded by two lunettes $PQ\cdot MH$

on $\bar{K}_3^{(5)}$ and $PQ\cdot ME$ on $\bar{K}_3^T$ yield a reoverlapping volume v_{HE}, which can be computed after the projective principle as

$$(14.32)\quad v_{HE}=2\int_{\beta_0}^{\beta_1}d\varphi\int_0^{\delta}\cos\vartheta\, d\vartheta\int_{\rho_0}^{\rho_1}\sec\gamma\cdot\rho^2 d\rho,\ \text{ where }\ \sec\gamma=\sqrt{5}s/\sqrt{5s^2-\rho^2},$$

whose limits are found by reference to Fig. 6a to be $\tan^{-1}\beta_0=h_2/h_3$ if $h_3=G_2G_3=5\bar{x}/2\sqrt{3}$, $h_2=5\bar{x}/\sqrt{6}$, so that $\beta_0=\tan^{-1}\sqrt{2}=\sec^{-1}\sqrt{3}$, $\beta_1=\sec^{-1}\sqrt{3}\tau'$ and $\delta=\sec^{-1}(\tau'\sqrt{3}\cos\varphi)$, as well as $\rho_1=2s'$, $\rho_0=\sqrt{5}s/\sqrt{1+0.6\cos^2\varphi\cos^2\vartheta}$, where $2s'=\sqrt{5(s^2-\bar{x}^2/4)}$, $\tau'=\sqrt{(4\tau^2-1)/5}$. Executing the integration, we get

$$(14.33)\quad v_{HE}=5\sqrt{5}s^3\Big[\sqrt{\frac{2\tau^2-3}{3}}J(\tau)-\frac{\pi}{6}\tan^{-1}\sqrt{\frac{2\tau^2-3}{3}}$$
$$+\frac{1}{4\tau^2}\Big(\sqrt{\frac{5}{3}}\tan^{-1}\sqrt{\frac{3}{5}(2\tau^2-3)}-\sqrt{4\tau^2-1}\tan^{-1}\sqrt{\frac{2\tau^2-3}{4\tau^2-1}}\Big)\Big].$$

where

$$(14.34)\quad J(\tau)=\int_0^1\tan^{-1}\sqrt{1/3+2(2\tau^2-3)(1-\zeta^2)/15\tau^2}\,d\zeta/(1+(2\tau^2-3)\zeta^2/3),$$

which could not be unfortunately expressed as attained in (25) in a finite form, but, if τ be given, it may be computed by Gauss' method of numerical integration. Since we get still a congruent one bounded by $\bar{K}_3^{(4)}$ and $\bar{K}_3^T$, and there are 10 sides and besides we may choose the vertex against a side in 3 ways, the total re-overlapping area is

$$(14.35)\quad O_{IV}=60v_{HE}=F_{3,I}L(\tau)\ \text{say},$$

which should be subtracted from F_{III} to obtain F_{IV}:

$$(14.36)\quad F_{3,IV}=F_{3,III}-O_{IV}=F_{3,I}[1-H(\tau)+K(\tau)-L(\tau)],$$

where

$$(14.37)\quad L(\tau)=O_{IV}/F_{3,I}=\frac{30}{\pi^2}\sqrt{\frac{2\tau^2-3}{3}}J(\tau)-\frac{5}{\pi}\tan^{-1}\sqrt{\frac{2\tau^2-3}{3}}$$
$$+\frac{15}{2\pi^2\tau^2}\Big(\sqrt{\frac{5}{3}}\tan^{-1}\sqrt{\frac{3}{5}(2\tau^2-3)}-\sqrt{4\tau^2-1}\tan^{-1}\sqrt{\frac{2\tau^2-3}{4\tau^2-1}}\Big).$$

When $\tau=2$, the s-sphere passes through all vertices, and consequently $F_{3,IV}(2)=0$. Hence by (36) we must have

$$(14.38)\quad 1-H(2)+K(2)-L(2)=0.$$

But, after (16) and (29) we find readily that

$$(14.39)\quad H(2)=\frac{5}{\pi}(\tan^{-1}\sqrt{15}-\sqrt{15}/16)=1.712594$$

and

$$(14.40) \qquad K(2) = \frac{10}{\pi}(\tan^{-1}\sqrt{15} - \tan^{-1}\sqrt{15}/3 - \sqrt{15}/24) = 0.779870.$$

So that by the above (38) also

$$(14.41) \quad L(2) = 1 - H(2) + K(2) = 1 + \frac{5}{\pi}\left(\tan^{-1}\sqrt{15} - 2\tan^{-1}\frac{\sqrt{15}}{3} - \frac{\sqrt{15}}{48}\right)$$
$$= 0.067276.$$

Lastly this value being substituted in (37) for $\tau = 2$, the uncompleted integral $J(2)$ can be evaluated as

$$(14.42) \qquad J(2) = \frac{\pi}{2\sqrt{15}}\left(\frac{\pi}{5} + \tan^{-1}\sqrt{15} - \tan^{-1}\frac{\sqrt{15}}{3}\right) = 0.4196499.$$

On the otherhand, calculating (34) for $\tau = 2$ by means of Gauss' five ordinates method of integration employing Chamber's Table, we obtain indeed

0.419651,

and thus the percentage error is only 0.002%, which evidences the excellence of Gauss' method.

15. *Student's Functions for the special Case* $n = 5$. Now that every volume element dV_N (N=I, II, III, IV) has been obtained, the Student's functions may be readily written down. If the parent fr. f. $f(x)$ be such that $\prod_1^n f(x_i) = g_n(\bar{x}, s)$, the partial joint-probability of $\bar{x}$ and s would be

$$(15.1) \quad f_N(\bar{x}, s)d\bar{x}ds = g_n(\bar{x}, s)dV_N = g_n(\bar{x}, s)F_N(\bar{x}, s)nd\bar{x}ds \quad (N = \mathrm{I}, \mathrm{II}, \cdots).$$

Whence we obtain, on replacing s by Student's ratio $t = \sqrt{n-1}(\bar{x} - m)/s$, where m is the parent mean,

$$(15.2) \quad f_N(\bar{x}, t)d\bar{x}dt = g_n\left(\bar{x}, s = \sqrt{n-1}(\bar{x} - m)/t\right)F_N\left(\bar{x}, s = \sqrt{n-1}(\bar{x} - m)/t\right)$$
$$\times n\sqrt{n-1}\,|x - m|\,d\bar{x}dt/t^2.$$

We shall detail the concrete shape regarding the special case $n = 5$. Substituting F_N obtained in the foregoing section, we have

I: $0 < \tau = s/\bar{x} < 1/2$:

$$(15.3) \qquad f_I(\bar{x}, t) = 800\sqrt{5}\pi^2(x - m)^4 g(\bar{x}, t)/t^5.$$

II: $1/2 < \tau < \sqrt{2/3}$:

$$(15.4) \qquad f_{II}(\bar{x}, t) = f_I(\bar{x}, t)(1 - H(\tau)),$$

where H as well as K and L below, as have been seen in section 14, are all functions of $\tau = s/\bar{x} = 2(\bar{x} - m)/\bar{x}t$ alone.

III: $\sqrt{2/3} < \tau < \sqrt{3/2}$:

$$f_{III}(\bar{x}, t) = f_I(\bar{x}, t)(1 - H(\tau) + K(\tau)). \tag{15.5}$$

IV: $\sqrt{3/2} < \tau < 2$:

$$f_{IV}(\bar{x}, t) = f_I(\bar{x}, t)(1 - H(\tau) + K(\tau) - L(\tau)). \tag{15.6}$$

Hence, the partial Student fr. f. is obtained by integrating these $f_N(\bar{x}, t)$ about $\bar{x}$:

$$s_N(t) = \int_{\bar{x}_0}^{\bar{x}_1} f_N(\bar{x}, t)\, d\bar{x}. \tag{15.7}$$

It remains to determine the limits of integrations for several subintervals. Because of $\tau = 2(\bar{x} - m)/\bar{x}t$, the four subcases are denoted in terms of t as

$$\begin{cases} \text{I:}\ \ 0 \gtrless \bar{x} - m \gtrless \bar{x}t/4, & \text{II:}\ \ \bar{x}t/4 \gtrless \bar{x} - m \gtrless \bar{x}t/\sqrt{6}, \\ \text{III:}\ \ \bar{x}t/\sqrt{6} \gtrless \bar{x} - m \gtrless \bar{x}t/\sqrt{8/3}, & \text{IV:}\ \ \bar{x}t/\sqrt{8/3} \gtrless \bar{x} - m \gtrless \bar{x}t, \end{cases} \tag{15.8}$$

where the double inequality signs must be taken all upper or all lower according as $t \lesseqgtr 0$. Further, if we put

$$x_1 = \frac{m}{1-t}, \quad x_2 = \frac{m}{1-t/\sqrt{8/3}}, \quad x_3 = \frac{m}{1-t/\sqrt{6}}, \quad x_4 = \frac{m}{1-t/4}, \tag{15.9}$$

their magnitudes are in order as follows:

1° if $-\infty < t < 0$, $\quad 0 < x_1 < x_2 < x_3 < x_4 < m$;

2° if $0 < t < 1$, $\quad m < x_4 < x_3 < x_2 < x_1 < \infty$;

3° if $1 < t < \sqrt{8/3}$, $\quad m < x_4 < x_3 < x_2 < \infty$ with $x_1 < 0$;

4° if $\sqrt{8/3} < t < \sqrt{6}$, $\quad m < x_4 < x_3 < \infty$ with $x_2, x_1 < 0$;

5° if $\sqrt{6} < t < 4$, $\quad m < x_4 < \infty$ with $x_3, x_2, x_1 < 0$;

6° if $4 < t < \infty$, $\quad m < \infty$ with all x_i's < 0.

Therefore, the inequalities (8) hold by adopting those non-negative x_1, x_2, x_3, x_4 as permissible limits of integrations:

$$\text{I: } m \gtrless \bar{x} \gtrless x_4, \quad \text{II: } x_4 \gtrless \bar{x} \gtrless x_3, \quad \text{III: } x_3 \gtrless \bar{x} \gtrless x_2, \quad \text{IV: } x_2 \gtrless \bar{x} \gtrless x_1, \tag{15.10}$$

where the double signs are to be taken all upper or all lower according as $t \lesseqgtr 0$. However, it must be noticed that for certain $t > 0$ if the left side becomes positive, yet the right side occurs to be negative, the latter should be replaced by ∞, and if both sides become negative, that interval shall naturally be abandoned. Hence, only the following integrals are permissible summands in each t-interval:

$1^\circ\ -\infty < t < 0$: $\int_{x_4}^{m} f_I d\bar{x} + \int_{x_3}^{x_4} f_{II} d\bar{x} + \int_{x_2}^{x_3} f_{III} d\bar{x} + \int_{x_1}^{x_2} f_{IV} d\bar{x}$;

$2^\circ\ 0 < t < 1$: $\int_{m}^{x_4} f_I dx + \int_{x_4}^{x_3} f_{II} d\bar{x} + \int_{x_3}^{x_2} f_{III} d\bar{x} + \int_{x_2}^{x_1} f_{IV} d\bar{x}$;

$3^\circ\ 1 < t < \sqrt{8/3}$: $\int_{m}^{x_4} f_I d\bar{x} + \int_{x_4}^{x_3} f_{II} d\bar{x} + \int_{x_3}^{x_2} f_{III} d\bar{x} + \int_{x_2}^{\infty} f_{IV} d\bar{x}$;

$4^\circ\ \sqrt{8/3} < t < \sqrt{6}$: $\int_{m}^{x_4} f_I d\bar{x} + \int_{x_4}^{x_3} f_{II} d\bar{x} + \int_{x_3}^{\infty} f_{III} d\bar{x}$;

$5^\circ\ \sqrt{6} < t < 4$: $\int_{m}^{x_4} f_I d\bar{x} + \int_{x_4}^{\infty} f_{II} d\bar{x}$,

$6^\circ\ 4 < t < \infty$: $\int_{m}^{\infty} f_I d\bar{x}$ only.

But, since, if we abbreviate $f_I(\bar{x}, t)$ of (3) by $f(\bar{x}, t)$,

$$f_I = f,\quad f_{II} = f - fH,\quad f_{III} = f - fH + fK,\quad f_{IV} = f - fH + fK - fL,$$

we get the full Student fr. f. $s(t)$ on summing up the alike integrals:

$1^\circ\ -\infty < t < 0$:

$$s(t) = \int_{x_1}^{m} f(\bar{x}, t)d\bar{x} - \int_{x_1}^{x_4} f(\bar{x}, t)H(\tau)d\bar{x} + \int_{x_1}^{x_3} f(\bar{x}, t)K(\tau)d\bar{x} - \int_{x_1}^{x_2} f(\bar{x}, t)L(\tau)d\bar{x}. \tag{15.11}$$

And whence the Student d. f. is given by

$$S(t_0) = \int_{-\infty}^{t_0} s(t)dt,\quad S(0) = \int_{-\infty}^{-0} s(t)dt.$$

$2^\circ\ 0 < t < 1$:

$$s(t) = \int_{m}^{x_1} f(\bar{x}, t)d\bar{x} - \int_{x_4}^{x_1} f(\bar{x}, t)H(\tau)d\bar{x} + \int_{x_3}^{x_1} f(\bar{x}, t)K(\tau)d\bar{x} - \int_{x_2}^{x_1} f(\bar{x}, t)L(\tau)d\bar{x}. \tag{15.12}$$

And $$S(t_0) = S(0) + \int_{0}^{t_0} s(t)dt.$$

$3^\circ\ 1 < t < \sqrt{8/3} = 1.6330$:

$$s(t) = \int_{m}^{\infty} f(\bar{x}, t)d\bar{x} - \int_{x_4}^{\infty} f(\bar{x}, t)H(\tau)d\bar{x} + \int_{x_3}^{\infty} f(\bar{x}, t)K(\tau)d\bar{x} - \int_{x_2}^{\infty} f(\bar{x}, t)L(\tau)d\bar{x}. \tag{15.13}$$

$$S(t_0) = S(1) + \int_{1}^{t_0} s(t)dt.$$

$4^\circ\ \sqrt{8/3} < t < \sqrt{6} = 2.4495$:

$$s(t) = \int_{m}^{\infty} f(\bar{x}, t)d\bar{x} - \int_{x_4}^{\infty} f(\bar{x}, t)H(\tau)d\bar{x} + \int_{x_3}^{\infty} f(\bar{x}, t)K(\tau)d\bar{x}. \tag{15.14}$$

$$S(t_0) = S(\sqrt{8/3}) + \int_{\sqrt{8/3}}^{t_0} s(t)dt.$$

5° $\sqrt{6}<t<4$:

(15.15) $$s(t)=\int_m^\infty f(\bar{x},t)d\bar{x}-\int_{x_4}^\infty f(\bar{x},t)H(\tau)d\bar{x}.$$

$$S(t_0)=S(\sqrt{6})+\int_{\sqrt{6}}^{t_0}s(t)dt.$$

6° $4<t<\infty$:

(15.16) $$s(t)=\int_m^\infty f(\bar{x},t)d\bar{x},$$

$$S(t_0)=S(4)+\int_4^{t_0}s(t)dt. \qquad \text{Or, else,} \qquad S(t_0)=1-\int_{t_0}^\infty s(t)dt.$$

If we take generally the conjugate d. f. $\bar{S}(t_0)=\int_{t_0}^\infty s(t)dt$, we have $S(t_0)=1-\bar{S}(t_0)$.

Here, for the sake of later reference, the correction-factors are recapitulated:

(15.17) $$H(\tau)=\frac{5}{\pi}\left[\sec^{-1}2\tau-\frac{\sqrt{4\tau^2-1}}{4\tau^2}\right]$$

$$=\frac{5}{\pi}\left[\cos^{-1}\frac{\bar{x}t}{4(\bar{x}-m)}-\frac{x|t|\sqrt{16(x-m)^2-x^2t^2}}{16(x-m)^2}\right],$$

where $16(x-m)^2>x^2t^2$ because of $\tau>1/2$, and

$$H(1/2)=0, \quad H(2)=1.71259.$$

(15.18) $$K(\tau)=\frac{10}{\pi}\left[\sec^{-1}2\tau-\frac{\sqrt{4\tau^2-1}}{4\tau^2}+\frac{\sqrt{15}}{12\tau^2}-\tan^{-1}\sqrt{\frac{5}{3}}\right]$$

$$=2H(\tau)+\frac{5\sqrt{15}}{24\pi(x-m)^2}-\frac{10}{\pi}\tan^{-1}\sqrt{\frac{5}{3}},$$

and

$$K(\sqrt{2/3})=0, \quad K(2)=0.77987.$$

(15.19) $$L(\tau)=\frac{30}{\pi^2}\sqrt{\frac{2\tau^2-3}{3}}J(\tau)-\frac{5}{\pi}\tan^{-1}\sqrt{\frac{2\tau^2-3}{3}}$$

$$+\frac{15}{2\pi^2\tau^2}\left(\sqrt{\frac{5}{3}}\tan^{-1}\sqrt{\frac{3}{5}(2\tau^2-3)}-\sqrt{4\tau^2-1}\tan^{-1}\sqrt{\frac{2\tau^2-3}{4\tau^2-1}}\right),$$

where

(15.20) $$J(\tau)=\int_0^1\frac{\tan^{-1}\sqrt{1/3+2(2\tau^2-3)(1-\zeta^2)/15\tau^2}}{1+(2/3\tau^2-1)\zeta^2}d\zeta,$$

and

$$L(\sqrt{3/2})=0, \quad L(2)=0.06728.$$

Furthermore the derivatives of these correction-factors are

$$(15.21)\quad H'(\tau)=\frac{5\sqrt{4\tau^2-1}}{2\pi\tau^3}=\frac{5x^2t^2\sqrt{16(x-m)^2-x^2t^2}}{16\pi|x-m|^3},\quad H'\left(\frac{1}{2}\right)=0.$$

$$(15.22)\quad K'(\tau)=2H'(\tau)-\frac{5\sqrt{15}}{3\pi\tau^3}=2H'-\frac{5\sqrt{15}x^3t^3}{24\pi(x-m)^3},\quad K'\left(\sqrt{\frac{2}{3}}\right)=0.$$

$$(15.23)\quad L'(\tau)=\frac{5}{\pi^2\tau}\left[\sqrt{\frac{3}{2\tau^2-3}}(4\tau^2J(\tau)-\pi)\right.$$

$$+6\sqrt{\frac{2\tau^2-3}{3}}\left(\frac{\pi}{4\tau}-\tau\tan^{-1}\sqrt{\frac{3\tau^2-2}{5\tau^2}}\right)$$

$$\left.+\frac{3(2\tau^2-1)}{\tau^2\sqrt{4\tau^2-1}}\tan^{-1}\sqrt{\frac{2\tau^2-3}{4\tau^2-1}}-\frac{\sqrt{15}}{\tau^2}\tan^{-1}\sqrt{\frac{3}{5}(2\tau^2-3)}\right],$$

where $4\tau^2J(\tau)-\pi$ tends to zero as $\tau\to\sqrt{3/2}$, so that divisible by $\tau-\sqrt{3/2}$ and still $L'(\sqrt{3/2})=0$ also.

But, what really wanted afterward, is $\int_{\sqrt{3/2}}^{2}\tau^4L'(\tau)d\tau$. This integrated by parts, yields

$$(15.24)\quad \int_{\sqrt{3/2}}^{2}\tau^4L'(\tau)d\tau=\tau^4L(\tau)\Big|_{\sqrt{3/2}}^{2}-4\int_{\sqrt{3/2}}^{2}\tau^3L(\tau)d\tau$$

$$=16L(2)-\frac{40\sqrt{3}}{\pi^2}\int_{\sqrt{3/2}}^{2}\tau^3\sqrt{2\tau^2-3}J(\tau)\,d\tau$$

$$+\frac{20}{\pi}\int_{\sqrt{3/2}}^{2}\tau^3\tan^{-1}\sqrt{\frac{2\tau^2-3}{3}}\,d\tau-\frac{10\sqrt{15}}{\pi^2}\int_{\sqrt{3/2}}^{2}\tan^{-1}\sqrt{\frac{3}{5}(2\tau^2-3)}\,d\tau$$

$$+\frac{30}{\pi^2}\int_{\sqrt{3/2}}^{2}\tau\sqrt{4\tau^2-1}\tan^{-1}\sqrt{\frac{2\tau^2-3}{4\tau^2-1}}\,d\tau$$

$$=(0)-(\mathrm{i})+(\mathrm{ii})-(\mathrm{iii})+(\mathrm{iv}).$$

The integral (i) being really a double integral, it can be evaluated by applying Gauss' method of numerical integrations iteratively, while all others are immediately found by integrations by parts. We get thus

$$(15.25)\quad \int_{\sqrt{3/2}}^{2}\tau^4L'(\tau)d\tau=1.0765-17.7331+16.0258-4.0169+5.2463$$

$$=0.5986.$$

Hence, we have also by (24)

$$(15.26)\quad \int_{\sqrt{3/2}}^{2}\tau^3L(\tau)d\tau=0.1195.$$

16. *The Truncated Laplace Distribution as Universe, Case* $n=5$. When the function $\prod_1^n f(x_i)=g(\bar{x}, s)$ does not contain s, the problem reduces to a somewhat simpler one. This is the case e.g. if the universe is a truncated Laplace distribution

$$f(x)=e^{-x} \quad (0<x<\infty). \tag{16.1}$$

We would discuss below this one to examplify concretely the foregoing investigation. An apparently more general case $f(x)=ae^{-bx}(a, b>0, x>c)$ can be written as $f(x)=ae^{-bc}e^{-b(x-c)}=a'e^{-bx'}$, where $a'>0$, $x'=x-c>0$; or, because of $1=\int_0^\infty a'e^{-bx'}dx'=a'/b$, it reduces to $f(x')=be^{-bx'}(x'>0)$, which becomes (1) by putting $bx'=x$. Hence, without loss of generality we may treat (1). Here the parent mean $m=1$ and $g(\bar{x}, s)=e^{-5\bar{x}}$. We obtain by (15.3)

$$f_I(\bar{x}, t)=c(\bar{x}-1)^4e^{-5\bar{x}}/t^5, \quad \text{where } c=800\sqrt{5}\pi^2=17655.3. \tag{16.2}$$

After (15.4)-, the remaining f_{II}, f_{III}, f_{IV} are obtained by the above f_I multiplied by the correction-factors $1-H$, $1-H+K$, $1-H+K-L$, respectively. Hence the Student partial fr. f. is given by (15.7)

$$s_N(t)=\frac{c}{|t|^5}\int_{\bar{x}_0}^{\bar{x}_1}(x-1)^4e^{-5x}\times(\text{the corresponding correction-factor})\,d\bar{x}, \tag{16.3}$$

$$N=\text{I, II, III, IV}.$$

For the sake of later convenience, let us put

$$\begin{aligned} G(x)&=\int_x^\infty-(x-1)^4e^{-5x}dx=\int_\infty^x(x-1)^4e^{-5x}dx=\int_\infty^x G'(x)dx \\ &=-e^{-5x}[0.2(x-1)^4+0.16(x-1)^3 \\ &\qquad +0.096(x-1)^2+0.0384(x-1)+0.00768] \\ &=-e^{-5x}[0.10528-0.4736x+0.816x^2-0.64x^3+0.2x^4]. \end{aligned} \tag{16.4}$$

So that

$$G'(x)=(x-1)^4e^{-5x}\geq 0, \tag{16.5}$$

$$G''(x)=-(5x-9)(x-1)^3e^{-5x}\gtrless 0. \tag{16.6}$$

The function $G(x)$ is negative and monotonic increasing throught the interval $0<x<\infty$ (the Table below and Fig. 7), its absolute value being small. Also,

$$G^{(\nu)}(1)=0, \quad \nu=1, 2, 3, 4, \quad \text{but} \quad G^{(5)}(1)=-24e^{-5}. \tag{16.7}$$

Consequently $G(x)$ osculates strongly its tangent at $x=1$, and we have

$$G(x)=G(1)+0.0013476(x-1)^5+O((x-1)^6), \tag{16.8}$$

x	$G(x)$	$G'(x)$
0	-0.10528	1
0.1	-0.00537	0.3979
1	$-0.0^4 517474$	0 (min)
1.5	$-0.0^4 462$	$0.0^4 346$
1.8	$-0.0^4 325$	$0.0^4 505$ (max)
2	$-0.0^4 228$	$0.0^4 454$
3	$-0.0^4 015$	$0.0^4 043$
4	$-0.0^4 004$	$0.0^4 017$
4.5	$-0.0^4 0001$	$0.0^4 0004$
∞	-0	$+0$

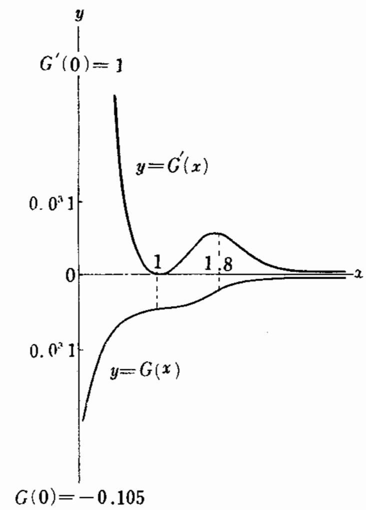

Fig. 7

while, if x itself be small enough, (4) yields

(16.9) $G(x) = G(0) + x - 4.5x^2 + 12.83x^3 - 26.2083x^4 + 41.075x^5 + O(x^6)$.

In fact, $G(z)$ being an entire transcendental function, it behaves regular in the whole z-plane, and its Taylor expansion at any finite point has the radius of convergence $=\infty$, so that the above O are really $\sum_6^\infty a_\nu (x-1)^\nu$ and $\sum_6^\infty b_\nu x^\nu$, respectively. With this $G(x)$, we can rewrite

(16.10) $$s_I(t) = \frac{c}{|t|^5}\int_{\bar{x}_0}^{\bar{x}_1} G'(x)\,dx$$

and s_{II}, s_{III}, s_{IV} are obtained by multiplying the above integrand by $(1-H)$, $(1-H+K)$, $(1-H+K-L)$, respectively.

Now according to (15.11)-, we can write down the full $s(t)$ as follows:

1° $-\infty < t < 0$:

(16.11) $$s(t) = \frac{c}{|t|^5}\left[\int_{x_1}^{1} G'dx - \int_{x_1}^{x_4} G'Hdx + \int_{x_1}^{x_3} G'Kdx - \int_{x_1}^{x_2} G'Ldx\right];$$

2° $0 < t < 1$:

(16.12) $$s(t) = \frac{c}{t^5}\left[\int_{1}^{x_1} G'dx - \int_{x_4}^{x_1} G'Hdx + \int_{x_3}^{x_1} G'Kdx - \int_{x_2}^{x_1} G'Ldx\right];$$

3° $1 < t < \sqrt{8/3} = 1.6330$:

(16.13) $$s(t) = \frac{c}{t^5}\left[\int_{1}^{\infty} G'dx - \int_{x_4}^{\infty} G'Hdx + \int_{x_3}^{\infty} G'Kdn - \int_{x_2}^{\infty} G'Ldx\right];$$

4° $\sqrt{8/3} < t < \sqrt{6} = 2.4495$:

(16.14) $$s(t) = \frac{c}{t^5}\left[\int_{1}^{\infty} G'dx - \int_{x_4}^{\infty} G'Hdx + \int_{x_3}^{\infty} G'Kdx\right];$$

5° $\sqrt{6}<t<4$:

$$s(t)=\frac{c}{t^5}\left[\int_1^\infty G'dx-\int_{x_4}^\infty G'Hdx\right]; \tag{16.15}$$

Lastly, for 6° $4<t<\infty$

$$s(t)=\frac{c}{t^5}\int_1^\infty G'dx=\frac{c}{t^5}[G(\infty)-G(1)]=-\frac{cG(1)}{t^5}=\frac{c_1}{t^5}, \tag{16.16}$$

where $c_1=-cG(1)=0.91361$.

Consequently the d. f. from right in 6° immediately can be obtained as

$$\bar{S}(t_0)=\int_{t_0}^\infty s(t)dt=c_1\int_{t_0}^\infty\frac{dt}{t^5}=\frac{c_1}{4t_0^4}=\frac{0.22840}{t_0^4}. \tag{16.17}$$

And particularly

$$\bar{S}(4)=0.00089. \tag{16.18}$$

Thus, $t=4$ lies above even the upper 1 percentage critical point.

We shall below detail about the first subinterval more minutely. Whereby the correction factors H, K and especially L come intricate. There are two ways in their employment: One is to express them by $\bar{x}$ and t on substituting $\tau=s/\bar{x}=2(\bar{x}-1)/\bar{x}t$ in them. Otherwise, retaining τ as it stands and rather to denote t(or $\bar{x}$) in τ and the remaining $\bar{x}$ (or t). The former appears obvious to look over and may be applied for H, K, although it is troublesome to do so for L, for which the latter method is convenient.

1° $-\infty<t<0$: Integrating (11) by parts, we obtain

$$\begin{aligned}s(t)=\frac{c}{|t|^5}\Big[G(1)-G(x_1)-G(x)H(\tau)\Big|_{x_1}^{x_4}+\int_{x_1}^{x_4}H_1(x,t)dx\\+G(x)K(\tau)\Big|_{x_1}^{x_3}-\int_{x_1}^{x_3}K_1(x,t)dx-G(x)L(\tau)\Big|_{x_1}^{x_2}+\int_{x_1}^{x_2}L_1(x,t)dx\Big],\end{aligned} \tag{16.19}$$

where, in view of (15.21) and (15.22),

$$H_1\equiv 2G(x)H'(\tau)/x^2t=\frac{5tG(x)}{8\pi|x-1|^3}\sqrt{16(x-1)^2-x^2t^2}\ (>0,\text{ since } tG(x)>0),$$

$$K_1\equiv 2G(x)K'(\tau)/x^2t=2H_1-\frac{5\sqrt{15}t^2xG(x)}{12\pi(x-1)^3} \tag{16.20}$$

$$\left(\text{the last term}<0,\text{ since}\frac{G(x)}{(x-1)^3}>0\right),$$

$$L_1\equiv 2G(x)L'(\tau)/x^2t\ \ (\text{cf. (15.23)}).$$

From $\tau=2(x-1)/xt$, e.g. for $x_4=1/(1-t/4)$ yields $\tau=1/2$, and similarly x_3, x_2, x_1 correspond to $\tau=\sqrt{2/3}, \sqrt{3/2}, 2$. Now that $H(1/2)=K(\sqrt{2/3})=L(\sqrt{3/2})=0$ and $1-H(2)+K(2)-L(2)=0$, all the integrated parts in (19) reduce to naught and we have

$$(16.21)\qquad s(t)=\frac{c_1}{t^5}+\frac{c}{|t|^5}\left[\int_{x_1}^{x_4}H_1dx-\int_{x_1}^{x_3}K_1dx+\int_{x_1}^{x_2}L_1dx\right]$$
$$=s_0(t)+s_1(t)+s_2(t)+s_3(t),$$

whose complete integrations to an explicit finite form however seem of less hope. Rather we conceive the d. f.

$$(16.22)\qquad S(t_0)=\int_{-\infty}^{t_0}s(t)dt=\sum_0^3\int_{-\infty}^{t_0}s_\nu(t)dt=\sum_0^3 S_\nu(t_0),$$

in which

$$(16.23)\qquad S_0(t)=-c_1/4t_0^4=-0.22840/t_0^4.$$

As to the remaining, first consider

$$S_1(t)=\frac{5\tau}{8\pi}\int_{-\infty}^{t_0}\frac{dt}{t^4}\int_{x_{10}}^{x_{40}}\frac{G(x)}{(x-1)^3}\sqrt{16(x-1)^2-x^2t^2}\,dx\,(>0),$$

where and below $x_{10}=1/(1-t_0)$, $x_{40}=1/(1-t_0/4)$, &c. Interchanging the order of integrations (Fig. 8) and making use of the formula

$$\int\frac{\sqrt{a^2-b^2t^2}\,dt}{t^4}=-\frac{\sqrt{a^2-b^2t^2}^3}{3a^2t^3},$$

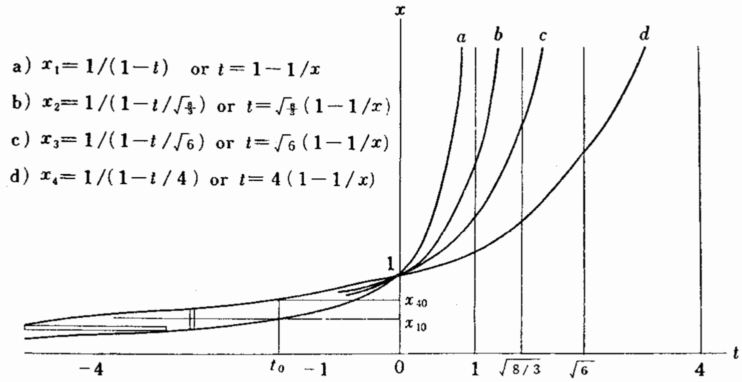

Fig. 8

we obtain

$$(16.24)\quad S_1(t_0)=c_2\int_0^{x_{10}}\frac{G(x)dx}{(x-1)^5}\left[\frac{\sqrt{16(x-1)^2-x^2t^2}^3}{-t^3}\right]_{4(1-1/x)}^{1-1/x}+\int_{x_{10}}^{x_{40}}\,''\,[\,''\,]_{4(1-1/x)}^{t_0}$$
$$=S_{1.1}+S_{1.2}\quad\text{say,}\qquad\text{where}\quad c_2=5c/384\pi=73.17515.$$

The inner integrands vanish when the lower limit $t=4(1-1/x)$ is substituted and consequently

(16.24.1) $$S_{1.1}=c_3\int_0^{x_{10}}\frac{G(x)x^3dx}{(x-1)^5}, \quad \text{where} \quad c_3=15\sqrt{15}c_2=4251.1.$$

Next, in $S_{1.2}$ replacing x by $x=1/(1-\theta t_0)$, we get

(16.24.2) $$S_{1.2}=\frac{c_2}{t_0^4}\int_{1/4}^1\left[-G\left(\frac{1}{1-\theta t_0}\right)\frac{\sqrt{16\theta^2-1}^3}{\theta^5}\right]d\theta.$$

Secondly, observing that $S_2(t_0)$ is obtainable similarly as in S_1, but now non-vanishing at inner limits:

(16.25.1) $$S_{2.1}=-2c_2\int_0^{x_{10}}\frac{G(x)}{(x-5)^5}\left[\frac{16(x-1)^2-x^2t^2}{-t^3}\right]_{\sqrt{6}(1-1/x)}^{1-1/x}=-c_4\int_0^{x_{10}}\frac{G(x)x^3}{(x-1)^5}dx,$$

where $c_4=2c_2(15\sqrt{15}-5\sqrt{15}/9)=8187.3$, and

(16.25.2) $$S_{2.2}=\frac{2c_2}{t_0^4}\int_{1/\sqrt{6}}^1 G\left(\frac{1}{1-\theta t_0}\right)\left(\sqrt{16\theta^2-1}^3-\frac{5\sqrt{15}}{9}\right)\frac{d\theta}{\theta^5}.$$

But there is still the additional part:

(16.25.3) $$S_{2.3}=c_5\int_0^{x_{10}}\frac{G(x)x^3dx}{(x-1)^5}+c_6\int_{x_{10}}^{x_{30}}\frac{xG(x)}{(x-1)^3}\left[\frac{1}{t_0^2}-\frac{x^2}{6(x-1)^2}\right]dx,$$

where $c_5=25\sqrt{15}c/144\pi=3778.75$ and $c_6=5\sqrt{15}c/24\pi=6c_5/5=4534.5$. The last half yields, when x is replaced by $x=1/(1-\theta t_0)$,

$$S_{2.3.2}=\frac{c_6}{t_0^4}\int_{1/\sqrt{6}}^1 G\left(\frac{1}{1-\theta t_0}\right)\left(\frac{1}{\theta^3}-\frac{1}{6\theta^5}\right)d\theta,$$

which may be combined with $S_{2.2}$ together.

Thirdly

(16.26) $$S_3(t_0)=-2c\int_{-\infty}^{t_0}\frac{dt}{t^6}\int_{x_1}^{x_2}\frac{G(x)}{x^2}L'(\tau)dx$$

$$=-2c\int_0^{x_{10}}\frac{G(x)dx}{x^2}\int_{\sqrt{8/3}(1-1/x)}^{1-1/x}L'(\tau)\frac{dt}{t^6}-2c\int_{x_{10}}^{x_{20}}{''}\int_{\sqrt{8/3}(1-1/x)}^{t_0}{''}.$$

Or, on transforming the inner integration-variable t into $\tau=2(x-1)/xt$, i.e. $t=2(x-1)/x\tau$, $dt=-2(x-1)d\tau/x\tau^2$, the first half becomes

$$S_{3.1}=\frac{c}{16}\int_0^{x_{10}}\frac{G(x)x^3dx}{(x-1)^5}\int_{\sqrt{3/2}}^2\tau^4L'(\tau)d\tau$$

whose inner integral reduces to the constant 0.5986 by (15.25). Hence on putting $c_7=0.5986c/16=660.5$, we have

(16.26.1) $$S_{3.1}=c_7\int_0^{1/(1-t_0)}\frac{G(x)x^3}{(x-1)^5}dx.$$

Similarly

$$S_{3.2}=\frac{c}{16}\int_{x_{10}}^{x_{20}}\frac{G(x)x^3}{(x-1)^5}dx\int_{\sqrt{3/2}}^{2(x-1)/xt_0}\tau^4L'(\tau)d\tau,$$

which yields if the order of integrations once more interchanged (Fig. 9)

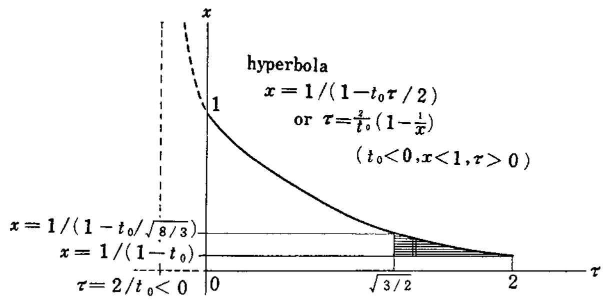

Fig. 9

$$S_{3.2}=\frac{c}{16}\int_{\sqrt{3/2}}^{2}\tau^4L'(\tau)d\tau\int_{1/(1-t_0)}^{1/(1-t_0\tau/2)}\frac{G(x)x^3dx}{(x-1)^5}.$$

So that if the inner integration-variable x be transformed into θ as before by $x=1/(1-\theta t_0)$, we get

$$S_{3.2}=-\frac{c}{16t_0^4}\int_{\sqrt{3/2}}^{2}\tau^4L'(\tau)d\tau\int_{\tau/2}^{1}G\left(\frac{1}{1-t_0\theta}\right)\frac{d\theta}{\theta^5}. \tag{16.26.2}$$

We are going to obtain the asymptotic forms for several expressions as $t_0\cong 0$, or $1/t_0\cong 0$, which shall be denoted by starring them once or twice, respectively. Thus, we find after (8)

$$G^*(1/(1-\theta t_0))\cong G(1)+O(t_0^5); \tag{16.27}$$

or, after (9)

$$G^{**}\left(\frac{1}{1-\theta t_0}\right)\cong G(0)-\frac{1}{t_0\theta}-\frac{5.5}{(t_0\theta)^2}-\frac{22.83}{(t_0\theta)^3}-\frac{79.2083}{(t_0\theta)^4}+O\left(\frac{1}{t_0^5}\right). \tag{16.28}$$

And consequently we have

$$S_{3.2}^*\cong-\frac{c}{16}\left[\frac{G(1)}{t_0^4}+O(t_0)\right]\int_{\sqrt{3/2}}^{2}\tau^4L'(\tau)d\tau\left[\frac{-1}{4\theta^4}\right]_{\tau/2}^{1}$$

$$\cong\frac{c_1}{16t_0^4}\int_{\sqrt{3/2}}^{2}\left(4-\frac{1}{4}\tau^4\right)L'(\tau)d\tau,$$

which being integrated by parts, the integrated parts do vanish because of $(4-\tau^4/4)_{\tau=2}=0$ and $L(\sqrt{3/2})=0$. Hence

$$S_{3.2}^*\cong\frac{c_1}{16t_0^4}\int_{\sqrt{3/2}}^{2}\tau^3L(\tau)d\tau,$$

whose integral was evaluated as 0.1195 at (15.26), so that

$$S^{*}_{3.2} \cong 0.00682/t_0^4. \tag{16.29}$$

But, we obtain similarly

$$S^{**}_{3.2} \cong \frac{-c}{16}\int_{\sqrt{3/2}}^{2} L(\tau)\Big[\frac{G(0)\tau^3}{t_0^4}+\frac{4}{5t_0^5}\Big(\tau^3+\frac{8}{\tau^2}\Big)+\frac{11}{3t_0^6}\Big(\tau^3+\frac{32}{\tau^3}\Big)+\cdots\Big]d\tau.$$

So that to know the coefficients of t_0^{-5}, $t_0^{-6}, \cdots$, we ought still further evaluate the integrals $\int_{\sqrt{3/2}}^{2} L(\tau)d\tau/\tau^{\nu}$, ($\nu=2, 3,\cdots$), e.g. by iterative employment of Gauss' method, which are never so easy going. No doubt the matter will go smoothly, if we could expand $L(\tau)$ by Laurent series. Actually the author tried to do so by taking powers up to the sixth, yet the results were unpleasing. Probably we ought to take terms up to τ^{-12} at least, but desirably to τ^{-24} or more, which needs extraordinary labours and is put off for a future work. Presently we shall neglect all powers of t_0^{-1} with indices ≥ 5, as $t_0^{-1} \cong 0$. Thus treating

$$S^{**}_{3.2} \cong -\frac{cG(0)}{16t_0^4}\int_{\sqrt{3/2}}^{2} \tau^3 L(\tau)d\tau \cong 13.88/t_0^4. \tag{16.30}$$

On picking up similar terms from the foregoing, we attain for $-\infty < t_0 < 0$

$$\begin{aligned} S(t) = & -\frac{0.22840}{t_0^4} + c_8\int_0^{x_{10}} \frac{G(x)x^3}{(x-1)^5}dx - \frac{c_2}{t_0^4}\int_{1/4}^{1} G\Big(\frac{1}{1-\theta t_0}\Big)\frac{\sqrt{16\theta^2-1}^3}{\theta^5}d\theta \\ & + 2c_2\int_{1/\sqrt{6}}^{1} G\Big(\frac{1}{1-\theta t_0}\Big)\Big[\frac{\sqrt{16\theta^2-1}^3}{\theta^5} - \frac{5\sqrt{15}}{9\theta^5} - 8\sqrt{15}\Big(\frac{1}{\theta^3}-\frac{1}{6\theta^5}\Big)\Big]d\theta + S_{3.2} \\ = & (0)+(\mathrm{i})+(\mathrm{ii})+(\mathrm{iii})+(\mathrm{iv}), \quad \text{where } c_8=c_3-c_4+c_5+c_7=503.1. \end{aligned} \tag{16.31}$$

We shall estimate these terms for $t_0 \cong 0$ as well as $1/t_0 \cong 0$, in the latter case taking powers of t_0^{-1} up to 4, as a rongh approximation.

First, to compute (i), replacing x by $y=1-x$, $y_0=-1$, $y_1=t_0/(1-t_0)$, we obtain easily by integration by parts

$$\begin{aligned} \int_0^{1/(1-t_0)} G(x)\frac{x^3dx}{(x-1)^5} = & -G(1+y)\Big[\frac{1}{y}+\frac{3}{2y^2}+\frac{1}{y^3}+\frac{1}{4y^4}\Big] (\equiv M(t_0)) \\ & + \int_{-1}^{y_1} G'(1+y)\Big[\quad '' \quad\Big]dy \ (\equiv N(t_0), \text{ say}). \end{aligned}$$

The integrated parts become

$$M(t_0) = -\frac{1}{4}G(0) + \frac{1}{4}\Big(1-\frac{1}{t_0^4}\Big)G\Big(\frac{1}{1-t_0}\Big), \tag{16.32}$$

so that availing (27) or (28), we find

$$M^* \cong 0.02631 + 0.00001/t_0^4,$$

$$M^{**} \cong -\frac{G(0)}{4t_0^4} - \frac{1}{4}\left(\frac{1}{t_0} + \frac{5.5}{t_0^2} + \frac{22.83}{t_0^3} + \frac{79.2083}{t_0^4}\right).$$

Since $G'(1+y)=y^4 \exp[-5(1+y)]$ after (5), the not-yet integrated parts yield by further integrations by parts

(16.33) $N(t_0) = 0.0244 - \exp[-5(1+y_1)]\,[0.2y_1^3 + 0.42y_1^2 + 0.368y_1 + 0.1236]$,

where $e^{-5y_1}y_1 \cong 0(1)$. So that

$$N^* \cong -0.02523,$$

$$N^{**} \cong \frac{1}{4}\text{(the same as the bracketed expression in } M^{**}\text{)}.$$

Hence, on multiplying M^*+N^* as well as $M^{**}+N^{**}$ by $c_8=503.1$, we get

$$\text{(i)}^* \cong 0.5432 + 0.00650/t_0^4, \qquad \text{(i)}^{**} \cong 13.24/t_0^4.$$

Next, making use of (27) and (28) for the factors in (ii), (iii) we heve the resulting integrals

(16.34) $$\int_{1/4}^{1} \sqrt{16\theta^2-1}\,d\theta/\theta^5 = 88.7776, \qquad \int_{1/\sqrt{6}}^{1} \;'' = 67.0914.$$

So that

$\text{(ii)}^* \cong -88.777c_2G(1)/t_0^4 \cong 0.33616/t_0^4$, $\quad \text{(ii)}^{**} \cong -88.777c_2G(0)/t_0^4 \cong 683.93/t_0^4$.

Further

$$\text{(iii)} = \frac{2c_2}{t_0^4}\int_{1/\sqrt{6}}^{1} G\left(\frac{1}{1-\theta t_0}\right)\left[\frac{\sqrt{16\theta^2-1}^{\,3}}{\theta^5} + \frac{7\sqrt{15}}{\theta^5} - \frac{8\sqrt{15}}{\theta^3}\right]d\theta$$

in which again availing (27) or (28) and integrating, we find

$$= \frac{31.979c_2}{t_0^4} \times \Big(G(1) \text{ or } G(0)\Big).$$

Hence

$$\text{(iii)}^* \cong -0.12109/t_0^4, \qquad \text{(iii)}^{**} \cong -246.36/t_0^4.$$

We have already seen that in (29), (30)

$$\text{(iv)}^* \cong 0.00682/t_0^4, \qquad \text{(iv)}^{**} \cong 13.88/t_0^4,$$

and by (23)

$$(0)^* = (0)^{**} = -0.22840/t_0^4.$$

Summing up all these, we obtain for $-\infty < t_0 < 0$

(16.35) $S^* = 0.5423 + 0.34948/t_0^4 - 0.34949/t_0^4 = 0.5423 - 0.00001/t_0^4$.

Here the second term deviates from the true value zero in consequence of calculations rounded at the fifth decimal place. Thus we see that

$$S(0) = 0.5423$$

and the median lying in the negative side, the distribution is skew.

On the other hand

(16.36) $$S^{**} \cong 464.46/t_0^4 \text{ roughly.}$$

This being equated to $\alpha/2 = 0.05, 0.025, 0.005$, we obtain roughly as the first approximation for the lower critical point with significant level α

(16.37) $$t_\alpha = -\sqrt[4]{464.5 \times 2/\alpha} = -\sqrt[4]{929/\alpha},$$

and thus

(16.38) $$t_{0.1} = -\sqrt[4]{9290} = -9.8,$$
$$t_{0.05} = -\sqrt[4]{18580} = -11.7,$$
$$t_{0.01} = -\sqrt[4]{92900} = -17.05,$$

respectively. However, these are only crude results, because the corresponding lower critical points for $n=4$ were after [III], -7.99, -10.91, -19.15, than which the present values must naturally be smaller in absolute value. To proceed more correctly, we shall still calculate coefficients of t_0^{-5}, t_0^{-6}, t_0^{-7}, &c. to exactify S^{**} and solve the equation $S^{**} = \alpha/2$ by Horner. The computation by means of series perhaps an appropriate method, as before mentioned, and the author expects to perform it in some future. The present note shall be ended with a few touching on the other subintervals and the upper critical points.

To get a foothold, if in the fr. f. the first term only be taken

(16.39) $$s(t) = \frac{c_1}{t^5}, \text{ and then, } \bar{S}(t_0) = c_1 \int_{t_0}^{\infty} \frac{dt}{t^5} = \frac{c_1}{4t_0^4} = \frac{0.2284}{t_0^4}.$$

This being equated to $\alpha/2 (\alpha = 0.01, 0.05, 0.1)$, we obtain

$$t_0 = \sqrt[4]{0.4568/\alpha}.$$

Hence, we have approximately

(16.40) $$t_{0.01} = 2.60, \quad t_{0.05} = 1.74, \quad t_{0.1} = 1.46, \quad \text{for the present size } n = 5.$$

These compared with those critical values obtained in [II], [III]:

$t_{0.01} =$ 3.10,	$t_{0.05} = 1.83$,	$t_{0.1} = 1.32$,	for $n=4$,
4.91,	2.19,	1.55,	for $n=3$,
27.07,	5.41,	2.71,	for $n=2$,

the present values seem not to be unplausible in outline, except the 10% point is probably overestimated. They should be exactified by taking into consideration the remaining terms further more.

We have seen for the end interval 6° $4 < t < \infty$

the area under $s(t) = P_6 = \bar{S}(4) = \int_4^\infty s(t)dt = 0.00089.$

In the neighbouring interval 5° $\sqrt{6} < t < 4$, we have to compute, for $t_0 = \sqrt{6}$ to get P_5 and for $t_0 = 2.6$ to check (40),

$$(16.41) \qquad \bar{S}(t_0) = \bar{S}(4) + \int_{t_0}^4 s(t)dt = \frac{c_1}{4t_0^4} + c\int_{t_0}^4 \frac{dt}{t^5}\int_{x_4}^\infty H_1(x, \tau)dx$$

after (15). The double integral, when the order of integrations interchanged (Fig. 10) and treated as before, yields

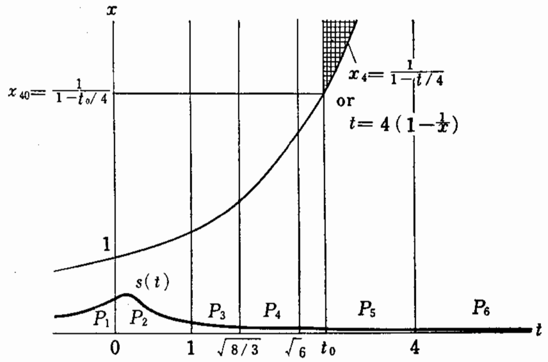

Fig. 10

$$(16.42) \quad c_2\int_{x_{40}}^\infty \frac{G(x)dx}{(x-1)^5}\left[\frac{\sqrt{16(x-1)^2 - x^2t^2}^3}{-t^3}\right]_{t_0}^{4(1-1/x)}$$

$$= c_2\int_{1/4}^{1/t_0} G\left(\frac{1}{1-\theta t_0}\right)\frac{\sqrt{16\theta^2-1}^3}{\theta^5}d\theta = \frac{c_2}{t_0^4}U(t_0) \quad \text{say,}$$

which may be also written by putting $\theta t_0 = \varphi$ as

$$(16.42.1) \qquad \frac{c_2}{t_0^4}U(t_0) = c_2\int_{t_0/4}^1 G\left(\frac{1}{1-\varphi}\right)\sqrt{\frac{16\varphi^2}{t_0^4}-1}^3\frac{d\varphi}{\varphi^5}.$$

Let us calculate the definite integral of the form with finite limits of integrations

$$(16.43) \qquad F(t_0) = \int_a^b f(\xi)d\xi,$$

where t_0, a parameter, but presently assumed as a known constant, may be contained in $f(\xi)$ or in the constants a, b also. For this purpose we have only after Gauss' method of numerical integration by means of 5 selected ordinates, simply to evaluate the expression

$$(16.44)\qquad (b-a)\sum_{\nu=1}^{5} A_\nu f(\xi_\nu) \quad \text{with} \quad \xi_\nu = \frac{1}{2}(b+a) + \frac{1}{2}(b-a)\alpha_\nu,$$

where A_ν, α_ν are Gaussian constants

$$(16.45)\qquad \begin{cases} \alpha_1 = -0.9061798 = -\alpha_5, \\ \alpha_2 = -0.5384693 = -\alpha_4, \\ \alpha_3 = 0, \end{cases} \qquad \begin{cases} A_1 = 0.1184634 = A_5, \\ A_2 = 0.2393143 = A_4, \\ A_3 = 0.2844444. \end{cases}$$

In this way it is found for (42)

$$U(\sqrt{6}) = -0.0^5 1877 \quad \text{and} \quad U(2.6) = -0.0^6 383.$$

So that

$$\bar{S}(\sqrt{6}) = 0.22840/36 - 0.0^5 1877 c_2/36 = 0.00634,$$

and accordingly the area under $s(t)$ between $t=\sqrt{6}$ and $t=4$ is given by

$$(16.46)\qquad P_5 = \bar{S}(\sqrt{6}) - \bar{S}(4) = 0.00545.$$

Also

$$\bar{S}(2.6) = 0.22840/2.6^4 - 0.0^6 383 c_2/2.6^4 = 0.004997.$$

Hence, the forecasted value for $t_{0.01}$ can be said to be just correct. For, when we interpolate such value as $\bar{S}(t_0) = 0.005$ from the above obtained two values $\bar{S}(\sqrt{6})$ and $\bar{S}(2.6)$ by law of P. P., we find $t_{0.01} = 2.600$ almost exactly.

Next, for the interval 4° $\sqrt{8/3} < t < \sqrt{6}$, we have in view of (14)

$$(16.47)\qquad \bar{S}(t_0) - \bar{S}(\sqrt{6}) = \int_{t_0}^{\sqrt{6}} s(t)dt = \frac{c_1}{4t_0^4} + c\int_{t_0}^{\sqrt{6}} \frac{dt}{t^5}\int_{x_4}^{\infty} H_1 dx - c\int_{t_0}^{\sqrt{6}} \frac{dt}{t^5}\int_{x_3}^{\infty} K_1 dx,$$
$$= (0) + (\text{i}) + (\text{ii}),$$

which becomes, when the order of integrations interchanged (Fig. 11) and in-

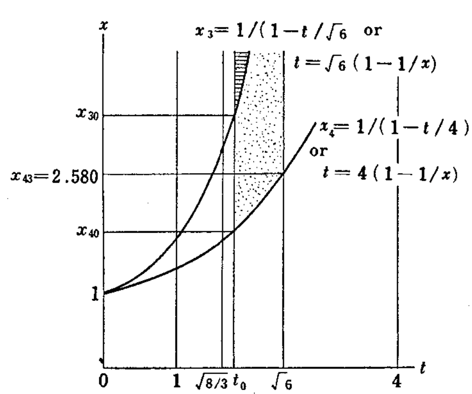

Fig. 11

tegrated about t,

$$(\mathrm{i}) = c_2 \int_{x_{40}}^{x_{43}} \frac{G(x)dx}{(x-1)^5} \left[\frac{\sqrt{16(x-1)^2 - x^2t^2}^3}{-t^3} \right]_{t_0}^{4(1-1/x)} + c_2 \int_{x_{43}}^{\infty} {}'' \left[{}'' \right]_{t_0}^{\sqrt{6}},$$

where $x_{43} = 1/(1-\sqrt{3/8}) = 2.5798$ and

$$(\mathrm{ii}) = 2c_2 \int_{x_{30}}^{\infty} \frac{G(x)dx}{(x-1)^5} \left[\frac{\sqrt{16(x-1)^2 - x^2t^2}^3}{t^3} - \frac{8x(x-1)^2}{t^2} \right]_{t_0}^{\sqrt{6}(1-1/x)}.$$

Or, if x be transformed into θ by $x=1/(1-\theta t_0)$ or into φ by $x=1/(1-\varphi)$, they become

$$(16.48)\ (\mathrm{i}) = \frac{c_2}{t_0^4} \int_{1/4}^{1/t_0} G\left(\frac{1}{1-\theta t_0}\right) \frac{\sqrt{16\theta^2-1}^3}{\theta^5} d\theta - \frac{c_2}{t_0^4} \int_{\sqrt{3/8}}^{1} G\left(\frac{1}{1-\varphi}\right) \sqrt{\frac{8}{3}\varphi^2 - 1}^3 \frac{d\varphi}{\varphi^5}$$

$$= \frac{c_2}{t_0^4} (U(t_0) - U_1)$$

whose second integral reduces to a constant, as well as

$$(16.49)\ (\mathrm{ii}) = \frac{c_6}{8t_0^4} \int_{1/\sqrt{6}}^{1/t_0} G\left(\frac{1}{1-\theta t_0}\right) \left(24\theta^2 - \frac{31}{9} - 15\sqrt{\frac{16\theta^2-1^3}{15}}\right) \frac{d\theta}{\theta^5} \equiv \frac{c_6}{8t_0^4} V(t_0).$$

These integrals U, U_1, V are again to be computed after Gauss' method (44) for $t_0 = \sqrt{8/3}$ to obtain the area P_4 and for $t_0 = 1.7$ to check the forecasted value of $t_{0.05}$, respectively. Hence there are needed $3 \times 2 = 6$ calculations, which however are left as students' exercise.

Finally, for the interval 3° $1 < t < \sqrt{8/3}$, we have in view of (13)

$$(16.50)\ \bar{S}(t_0) - \bar{S}(\sqrt{8/3}) = \int_{t_0}^{\sqrt{8/3}} s(t)dt$$

$$= \frac{c_1}{4t_0^4} + c \int_{t_0}^{\sqrt{8/3}} \frac{dt}{t^5} \left[\int_{x_4}^{\infty} H_1 dx - \int_{x_3}^{\infty} K_1 dx + \int_{x_2}^{\infty} L_1 dx \right]$$

$$= (0) + (\mathrm{i}) + (\mathrm{ii}) + (\mathrm{iii}).$$

We interchange the order of double integrations (Fig. 12). As to (i) and (ii)

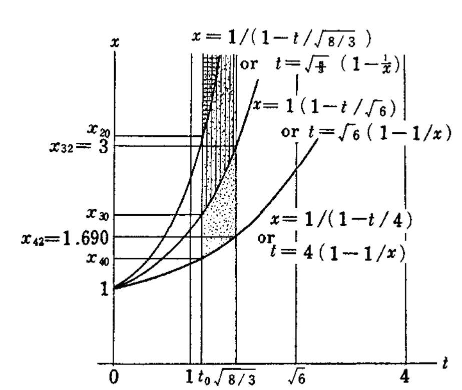

Fig. 12

almost similar as in (48) and (49), we get

$$(16.51)\quad \text{(i)}=\frac{c_2}{t_0^4}\int_{1/4}^{1/t_0} G\Big(\frac{1}{1-\theta t_0}\Big)\sqrt{16\theta^2-1}^3\frac{d\theta}{\theta^5}-\frac{c_2}{t_0^4}\int_{1/\sqrt{6}}^{1} G\Big(\frac{1}{1-\varphi^2}\Big)\sqrt{6\varphi^2-1}^3\frac{d\varphi}{\varphi^5}$$

$$=\frac{c_2}{t_0^4}[U(t_0)-U_2],$$

where U_2=const., as well as

$$(16.52)\quad \text{(ii)}=\frac{c_6}{8t_0^4}\int_{1/\sqrt{6}}^{2/3t_0} G\Big(\frac{1}{1-\theta t_0}\Big)\Big[8\theta_2-\frac{7}{9}-15\sqrt{\frac{16\theta^2-1}{15}}^3\Big]\frac{d\theta}{\theta^5}$$

$$+\frac{c_6}{8t_0^4}\int_{2/3t_0}^{1/t_0} G\Big(\frac{1}{1-\theta t_0}\Big)\Big[8\theta^2-3-\frac{1}{\sqrt{15}}(\sqrt{16\theta^2-1}^3+\sqrt{6\theta^2-1}^3)\Big]\frac{d\theta}{\theta^5}$$

$$=\frac{c_6}{8t_0^4}[V_1(t_0)+V_2(t_0)].$$

These four integrals U, U_2, V_1, V_2 should be integrated also by Gauss' method for $t_0=1$ or $t_0=1.2$, respectively, in order to obtain P_3 or to correct the foreshown values of $t_{0.1}$, which 8 computations, however, are again left to students' exercises.

But, there remains still the third part, which, when written in the original form (13) is

$$\text{(iii)}=-c\int_{t_0}^{\sqrt{8/3}}\frac{dt}{t^5}\int_{x_2}^{\infty}G'Ldx=-c\int_{x_{20}}^{\infty}e^{-5x}(x-1)^4dx\int_{t_0}^{\sqrt{8/3}(1-1/x)}L(\tau)dt/t^5.$$

Here t transformed into τ by $t=2(x-1)/x\tau$, $dt=-2(x-1)d\tau/x\tau^2$, and x into ξ by $x=1/(1-\xi)$, yields

$$(16.53)\qquad \text{(iii)}=-\frac{c}{16}\int_{x_{20}}^{\infty}e^{-5x}x^4dx\int_{\sqrt{3/2}}^{\tau_1}\tau^3L(\tau)d\tau$$

$$=-\frac{c}{16}\int_{\xi_0}^{1}f(\xi)d\xi\int_{\sqrt{3/2}}^{\tau_1}\tau^3L(\tau)d\tau,$$

where

$$(16.54)\quad f(\xi)=\exp\Big(\frac{-5}{1-\xi}\Big)\cdot\frac{1}{(1-\xi)^6},\quad\text{and}\quad \xi_0=\sqrt{\frac{3}{8}}t_0,\ \tau_1=\frac{2(x-1)}{xt_0}=\frac{2\xi}{t_0}.$$

Further, if the inner integration-variable τ be transformed into η by $\tau^2=\eta$ and $\tau_1^2=4\xi^2/t_0^2=y_1$, the last inner integral becomes after (15.19)

$$\int_{\sqrt{3/2}}^{\tau_1}\tau^3L(\tau)d\tau=\frac{5\sqrt{3}}{\pi^2}\int_{3/2}^{y_1}g(\eta)d\eta\int_0^1 h(\eta,\zeta)d\zeta-\frac{5}{4\pi^2}\int_{3/2}^{y_1}\psi(\eta)d\eta,$$

where

$$(16.55)\quad \begin{cases} g(\eta) = \eta\sqrt{2\eta-3}, \\ h(\eta,\zeta) = \tan^{-1}\sqrt{\dfrac{1}{3}+\dfrac{4\eta-6}{15\eta}(1-\zeta^2)\Big/\Big(1+\Big(\dfrac{2}{3}\eta-1\Big)\zeta^2\Big)}, \\ \psi(\eta) = 2\pi\eta\tan^{-1}\sqrt{\dfrac{2\eta-3}{3}} - \sqrt{15}\tan^{-1}\sqrt{\dfrac{3}{5}(2\eta-3)} \\ \qquad\qquad + 3\sqrt{4\eta-1}\tan^{-1}\sqrt{\dfrac{2\eta-3}{4\eta-1}}. \end{cases}$$

Thus, we obtain

$$(16.56)\quad \text{(iii)} = -c_9\int_{\xi_0}^1 f(\xi)d\xi\int_{3/2}^{y_1} g(\eta)d\eta\int_0^1 h(\eta,\zeta)d\zeta + c_{10}\int_{\xi_0}^1 f(\xi)d\xi\int_{3/2}^{y_1}\psi(\eta)d\eta$$
$$= -c_9 W_3 + c_{10} W_2,$$

where $c_9 = 5\sqrt{3}c/16\pi^2 = 96.825$ and $c_{10} = 5c/64\pi^2 = 13.975$. Hence, to integrate (56) for $t_0=1$ and $t_0=1.2$, we must iterate Gauss' method thrice or twice.

The triple integral in (56) can be computed by iteration of (44) as

$$(16.57)\quad W_3 = (1-\xi_0)\sum_{\lambda=1}^{5} A_\lambda(4\xi_\lambda^2/t_0^2 - 3/2)f(\xi_\lambda)\sum_{\mu=1}^{5} A_\mu g(\eta_{\lambda\mu})\sum_{\nu=1}^{5} A_\nu h(\eta_{\lambda\mu}, \zeta_{\lambda\mu\nu}),$$

where

$$(16.58)\quad \begin{cases} \xi_\lambda = \dfrac{1}{2}(1+\xi_0) + \dfrac{1}{2}(1-\xi_0)\alpha_\lambda, \\ \eta_{\lambda\mu} = (2\xi_\lambda^2/t_0^2 + 3/4) + (2\xi_\lambda^2/t_0^2 - 3/4)\alpha_\mu, \\ \zeta_{\lambda\mu\nu} = \dfrac{1}{2}(1+\alpha_\nu), \end{cases}$$

where α_λ, α_μ, α_ν and A_λ, A_μ, A_ν are those Gaussian constants in (45). Thus, to integrate the triple integral by iterated Gauss' method, with n abscissas, we must compute $5^3 = 125$ ordinates, if $n = 5$, or 1000 ordinates if $n = 10$, which requires a pretty labourious work. This method applied, we obtain for $t_0 = 1$, $W_3 = 0.0^5 2383449642$ and for $t_0 = 1.2$, $W_3 = 0.0^8 5292247$[1]. The function $W_3(t_0)$ does indeed vanish for $t_0 = \sqrt{8/3}$ (=1.6330), since then the lower limit $\xi_0 = \sqrt{3/8}t_0$ tends to unity. Moreover, its left-sided derivatives of every order vanishing at $t_0 = \sqrt{8/3} - 0$, the curve osculates the t_0-axis very strongly there. Hence, we may interpolate $W_3(t_0)$ e.g. by

$$Y(t_0) = \Big[A + B\Big/\Big(\frac{8}{3} - t_0^2\Big)\Big]\exp\Big[-1\Big/\Big(\frac{8}{3} - t_0^2\Big)\Big]$$

and obtain $A = 0.0^4 1641715$, $B = -0.0^4 2012370$ on availing the above data.

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1) About these calculations, the author is indebted to Messrs. Members in the Institute of Industrial Science, Tokyo University, for the achievement accomplished by electronic calculator.

Also the double integral in (56) will be similarly obtained by

$$(16.59) \qquad W_2=(1-\xi_0)\sum_{\lambda=1}^{5}A_\lambda(4\xi_\lambda^2/t_0^2-3/2)f(\xi_\lambda)\sum_{\mu=1}^{5}A_\mu\psi(\eta_{\lambda\mu}).$$

However, as the inner integrand $\psi(\eta)$ can be integrated finitely, we shall be able to evaluate W_2 by a single Gaussian process as in (44). In fact

$$(16.60) \quad \int_{3/2}^{y_1}\psi(\eta)d\eta=\pi y_1^2\tan^{-1}\sqrt{2y_1-3}-\frac{17}{2}\sqrt{\frac{5}{3}}\tan^{-1}\sqrt{\frac{3}{5}(2y_1-3)}$$

$$+\frac{1}{2}\sqrt{4y_1-1}^3\tan^{-1}\sqrt{\frac{2y_1-3}{4y_1-1}}-\frac{1}{6}[\pi(y_1+3)+18y_1-35]\sqrt{2y_1-3}$$

$$=\Psi(\xi,t_0)$$

because of $y_1=4\xi^2/t_0^2$. So that

$$(16.61) \qquad W_2=\int_{\xi_0=\sqrt{3/8}t_0}^{1}\exp\left(\frac{-5}{1-\xi}\right)\frac{\Psi(\xi,t_0)}{(1-\xi)^6}d\xi.$$

Students would have the exercise to compute W_2 in two ways (59), (61) and compare the results.