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NOTE ON SEMI-LOCAL RINGS

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We know that the notion of a system of parameters of a local ring can be extended to the case of a semi-local ring (cf. [4][1]). In this note, we introduce the concept of a regular semi-local ring, naturally extending the definition from regular local ring to semi-local ring. For regular semi-local rings we shall show some properties which are analogous to the fundamental properties of regular local rings. Throughout this note the term "local or semi-local ring" will mean commutative Noetherian local or semi-local ring with identity.

1. Regular semi-local rings

Let R be a semi-local ring and $\mathfrak{q}$ be a defining ideal of R. We denote by $dim\,R$ the dimension of R, by $l(R/\mathfrak{q}^n)$ the length of $R/\mathfrak{q}^n$ and by $e(\mathfrak{q})$ the multiplicity of $\mathfrak{q}$. The integer $e(\mathfrak{m})$ is called the multiplicity of R, where $\mathfrak{m}$ is the J-radical[2] of R.

Given a semi-local ring R of dimension d, a system $\{x_1, \cdots, x_d\}$ of d elements of R which generates a defining ideal is called a system of parameters of R.

The well known relation between multiplicities and lengths is given as follows:

PROPOSION 1. *Let R be a semi-local ring of dimension d, $\{x_1, \cdots, x_d\}$ a system of parameters of R and $\mathfrak{q}$ the ideal $\sum_{i=0}^{d} Rx_i$. Then we have $e(\mathfrak{q}) \leq l(R/\mathfrak{q})$. The equality holds if and only if the form ring $F(\mathfrak{q}) = \sum_{n=0}^{\infty} \mathfrak{q}^n/\mathfrak{q}^{n+1}$ is isomorphic to the polynomial ring $(R/\mathfrak{q})[X_1, \cdots, X_d]$.*

For the proof, see [4].

Let R be a semi-local ring having maximal ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_t$ and $\mathfrak{m}$ be the J-rabical of R (i.e. $\mathfrak{m} = \bigcap_{j=1}^{t} \mathfrak{p}_j$).

DEFINITION. We say that R is a *regular semi-local ring* if $\mathfrak{m}$ is generated by a system of parameters of R and the dimention of every quotient ring $R_{\mathfrak{p}_j}$ $(j=1, \cdots, t)$ is equal to that of R.

The system of parameters of a regular semi-local ring which generates

1) Numbers in brackets refer to the bibliography at the end of this note.
2) We mean by the J-radical the intersection of all maximal ideals of R.

the J-radical will be called a regular system of parameters.

LEMMA 1.[3] *Let R be a semi-local ring and $\mathfrak{q}$ be a primary ideal belonging to an arbitrary maximal ideal $\mathfrak{p}$ of R. Then $\mathfrak{q}$ is generated by a finite number of elements such that none of them is in any maximal ideal different from $\mathfrak{p}$.*

Proof. Let $\mathfrak{p}=\mathfrak{p}_1, \cdots, \mathfrak{p}_t$ be maximal ideals of R. We consider the non-empty set $Q=\{a \mid a\in\mathfrak{q}$ and $a\notin\bigcup_{j=2}^{t}\mathfrak{p}_j\}$ and let $\mathfrak{q}'$ be the ideal generated by the set Q. Since R is a Noetherian ring, $\mathfrak{q}'$ is generated by a finite number of elements of Q. For the proof of this lemma it is enough to show $\mathfrak{q}\subseteq\mathfrak{q}'$. Let a be an arbitrary element of $\mathfrak{q}\cap\mathfrak{p}_2\cap\cdots\cap\mathfrak{p}_t$ and let b be an element of Q. Then $a+b\in Q$, whence $a=(a+b)-b\in\mathfrak{q}'$. Thus $\mathfrak{q}\cap\mathfrak{p}_2\cap\cdots\cap\mathfrak{p}_t\subseteq\mathfrak{q}'$. Since $\mathfrak{q}'\nsubseteq\mathfrak{p}_j$ $(j\geq 2)$, we see that $\mathfrak{p}$ is the unique prime ideal containing $\mathfrak{q}'$. Therefore $\mathfrak{q}'$ is $\mathfrak{p}$-primary. Since $\mathfrak{q}\cap\mathfrak{p}_t\cap\cdots\cap\mathfrak{q}\subseteq\mathfrak{q}'$, we have $\mathfrak{q}R_{\mathfrak{p}}=(\mathfrak{q}\cap\mathfrak{p}_2\cap\cdots\cap\mathfrak{p}_t)R_{\mathfrak{p}}\subseteq\mathfrak{q}'R_{\mathfrak{p}}$. Therefore we have $\mathfrak{q}\subseteq\mathfrak{q}'$.

PROPOSITION 2. *Let R be a semi-local ring and $\mathfrak{p}_1, \cdots, \mathfrak{p}_t$ be maximal ideals of R. Then R is a regular semi-local ring if and only if all the quotient rings $R_{\mathfrak{p}_1}, \cdots, R_{\mathfrak{p}_t}$ are regular local rings having the same dimension.*

Proof. The if part: Since $R_{\mathfrak{p}_j}$ is a regular local ring of dimension d (where $d=dim\ R$), by Lemma 1 there exist d elements $a_1^{(j)}, \cdots, a_d^{(j)}$ *in* $\mathfrak{p}_j$ such that none of them is in any maximal ideal different from $\mathfrak{p}_j$ and such that $\mathfrak{p}_jR_{\mathfrak{p}_j}=\sum_{i=1}^{d}a_i^{(j)}R_{\mathfrak{p}_j}$. Let $\mathfrak{a}_j$ be the ideal which is generated by these d elements. Since $\mathfrak{a}_j\nsubseteq\mathfrak{p}_k$ $(k\neq j)$, all of the prime divisors of $\mathfrak{a}_j$ are contained in $\mathfrak{p}_j$. Hence we have $\mathfrak{p}_j=\mathfrak{a}_j$ for every j. Considering the d elements $a_i=\prod_{i=1}^{t}a_i^{(j)}$ $(i=1, \cdots, d)$ and the ideal $\mathfrak{a}=\sum_{i=1}^{d}Ra_i$, we have $\mathfrak{a}\subseteq\mathfrak{m}$ and $\mathfrak{a}R_{\mathfrak{p}_j}=\mathfrak{m}R_{\mathfrak{p}_j}$ for every j, where $\mathfrak{m}$ is the J-radical of R. Therefore we have $\mathfrak{a}=\mathfrak{m}$.

The only if part follows immediately from the definition of a regular semi-local ring.

LEMMA 2. *Let R be are regular semi-local ring and $\mathfrak{m}$ the J-radical. Then $e(\mathfrak{m})=l(R/\mathfrak{m})$.*

Proof. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_t$ be maximal ideals of R. Since

$$R/\mathfrak{m}^n\cong(R_{\mathfrak{p}_1}/\mathfrak{p}_1^nR_{\mathfrak{p}_1})\oplus\cdots\oplus(R_{\mathfrak{p}_t}/\mathfrak{p}_t^nR_{\mathfrak{p}_t})\quad\text{(direct sum)}$$

and since $dim\ R_{\mathfrak{p}_j}=dim\ R$ for every j, we have $e(\mathfrak{m})=\sum_{j=1}^{t}e(\mathfrak{p}_jR_{\mathfrak{p}_j})=t$. On the other hand, $l(R/\mathfrak{m})=\sum_{j=1}^{t}l(R/\mathfrak{p}_j)=t$ since $R/\mathfrak{m}\cong(R/\mathfrak{p}_1)\oplus\cdots\oplus(R/\mathfrak{p}_t)$. Q.E.D.

3) This lemma and Proposition 2 have already been obtained by S. Endo (cf. [1]).

THEOREM 1. *Let R be a semi-local ring of dimension d, $\mathfrak{p}_1, \cdots, \mathfrak{p}_t$ be maximal ideals and $\mathfrak{m}$ be the J-radical. Then the following three conditions are equivalent*:

(a) *R is a regular semi-local ring.*

(b) *The form ring $F(\mathfrak{m})=\sum_{n=0}^{\infty}\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a polynomial ring in d variables.*

(c) *The maximum number of linearly independent elements of the $R/\mathfrak{m}$-module $\mathfrak{m}/\mathfrak{m}^2$ is equal to d and the dimension of every local ring $R_{\mathfrak{p}_j}$ ($j=1, \cdots, t$) is equal to d.*

Proof. That (a) implies (b) follows directly from Proposition 1 and Lemma 2. Assume next that (b) is true and we want to show that (c) is true. The first part of (c) is evident. And hence there are d elements $x_1, \cdots, x_d$ in $\mathfrak{m}$ such that $\mathfrak{m}=\sum_{i=1}^{d}Rx_i+\mathfrak{m}^2$. Sinec R is a Zariski ring with respect to the $\mathfrak{m}$-topology, we have $\mathfrak{m}=\sum_{i=1}^{d}Rx_i$. This shows that the system $\{x_1, \cdots, x_d\}$ is a system of parameters of R. Applying this and the condition (b) to Proposition 1, we have $e(\mathfrak{m})=l(R/\mathfrak{m})=t$. We may assume that $\dim R_{\mathfrak{p}_j}=\dim R$ if and only if $j\leq s$. Then $e(\mathfrak{m})=\sum_{j=1}^{s}e(\mathfrak{p}_jR_{\mathfrak{p}_j})$. Since $\mathfrak{m}=\sum_{i=1}^{d}Rx_i$, $\mathfrak{p}_jR_{\mathfrak{p}_j}=\mathfrak{m}R_{\mathfrak{p}_j}$ and $\dim R_{\mathfrak{p}_j}=d$ for $j=1, \cdots, s$, each of these $R_{\mathfrak{p}_j}$ is a regular local ring. Hence $e(\mathfrak{p}_jR_{\mathfrak{p}_j})=1$ for $j=1, \cdots, s$ and therefore we have $e(\mathfrak{m})=s$. This shows $s=t$. That (c) implies (a) is included in the proof above given.

COROLLARY. *Let R be a semi-local ring, $\mathfrak{m}$ be the J-radical and $\hat{R}$ be the $\mathfrak{m}$-adic completion of R. Then $\hat{R}$ is a semi-local ring having the same dimension of R and $\mathfrak{m}\hat{R}$ is the J-radical of $\hat{R}$. Furthermore R is a regular semi-local ring if and only if $\hat{R}$ is so.*

The first part of this corollary is well known (see [5]). Since the form ring of R and that of $\hat{R}$ are the same (see [5]), the second part follows immediately from Theorem 1.

2. Unmixed semi-local rings

The properties of unmixed local rings were studied by M. Nagata (cf. [3]). In this paragraph we show some results which are directly followed from Nagata's results and apply these to regular semi-local rings.

Let R be a semi-local ring and $\mathfrak{m}$ be the J-radical. We denote by $\hat{R}$ the $\mathfrak{m}$-adic completion of R.

DEFINITION We say that a semi-local ring R is *unmixed* if $\dim \hat{R}/\hat{\mathfrak{p}}=\dim R$ for any prime divisor $\hat{\mathfrak{p}}$ of zero in $\hat{R}$.

PROPOSITION 3. *Let R be an unmixed semi-local ring. Then we have* $\dim R/\mathfrak{p}+\dim R_{\mathfrak{p}}=\dim R$ *for any prime ideal* $\mathfrak{p}$ *of* R.

For the proof, see [3].[4)]

PROPOSITION 4. *If a semi-local ring R is unmixed, then for every prime ideal $\mathfrak{p}$ of R the quotient ring $R_{\mathfrak{p}}$ is also unmixed.*

Proof. In the local case, this proposition is true (see [3]). If $\mathfrak{q}$ and $\mathfrak{q}'$ are arbitrary prime ideals such that $\mathfrak{q}\subseteq\mathfrak{q}'$, then $R_{\mathfrak{q}}\cong(R_{\mathfrak{q}'})_{\mathfrak{q}R_{\mathfrak{q}'}}$. Therefore we may assume that $\mathfrak{p}$ is a maximal ideal of R. Let $\mathfrak{p}=\mathfrak{p}_1, \cdots, \mathfrak{p}_t$ be maximal ideals of R and R_j be the $\mathfrak{p}_jR_{\mathfrak{p}_j}$-adic completion of $R_{\mathfrak{p}_j}$. Then $\hat{R}=R_1\oplus\cdots\oplus R_t$ (direct sum). Let $\mathfrak{q}_1$ be any prime divisor of zero in R_1. We consider the prime ideal $\hat{\mathfrak{p}}=\mathfrak{q}_1\oplus R_2\oplus\cdots\oplus R_t$ of $\hat{R}$. Since $(0):\mathfrak{q}_1\neq(0)$ in R_1, we have $(0):\hat{\mathfrak{p}}\neq(0)$ in $\hat{R}$, which shows that $\hat{\mathfrak{p}}$ is a prime divisor of zero in $\hat{R}$ (because R is unmixed). Let $e_1, \cdots, e_t$ be the orthogonal idempotents corresponding to the decomposition $\hat{R}=R_1\oplus\cdots\oplus R_t$ and $\hat{\mathfrak{p}}\subset\mathfrak{P}_1\subset\cdots\subset\mathfrak{P}_t$ be a chain of prime ideals in $\hat{R}$. Then $\mathfrak{q}_1=\hat{\mathfrak{p}}e_1\subset\mathfrak{P}_1e_1\subset\cdots\subset\mathfrak{P}_se_1$ is a chain of prime ideals in R_1. Hence we have

$$\dim R_{\mathfrak{p}}=\dim R_1\geq\dim R_1/\mathfrak{q}_1$$

$$\geq\dim \hat{R}/\hat{\mathfrak{p}}=\dim R\geq\dim R_{\mathfrak{p}}. \qquad \text{Q.E.D.}$$

The following characterization of a regular local ring was given by M. Nagata (see [2]).

PROPOSITION 5. *A local ring R is a regular local ring if and only if it is of multilicity one and unmixed.*

In the semi-local case, this is generalized as follows:

THEOREM 2. *A semi-local ring R is a regular semi-local ring if and only if the multiplicity of R is equal to the number of maximal ideals of R and R is unmixed.*

Proof. The if part: Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_t$ be maximal ideals of R and $\mathfrak{m}$ be the J-radical. By Proposition 3 the equality $\dim R_{\mathfrak{p}_j}=\dim R$ holds for every j. Then we have $e(\mathfrak{m})=\sum_{j=1}^{t}e(\mathfrak{p}_jR_{\mathfrak{p}_j})$, whence the multiplicity of $R_{\mathfrak{p}_j}$ is one. On the other hand, $R_{\mathfrak{p}_j}$ is unmixed by Proposition 4. Therefore R is a regular semi-local ring by Proposition 5 and 2.

The only if part: The equality $e(\mathfrak{m})=l(R/\mathfrak{m})$ has already seen in Lemma 2. This shows that $e(\mathfrak{m})$ is equal to the number of maximal ideals. For the proof of the unmixedness, by the definition of unmixedness and by

4) In [3] M. Nagata has proved this proposition in the quasi-unmixed case which is a weaker condition than the unmixedness.

Corollary to Theorem 1 we may assume that R is complete. Let $\mathfrak{p}'$ be an arbitrary prime divisor of zero in R and $\mathfrak{p}$ be any maximal ideal such that $\mathfrak{p}' \subseteq \mathfrak{p}$. Since $R_{\mathfrak{p}}$ is an integral domain (because $R_{\mathfrak{p}}$ is a regular local ring), we have $\mathfrak{p}' R_{\mathfrak{p}} = 0$. Hence $(R/\mathfrak{p}')_{\mathfrak{p}/\mathfrak{p}'}$ is isomorphic to $R_{\mathfrak{p}}$. Therefore, since $\dim R_{\mathfrak{p}} = \dim R$, we have $\dim R/\mathfrak{p}' = \dim R$. Thus the proof is completed.

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