Journal of Gakugei, Tokushima University
Vol. XII. 1961

THE STUDENT'S DISTRIBUTION FOR A UNIVERSE BOUNDED AT ONE OR BOTH SIDES (Continued)

By

Yoshikatsu WATANABE

(Received September 30, 1961)

The author schemed first to investigate fully the distribution of Student's ratio for the χ_k^2- and B_{pq}-distribution taken as universe, which for the sample size $n=2$ but with all positive k and p, q, so simple yet their various remarkable features may be comprehensibly and interestingly grasped. However, this pleasing plan being kept for students' heuristic self study, the present author did continue the previous work and intended especially to treat the complicated cases $n=3, 4, \cdots$, possibly the general case, since he deems it his duty to clarify and supplement his preceding papers in this journal.[1)]

8. *The χ_k^2-Distribution as Universe, when the Sample Size $n=3$.*

Let a sample taken from a χ_k^2-universe[2)]

$$(8.0)\qquad f_k(x)=c_k e^{-x/2}x^{k/2-1},\ \text{where}\ c_k=1/2^{k/2}\Gamma\left(\frac{k}{2}\right),\ x>0,\ E(x)=k$$

be $\{x_1, x_2, x_3\}$ with mean $\bar{x}$ and S. D. s. Its probability is

$$dp=d_k e^{-3\bar{x}/2}(x_1x_2x_3)^{k/2-1}dx_1dx_2dx_3/8^{k/2}\Gamma\left(\frac{k}{2}\right)^3.$$

First we transform, as usual, x_1, x_2, x_3 into ξ, η, ζ orthogonally with $\zeta=\sqrt{3}\,\bar{x}$:

	x_1	x_2	x_3
ξ	$-1/\sqrt{2}$	$1/\sqrt{2}$	0
η	$-1/\sqrt{6}$	$-1/\sqrt{6}$	$2/\sqrt{6}$
ζ	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$

$$\begin{cases} x_1=-\xi/\sqrt{2}-\eta/\sqrt{6}+\zeta/\sqrt{3}\ (=\bar{x}) \\ x_2=+\xi/\sqrt{2}-\zeta/\sqrt{6}+\bar{x} \\ x_3=\qquad\qquad 2\zeta/\sqrt{6}+\bar{x}, \end{cases}$$

and secondly into polar co-ordinates $\xi=\rho\cos\theta$, $\eta=\rho\sin\theta$ with $\rho=\sqrt{3}\,s$, so that

$$x_1x_2x_3=\bar{x}^3-\frac{1}{2}(\xi^2+\eta^2)\bar{x}-\left(\xi^2-\frac{\eta^2}{3}\right)\frac{\eta}{\sqrt{6}}=\bar{x}^3-\frac{1}{2}\rho^2\bar{x}-\frac{\rho^3}{3\sqrt{6}}\sin 3\theta$$
$$=\bar{x}^3-\frac{3}{2}s^2\bar{x}-\frac{s^3}{\sqrt{2}}\sin 3\theta.$$

1) Y. Watanabe, The Student's Distribution for a Universe bounded at one or both Sides, Journal of Tokushima University, vol. XI (1960), pp. 11—, which will be below cited as [II]. Also the same author's first paper: Y. Watanabe, Some Exceptional Examples to Student's Distribution, ditto, vol. X (1959), pp. 11— is referred as [I].

2) H. Cramér, Mathematical Methods of Statistics, p. 233.

Now taking Student's ratio $t=(\bar{x}-k)\sqrt{2}/s$ and writing $\tau=(\bar{x}-k)/\bar{x}t$ $(=s/\sqrt{2}\,\bar{x}>0)$ for brevity, we have

$$x_1x_2x_3=\bar{x}^3(1-3\tau^2-2\tau^3\sin3\theta).$$

Also

$$dx_1dx_2dx_3=d\xi d\eta d\zeta=\sqrt{3}\,d\bar{x}\rho d\rho d\theta=3\sqrt{3}\,sdsd\bar{x}d\theta=6\sqrt{3}\,(\bar{x}-k)^2d\bar{x}d\theta dt/|t|^3.$$

So that the elementary probability reduces to

$$dP=d_ke^{-3\bar{x}/2}(k-\bar{x})^2\bar{x}^{3k/2-1}(1-3\tau^2-2\tau^3\sin3\theta)^{k/2-1}d\bar{x}d\theta dt/|t|^3,$$

where

$$d_k=6\sqrt{3}/8^{k/2}\Gamma(k/2)^3.$$

Consequently Student's fr. f. is given by

$$(8.1)\qquad s(t)=\frac{d_k}{|t|^3}\int_{x_0}^{x_1}e^{-3\bar{x}/2}(\bar{x}-k)^2\bar{x}^{3k/2-3}Jd\bar{x}$$

where

$$(8.2)\qquad J=\int_{\theta_0}^{\theta_1}(1-3\tau^2-2\tau^3\sin3\theta)^{k/2-1}d\theta\qquad\left(\tau=\frac{\bar{x}-k}{\bar{x}t}=\frac{s}{\sqrt{2}\,\bar{x}}\right)$$

and the limits of integrations are determined after several subcases.

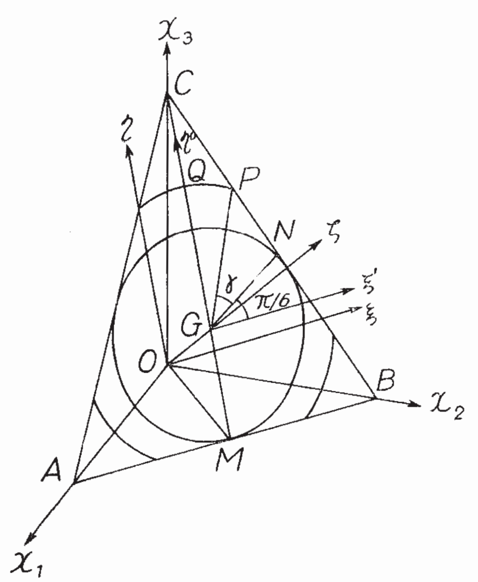

Fig. 1

In Fig. 1 the simplex S_2 is given by the equilateral triangle ABC, where its side $a=3\sqrt{2}\,\bar{x}$, height $h=3\sqrt{\frac{3}{2}}\,\bar{x}$, $GM=GN=\sqrt{\frac{3}{2}}\,\bar{x}$, $GC=\sqrt{6}\,\bar{x}$, &c. The relation between the radius of s-circle $(=\sqrt{3}\,s)$ and $OG(=\sqrt{3}\,\bar{x})$ yields the following subcases:

I: $0<s<\bar{x}/\sqrt{2}\quad(0<\tau<1/2)$.

When the radius is less than GM, i.e. $0<\sqrt{3}\,s<\sqrt{\frac{3}{2}}\,\bar{x}$, the whole arc of s-circle can be adopted, so that $\theta_0=0$, $\theta_1=2\pi$. Also inequalities $0<2(\bar{x}-k)/t<\bar{x}$ follow. Accordingly (i) if $0<t<2$, we obtain $k<\bar{x}<2k/(2-\bar{x})$, (ii) $2<t<\infty$, $k<\bar{x}<\infty$ and (iii) if $-\infty<t<0$, $\frac{2k}{2-t}<\bar{x}<k$.

II: $\bar{x}/\sqrt{2}<s<\bar{x}\sqrt{2}\quad(1/2<\tau<1)$.

When the s-circle's radius is between $GM=GN=\sqrt{\frac{3}{2}}\bar{x}$ and $GC=\sqrt{6}\,\bar{x}$, i.e. if $\bar{x}/\sqrt{2}<s<\sqrt{2}\,\bar{x}$, the contribution from its arc is 6 times PQ (Fig. 1). Consequently $\angle\xi'GQ=\theta_1=\frac{\pi}{2}$ and $\angle\xi'GP=\theta_0=\gamma+\frac{\pi}{6}$, where $\cos\gamma=\frac{GN}{GP}=\frac{\bar{x}}{\sqrt{2}\,s}=\frac{\bar{x}t}{2(\bar{x}-k)}=\frac{1}{2\tau}$. Also from the inequalities $\bar{x}/\sqrt{2}<(\bar{x}-k)\sqrt{2}/t<2\bar{x}$, the limits x_0 and x_1 are decided as: (i) if $0<t<1$, $\frac{2k}{2-t}<\bar{x}<\frac{k}{1-t}$, (ii) if $1<t<2$, $\frac{2k}{2-t}<\bar{x}<\infty$; however the case $2<t<\infty$ is impossible in II, and lastly (iii) if $-\infty<t<0$, $\frac{k}{1-t}<\bar{x}<\frac{2k}{2-t}$.

The behaviour of $s(t)$ in a neighbourhood of the origin shall be investigated by means of (i) and (iii), while the significant limits are to be determined from (ii) or (iii).

To express (2), the inner integral J, more suitably, we put $3\theta=\frac{3}{2}\pi+2\varphi$ in I, and $3\theta=\frac{3}{2}\pi-2\varphi$ in II, respectively. Then (2) becomes

$$J=\int_{\varphi_0}^{\varphi_1}(1-3\tau^2+2\tau^3-4\tau^3\sin^2\varphi)^{k/2-1}\frac{2}{3}d\varphi\equiv\int_{\varphi_0}^{\varphi_1}K(\varphi,\tau)\frac{2}{3}d\varphi.$$

Since the range of integration in I: $0<\theta<2\pi$ can be replaced by $\frac{\pi}{2}<\theta<\frac{5}{2}\pi$ by periodicity, it follows that $\varphi_0=0$ and $\varphi_1=3\pi$, and thus

$$J_I=\int_0^{2\pi}Kd\theta=\int_0^{3\pi}\frac{2}{3}Kd\varphi=4\int_0^{2/\pi}Kd\varphi.$$

Similarly in II, the corresponding values of φ being $\varphi_0=0$ for $\theta_1=\frac{\pi}{2}$ and $\varphi_1=\frac{\pi}{2}-\frac{3}{2}\gamma$ for $\theta_0=\gamma+\frac{\pi}{6}$,

$$J_{II}=6\int_{\theta_0}^{\theta_1}Kd\theta=6\int_{\varphi_1}^{0}-\frac{2}{3}Kd\varphi=4\int_0^{\varphi_1=\pi/2-3\gamma/2}Kd\varphi.$$

Also putting

$$1-3\tau^2+2\tau^3=(1-\tau)^2(1+2\tau)=g(\tau),\quad \sqrt{4\tau^3/g(\tau)}=\mu(\tau)$$

as auxiliaries, we see that in the whole course $0<\tau<1$, g and μ are positive and monotonic decreasing or increasing. Namely, $1>g(\tau)>1/2$ and $0<\mu(\tau)<1/2$ in I: $0<\tau<1/2$, while $1/2>g(\tau)>0$ and $1/2<\mu(\tau)<\infty$ in II: $1/2<\tau<1$ (Fig. 2). By means of these auxiliaries we may write

$$K(\mu,\tau)=[(1-\tau)^2(1+2\tau)-4\tau^3\sin^2\varphi]^{k/2-1}=g(\tau)^{k/2-1}(1-\mu^2\sin^2\varphi)^{k/2-1},$$

so that

(8.4) $$J=4g(\tau)^{k/2-1}\int_0^{\varphi_1}(1-\mu^2\sin^2\varphi)^{k/2-1}d\varphi$$

where $\varphi_1=\frac{\pi}{2}$ for I, but $\varphi_1=\frac{\pi}{2}-\frac{3}{2}\gamma\left(0<\gamma=\cos^{-1}\frac{1}{2\tau}<\frac{\pi}{3}\right)$ for II.

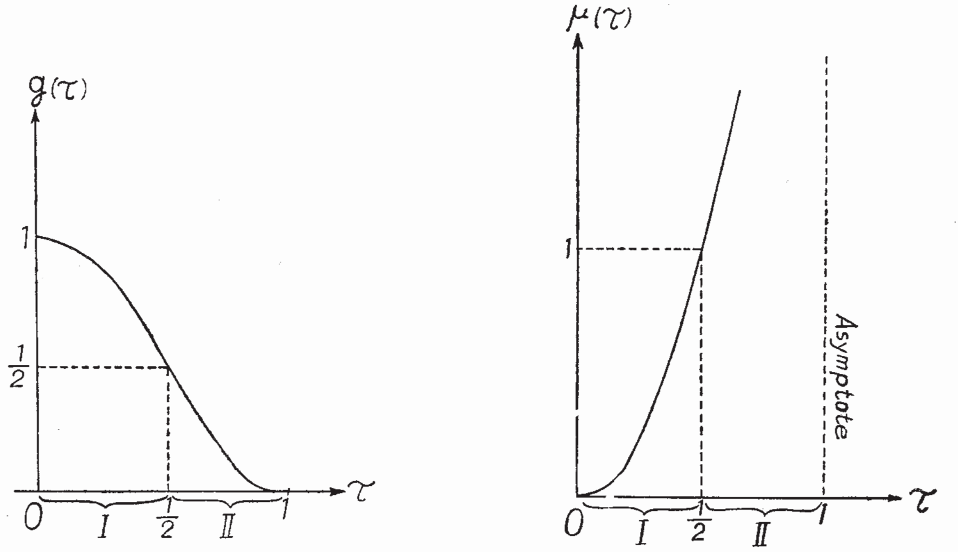

Fig. 2

Hence, for I (ii): $2<t<\infty$, we have

(8.5) $$s(t)\equiv s_I(t)=\frac{4d_k}{t^3}\int_k^\infty e^{-3\bar{x}/2}(\bar{x}-k)^2\bar{x}^{3k/2-3}g(\tau)^{k/2-1}d\bar{x}\int_0^{\pi/2}(1-\mu^2\sin^2\varphi)^{k/2-1}d\varphi\,,$$

where the inner integral being the complete elliptic integral $F\left(\mu,\frac{\pi}{2}\right)$ with modulus $\mu(\lessapprox 1$ in I), it can be expanded into a power series:

(8.6) $$F_k(\mu)\equiv F_k\left(\mu,\frac{\pi}{2}\right)=\int_0^{\pi/2}(1-\mu^2\sin^2\varphi)^{k/2-1}d\varphi \quad (0<\mu<1)$$

$$=\sum_{\nu=0}^\infty\binom{k/2-1}{\nu}(-1)^\nu\mu^{2\nu}\int_0^{\pi/2}\sin^{2\nu}\varphi d\varphi$$

$$=\frac{\pi}{2}\sum_{\nu=0}^\infty(-1)^\nu\frac{(k-2)(k-4)\cdots\cdots(k-2\nu)}{(2^\nu\nu!)^3}(2\nu)!\,\mu^{2\nu}$$

$$=\frac{\pi}{2}\left[1-\frac{k-2}{4}\mu^2+\frac{3(k-2)(k-4)}{64}\mu^4-\frac{5(k-2)(k-4)(k-6)}{768}\mu^6+\cdots\right],$$

in which $(1-\mu^2\sin^2\varphi)^{k/2-1}$ is certainly absolutely convergent in the open interval $\left(0\leq\tau<\frac{1}{2},\ 0\leq\mu<1\right)$, so that the termwise integration is legitimate there. However, since the ultimate series converges absolutely even at $\mu=1$, it is absolutely and uniformly convergent in the closed interval $(0\leq\mu\leq1)$. Further we have

$$\mu^2 = \frac{4\tau^3}{(1-\tau)^2(1+2\tau)} = 4\tau^3(1+3\tau^2-2\tau^3+9\tau^4-\cdots\cdots)$$

which is absolutely and uniformly convergent in $\left(0 \leq \tau < \frac{1}{2}\right.$. Hence, if this be substituted in (6), the resulting series of $F_k(\mu)$ shall behave similarly regular in $0 \leq \tau < \frac{1}{2}$. And we can say the same about $g(\tau)^{k/2-1} = (1-\tau)^{k-2}(1+2\tau)^{k/2-1}$ and consequently so also about their product series

$$g^{k/2-1}(\tau)F_k(\mu) = \frac{\pi}{2}\left[1-\frac{3}{2}(k-2)\tau^2+\frac{9}{8}(k-2)(k-4)\tau^4+\cdots\cdots\right] \equiv \frac{\pi}{2}H(\tau).$$

Therefore, on transforming the variable $\bar{x}$ in (5) into $\tau = \frac{\bar{x}-k}{\bar{x}t}$, i.e. $\bar{x} = \frac{k}{1-t\tau}$, $s(t)$ reduces to

$$(8.7) \qquad s(t) = 2\pi d_k k^{3k/2}\int_0^{1/t}\exp\left(\frac{-3k/2}{1-t\tau}\right)\frac{H(\tau)\tau^2 d\tau}{(1-t\tau)^{3k/2+1}},$$

which is surely integrable so far $2 < t < \infty$.

We are going to express $s(t)$ in $(2 < t < \infty)$ by a power series of t^{-1} up to a certain degree, say t^{-4}, so that up to τ^4. For this purpose, again replacing τ in (7) by $\tau = u/t$, it yields

$$(8.8) \qquad s(t) = 2\pi d_k \frac{k^{3k/2}}{t^3}\int_0^1 \exp\left(\frac{-3k/2}{1-u}\right)\frac{u^2}{(1-u)^{3k/2+1}} \times$$
$$\times\left[1-\frac{3}{2}(k-2)\left(\frac{u}{t}\right)^2+\frac{9}{8}(k-2)(k-4)\left(\frac{u}{t}\right)^4+\cdots\cdots\right]du.$$

Accordingly we attain finally the following d. f.

$$(8.9) \qquad S(t) = \int_{t_1}^{\infty} s(t)dt$$
$$= \pi d_k k^{3k/2}\exp\left(\frac{-3k/2}{1-u}\right)\left[\frac{u^2}{t_1^2}-\frac{3}{4}(k-2)\frac{u^4}{t_1^4}+\frac{3}{8}\frac{(k-2)(k-4)}{t_1^6}+\cdots\right]du/(1-u)^{3k/2+1}$$
$$= \pi d_k k^{3k/2}\left[J_2/t_1^2-\frac{3}{4}(k-2)J_4/t_1^4+\frac{3}{8}(k-2)(k-4)J_6/t_1^6+\cdots\cdots\right] \quad (t_1 > 2),$$

where

$$(8.10) \qquad J_n = \int_0^1 \exp\left(\frac{-3k/2}{1-u}\right)u^n du/(1-u)^{3k/2+1} \quad (n = 2, 4, 6, \cdots).$$

Hence, if these J_n be computed, we can obtain an approximate equation

$$(8.11) \qquad S(t_1) = A/t_1^2 + B/t_1^4 + C/t_1^6.$$

To compute (10) actually, Gauss' method of numerical integration may be applied. However, the variations of its integrand being somewhat enormous, it is rather preferable to calculate the exact value directly, somewhat cumbrous as it is. For this purpose, on writing $v=1/(1-u)$ and $l=3k/2$, the integral (10) becomes

$$(8.12)\qquad J_n=\int_1^\infty e^{-lv}v^{l-1}\left(1-\frac{1}{v}\right)^n dv=\int_1^\infty e^{-lv}v^{l-1}\sum_{\nu=0}^{n}(-1)^\nu\binom{n}{\nu}v^{-\nu}dv.$$

Thus, e. g. $J_2=\int_1^\infty e^{-lv}(v^{l-1}-2v^{l-2}+v^{l-3})dv$. By successive applications of recurring formulas

$$\int v^m e^{-av}dv=-\frac{v^m}{a}e^{-av}+\frac{m}{a}\int v^{m-1}e^{-av}dv,\qquad \int\frac{e^{-av}}{v^m}dv=\frac{-1}{m-1}\frac{e^{-av}}{v^{m-1}}-\frac{a}{m-1}\int\frac{e^{-av}}{v^{m-1}}dv,$$

we can make v's exponent possibly small in the absolute value: If k be even, $l=3k/2$ becomes a positive integer, so that the final exponent reduces to zero. But, if k be odd, we get, as the final integral $\int e^{-av}dv/\sqrt{v}$, which on putting $av=\frac{1}{2}x^2$ yields

$$\sqrt{\frac{2}{a}}\int_{\sqrt{2a}}^\infty e^{-x^2/2}dx=\sqrt{\frac{\pi}{a}}\left[1-\Phi(\sqrt{2a})\right],$$

where

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_0^x e^{-t^2/2}dt,\ \text{the usual probability integral.}$$

For example, if $k=1$, $d_1=6\sqrt{3}/\sqrt{8\pi^3}=0.659845$, we have for $t>2$ after (12), e.g.

$$J_2=\int_1^\infty e^{-3v/2}(v^{1/2}-2v^{-1/2}+v^{-3/2})dv=\frac{8}{3}e^{-1\cdot5}-\frac{14}{9}\sqrt{6\pi}\,[1-2\Phi(\sqrt{3})]=0.03265.$$

So that $A=\pi d_1J_2=0.06769$, and similarly $B=\pi d_1J_4=0.01578$, $C=\pi d_1J_6=0.00770$.

And thus the approximate equation (11) is gained. This being equated to $\alpha/2(\alpha=0.1,\ 0.05,\ 0.01)$ and solved for t^{-2}, we find the upper significant limits $t_{0\cdot1}=1.27$, $t_{0\cdot05}=1.72$, $t_{0\cdot01}=3.71$, which, compared with the corresponding classical Student's values 2.920, 4.303, 9.925, are remarkably small.

Also the lower significant limits can be treated just similarly. Now after I (iii): $-\infty<t<0$, we get by (1) and similarly to (5)

$$(8.13)\quad s_I(t)=\frac{4d_k}{-t^3}\int_{2k/(2-t)}^{k}e^{-3\bar{x}/2}(k-\bar{x})^2\bar{x}^{3k/2-1}g(\tau)^{k/2-1}d\bar{x}\int_0^{\pi/2}(1-\sin^2\varphi)^{k/2-1}d\varphi\,(\mu<1)$$

$$=4d_kk^{3k/2}\int_0^{1/2}\exp\left(\frac{-3k/2}{1-t\tau}\right)g(\tau)^{k/2-1}F_k\left(\mu,\frac{\pi}{2}\right)\tau^2d\tau/(1-t\tau)^{3k/2+1}\,(t<0).$$

Besides however this time comes the additional contribution from II (iii): $-\infty<t<0$, $1/2<\tau<1$, as

$$(8.14)\quad s_{II}(t)=\frac{4d_k}{-t^3}\int_{k/(1-t)}^{2k/(2-t)}e^{-3\bar{x}/2}(k-\bar{x})^2\bar{x}^{3k/2-3}g(\tau)^{k/2-1}dx\int_0^{\varphi_1=\pi/2-3\gamma/2}(1-\mu^2\sin^2\varphi)^{k/2-1}d\varphi$$

$$=4d_kk^{3k/2}\int_{1/2}^{1}\exp\left(\frac{-3k/2}{1-t\tau}\right)\frac{g(\tau)^{k/2-1}\tau^2d\tau}{(1-t\tau)^{3k/2+1}}\int_0^{\varphi_1}(1-\mu^2\sin^2\varphi)^{k/2-1}d\varphi.$$

Now that $1/2<\tau<1$, it is $1<\mu^2<\infty$, and $\varphi_1=\pi/2-3\gamma/2$, $\cos\gamma=1/2\tau$, so that $\sin\varphi_1=\cos 3\gamma/2=\sqrt{g/4\tau^3}=1/\mu$. Hence, if we put $\mu\sin\varphi=\sin\psi$, the inner integral reduces to

$$(8.15)\qquad \int_0^{\varphi_1}(1-\mu^2\sin^2\varphi)^{2/k-1}d\varphi=\frac{1}{\mu}\int_0^{2/\pi}\frac{\cos^{k-1}\psi d\psi}{\sqrt{1-\mu^2\sin^2\psi}}\equiv G_k(\mu),\ \text{say.}$$

In particular, for $k=1$

$$G_1(\mu)=\frac{1}{\mu}\int_0^{\pi/2}\frac{d\psi}{\sqrt{1-\mu^2\sin^2\psi}}=\frac{1}{\mu}F_1\left(\frac{1}{\mu},\frac{\pi}{2}\right).$$

Thus $G_1(\mu)$ reduces to an ordinary complete elliptic integral of the first kind with modulus $1/\mu(<1)$, $F_1(1/\mu)$, multiplied by $1/\mu$.

In general we obtain from (13) and (14) the d. f, for $t_0<0$

$$(8.16)\quad S(t_0)=\int_{-\infty}^{t_0}[s_I(t)+s_{II}(t)]dt$$

$$=4d_kk^{3/2}\left[\int_0^{1/2}g(\tau)^{k/2-1}F_k(\mu)J^*(\tau,t_0)\tau d\tau+\int_{1/2}^{1}g(\tau)^{k/2-1}G_k(\mu)J^*(\tau,t_0)\tau d\tau\right],$$

where

$$(8.17)\qquad G_k(\mu)=\frac{1}{\mu}\int_0^{\pi/2}\frac{\cos^{k-1}\psi d\psi}{\sqrt{1-\frac{1}{\mu^2}\sin^2\varphi}}\left(\frac{1}{\mu}<1\right),$$

and

$$(8.18)\qquad J^*(\tau,t_0)=\int_{-\infty}^{t^0}\exp\left(\frac{-3k/2}{1-t\tau}\right)\frac{\tau dt}{(1-t\tau)^{3k/2+1}}\,.$$

Since the lower significant limit t_0 is rather large in magnitude, we have merely to take few terms, or roughly the first term only, when the integrand of (18) expanded in powers of t^{-1}. Thus we have

$$(8.19)\qquad J^*(\tau,t_0)=\sum_{\nu=0}^{\infty}\int_{-\infty}^{t_0}\frac{(-3k/2)^\nu\tau dt}{\nu!\,(1-t\tau)^{\nu+3k/2+1}}\cong\frac{2}{3k}(-t_0\tau)^{-3k/2}(t_0<0).$$

Accordingly it remains only to compute two integrals in

$$(8.20)\quad S(t_0)=\frac{8}{3}d_k\frac{k^{3k/2-1}}{(-t_0)^{3k/2}}\left[\int_0^{1/2}\frac{g(\tau)^{k/2-1}F_k(\mu)dt}{\tau^{3k/2-1}}+\int_{1/2}^{1}\frac{g(\tau)^{k/2-1}G_k(\mu)d\tau}{\tau^{3k/2-1}}\right].$$

For example if $k=1$

$$(8.21)\qquad S(t_0)=\frac{8}{3}d_1\left(\frac{1}{-t_0}\right)^{3/2}\left[\int_0^{1/2}\frac{F_1(\mu)d\tau}{\sqrt{\tau g(\tau)}}+\int_{1/2}^{1}\frac{F_1(1/\mu)d\tau}{\mu\sqrt{\tau g(\tau)}}\right]$$

$$=\frac{8}{3}d_1\left(\frac{1}{-t_0}\right)^{3/2}\left[\int_0^{1/2}\frac{F_1(\mu)d\mu}{(1-\tau)\sqrt{\tau(1+2\tau)}}+\frac{1}{2}\int_{1/2}^1 F_1\left(\frac{1}{\mu}\right)\frac{d\tau}{\tau^2}\right],$$

where $g=(1-\tau)^2(1+2\tau)$, $\mu=\sqrt{4\tau^3/g}$ and $F_1(\mu)$, $F_1(1/\mu)$ denote the complete elliptic integrals of the first kind with modulus μ and $1/\mu$, respectively, both of which are <1 in their respective integral interval. On calculating after Gauss' method of numerical integration, we get approximately

$$S(t_0)\cong\frac{8}{3}d_1(-t_0)^{-3/2}(1.1731+0.4178)=2.799/(-t_0)^{3/2}.$$

This being equated to $\alpha/2$ ($\alpha=0.1$, 0.05, 0.01), we obtain as the lower significant limits $t_{0.1}=-14.6$, $t_{0.05}=-23.2$, $t_{0.01}=-67.9$, which are in magnitude considerably greater than the corresponding classical values -2.920, -4.303, -9.925.

To investigate the behaviour of $s(t)$ at the origin, we need to refer to I (i), II (i) for $t>0$ and I (ii), II (iii) for $t<0$. After transforming $\bar{x}$ into τ, we obtain for $t\gtrless 0$ both the same form

$$(8.22)\qquad s(t)=4d_k k^{3k/2}\int_0^{1/2}\exp\left(\frac{-3k/2}{1-t\tau}\right)g(\tau)^{k/2-1}F_k(\mu)d\tau/(1-t\tau)^{3k/2+1}$$
$$+\ \text{''}\int_{1/2}^1\ \text{''}\quad\text{''}\quad G_k(\mu)\quad\text{''}\quad,$$

and as their derivatives

$$(8.23)\qquad s'(t)=4d_k k^{3k/2}\int_0^{1/2}\exp\left(\frac{-3k/2}{1-t\tau}\right)\left[\tau-\left(1+\frac{3}{2}k\right)t\tau^2\right]g^{k/2-1}F_k(\mu)d\tau/(1-t\tau)^{3k/2+3}$$
$$+\ \text{''}\int_{1/2}^1\ \text{''}\quad\text{''}\quad\text{''}\quad G_k(\mu)\quad\text{''}\quad.$$

Therefore

$$(8.24)\qquad s(\pm 0)=4d_k k^{3k/2}e^{-3k/2}\left[\int_1^{1/2}g^{k/2-1}F_k(\mu)d\tau+\int_{1/2}^1 g^{k/2-1}G_k(\mu)d\tau\right],$$

$$(8.25)\qquad s'(\pm 0)=\quad\text{''}\quad\left[\int_0^{1/2}\ \text{''}\quad\tau d\tau+\int_{1/2}^1\ \text{''}\quad\tau d\tau\right],$$

all of which are positive finite. Accordingly $s(t)$ is regular at the origin and increasing there, so that the mode should lie on the positive axis. All these asymmetric feature is quite different from the ordinary Student's fr. f., that is symmetric about the origin, which is also its mode.

The special case $k=1$ illustrated above materially corresponds to a truncated N. D., as is seen by putting $x=y^2$ in (8.0). The investigation of $s(t)$ about various values of k, even confined as $n=3$ only, becomes enough diverse, which however is left for students' exercise.

9. *Continued (Case $n=4$).* The probability element is now

$$dp=\frac{e^{-2\bar{x}}}{4^k\Gamma(k/2)^4}\int(x_1x_2x_3x_4)^{k/2-1}dx_1dx_2dx_3dx_4\quad(x's>0,\text{ and }\bar{x},\ s\text{ given}).$$

Let the co-ordinates x_1, x_2, x_3, x_4 be first transformed orthogonally into ξ, η, ζ, χ in conformity with those in paper [I], as

	x_1	x_2	x_3	x_4
ξ	0	$1/\sqrt{6}$	$1/\sqrt{6}$	$-2/\sqrt{6}$
η	0	$-1/\sqrt{2}$	$1/\sqrt{2}$	0
ζ	$3/\sqrt{12}$	$-1/\sqrt{12}$	$-1/\sqrt{12}$	$-1/\sqrt{12}$
$\chi(=2\bar{x})$	1/2	1/2	1/2	1/2

$$x_1= 3\zeta/\sqrt{12}+\chi/2(=\bar{x})$$
$$x_2=\xi/\sqrt{6}+\eta/\sqrt{2}-\zeta/\sqrt{12}+\bar{x}$$
$$x_3=\xi/\sqrt{6}+\eta/\sqrt{2}-\zeta/\sqrt{12}+\bar{x}$$
$$x_4=-2\xi/\sqrt{6} -\zeta/\sqrt{12}+\bar{x}$$

and next into polar co-ordinates $\xi=\rho\sin\theta\cos\varphi$, $\eta=\rho\sin\theta\sin\varphi$, $\zeta=\rho\cos\theta$ with $\rho=2s$, $s=(\bar{x}-k)\sqrt{3}/t$. Further, for convenience' sake, putting $(\bar{x}-k)/\bar{x}t=s/\sqrt{3}\,\bar{x}=\tau$, we get

$$x_1x_2x_3x_4=\bar{x}^4[1-6\tau^2-2\tau^3(2\sqrt{2}\sin^3\theta\cos3\varphi+3\sin^2\theta\cos\theta(3-\cos2\varphi)-4\cos^3\theta)$$
$$-3\tau^4(4\sqrt{2}\sin^3\theta\cos3\varphi-6\sin^2\theta\cos\theta+\cos^3\theta)\cos\theta]\equiv\bar{x}g(\theta,\varphi,\tau),$$

and the volume element

$$dv=dx_1dx_2dx_3dx_4=16s^2dsd\bar{x}.\sin d\theta d\varphi=\frac{48\sqrt{3}}{t^4}dt.|x-k|^3d\bar{x}.\sin\theta d\theta d\varphi.$$

Therefore, Student's fr. f. becomes

$$(9.1)\qquad s(t)=\frac{a_k}{t^4}\int_{x_0}^{x_1}e^{-2x}|\bar{x}-k|^3\bar{x}^{2k-4}d\bar{x}\int_\varphi\int_\theta g^{k/2-1}(\theta,\varphi,\tau)\sin\theta d\theta d\varphi$$

where $a_k=48\sqrt{3}/4^k\Gamma(k/2)^4$ and the limits of integrations shall be determined below according to several subcases.

We have in the simplex S_3 (tetrahedron, Fig. 3), $a=BC=4\bar{x}\sqrt{2}$,

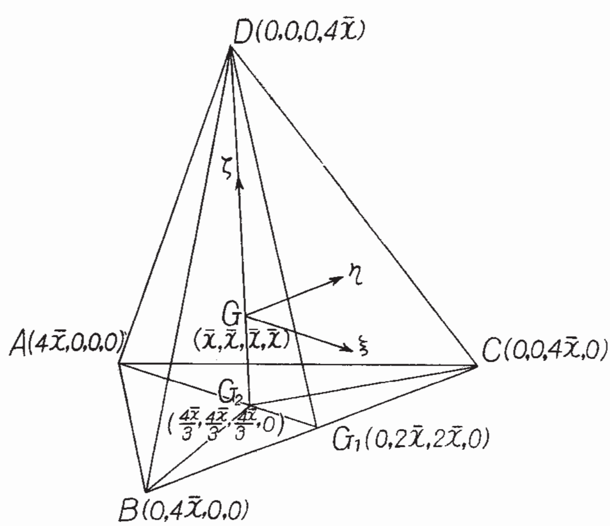

Fig. 3

$DG_1=2\sqrt{6}\,\bar{x},\ G_1G_2=\frac{2}{3}\sqrt{6}\,\bar{x},\ DG_2=\frac{8}{\sqrt{3}}\bar{x},\ GG_2=\frac{2}{\sqrt{3}}\bar{x},\ GD=\frac{6}{\sqrt{3}}\bar{x},\ GG_1=2\bar{x},$ &c.

Subcase I: $0<s=(\bar{x}-k)\sqrt{3}/t<\bar{x}/\sqrt{3}$ $(0<\tau=(x-k)/\bar{x}t<1/3)$. Here $0<2s<GG_2$, and the whole s-sphere lies in S_3, so that $0<\varphi<2\pi$, $0<\theta<\pi$. As to x_0, x_1 (i) if $-\infty<t<0$, then $\frac{3k}{3-t}<\bar{x}<k$ (ii) if $0<t<3$, $k<\bar{x}<\frac{3k}{3-t}$ (iii) if $3<t<\infty$, $k<\bar{x}<\infty$, since one half of the condition holds by itself, $\bar{x}$ cannot be upperly bounded.

II: $\bar{x}/\sqrt{3}<s<\bar{x}$ $(1/3<\tau<1/\sqrt{3})$. Then $GG_2<2s<GG_1$ and 4 calottes (spherical segments outside S_3) should be subtracted from the whole s-sphere. Also (i) if $-\infty<t<0$, $\frac{\sqrt{3}\,k}{\sqrt{3}-k}<\bar{x}<\frac{3k}{3-t}$ (ii) if $0<\bar{x}<\sqrt{3}$, $\frac{3k}{3-t}<\bar{x}<\frac{\sqrt{3}\,k}{\sqrt{3}-k}$ (iii) if $\sqrt{3}<t<3$, $3k/(3-t)<\bar{x}<\infty$.

III: $\bar{x}<s<\sqrt{3}\,\bar{x}$ $(1/\sqrt{3}<\tau<1)$. Thus $GG_1<2s<GD$ and the spherical portion inside S_3 shall be actually computed. About x_0, x_1, we have (i) if $-\infty<t<0$, $\frac{k}{1-t}<\bar{x}<\frac{\sqrt{3}\,k}{\sqrt{3}-t}$ (ii) if $1<t<\sqrt{3}$, $\frac{k\sqrt{3}}{\sqrt{3}-t}<\bar{x}<\frac{k}{1-t}$ (iii) if $1<t<\sqrt{3}$, $\frac{k\sqrt{3}}{\sqrt{3}-t}<\bar{x}<\infty$. The case $t>\sqrt{3}$ in III and $t<3$ in II do not take place.

Also, if $\bar{x}$ be transformed into $\tau=\frac{\bar{x}-k}{\bar{x}t}$, (1) becomes

$$s(t)=a_k k^{2k}\int_{\tau_1}^{\tau_0}\exp\left(\frac{-2k}{1-t\tau}\right)\frac{\tau^3 d\tau}{(1-t\tau)^{2k+1}}\int_\varphi\int_\theta g^{k/2-1}(\theta,\varphi,\tau)\sin\theta d\theta d\varphi, \tag{9.2}$$

where it holds I: $\tau_0=0$, $\tau_1=1/3$: II: $\tau_0=1/3$, $\tau_1=1/\sqrt{3}$ and III: $\tau_0=1/\sqrt{3}$, $\tau_1=1$ respectively.

In the general case we need to investigate the inner double integral, what however being far intricate than that of case $n=3$, is here left over. Presently we shall simply consider about the special case $k=2$, i.e. the truncated Laplace distribution. In this case we can ignore $g(\theta,\varphi,\tau)$ and put $dP=e^{-2x}dV/16$, where the volume element dV can be written after [I] as follows:

I: $dV/16=4\pi s^2 ds d\bar{x}=c\,|\bar{x}-2|^3 d\bar{x}dt/t^4$, where $c=12\sqrt{3}\,\pi=65.2968.$

II: $$\frac{dV}{16}=4\pi s^2\left(\frac{2\bar{x}}{\sqrt{3}\,s}-2\right)ds d\bar{x}=c\,|\bar{x}-2|^3\left[\frac{2\bar{x}\cdot t}{3(\bar{x}-2)}-2\right]\frac{d\bar{x}dt}{t^4}$$
$$=\frac{2}{3}c\bar{x}(\bar{x}-2)^2 d\bar{x}dt/|t|^3-c\,|\bar{x}-2|^3 d\bar{x}dt/t^4,$$

III: $$\frac{dV}{16}=6\left\{\frac{2\pi s}{3}\left(\frac{2\bar{x}}{\sqrt{3}}-s\right)+4s^2\tan^{-1}\sqrt{\frac{1}{2}\left(1-\frac{\bar{x}^2}{s^2}\right)}-\frac{4s\bar{x}}{\sqrt{3}}\tan^{-1}\sqrt{\frac{3}{2}\left(\frac{s^2}{\bar{x}^2}-1\right)}\right.ds d\bar{x}$$

$$=\frac{2}{3}\frac{c}{|t|^3}\bar{x}(\bar{x}-2)^2 d\bar{x}dt-\frac{c}{t^4}|\bar{x}-2|^3 d\bar{x}dt+\frac{2c}{|t|^3}\bar{x}(\bar{x}-2)^2 T(\tau)d\bar{x}dt,$$

where

$$T=\frac{3\tau}{\pi}\tan^{-1}\sqrt{\frac{1}{2}\left(1-\frac{1}{3\tau^2}\right)}-\frac{1}{\pi}\tan^{-1}\sqrt{\frac{3}{2}(3\tau^2-1)}\,,\quad \tau=\frac{\bar{x}-2}{\bar{x}t}.$$

Or, else, replacing $\bar{x}$ by τ, we obtain as the second forms:

I: $dV/16=16c\tau^3 d\tau dt/(1-t\tau)^5$, for $0<\tau<1/3$,

II: $dV/16=16c\tau^2(2/3-\tau)d\tau dt/(1-t\tau)^5$, for $1/3<\tau<1/\sqrt{3}$,

III: $dV/16=16c\tau^2[(2/3-\tau)+2T]d\tau dt/(1-t\tau)^5$ for $1/\sqrt{3}<\tau<1$.

By means of these dV we may write Student's fr. f. for $n=4$

(9.3) $$s(t)=\frac{1}{16}\int_{x_0}^{x_1} e^{-2\bar{x}}\frac{dV}{dt}d\bar{x}\,,$$

which yields concrete expressions for several subcases.

We require to find the partial probabilities P_i, that t falls in the i-th portion of the following successive intervals:

1° $-\infty<t<0$, 2° $0<t<1$, 3° $1<t<\sqrt{3}$, 4° $\sqrt{3}<t<3$, 5° $3<t<\infty$,

and incidentally to get lower and upper significant limits also.

For the sake of brevity we shall use as auxiliary functions

(9.4) $$ce^{-2k}|x-2|^3=g(x),\qquad ce^{-2x}(x-2)^2=h(x).$$

Further putting

(9.5) $$x_1=\frac{2}{1-t},\quad x_2=\frac{2\sqrt{3}}{\sqrt{3}-t},\quad x_3=\frac{6}{3-t},$$

we have for $t<0$, $0<x_1<x_2<x_3<2$, but for $t>0$, $x_i>x_j$ as $i<j$, so that $2<x_j<x_i<\infty$, the negative valued x_k, if any, being abandoned. On summing up those with the same integrands we may rewrite $s(t)$ as follows:

1° $-\infty<t<0$. Here $0<x_1<x_2<x_3<2$ and

(9.6) $$s(t)=\frac{1}{t^4}\left(\int_{x_1}^{2} g\,dx-2\int_{x_1}^{x_3} g\,dx\right)+\frac{2}{3}\frac{1}{|t|^3}\int_{x_1}^{x_3} h\,dx+\frac{2}{|t|^3}\int_{1x}^{x_2} hT\,dx.$$

2° $0<t<1$. Here $2<x_3<x_2<x_1<\infty$ and

(9.7) $$s(t)=\frac{1}{t^4}\left(\int_{2}^{x_1} g\,dx-2\int_{x_3}^{x_1} g\,dx+\frac{2}{3}\frac{1}{t^3}\int_{x_3}^{\infty} h\,dx+\frac{2}{t^3}\int_{x_2}^{\infty} hT\,dx.\right.$$

3° $1<t<\sqrt{3}$. Then $2<x_3<x_2<\infty$ but $x_1<0$ useless, and

(9.8) $$s(t)=\frac{1}{t^4}\left(\int_2^\infty g\,dx-2\int_{x_3}^\infty g\,dx\right)+\frac{2}{3}\,\frac{1}{t^3}\int_{x_3}^\infty h\,dx+\frac{2}{t^3}\int_{x_2}^\infty hT\,dx.$$

4° $\sqrt{3}<t<3$. Then $2<x_3<\infty$ and

(9.9) $$s(t)=\frac{1}{t^4}\left(\int_2^\infty g\,dx-2\int_{x_3}^\infty g\,dx\right)+\frac{2}{3t^3}\int_{x_3}^\infty h\,dx.$$

5° $3<t<\infty$. Here all x'_is<0 are of no use and we have simply

(9.10) $$s(t)=\frac{1}{t^4}\int_2^\infty g\,dx=\frac{c}{t^4}\int_2^\infty e^{-2x}(x-2)^3dx.$$

We shall discuss each case in a further details.

1° $t<0,\ 0<x<2$. For convenience' sake let us write

(9.11) $$2x=y+4,\quad y_1=\frac{4t}{1-t},\quad y_2=\frac{4t}{\sqrt{3}-t},\quad y_3=\frac{4t}{3-t}.$$

so that $-4<y_1<y_2<y_3<0$. Then (6) becomes

(9.12) $$s(t)=\frac{c_1}{t^4}\left(\int_0^{y_1}e^{-y}y^3dy-2\int_{y_3}^{y_1}e^{-y}y^3dy\right)+\frac{2}{3}t\int_{y_0}^{y_1}e^{-y}y^2(y+4)dy$$
$$+2t\int_{y_2}^{y_1}e^{-y}(y+4)T\,dy\equiv s_{(I)}(t)+s_{(II)}(t)+s_{(III)}(t),$$

where $c_1=ce^{-4}/16=0.074747$. But, since

$$\int e^{-y}y^3dy=-e^{-y}(y^3+3y^2+6y+6),\quad \int e^{-y}y^2(y+4)dy=-e^{-y}(y^3+7y^2+14y+14),$$

we obtain

$$s_{(I)}+s_{(II)}=\frac{c_1}{t^4}\left\{e^{-y}(y^3+3y^2+6y+6)\Big|_{y_1}^{0}-2e^{-y}(y^3+3y^2+6y+6)\Big|_{y_1}^{y_3}\right.$$
$$\left.+\frac{2}{3}te^{-y}(y^3+7y^2+14y+14)\Big|_{y_1}^{y_3}\right\}$$
$$=\frac{c_1}{t^4}\left\{6+e^{-y_1}[y_1^3+3y_1^2+6y_1+6-\frac{2}{3}t(y_1^3+7y_1^2+14y_1+14)\right.$$
$$\left.-2e^{-y_3}[y_3^3+3y_3^2+6y_3+6-\frac{1}{3}t(y_3^3+7y_3^2+14y_3+14)]\right\}.$$

Now, required to find the d. f. $S(t_0)=\int_{-\infty}^{t_0}s(t)dt$, we transform the variable t into $y_v=4t/(\tau_v^{-1}-t)$ ($\tau_1=1,\ \tau_3=1/3$) by means of $t=y_v/(y_v+4)\tau_v$. Consequently

$$S_{(I)}(t_0)+S_{(II)}(t_0)=\int_{-\infty}^{t_0}(s_{(I)}+s_{(II)})dt$$
$$=6c_1\int_{-\infty}^{t_0}\frac{dt}{t^4}+\frac{4}{3}c_1\int_{-4}^{y_{1,0}}e^{-y}\left(y+11+\frac{54}{y}+\frac{116}{y^2}+\frac{320}{y^3}+\frac{288}{y^4}\right)dy$$

$$-\frac{32}{27}c_1\int_{-4}^{y_{3,0}} e^{-y}\left(\frac{1}{y}+\frac{8}{y^2}+\frac{22}{y^3}+\frac{24}{y^4}\right)dy,$$

where $y_{1,0}=4t_0/(1-t_0)$, $y_{3,0}=4t_0/(3-t_0)$. On applying the recurring formulas

$$\int e^{-y}y^n dy=-e^{-y}y^n+n\int e^{-y}y^{n-1}dy,\quad \int\frac{e^{-y}}{y^n}dy=-\frac{e^{-y}}{(n-1)y^{n-1}}-\frac{1}{n-1}\int\frac{e^{-y}}{y^{n-1}}dy$$

repeatedly, we obtain

$$\begin{aligned}(9.13)\qquad S_{(I)}(t_0)+S_{(II)}(t_0)&=-\frac{2c_1}{t_0^3}-\frac{4c_1}{3}e^{-y_{1,0}}\left[y_{1,0}+12+\frac{54}{y_{1,0}}+\frac{112}{y_{1,0}^2}+\frac{96}{y_{1,0}^3}\right]\\&\quad+\frac{32}{27}c_1e^{-y_{3,0}}\left[\frac{1}{y_{3,0}}+\frac{7}{y_{3,0}^2}+\frac{8}{y_{3,0}^3}-\frac{e^4}{16}\right]\\&=(0)+(\mathrm{i})+(\mathrm{ii})\quad\text{say.}\end{aligned}$$

We want their asymptotic forms for $t_0\cong0$ as well as $1/t_0\cong0$, denoting those by dashing once or twice, respectively. On availing Laurent's expansion about $t_0=0$, we get for $t_0\cong0$

$$(\mathrm{i})'\cong c_1\left(-\frac{2}{t_0^3}+\frac{14}{3t_0^2}\right),\quad (\mathrm{ii})'\cong c_1\left(\frac{4}{t_0^3}-\frac{14}{3t_0^2}+\frac{142}{81}\right)-\frac{2}{27}c_2,$$

where $c_2=c_1e^4=4.08105$, so that we have

$$(9.13.1)\qquad S_{(I)}(-0)+S_{(II)}(-0)\cong(0)+(\mathrm{i})'+(\mathrm{ii})'=\frac{142}{81}c^{\mathrm{I}}-\frac{2}{27}c_2=-0.17126.$$

Also, for $1/t_0\cong0$,

$$(\mathrm{i})''\cong c_2\left[\frac{2}{t_0^2}+\frac{34}{3t_0^3}+\frac{128}{t_0^4}+\frac{128}{t_0^5}+\frac{3008}{6t_0^6}+\cdots\cdots\right],$$

$$(\mathrm{ii})''\cong c_2\left[-\frac{2}{t_0^2}-\frac{68}{3t_0^3}-\frac{192}{t_0^4}-\frac{6912}{t_0^5}-\frac{4424}{t_0^6}-\cdots\cdots\right],$$

So that we have for $1/t_0\cong0$

$$(9.13.2)\qquad S_{(I)}(t_0)+S_{(II)}(t_0)\cong-\frac{46.4014}{t_0^3}-\frac{609.4366}{t_0^4}-\frac{5119.2674}{t_0^5}-\frac{24551.5889}{t_0^6},$$

which is of use for the determination of the lower significant limit.

It remains to compute

$$s_{(III)}(t)=\frac{2c_1}{-t^3}\int_{y_1}^{y_2}e^{-y}y^2(y+4)\,Tdy.$$

However this being somewhat intricate, rather we consider its integral, the d. f.

$$S_{(III)}(t_0)=\int_{-\infty}^{t_0}s_{(III)}dt=2c_1\int_{-\infty}^{t_0}\frac{dt}{-t^3}\int_{y_1=4t/(1-t)}^{y_2=4t/(\sqrt{3}-t)}e^{-y}y^2(y+4)\,Tdy,$$

where

$$T=\frac{3\tau}{\pi}\tan^{-1}\sqrt{\frac{1}{2}\left(1-\frac{1}{3\tau^2}\right)}-\frac{1}{\pi}\tan^{-1}\sqrt{\frac{3}{2}(3\tau^2-1)},\quad \tau=\frac{y}{(y+4)t},$$

which is non-negative, because $T'(\tau)=\frac{3}{\pi}\tan^{-1}\sqrt{\frac{1}{2}\left(1-\frac{1}{3\tau^2}\right)}\geqq 0$ in $\frac{1}{\sqrt{3}}<\tau<1$ and $T(1/\sqrt{3})=0$. First, interchanging the order of integrations, we have

$$(9.14)\quad S_{(III)}(t_0)=2c_1\int_{-\infty}^{t_0}\frac{dt}{-t^3}\int_{y_1}^{y_2}e^{-y}y^2(y+4)dy$$
$$=2c_1\int_{-4}^{y_{1,0}}e^{-y}y^2(y+4)dy\int_{t_2}^{t_1}\frac{Tdt}{-t^3}+2c_1\int_{y_{1,0}}^{y_{2,0}}e^{-y}(y^2+4)dy\int_{t_2}^{t_0}\frac{T}{-t^3}dt$$
$$=(\text{iii})+(\text{iv}),$$

where $t_1=y/(y+4)$, $y_2=y\sqrt{3}/(y+4)$ and $y_{1,0}=4t_0/(1-t_0)$, $y_{2,0}=4t_0/(\sqrt{3}-t_0)$. Next, replaced t by $\tau=y/(y+4)t$, the inner integral of (iii) yields

$$\int_{t_2}^{t_1}T(t,y)\frac{dt}{-t^3}=\left(\frac{y+4}{y}\right)^2\int_{1/\sqrt{3}}^{1}T(\tau)\tau d\tau,$$

in which the definite integral after integration reduces to

$$c_3=\int_{1/\sqrt{3}}^{1}T(\tau)\tau d\tau=\frac{1}{27}\left(\frac{\sqrt{3}}{\pi}+\frac{1}{6}\right)=0.02659.$$

Hence, on putting $c_4=2c_1c_3=0.21705e^{-4}=0.003975$, $c_5=c_4e^4=0.21705$, we obtain

$$(\text{iii})=c_4\int_{-4}^{y_{1,0}}e^{-y}(y+4)^3dy=c_5\left\{6-\exp\left(\frac{-4}{1-t_0}\right)\left[6+\frac{24}{1-t_0}+\frac{48}{(1-t_0)^2}+\frac{64}{(1-t_0)^3}\right]\right\}.$$

Therefore, as $t_0\cong 0$,

$$(\text{iii})'=(6-142e^{-4})c_5=6c_5-142c_4=0.73779,$$

while, as $1/t_0\cong 0$,

$$(\text{iii})''\cong\frac{-13.8912}{t_0^4}-\frac{100.0156}{t_0^5}-\frac{445.6760}{t_0^6}.$$

Lastly, as to (iv), its inner integral becomes

$$\left(\frac{y+4}{y}\right)^2\int_{1/\sqrt{3}}^{\tau_0}T\tau d\tau,\qquad \tau_0=\frac{y}{(y+4)t_0}.$$

This integral yields after integrations by parts

$$(9.15)\quad \int_{1/\sqrt{3}}^{\tau_0}T\tau d\tau=\frac{\tau_0^3}{\pi}\tan^{-1}\sqrt{\frac{1}{2}\left(1-\frac{1}{3\tau_0^2}\right)}-\frac{1}{2\pi}\left(\tau_0^2-\frac{1}{27}\right)\tan^{-1}\sqrt{\frac{3}{2}(3\tau_0^2-1)}$$
$$+\frac{\sqrt{6}}{54\pi}\sqrt{3\tau_0^2-1}\equiv\mathcal{T}(\tau_0)\quad\text{say}$$

Hence

$$(\text{iv}) = 2c_1\int_{y_{1,0}}^{y_{2,0}} e^{-y}(y+4)^3\mathcal{T}(\tau_0)dy.$$

Here on transforming y into $\eta = y/(y+4)t_0$, i.e. by replacing y by $4t_0\eta/(1-t_0\eta)$, we obtain

$$(\text{iv}) = -c_6t_0\int_{1/\sqrt{3}}^{1}\exp\left(\frac{-4}{1-t_0\eta}\right)\mathcal{T}(\eta)\frac{d\eta}{(1-t_0\eta)^5} \quad (c_6 = 512c_2 = 2089.50).$$

Therefore, when $t_0 \to 0$, $(\text{iv})' = O(t_0) = 0(1)$, so that

(9. 14. 1) $$S_{(III)}(0) = 0.73779.$$

This together with (13.1) yields

(9. 16) $$S(0) = S_{(I)}(0) + S_{(II)}(0) + S_{(III)}(0) = 0.56653 = P_1,$$

which also shows that the distribution is unsymmetrical about the origin and the median is negative.

Finally, required the asymptotic expression for (iv) as $1/t_0 \cong 0$. Expanding the integrand of (iv) in powers of t_0^{-1}, we get

$$(\text{iv}) = \frac{c_6}{t_0^4}\int_{1/\sqrt{3}}^{1}\left(1 + \frac{9}{t_0\eta} + \frac{47}{t_0^2\eta^2} + \cdots\cdots\right)\mathcal{T}(\eta)\frac{d\eta}{\eta^5}.$$

Whence the successive coefficients of t_0^{-m}, as $\int\mathcal{T}(\eta)d\eta/\eta^m$ $(m = 5, 6, 7)$ are found by successive integrations by parts, and thus

$$(\text{iv})'' \cong \frac{207.4682}{t_0^4} + \frac{124.8300}{t_0^5} + \frac{769.1497}{t_0^6}.$$

Hence, by the superposition of (iii)″ and (iv)″ we get

(9. 14. 2) $$S_{(III)}(t_0) = \frac{183.5770}{t_0^4} + \frac{24.8144}{t_0^5} + \frac{323.4737}{t_0^6}.$$

Combining (13.2), (14.2) together, we attain finally

(9. 17) $$S(t_0) = -\frac{46.4014}{t_0^3} - \frac{425.8596}{t_0^4} - \frac{5094.4536}{t_0^5} - \frac{244228}{t_0^6},$$

when $t_0 < 0$, $|t_0| \gg 1$. This equated to $\alpha/2$ ($\alpha = 0.1$, 0.05, 0.01) and solved by Horner, we find as the lower significant limits $t_{0,1} = -7.99$, $t_{0,05} = -10.91$, $t_{0,01} = -19.15$.

2° $0 < t < 1$ $(2 < \bar{x} < \infty)$. Starting from (7), we use the same notations as in 1°, but with the alternation of the upper- and lower-limits of integrations after I–, II–, III–(ii). Also transforming the integration variable x into $\tau = (x-2)/xt$, i.e. replacing x by $2/(1-t\tau)$ and denoting

$$\exp\left(\frac{-4}{1-t\tau}\right)\Big/(1-t\tau)^5=G(t,\tau),$$

we have

(9. 18) $$s(t)=16c\left[\int_0^1 G\tau^3 d\tau-2\int_{1/3}^1 G\tau^3 d\tau+\frac{2}{3}\int_{1/3}^1 G\tau^2 d\tau+2\int_{1/\sqrt{3}}^1 G\tau^2 T d\tau\right].$$

When $t=0$, $G=e^{-4}$ and we get readily $s(0)=0.29025$. Further to compute $P_2=\int_0^1 s(t)dt$, substituting $s(t)$ by (18), interchanging the order of integration and then replacing t by $u=4/(1-t\tau)$, we get

(9. 19) $$P_2=\frac{c}{16}\left[\int_0^1\tau^2 d\tau-2\int_{1/3}^1\tau^2 d\tau+\frac{2}{3}\int_{1/3}^1\tau d\tau+2\int_{1/\sqrt{3}}^1 T\tau d\tau\right]\int_4^{v=4/(1-\tau)} e^{-u}u^3 du.$$

The new inner integral becomes after integrations by parts

(9. 20) $142e^{-4}-e^{-v}(v^3+3v^2+6v+6)\equiv K-V(v)$, where $v=4/(1-\tau)$.

Hence, the part of P_2, which comes from K, is

(9. 21) $$\frac{142}{16}ce^{-4}\left[\int_0^1\tau^2 d\tau-2\int_{1/3}^1\tau^2 d\tau+\frac{2}{3}\int_{1/3}^1\tau d\tau+2\int_{1/\sqrt{3}}^1\tau T d\tau\right]=\frac{71\sqrt{3}}{108}ce^{-4}=0.43347,$$

while the remaining part of P_2, due to V, reduces to

(9. 22) $$-\frac{1}{8}ce^{-4}+\frac{55}{54}ce^{-6}-\frac{7}{9}ce^{-6}-0.00023=-0.11075.$$

In fact on replacing τ by $(v-4)/v$ and integrating by parts repeatedly, $\int\tau V d\tau$, $\int\tau^2 V d\tau$ are easily found, while the last one $\int\tau T V d\tau$ becomes

$$j=-\frac{c}{8}\int_{1/\sqrt{3}}^1 e^{-v}(v^3+3v^2+6v+6)\,T\tau d\tau$$
$$=\frac{-6c}{\pi}\int_{v_0=9.4641}^{\infty}\tan^{-1}\sqrt{\frac{1}{2}\left(1-\frac{1}{3}\left(\frac{v}{v-4}\right)^2\right)}e^{-v}\left(\frac{1}{v}-\frac{6}{v^3}-\frac{12}{v^4}\right)dv,$$

which ought to be numerically computed. Upon expanding by known formulas $\tan^{-1}\sqrt{\frac{1}{2}\left(1-\frac{1}{3}\left(\frac{v}{v-4}\right)^2\right)}=\frac{1}{2}\sec^{-1}X=\frac{1}{2}\left[\frac{\pi}{2}-\frac{1}{X}-\frac{1}{6X^3}-\frac{3}{40X^5}-\frac{5}{112X^7}-\cdots\cdots\right]$, where $X=\left[9\left(1-\frac{4}{v}\right)^2-1\right]\Big/\left[3\left(1-\frac{4}{v}\right)^2+1\right]$ and further writing $w=\frac{6}{v}\left(1-\frac{2}{v}\right)$, $X=\frac{2-3w}{1-w}=2-w-w^2\cdots=2\left[1-\frac{3}{v}\left(1+\frac{4}{v^2}+\cdots\right)\right]=2(1-\zeta)$ &c., we get at length $\sec^{-1}X=1.0472-\frac{1.7289}{v}-\frac{12.9231}{v^2}-\cdots\cdots$, which being stbstituted in j and integrated about v, yields the last figure of (22):

(9. 23) $$j=-\frac{3c}{\pi}\int_{v_0}^{\infty}\sec^{-1}X\cdot e^{-v}\left(\frac{1}{v}-\frac{6}{v^3}-\frac{12}{v^4}\right)dv=-0.000228.$$

Therefore (21), (22) together amount to

(9. 24) $$P_2=0.32272.$$

3° $1<t<\sqrt{3}$. From (8) and (11) we get the fr. f. in $1<t<\sqrt{3}$, similarly as (12),

$$s(t)=\frac{c_1}{t^4}\int_0^{\infty}e^{-y}y^3dy-\frac{2c_1}{t^4}\int_{y_3}^{\infty}e^{-y}y^3dy+\frac{2}{3}\frac{c_1}{t^3}\int_{y_3}^{\infty}e^{-y}y^2(y+4)dy$$
$$+\frac{2c_1}{t^3}\int_{y_2}^{\infty}Te^{-y}y^2(y+4)dy.$$

The integrations are easy except the last one. Further, writing $u=y_3=4t/(3-t)$, we have the d.f. in $1<t<\sqrt{3}$

(9. 25) $$S(t)=\int_{t_1}^{\sqrt{3}}s(t)dt=2c_1\left(\frac{1}{t_1^3}-\frac{1}{\sqrt{27}}\right)-2c_1\int_{t_1}^{\sqrt{3}}e^{-v}(u^3+3u^2+6u+6)\frac{dt}{t^4}$$
$$+\frac{2}{3}c_1\int_{t_1}^{\sqrt{3}}e^{-u}(u^3+7u^2+14u+14)\frac{dt}{t^3}+2c_1\int_{t_1}^{\sqrt{3}}\frac{dt}{t^3}\int_{y_2=4t/(\sqrt{3}-t)}^{\infty}Te^{-y}y^2(y+4)dy$$
$$=(\text{i})-(\text{ii})+(\text{iii})+(\text{iv})$$

Considered the second and third together, replacing the variable t by u as $t=3u/(u+4)$, they become after integrations by parts

$$-(\text{ii})+(\text{iii})=-\frac{32c_1}{27}\int_{u_1}^{5.4641}e^{-u}\left(\frac{1}{u}+\frac{8}{u^2}+\frac{22}{u^3}+\frac{24}{u^4}\right)du$$
$$=\frac{32c_1}{27}e^{-u}\left[\frac{1}{u}+\frac{7}{u^2}+\frac{8}{u^3}\right]\Bigg|_{u_1}^{5.4641}$$
$$=0.000175-0.088589e^{-u_1}\left(\frac{1}{u_1}+\frac{7}{u_1^2}+\frac{8}{u_1^3}\right),$$

where $u_1=4t_1/(3-t_1)$. In particular for $t_1=1$, $u_1=2$, we get

$$(\text{i})-(\text{ii})+(\text{iii})=0.081934.$$

As to (iv), we first interchange the order of integrations and then replace t by τ $(=y/(y+4)t)$

$$(\text{iv})=2c_1\int_1^{\sqrt{3}}\frac{dt}{t^3}\int_{y=4t/(\sqrt{3}-t)}^{\infty}Te^{-y}y^2(y+4)dy=2c_1\int_{y_0=2(\sqrt{3}+1)}^{\infty}e^{-y}y^2(y+4)dy\int_1^{\sqrt{3}\,y/(y+4)}Tdt/t^3$$
$$=2c_1\int_{y_0=5.4641}^{\infty}e^{-y}(y+4)^3dy\int_{1/\sqrt{3}}^{y/(y+4)}T\tau d\tau=2c_1e^4\int_{z_0=y_0+4}^{\infty}e^{-z}z^3dz\int_{1/\sqrt{3}}^{(z-4)/z=\tau_1}T\tau d\tau\,(z=y+4).$$

But $\int_{1/\sqrt{3}}^{\tau_1}T\tau d\tau=\mathcal{T}(\tau_1)$ after (15). So that

$$(\text{iv}) = 2c_1 e^4 \int_{z_0}^{\infty} e^{-z} z^3 \mathcal{T}(\tau_1) dz, \quad \tau_1 = 1 - \frac{4}{z}.$$

Integrating by parts and observing that

$$\int e^{-z} z d^3 z = -e^{-z}(z^3 + 3z^2 + 6z + 6), \quad \mathcal{T}'(\tau_1) = \tau_1 T(\tau_1), \quad \frac{d\tau_1}{dz} = \frac{4}{z^2},$$

we obtain

$$\begin{aligned}(\text{iv}) &= 8c_1 e^4 \int_{z_0}^{\infty} e^{-z}(z^3 + 3z^2 + 6z + 6)\, T(\tau_1)\tau_1 \frac{dz}{z^2} \\ &= 8c_1 e^4 \int_{9.4641}^{\infty} e^{-z}\left(z - 1 - \frac{6}{z} - \frac{18}{z^2} - \frac{24}{z^3}\right) T(\tau_1) dz.\end{aligned}$$

And once more integration by parts yields

$$(\text{iv}) = \frac{6\iota}{\pi} \int_{9.4641}^{\infty} \tan^{-1} \sqrt{\frac{1}{2}\left(1 - \frac{1}{3\tau_1^2}\right)} e^{-z}\left(\frac{1}{z} - \frac{6}{z^3} - \frac{12}{z^4}\right) dz = 0.00028,$$

because it is noting but j of (23) only the sign changed. Therefore we attain

(9.26) $$P_3 = (\text{i}) - (\text{ii}) + (\text{iii}) + (\text{iv}) = 0.08216.$$

But $P_1 + P_2 = 0.88925$, so that $P_1 + P_2 + P_3 = 0.97141$. Hence the upper significant limit $t_{0.1}$, such as $\int_{-\infty}^{t_{0.1}} s(t)dt = 0.95$, lies in the interval 3°: $1 < t < \sqrt{3}$. Writing $t_{0.1} = 1 + \delta$ we get by (25) &c.,

$$S(\delta) = \int_{1+\delta}^{\sqrt{3}} s(t)dt = 2c_1\left[\frac{1}{(1+\delta)^3} - \frac{1}{\sqrt{27}}\right] + 0.088589 e^{-u_1}\left(\frac{1}{u_1} + \frac{7}{u_1^2} + \frac{8}{u_1^3}\right) + (\text{iv}),$$

where $u_1 = 4(1+\delta)/(2-\delta)$ and after the foregoing, the present (iv) may be put $0.00023(1-\delta)$. Expanding in powers of δ, we obtain

$$S(\delta) = 0.08216 - 0.243301\delta + 0.74858\delta^2 - 1.854558\delta^3 + \cdots,$$

which, when subtracted from $P_1 + P_2 + P_3 = 0.97141$, yields just 0.95. Thus we obtain

$$0.88975 + 0.243301\delta - 0.74858\delta^2 + 1.8548\delta^3 = 0.95,$$

i.e. $$1.854558\delta^3 - 0.74858\delta^2 + 0.243301\delta - 0.06025 = 0.$$

This equation solved by Horner, we get $\delta = 0.317$, and therefore $t_{0.1} = 1.32$.

4° $\sqrt{3} < t < 3$. In this subinterval, the calculation becomes an easy task, now that III does not matter. By (9) we obtain

$$\begin{aligned}s(t) &= \frac{c_1}{t^4}\left[\int_0^{\infty} e^{-y} y^3 dy - 2\int_{y_3}^{\infty} e^{-y} y^3 dy + \frac{2}{3} t \int_{y_3}^{\infty} e^{-y} y^2 (y+4) dy\right] \\ &= \frac{2c_1}{t^4}\{3 - e^{-u}[u^3 + 3u^2 + 6u + 6 + \frac{t}{3}(u^3 + 7u^2 + 14u + 14)]\},\end{aligned}$$

where $u=y_3=4t/(3-t)$. To integrate about t, we may replace t by u, and thus obtain for $\sqrt{3}<t_1<3$

$$(9\cdot27)\qquad S(t_1)=\int_{t_1}^{3}s(t)dt=2c_1\left[\frac{1}{t_1^3}-\frac{1}{27}-\frac{16}{27}e^{-u_1}\left(\frac{1}{u_1}+\frac{7}{u_1^2}+\frac{8}{u_1^3}\right)\right],$$

where $u_1=4t_1/(3-t_1)$. In particular, for $t_1=\sqrt{3}$, $u_1=5.4641$, we get

$$(9.\ 28)\qquad P_4=\int_{\sqrt{3}}^{3}s(t)dt=0.02306.$$

Hence $P_1+P_2+P_3+P_4=0.99447$, while $P_1+P_2+P_3=0.97141$, so that the upper critical value, such that $\int_{-\infty}^{t_{0.05}}s(t)dt=0.975$, lies near $t=\sqrt{3}$. Putting $t_1=\sqrt{3}(1+\zeta)$, $u_1=4(1+\zeta)/(\sqrt{3}-1-\zeta)$ in (27), it yields

$$\int_{t_1}^{3}s(t)dt=0.02860-0.08357\zeta+0.15255\zeta^2-0.20554\zeta^3,$$

to which $P_5=1-0.99447=0.00553$ added up, the result should be equal to 0.025. Solving that equation, we obtain $\zeta=0.055$. so that $t_{0.05}=1.055\sqrt{3}=1.83$, which gives the upper significant limit for level $\alpha=0.05$.

5° $3<t<\infty$. After (10) we have simply

$$(9.\ 29)\qquad s(t)=\frac{c}{t^4}\int_1^{\infty}e^{-2x}(x-2)^3dx=\frac{3ce^{-4}}{8t^4}=\frac{0.44848}{t^4}.$$

so that

$$(9.\ 30)\qquad S(t_1)=\frac{3ce^{-4}}{8}\int_{t_1}^{\infty}\frac{dt}{t^4}=\frac{ce^{-4}}{8t_1^3}=\frac{0.149494}{t_1^3}.$$

Hence $s(3)=0.00554$, as well as

$$(9.\ 31)\qquad P_5=0.00554.$$

We obtain

$$(9.\ 32)\qquad P_1+P_2+P_3+P_4+P_5=1.00001,$$

where the error 0.00001 came from having rounded at the fifth decimal place the further following figures as 1 or 0 according as they are $\gtrless 0.5$ at that place. Lastly by equating (30) to 0.005, the upper significant limit is immediately found to be

$$t_{0.01}=\sqrt{0.149494/0.005}=\sqrt{29.90}=3.10.$$

10. *The Beta-Distribution as Universe* (*Case* $n=3$).

The general $B(p,q)$ distribution[1)]

---

1) H. Cramer, loc. cit., p. 126 and p. 243.

(10. 1) $$f(x)=\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}x^{p-1}(1-x)^{q-1}, \quad (0<x<1,\ p,q>0)$$

whose mean is $m=p/(p+q)$, so that $0<m<1$ and $m\lesseqgtr 1/2$ according as $p\lesseqgtr q$, but $m=1/2$ if $p=q$. At the lower end $x=0$, $f(x)$ tends 0 or ∞, according as $p\gtrless 1$ and similarly also at the upper end $x=1$, according as $q\gtrless 1$. Therefore, if p, q are both >1, its graph shows a bell-shaped one, but, if both or one of $p,q<1$, a U- or J-figured. When and only when $p=q=k$, the graph becomes symmetrical and bell or U-shaped according as $k\gtrless 1$. For the sake of simplicity, we shall only confine ourselves to this symmetrical case:

(10. 2) $$f(x)=c_k[x(1-x)]^{k-1}, \quad c_k=\Gamma(2k)/\Gamma(k)^2, \quad 0<x<1,$$

where the mean is always 1/2. More particularly if $k=1$, it reduces to the rectangular distribution, which however specially discussed in sections 11, 12.

Now from (2) a 3-sized sample (x_1, x_2, x_3) being drawn, with mean $\bar{x}$ and variance s^2, where $0<\bar{x}<1$, $0<s<\infty$, its probability element is given by

(10. 3) $$dp=c_k^3[x_1x_2x_3(1-x_1)(1-x_2)(1-x_3)]^{k-1}dx_1dx_2dx_3\,.$$

Transformed the co-ordinates orthogonally as in section 8 again:

	x_1	x_2	x_3
ξ	$-1/\sqrt{2}$	$1/\sqrt{2}$	0
η	$-1/\sqrt{6}$	$-1/\sqrt{6}$	$2/\sqrt{6}$
$\zeta=\sqrt{3}\bar{x}$	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$

$$x_1=-\xi/\sqrt{2}-\eta/\sqrt{6}+\zeta/\sqrt{3}\ (=\bar{x})$$
$$x_2=\ \ \xi/\sqrt{2}-\eta\sqrt{6}+x$$
$$x_3=\qquad 2\eta/\sqrt{6}+\bar{x},$$

and further into polar co-ordinates: $\xi=\rho\cos\theta$, $\eta=\rho\sin\theta$ with $\rho=\sqrt{3}\,s$, we get

$$dx_1dx_2dx_3=d\xi d\eta d\zeta=\rho d\rho d\theta d\sqrt{3}\,\bar{x}=3\sqrt{3}\,sdsd\bar{x}d\theta.$$

Hence the joint probability of $\bar{x}$, s becomes

(10. 4) $$dP=3\sqrt{3}\,c_k^3dsd\bar{x}\int Q^{k-1}d\theta,$$

where

$$Q=\left[\bar{x}^3-\frac{3}{2}s^2\bar{x}-\frac{s^3}{\sqrt{2}}\sin3\theta\right]\left[(1-\bar{x})^3-\frac{3}{2}s^2(1-\bar{x})+\frac{s^3}{\sqrt{2}}\sin3\theta\right].$$

Or, on writing $(x-1/2)\sqrt{2}/s=t$, the Student's fr. f. is given by

(10. 5) $$s(t)=\frac{c}{|t|^3}\int_{x_0}^{x_1}\left(\bar{x}-\frac{1}{2}\right)^2d\bar{x}\int_\theta Q^{k-1}d\theta, \quad c=6\sqrt{3}\,c_k^3,$$

where

(10. 6) $$Q=\left[\bar{x}^3-\frac{3}{t^2}\bar{x}\left(\bar{x}-\frac{1}{2}\right)^2-\frac{2}{t^3}\left(\bar{x}-\frac{1}{2}\right)^3\sin3\theta\right]$$
$$\left[(1-\bar{x})^3-\frac{3}{t^2}(1-\bar{x})\left(\bar{x}-\frac{1}{2}\right)^2+\frac{2}{t^3}\left(\bar{x}-\frac{1}{2}\right)^3\sin3\theta\right].$$

By reason of symmetry we may only conceive the case $t<0$, $\bar{x}<1/2$. The six planes $x_\nu=0, 1$ ($\nu=1, 2, 3$) enclose a cube $\mathfrak{C}$ with center $G_0(1/2, 1/2, 1/2)$ and side 1, which forms the entire domain of sample points $\{x_1, x_2, x_3\}$ (Fig. 4).

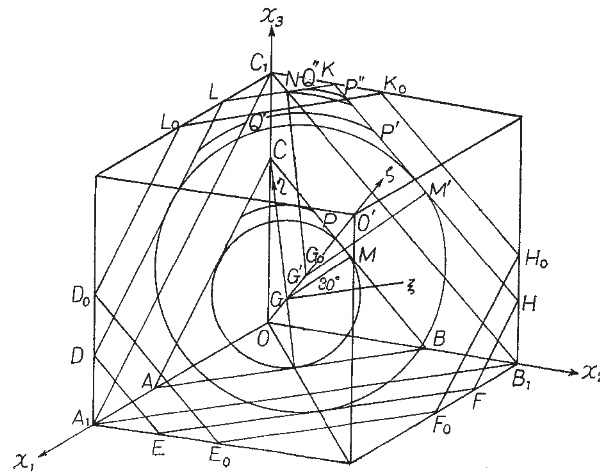

Fig. 4

The 3 sides of the cube, OA_1, OB_1, OC_1 being the x_1-, x_2-, x_3-axis, the ζ-axis is the diagonal OG_0O', through whose any point $G(\bar{x}, \bar{x}, \bar{x})$ with $OG=\sqrt{3}\,\bar{x}$, the plane $x_1+x_2+x_3=3\bar{x}$ passes perpendicularly to the ζ-axis and its section of the cube yields the region for $\bar{x}=\bar{x}$.

Primarily we consider the subcase 1° $0<\bar{x}<1/3$, where the section becomes an equilateral triangle ABC with G as centroid. In particular for $\bar{x}=1/3$ the section by the plane $x_1+x_2+x_3=1$ becomes a maximal triangle $A_1B_1C_1$, while, for $\bar{x}=0$ it degenerates into a single point O.

Secondly 2° if $1/3<\bar{x}<1/2$ the section becomes an hexagon $DEFHKL$, whose sides 3 by 3 are equal. In particular, if $\bar{x}=1/2$, the section becomes a maximal equilateral hexagon $D_0E_0F_0H_0K_0L_0$ corresponding to $t=0$, whose vertices coincides with the middle points of the six sides of cube.

For the general (triangular or hexagonal) section made by the plane $x_1+x_2+x_3=3\bar{x}$, the radius of inscribed circle being $GM=\sqrt{3/2}\,\bar{x}$ (or $G'M'=\sqrt{3/2}\,\bar{x}$), we get firstly I: $0<\sqrt{3}\,s<\sqrt{3/2}\,\bar{x}$, i.e. $0<\sqrt{2}\,s<\bar{x}$, where the s-circle lies wholly inside $\mathfrak{C}$ so that $0<\theta<2\pi$ and $0<(2\bar{x}-1)/t<2\bar{x}$. Hence we get

$$(10.7)\qquad s_I(t)=\frac{c}{-t^3}\int_{1/(2-t)}^{1/2}\left(\bar{x}-\frac{1}{2}\right)^2 d\bar{x}\int_0^{2\pi} Q^{k-1}d\theta \qquad (-\infty<t<0).$$

Next, when $0<\bar{x}<1/3$ with the triangular section, if $GM=\sqrt{3/2}\,\bar{x}<\sqrt{3}\,s<GC=\sqrt{6}\,\bar{x}$, i.e. II: $\bar{x}<\sqrt{2}\,s=(2\bar{x}-1)/t<2\bar{x}$, we get $x_0=1/2(1-t)$, but $x_1=\min(1/3, 1/(2-t))$, In these cases the s-circle splits into 3 pieces and their contributions amount to 6 times $\widehat{PQ}$ (Fig. 4) with $\theta_Q=\pi/2$ and $\theta_P=\pi/6+\gamma$, where $\cos\gamma=\bar{x}/\sqrt{2}\,s=\bar{x}t/(2\bar{x}-1)$. Hence

$$(10.8)\qquad s_{II}(t)=\frac{c}{-t^3}\int_{1/2(1-t)}^{1/(2-t)}\left(\bar{x}-\frac{1}{2}\right)^2 d\bar{x}\int_{\pi/6+\gamma}^{\pi/2}6Q^{k-1}d\theta \qquad (-\infty<t<-1)$$

$$=\frac{c}{-t_3}\int_{1/2(1-t)}^{1/3} \quad ,, \qquad ,, \qquad (-1<t<-1/2)$$

since in the latter $1/2(1-t)$ needs to be $<1/3$.

But, when $1/3<\bar{x}<1/2$ for the hexagonal section, the two adjacent sides being $HK>KL$ their central distances are $GM'=\sqrt{3/2}\,\bar{x}<GN=\sqrt{3/2}\,(1-\bar{x})$, $(\bar{x}<1/2)$. Hence, if $GM'<\sqrt{3}\,s<GN$, i.e. II′: $\bar{x}<\sqrt{2}\,s<1-\bar{x}(<2\bar{x})$, and accordingly $(1+t)/(2+t)<x<1/(2-t)$, so still holds $x_1=1/(2-t)$, but $x_0=\max(1/3, (1+t)/(2+t))$. And the contribution from s-circle is again 6 times the arc $P'Q'$ with $\theta_{Q'}=\pi/2$ and $\theta_{P'}=\pi/6+\gamma$, where $\cos\gamma=\bar{x}t/(2\bar{x}-1)$, as before. Hence

$$(10.9)\qquad s_{II}'(t)=\frac{c}{-t^3}\int_{1/3}^{1/(2-t)}\left(\bar{x}-\frac{1}{2}\right)^2 d\bar{x}\int_{\pi/6+\gamma}^{\pi/2}6Q^{k-1}d\theta \qquad (-1<t<-1/2)$$

$$=c/(-t)^3\int_{(1+t)/(2+t)}^{1/(2-t)} \quad ,, \qquad ,, \qquad (-1/2<t<0).$$

Furthermore we need to consider the s-circle with the remaining arc-pieces, such that $GN'=\sqrt{3/2}\,(1-\bar{x})<\sqrt{3}\,s<GK=\sqrt{6(\bar{x}^2-\bar{x}+1/3}$, i.e. II″: $1-\bar{x}<\sqrt{2}\,s<2\sqrt{\bar{x}^2-\bar{x}+1/3}\,(<2\bar{x})$. They are $6P''Q''$ (Fig. 4), where $\theta_{P''}=\cos^{-1}\bar{x}t/(2\bar{x}-1)=\gamma$ and $\theta_{Q''}=\cos^{-1}(1-\bar{x})t/(2\bar{x}-1)=\delta$. Also inequalities II″ yield $x_1=(1+t)/(2+t)$, $x_0=(1+t/\sqrt{3(1-t^2)})/2$. With these limits we have

$$(10.10)\qquad s_{II''}(t)=\frac{c}{-t^3}\int_{x_0}^{x_1}\left(\bar{x}-\frac{1}{2}\right)^2 d\bar{x}\int_{\gamma}^{\delta}6Q^{k-1}d\theta.$$

We shall treat as example the case $k=2$, $f(x)=6x(1-x)$, a parabolic distribution. Then, since the inner integrand reduces to Q itself, we have

$$(10.11)\qquad s_I(t)=\frac{c}{|t|^3}\int_{1/(2-t)}^{1/2}\left(\bar{x}-\frac{1}{2}\right)^2 d\bar{x}\int_0^{2\pi}Q\,d\theta \qquad (-\infty<t<0)$$

$$(10.12)\qquad s_{II}(t)=\frac{c}{|t|^3}\int_{1/2(1-t)}^{1/(2-t)}\left(\bar{x}-\frac{1}{2}\right)d\bar{x}\int_{\pi/6+\gamma}^{\pi/2}6Q\,d\theta \qquad (-\infty<t<-1)$$

&c., where Q denotes expression (6) and $\gamma=\cos^{-1}[\bar{x}t/(2\bar{x}-1)]$, $c=1296\sqrt{3}=2244.7378$.

To evaluate the critical values t_α ($\alpha=.1$, $.05$, 0.01) as $|t_\alpha|\gg 1$, it suffices merely to investigate the above two formulas and to compute approximately by the expansion in powers of t^{-1}, neglecting those powers of higher degree than e.g. 7. First (11) reduces to

$$(10.13)\qquad s_I(t)=\frac{2\pi c}{|t|^3}\int_{1(2-t)}^{1/2}\Big[\frac{x^3}{4}(1-x)^3(1-2x)^2$$
$$+\frac{3}{16t^2}(x-x^2)((1-x)^2+x^2)(1-2x)^4$$
$$+\frac{9}{64t^4}(x-x^2)(1-2x)^6-\frac{1}{128t^6}(1-2x)^8\Big]dx\qquad (t<-1),$$

whose last term may be neglected, since it becomes $O(t^{-9})$. Numerically it becomes

$$(10.14)\qquad s_I(t)=-\frac{50.3718}{t^3}+\frac{125.929}{t^5}-\frac{1819.68}{t^7}$$

approximately.

Similarly (12) reduces to

$$(10.15)\qquad s_{II}(t)=\frac{2c}{|t|^3}\int_{1/2(1-t)}^{1/(2-t)}\left(\pi-3\cos^{-1}\frac{xt}{2x-1}\right)\left[x^3(1-x)^3\left(\frac{1}{2}-x\right)^2\right.$$
$$\left.-\frac{3}{t^2}(x-x^2)((1-x)^2+x^2)\left(\frac{1}{2}-x\right)^4+\frac{9}{t^4}(x-x^2)\left(\frac{1}{2}-x\right)^6\right]dx$$
$$-\frac{c}{12t^6}\int_{1/2(1-t)}^{1/(2-t)}\left(\frac{1}{2}-x\right)^2((1-x)^3-x^3)((1-2x)^2-4x^2t^2)\sqrt{(1-2x)^2-x^2t^2}\,dx$$
$$=\ \text{(i)}-\text{(ii)}\quad \text{say.}$$

To rationalize the integrand in (ii), we transform x into y by $\sqrt{(1-2x)^2-x^2t^2}=y\sqrt{1-2x-xt}>0$, $x=(1-y^2)/D$, $dx=4tydy/D^2$, where $D=2(1-y^2)-t(1+y^2)>0$, because of $t<0$, $x<1/2$, so that $y^2<1$. We notice also that for $t<0$, $|t|\gg1$, $D\cong-t(1+y^2)>0$, $x\cong-(1-y^2)/(1+y^2)t>0$ and consequently $\sqrt{(1-2x)^2-x^2t^2}\cong2y/(1+y^2)>0$, but $(1-2x)^2-4x^2t^2=1-4(1-y^2)^2/(1+y^2)^2\leqq0$. To calculate (ii) up to the order of t^{-7}, we ought to take only those terms of $O(1)$ in its integrand. Hence

$$\text{(ii)}\cong\frac{2c}{3(t)^7}\int_0^{1/\sqrt{3}}\left[1-4\left(\frac{1-y^2}{1+y^2}\right)^2\right]\frac{y^2dy}{(1+y^2)^3}=\frac{3\sqrt{3}\,c}{64|t|^7}\quad (t<0).$$

Also, integrating (i) by parts, we have

$$\text{(i)}=\frac{2c}{|t|^3}\left(\pi-3\cos^{-1}\frac{xt}{2x-1}\right)I(x)\Big|_{1/2(1-t)}^{1/(2-t)}-\frac{2c}{t^2}\int_{1/2(1-t)}^{1/(2-t)}\frac{3I(x)dx}{(1-2x)\sqrt{(1-2x)^2-x^2t^2}}$$
$$=\text{(i)}'-\text{(i)}'',$$

where

$$I(x)=\int\left[x^3(1-x)^3\left(\frac{1}{2}-x\right)^2-\frac{3}{t^2}(x-x^2)((1-x)^2+x^2)\left(\frac{1}{2}-x\right)^4\right]dx\equiv\int E(x,t)dx.$$

Since the integrated part (i)$'$ vanishes for the lower limit, it reduces to

$$\text{(i)}' = \frac{2c\pi}{|t|^3} I\left(\frac{1}{2-t}\right) = \frac{2c\pi}{|t|^3}\int^{1/(2-t)} E(x, t)\,dx.$$

Hence also, as noticed above, we are only to take those terms in $E(x, t)$ of degree not higher than x^3, t^{-3}. Thus

$$\text{(i)}' \cong \frac{2c\pi}{|t|^3}\int^{1/(2-t)}\left(\frac{x^3}{4} - \frac{3}{t^2}\frac{x}{16}\right)dx = \frac{2c\pi}{|t|^3}\left[\frac{x^4}{16} - \frac{3x^2}{32t^2}\right]_{1/(2-t)} \cong \frac{c\pi}{16t^7}.$$

Lastly in the not yet integrated part (i)$''$, again equating the radical to $y[(1-2x)-xt]$, we obtain

$$\text{(i)}'' = \frac{4c}{t^3}\int_0^{1/\sqrt{3}} \frac{dy}{1+y^2}\left[\frac{x^4}{16} - \frac{3x^2}{32t^2}\right]_{x=-(1-y^2)/(1+y^2)t}$$

$$\cong \frac{c}{4t^7}\int_0^{1/\sqrt{3}}\left[\left(\frac{1-y^2}{1+y^2}\right)^4 - \frac{3}{2}\left(\frac{1-y^2}{1+y^2}\right)^2\right]\frac{dy}{1+y^2} = \frac{c}{64|t|^7}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{8}\right).$$

Therefore

$$(10.16) \qquad s_{II}(t) = \text{(i)}' - \text{(i)}'' - \text{(ii)}$$

$$= \frac{c\pi}{16t^7} + \frac{c}{64t^7}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{8}\right) = \frac{366.87}{t^7}.$$

Hence (14) and (16) together yield

$$(10.17) \qquad s(t) = \frac{50.3718}{t^3} + \frac{125.929}{t^5} - \frac{1452.81}{t^7} \qquad (t<0,\ |t|\gg 1).$$

Consequently the d. f. is

$$(10.18) \qquad S(t) = \int_{-\infty}^{t} s(t)dt = \frac{25.1859}{t^2} - \frac{31.482}{t^4} + \frac{242.135}{t^6} \qquad (\qquad " \qquad).$$

This is to be equated to $\alpha/2$ ($\alpha=0.1$, 0.05, 0.01) and solved for $t^{-2}=x$ by Horner, so that $t_\alpha = \pm 1/\sqrt{x}$. Thus we find $t_{0.1} = \pm 22.42$, $t_{0.05} = \pm 31.72$, $t_{0.01} = \pm 100.7$, which are considerably large in magnitude, compared with those for the corresponding classical Student ratios: 2.358, 3.183, 5.841.

The investigation for those cases that k takes other values than 2, especially the case that $k=1/2$, as a continuation of section 5 in [II], is recommended to students' self study.

11. *The Rectangular Distribution $f(x)=1$, $x>0$, as Universe, Case $n=4$.*

By symmetry we shall only consider the lower half: $0<\bar{x}<1/2$ ($t<0$), which shall be further separated into two subcases: 1° $0<x<1/4$ and 2° $1/4<\bar{x}<1/2$. For 1° the simplex S_3 (tetrahedron, Fig. 5) lies wholly in the 4-dimensional cube $\mathfrak{C}$ formed by 4 unit x_ν-semiaxis ($\nu=1, 2, 3, 4$), since the four vertices $A(4\bar{x}, 0, 0, 0)$, $B(0, 4\bar{x}, 0, 0)$, $C(0, 0, 4\bar{x}, 0)$, $D(0, 0, 0, 4\bar{x})$ all lie in $\mathfrak{C}$ because of $4\bar{x}<1$. However, for 2°, the vertices of S_3 are outside $\mathfrak{C}$, e.g. D's $x_4=4\bar{x}>1$ (Fig. 6). But the centroid G surely lies inside $\mathfrak{C}$ because, its co-ordinates all $=\bar{x}<1$. Hence, there exists a point M^* on DG with $x_4=1$, $0<x_\nu<1$ ($\nu=1, 2, 3$), which is also a boundary point of $\mathfrak{C}$. Indeed, an intersection of S_3 with $\mathfrak{C}$ is the plane $A^*B^*C^*$ drawn through M^* parallel to the base ABC. For, if P be any point inside $A^*B^*C^*$, and Q, R be the point of intersection of DP produced with the base, and AQ proedduc with BC, respectively, then the co-ordinates of the successive points R, Q, P are $R(0, 4\bar{x}/(1+\nu), 4\bar{x}\nu/(1+\nu), 0)$, $Q\left(\frac{4x}{1+\mu}, \frac{4\bar{x}\mu}{(1+\mu)(1+\nu)}, \frac{4\bar{x}\mu\nu}{(1+\mu)(1+\nu)}, 0\right)$, $P\left(\frac{4\bar{x}\lambda}{(1+\lambda)(1+\mu)}, \frac{4\bar{x}\lambda\mu}{(1+\lambda)(1+\mu)(1+\nu)}, \frac{4\bar{x}\lambda\mu\nu}{(1+\lambda)(1+\mu)(1+\nu)}, \frac{4\bar{x}}{1+\lambda}\right)$, where $\lambda=DP/PQ=DM^*/M^*M$, $\mu=AQ/QR$, $\nu=BR/RC$. Hence $\lambda=\text{const.}=DM^*/M^*M$ for all points on $A^*B^*C^*$, so that their 4-th

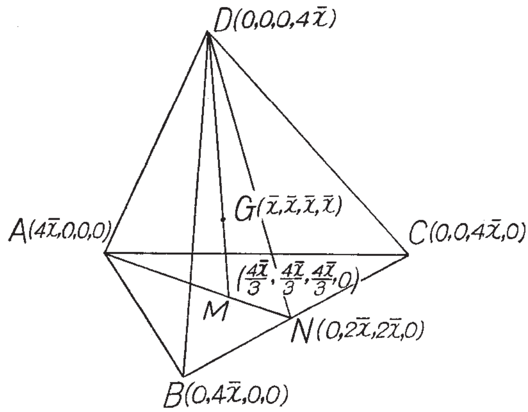

Fig. 5

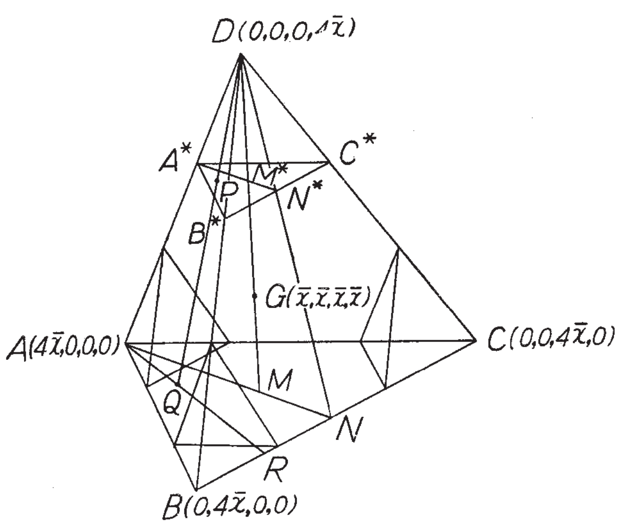

Fig. 6

ordinate $x_4=4\bar{x}/(1+\lambda)$ are also constant$=1$, because that of M^* is unity. Thus all points on $A^*B^*C^*$ have $x_4=1$ and lie on a boundary of $\mathfrak{C}$. In particular, if P coincides with M^*, $\lambda=4\bar{x}-1$, $\mu=2$, $\nu=1$, so that the co-ordinates of M^* are $x_1=x_2=x_3=(4\bar{x}-1)/3$, $x_4=1$). Similarly on any plane paralell to base, the 4-th ordinate x_4 is constant, and in particular $x_4=0$ on the baseplane ABC.

As the fourth ordinate x_4 of the inner point of the smaller tetrahedron

$A^*B^*C^*D$ is much more >1, the simplex S_3 is beheaded at D, and similarly at other 3 vertices also. Thus the tetrahedron becomes now an octahedron with 4 triangular and 4 hexagonal faces, the latters being the degenerates of 4 old triangular faces. Every point on the new triangular faces has 4 co-ordinates, one of which $=1$ and the remaining three $<1/2$, because their sum$=4\bar{x}-1<1$. In the limit that $\bar{x}=1/2$, however, it reduces to a regular octahedron (Fig. 7), whose faces are all equilateral triangle with side 1, where either $x_\nu=1$ or 0 $(\nu=1,2,3,4)$.

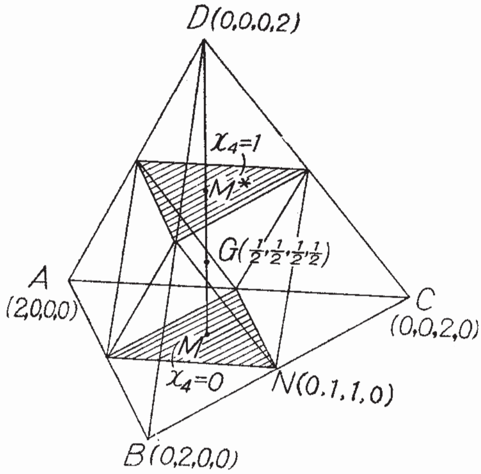

Fig. 7

Remark. For the case $1/2<\bar{x}<1$, geometrically we should consult with the tetrahedron, whose 4 vertices have 3 ordinates, each$=1$, and the remaining one$=4\bar{x}-3$. This tetrahedron is complete (unbeheaded) or beheaded, according as 3° $1>\bar{x}>3/4$ or 4° $3/4>\bar{x}>1/2$ and these two snbcases just correspond to the foregoing 1° and 2°, respectively.

Now we shall examine each subcase separately.

I $0<s<\bar{x}/\sqrt{3}$. After paper [I], we see that $dV_I=dP_I=64\pi s^2 ds dx=24\sqrt{3}\,\pi|2\bar{x}-1|^3 d\bar{x}dt/t^4$, so that the contribution from this portion to $s(t)$, is

$$s_I(t)=\frac{24\sqrt{3}\,\pi}{t^4}\int_{x_0}^{x_1}|2\bar{x}-1|^3 d\bar{x}$$

$$=\left[\frac{3\sqrt{3}\,\pi(2\bar{x}-1)^4}{-t^4}\right]_{x_0}^{x_1}\equiv\Big[J_1\Big]_{x_0}^{x_1},$$

where J_1 denotes the left standing indefinite integral. Condition I means that the radius of s-sphere, $2s$, is such as $0<2s=(2\bar{x}-1)\sqrt{3}/t<2x/\sqrt{3}=GM$ (Fig. 8), which inequalities solved for x yield $1/2>\bar{x}>3/2(3-t)\equiv a=a(t)$ for $t<0$, so that $x_0=a$, $x_1=1/2$. Hence

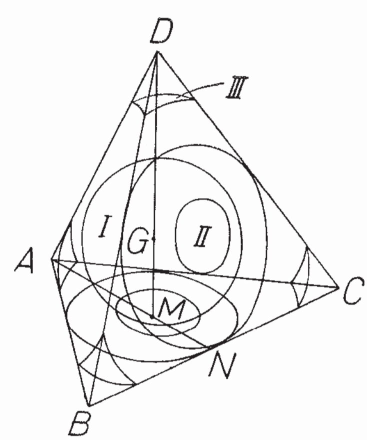

Fig. 8

$$(11.1)\quad s_I(t)=\Big[J_1\Big]_a^{1/2}=\frac{3\sqrt{3}\,\pi}{(3-t)^4}\quad(-\infty<t<0).$$

II $x/\sqrt{3}<s<\bar{x}$. After [I], four calottes being rejected, $dV_{II}=dP_{II}=64\pi(2\bar{x}/\sqrt{3}-s)sdsd\bar{x}=32\sqrt{3}\,\pi\bar{x}(2\bar{x}-1)^2 d\bar{x}dt/|t|^3-24\sqrt{3}\,\pi|2x-1|^3 dxdt/t^4$.

So that

(11. 2) $$s_{II}(t)=[J_2-J_1]_{x_0}^{x^1}$$

where

$$J_2=\frac{32\sqrt{3}}{|t|^3}\int \bar{x}(2\bar{x}-1)^2 d\bar{x}=\frac{2\sqrt{3}\,\pi}{|t|^3}\left[(2x-1)^4+\frac{4}{3}(2x-1)^3\right].$$

Condition II means that s-sphere's radius $2s$ is between $GM=2\bar{x}/\sqrt{3}$ and $GN=2\bar{x}$, and whence follows that $0<b\equiv\sqrt{3}/2(\sqrt{3}-t)<\bar{x}<3/2(3-t)=a$. Firstly, for 1° $0<\bar{x}<1/4$, we shall take $x_0=b$ and $x_1=\min(a, 1/4)$. Hence $x_1=a$, if $-\infty<t<-3$, but $x_1=1/4$ if $t>-3$, in which case however it needs $b<1/4$, so that $-3<t<-\sqrt{3}$. Thus

(11. 3) $$S_{II}(t)=[J_2-J_1]_b^a \qquad \text{if} \quad -\infty<t<-3,$$
$$=[\ \text{,,}\ ]_b^{1/4} \qquad \text{if} \quad -3<t<-\sqrt{3}.$$

II′ Next for 2° $1/4<\bar{x}<1/2$, the bounds of the II-typed s-sphere's radii $2s$, may be affected by beheading of S_3 (Fig. 6). Here it is clear that the central distance of the new face $A^*B^*C^*=GM^*=2(1-\bar{x})\sqrt{3}>GM=2\bar{x}/\sqrt{3}$, because of $\bar{x}<1/2$. Hence, $GM^*\gtreqless GN=2\bar{x}$, according as $\bar{x}\lesseqgtr(\sqrt{3}-1)/2=0.366$. First assuming the upper sign, if $1/4<a<0.366$, i.e. $-3<t<3(1-\sqrt{3})/2=-1.098$, still with the same bounding $MG<2s<GN$, we get $a>\bar{x}>b$ and $\bar{x}_1=a$. Yet now x_0 shall be $\max(b, 1/4)$, so that $x_0=1/4$, if $-3<t<-\sqrt{3}$, while $x_0=b$, if $-\sqrt{3}<t-1.098$. Next, when $0.366<a<1/2$, $-1.098<t<0$, $GM^*<GN$ and the II-typed s-bounding makes $MG<2s<GM^*$, which yields $a>\bar{x}>(3+2t)/2(3+t)=h>0$, and thus still $x_1=a$, but $x_0=\max(h, 1/4)$. Now $h\lesseqgtr 1/4$ according as $t\lesseqgtr -1$. Hence $x_0=1/4$ if $-1.098<t<-1$, while $x_0=h$ if $-1<t<0$. Therefore we obtain

(11. 4) $$s_{II}'(t)=[J_2-J_1]_{1/4}^a \qquad \text{if} \quad -3<t<-\sqrt{3},$$
$$=[\ \text{,,}\ ]_b^a \qquad \text{if} \quad -\sqrt{3}<t<-1.098,$$
$$=[\ \text{,,}\ ]_{1/4}^a \qquad \text{if} \quad -1.098<t<-1$$
$$=[\ \text{,,}\ ]_h^a \qquad \text{if} \quad -1<t<0.$$

However, the first of (4) combined with the second of (3), it enlarges the upper limit -3 into $-\sqrt{3}$ in the first of (3), which further combined with the second of (4) makes the upper limit -1.098 and we attain

(11. 5) $$s_{II}(t)=[J_2-J_1]_b^a \qquad \text{for} \quad -\infty<t<-1.098.$$

II″ The case $GM^*<2s<GN$ belongs also to II (Fig. 9). In this case however we ought to reject the s-spher's portion swelled over the new faces, whose area after Archimedes is $2\pi 2s.(2s-GM^*)\times 4$, so that the corresponding

elementary volume now becomes

$$dV_{II}=\left\{\frac{12\sqrt{3}\pi}{t^4}|2\bar{x}-1|^3+\frac{8\sqrt{3}\pi}{|t|^3}\bar{x}(2\bar{x}-1)^2-\frac{8\sqrt{3}\pi}{|t|^3}(2\bar{x}-1)^2\right\}4d\bar{x}dt.$$

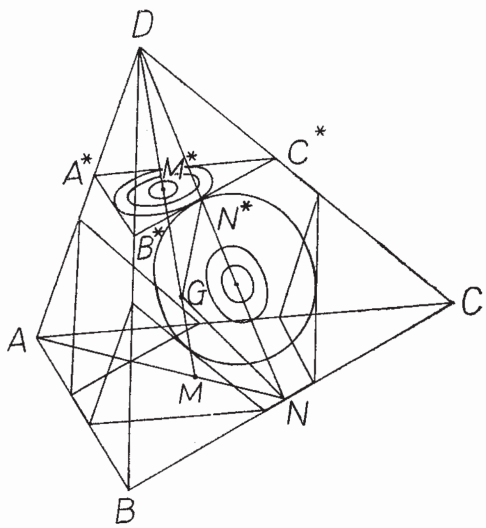

Fig. 9

Accordingly the integral $[2J_1+J_2-J_4]$, where $J_4=32\sqrt{3}\pi\int(2x-1)^2dx/|t|^3$, should be subtracted from the ordinary II-constituent $[J_2-J_1]$. Hence, the contribution now reduces to

(11. 6) $\qquad s_{II}''(t)=[J_4-3J_1]_{x_0}^{x_1}.$

As to the limits, it may happen that $GN\lesseqgtr GN^*$, which occurs according as $1/4<\bar{x}\lesseqgtr 3/8=0.375$. Hence, if $3/8<\bar{x}<1/2$, $GN<GN^*$, we should consider the contributions from
(i) $GM^*=2(1-\bar{x})/\sqrt{3}<2s<GN^*$
$=2\sqrt{(x-1/2)^2+1/8}<GN$
and (ii) $GN^*<2s<GN=2\bar{x}$.
For (i) we get

$$s_{II}''_{(i)}(t)=[J_4-3J_1]_{3/8}^{h} \qquad \text{if} \quad -0.6<t<-0.577,$$
$$=[\ \text{,,}\ ]_{k}^{h} \qquad \text{if} \quad -0.577<t<0,$$

where $h=\dfrac{3+2t}{2(3+t)}$, $k=\dfrac{1}{2}\left[1+\dfrac{t}{\sqrt{2(3-t^2)}}\right]$, and the lowest limit -0.6 is obtained by solving the inequality after t. Also for (ii)

$$s_{II}''_{(ii)}(t)=[J_{44}-3J_1+X]_{b}^{k} \qquad \text{for} \quad -0.577<t<0,$$

where the rejected calottes over the new faces overlap, so that some $X>0$ must be added in the integrand, of which the exact evaluation shall be performed by the direct method, as done in III of [I]. However, our main purpose being to detect the significant limits $t_\alpha(\alpha\leq 0.1)$ and they are really $>\sqrt{3}$ in absolute value. Therefore we may ignore those $s(t)$ for $|t|\leq 1$. Hence, the above obtained partial $s(t)$, or also those in below, whose argument t is ≤ 1 in absolute value, need not be investigated.

III $\bar{x}<s<\sqrt{3}\,\bar{x}$. We consider first the case $0<\bar{x}<1/4$ with unbeheaded S_3 (Fig. 8). Then III implies that $2\bar{x}=GN<2s<2\sqrt{3}\,\bar{x}=GA$ and we have by [1]

$$dV_{III}=dP_{III}=128\bar{x}s\left\{\frac{\pi}{2}\left(\frac{2}{\sqrt{3}}-\frac{s}{\bar{x}}\right)+\frac{3s}{\bar{x}}\tan^{-1}\sqrt{\frac{1}{2}\left(1-\frac{\bar{x}^2}{s^2}\right)}\right.$$

$$-\sqrt{3}\tan^{-1}\sqrt{\frac{3}{2}\left(\frac{s^2}{x^2}-1\right)}\Big\}d\bar{x}ds$$

$$=\{32\sqrt{3}\,\pi\bar{x}(2\bar{x}-1)^2/|t|^3-24\sqrt{3}\,\pi|2\bar{x}-1|^3/t^4$$
$$+96\sqrt{3}\,\pi(2\bar{x}-1)^2T/|t|^3\}d\bar{x}dt\,,$$

where

$$T=T(\tau)=\frac{3\tau}{\pi}\tan^{-1}\sqrt{\frac{1}{2}\left(1-\frac{1}{3\tau^2}\right)}-\frac{1}{\pi}\tan^{-1}\sqrt{\frac{3}{2}(3\tau^2-1)}\,,\ \tau=\frac{2\bar{x}-2}{2\bar{x}t}=\frac{s}{\bar{x}\sqrt{3}}\,.$$

Hence we obtain

$$(11.7)\qquad s_{III}(t)=[J_2-J_1+J_3]_{x_0}^{x_1},\quad J_3=\frac{96\sqrt{3}\,\pi}{|t|^3}\int x(2x-1)^2Tdx\,,$$

where $x_0=1/2(1-t)=g>0$, $x_1=\min(b,\ 1/4)$, so that

$$(11.8)\qquad s_{III}(t)=[J_2-J_1+J_3]_g^b \qquad \text{for}\quad -\infty<t<-\sqrt{3}\,,$$
$$=[\quad ,,\quad]_g^{1/4} \qquad \text{for}\quad -\sqrt{3}<t<-1\,.$$

III′ Secondly for $1/4<\bar{x}<1/2$ with the beheaded S_3 there arise 3 subcases:

(i) $\frac{1}{4}<\bar{x}<\frac{\sqrt{3}-1}{2}=0.366,$

$GN<2s<GM^*<GN^*$

(ii) ,, $GN<GM^*<2s<GN^*$

(Fig. 10)

(iii) $0.366<\bar{x}<3/8=0.375,$

$GM^*<GN<2s<GN^*.$

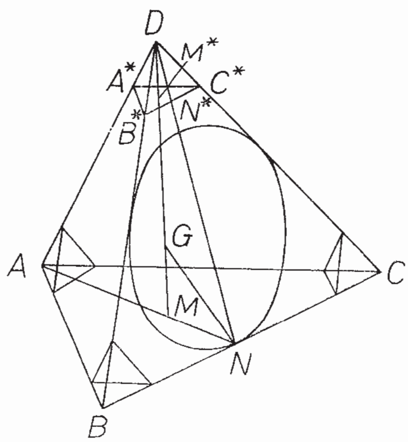

Fig. 10

We get for (i)

$$(11.9)\qquad s_{III'}(t)=[J_2-J_1+J_3]_{1/4}^b \qquad \text{if}\quad -\sqrt{3}<t<-1,$$
$$=[\quad //\quad]_h^b \qquad \text{if}\quad -1<t<-0.634,$$
$$=[\quad //\quad]_h^{0.366} \qquad \text{if}\quad -0.634<t<0.$$

Hence, summing up the second of (8) and the first of (9), we get

$$(11.10)\qquad s_{III}(t)=[J_2-J_1+J_3]_g^b \qquad \text{for}\quad -\infty<t<-1.$$

However, for (ii) (iii), we ought to reject the calottes over the new faces, as in (6), so that e.g. for (ii)

$$s_{III}''(t)=[J_4-3J_1+J_3]_{1/4}^{k} \qquad \text{if} \quad -1<t<-1(3-\sqrt{3})/2=-0.634,$$
$$=[\quad \text{〃} \quad]_{1/4}^{0.366} \qquad \text{if} \quad -0.634<t<-1/\sqrt{3}=-0.577,$$
$$=[\quad \text{〃} \quad]_{k}^{0.366} \qquad \text{if} \quad -0.577<t<0.$$

Thus these in (ii), and also those in (iii) (in which really $|t|<0.634$), all of them may be ignored.

So far we have exhausted all $2s<\max(GN, GN^*)$. To speak minutely, there still remain some corner cases with those $2s$, such that (iv) $GN<GN^*<2s<GC^*$ for $1/4<\bar{x}<3/8$ and (v) $GN^*<GN<2s<GC^*$ for $3/8<\bar{x}<1/2$. However here the upper bound of $|t|$ is 1 in (iv) and 0.577 in (v). So that all partial $s(t)$ for these argument values may be neglected.

Now we proceed to evaluate $s_4(t)=s(t)$, confining the interval as $-\infty<t<1.098$ or $<-\sqrt{3}$. We get in view of (1), (5), (8)

$$(11.11) \quad s(t)=\frac{3\sqrt{3}\pi}{(3-t)^4}+[J_2-J_1]_b^a+[J_2-J_1+J_3]_g^b$$
$$= \quad \text{〃} \quad +[J_2-J_1]_g^a+[J_3]_g^b$$
$$=c_0\left[\frac{2}{(1-t)^3}-\frac{2}{(3-t)^3}-\frac{3}{(1-t)^4}\right]+\frac{c_1}{|t|^3}\int_g^b x(2x-1)^2 T dx$$
$$=p(t)+q(t),$$

where $c_0=\pi/\sqrt{3}=1.81380$, $c_1=96\sqrt{3}\pi=522.3742$. Also $p(t)$ being a sum of plain binomials, it can be immediately integrated:

$$(11.12) \quad P(t)=\int_{-\infty}^{t} p(t)dt=c_0\left[\frac{1}{(1+t)^2}-\frac{1}{(3-t)^2}-\frac{1}{(1-t)^3}\right] \quad (-\infty<t<-\sqrt{3}).$$

And this may be also expanded in powers of t^{-1} as Laurent series when $|t|<3$. In fact

$$(11.13) \quad P(t)=-c_0\left[\frac{3}{t^3}+\frac{21}{t^4}+\frac{98}{t^5}+\frac{390}{t^6}+\frac{1437}{t^7}+\frac{5075}{t^8}+\frac{17460}{t^9}+\cdots\cdots\right] \quad (|t|>3)$$

which however is invalid for $|t|\lessapprox 3$. The second part $Q(t)=\int_{-\infty}^{t} q(t)dt$ is comparatively small, yet enough complicate to compute. Upon writing

$$(11.14) \qquad q(t)=[J_3]_g^b=\frac{c_1}{|t|^3}j=\frac{c_1}{|t|^3}\int_g^b x(2x-1)T(\tau)dx,$$

where

$$T(\tau)=\frac{3\tau}{\pi}\tan^{-1}\sqrt{\frac{1}{2}\left(1-\frac{1}{3\tau^2}\right)}-\frac{1}{\pi}\tan^{-1}\sqrt{\frac{3}{2}(3\tau^2-1)}, \quad T'(\tau)=\frac{3}{\pi}\tan^{-1}\sqrt{\frac{1}{2}\left(1-\frac{1}{3\tau^2}\right)},$$

$$T''(\tau)=\frac{\sqrt{6}}{(9\tau^2-1)\sqrt{3\tau^2-1}}, \qquad \tau=\frac{2xt}{2x-1}, \qquad \frac{d\tau}{dx}=\frac{1}{2tx^2}.$$

We have $\tau=1/\sqrt{3}$ or 1, when $x=b(t)$ or $g(t)$ and $T(1/\sqrt{3})=T'(1/\sqrt{3})=0$, $T(1)=1/6$, $T'(1)=1/2$. Hence, on putting

$$u(x)=\frac{1}{6}\left(x^4-\frac{4}{3}x^3+\frac{1}{2}x^2\right),\quad v(x)=\frac{1}{12t}\left(x^3-2x^2+\frac{3}{2}x\right),\quad c_2=\frac{\sqrt{6}}{4\pi}$$

and integrating by parts, we get

$$(11.15)\quad j=-u(g)+v(g)+\frac{c_2}{t^2}\int_g^6\left(x-2+\frac{3}{2x}\right)\frac{dx}{(9\tau^2-1)\sqrt{3\tau^2-1}}\left(=w,\ \text{say}\right).$$

To compute the last integral w, we transform x into $\tau=(2x-1)/2xt$ so that $x=1/\ 2(1-t\tau)$ and obtain

$$(11.16)\quad w=\frac{c_2}{4|t|}\int_{1/\sqrt{3}}^1\left[\frac{6}{1-t\tau}-\frac{4}{(1-t\tau)^2}+\frac{1}{(1-t\tau)^3}\right]\frac{d\tau}{(9\tau^2-1)\sqrt{3\tau^2-1}}.$$

We desire to rationalize this w's integrand. For this purpose we write its factor as a sum of odd and even functions, so as

$$w_1=\frac{c_2}{-4}\int_{1/\sqrt{3}}^1\left[\frac{6}{1-t^2\tau^2}-\frac{9}{(1-t^2\tau^2)^2}+\frac{4}{(1-t^2\tau^2)^3}\right]\frac{\tau d\tau}{(9\tau^2-1)\sqrt{3\tau^2-1}},$$

$$w_0=\frac{c_2}{-4t}\int_{1/\sqrt{3}}^1\left[\frac{10}{1-t^2\tau^2}-\frac{11}{(1-t^2\tau^2)^3}+\frac{4}{(1-t^2\tau^2)^3}\right]\frac{d\tau}{(9\tau^2-1)\sqrt{3\tau^2-1}}.$$

Further, puttiing $\sqrt{3\tau^2-1}=z$ or $z\tau$ in w_1 or w_0, we get

$$w_1=\frac{3c_2}{4}\int_0^{\sqrt{2}}\left[\frac{2}{t^2z^2+t^2-3}+\frac{9}{(t^2z^2+t^2-3)^2}+\frac{12}{(t^2z^2+t^2-3)^3}\right]\frac{dz}{3z^2+2},$$

$$w_0=\frac{c_2}{-4t}\int_0^{\sqrt{2}}\left[3+\frac{t^4}{(z^2+t^2-3)^2}\right]\frac{dz}{z^2+6}.$$

These can be formally integrated, which are of use to obtain the fr. f. $s(t)$. However, what is more required, is their integrals

$$(11.17)\quad W_1(t)=c_3\int_{-\infty}^t\frac{w_1(t)dt}{|t|^3}=c_2\int_{-\infty}^t\frac{3dt}{|t|^3}\int_0^{\sqrt{2}}\left[\frac{2}{t^2z^2+t^2-3}+\frac{9}{(t^2z^2+t^2-3)^2}\right.$$
$$\left.+\frac{12}{(t^2z^2+t^2-3)^3}\right]\frac{dz}{3z^2+2}$$

$$(11.18)\quad W_0(t)=c_3\int_{-\infty}^t\frac{w_0(t)dt}{|t|^3}$$
$$=c_3\int_{-\infty}^t\frac{dt}{t^4}\int_0^{\sqrt{2}}\left[3+\frac{t^4}{(z^2+t^2-3)^2}-\frac{4t^6}{(z^2+t^2-3)^3}\right]\frac{dz}{z^2+6},$$

where $c_3=c_1c_2/4=18\sqrt{2}$. But, e.g. in the domain $\mathscr{B}(-\infty<t<-2,\ 0<\tau<\sqrt{2})$ all these integrands are of magnitudes with definite sign $\gtrless 0$ respectively (the latter being factorized by $1-1/(z^2+t^2-3)$) and uniformly bounded (more in details $<69c_3/16$ and $<10c_3/16$, respectively. Therefore after Fubini the order of the repeated integrals may be interchanged, what is also true for the whole $W=W_1+W_0$. But, we are concerned with the d. f.

$$(11.19)\quad Q(t)=\int_{-\infty}^{t} q(t)dt=c_1\int_{-\infty}^{t}(-u+v+w)\frac{dt}{|-t|^3}=-U(t)+V(t)++W(t),$$

where

$$U(t)=c_1\int_{-\infty}^{t} u(g(t)\frac{dt}{|t|^3},\qquad V(t)=c_1\int_{-\infty}^{t} v(g(t))\frac{dt}{|t|^3}\quad\text{and}$$

$$(11.20)\quad W(t)=c_1\int_{-\infty}^{t} w\frac{dt}{|t|^3}$$

$$=c_3\int_{-\infty}^{t}\frac{dt}{t^4}\int_{1/\sqrt{3}}^{1}\left(\frac{6}{1-t\tau}-\frac{4}{(1-t\tau)^2}+\frac{1}{(1-t\tau)^3}\right)\frac{d\tau}{(9\tau^2-1)\sqrt{3\tau^2-1}}.$$

It is easy to integrate the first two: In fact,

$$U(t)=\frac{c_1}{6}\int_{-\infty}^{t} g^2\left(g^2-\frac{4}{3}g+\frac{1}{2}\right)\frac{dt}{|t|^3}$$

$$=-\frac{c_1}{24}\int_{-\infty}^{t}\left[\frac{1}{2(1-t)^2}-\frac{2}{3(1-t)^3}+\frac{1}{4(1-t)^4}\right]\frac{dt}{t^3},$$

$$V(t)=\frac{c_1}{12}\int_{-\infty}^{t}\left(g^3-2g^2+\frac{3}{2}g\right)\frac{dt}{t^4}=-\frac{c_1}{48}\int_{-\infty}\left[\frac{3}{1-t}-\frac{2}{(1-t)^2}+\frac{1}{2(1-t)^3}\right]\frac{dt}{t^4}.$$

Thereby making use of simple identities

$$\frac{1}{t(1-t)}=\frac{1}{1-t}+\frac{1}{t},\qquad \frac{1}{t^2(1-t)}=\frac{1}{1-t}+\frac{1}{t}+\frac{1}{t^2},$$

$$\frac{1}{t^3(1-t)^2}=\frac{1}{(1-t)^2}+\frac{3}{1-t}+\frac{3}{t}+\frac{2}{t^2}+\frac{1}{t^3},$$

$$\frac{1}{t^3(1-t)^3}=\frac{1}{(1-t)^3}+\frac{3}{(1-t)^2}+\frac{6}{1-t}+\frac{6}{t}+\frac{3}{t^2}+\frac{1}{t^3},\quad \&c.,$$

we obtain

$$(11.21)\qquad U(t)=c_0\left[\frac{1}{2t^2}-\frac{1}{2(1-t)^2}-\frac{1}{(1-t)^3}\right],\qquad (-\infty<t<-\sqrt{3})$$

$$(11.22)\qquad V(t)=3c_0\left[\frac{1}{2t^2}-\frac{1}{2(1-t)^2}+\frac{1}{t^3}\right]\qquad (\quad\text{''}\quad).$$

These together with (12) yields

$$(11.23)\quad P(t)-U(t)+V(t)=c_0\left[\frac{1}{t^2}-\frac{1}{(3-t)^2}+\frac{3}{t^3}\right]\qquad(-\infty<t<-\sqrt{3})$$

For example, if $t=-3$,

$$(11.24) \qquad P(-3)-U(-3)+V(-3)=-\frac{c_0}{36}=-0.05038<0,$$

which should be compensated by $W(-3)$. Returing to (20), interchange its order of integrations

$$W(t)=c_3\int_{1/\sqrt{3}}^{1}\frac{d\tau}{(9\tau^2-1)\sqrt{3\tau^2-1}}\int_{-\infty}^{t}\left[\frac{6}{1-t\tau}-\frac{4}{(1-t\tau)^2}+\frac{1}{(1-t\tau)^3}\right]\frac{dt}{t^4}.$$

Again availed here simple identities, similar to those before used,

$$\frac{1}{(1-t\tau)t^4}=\frac{\tau^4}{1-t\tau}+\frac{\tau^3}{t}+\frac{\tau^2}{t^2}+\frac{\tau}{t^3}+\frac{1}{t^4},$$

$$\frac{1}{(1-t\tau)^2t^4}=\frac{\tau^4}{(1-t\tau)^2}+\frac{4\tau^4}{1-t\tau}+\frac{4\tau^3}{t}+\frac{3\tau^2}{t^2}+\frac{2\tau}{t^3}+\frac{1}{t^4} \quad \&c.,$$

the above inner integral reduces to

$$\int_{-\infty}^{t}\left[\frac{\tau^4}{(1-t\tau)^3}+\frac{\tau}{t^3}+\frac{3}{t^4}\right]dt=\frac{\tau^3}{2(1-t\tau)^2}-\frac{\tau}{2t^2}-\frac{1}{t^3}.$$

Hence

$$W(t)=c_3\int_{1/\sqrt{3}}^{1}\left(\frac{\tau^3}{2(1-t\tau)^2}-\frac{\tau}{2t^2}-\frac{1}{t^3}\right)\frac{d\tau}{(9\tau^2-1)\sqrt{3\tau^2-1}}.$$

Or, writing the integrand as a sum of odd and even functions, we get

$$W(t)=W_1(t)+W_0(t),$$

where

$$W_1(t)=\frac{c_3}{2}\int_{1/\sqrt{3}}^{1}\left(\frac{\tau^2(1+t^2\tau^2)}{(1-t^2\tau^2)^2}-\frac{1}{t^2}\right)\frac{\tau d\tau}{(9\tau^2-1)\sqrt{3\tau^2-1}},$$

$$W_0(t)=c_3\int_{1/\sqrt{3}}^{1}\left(\frac{t\tau^4}{(1-t^2\tau^2)^2}-\frac{1}{t^3}\right)\frac{\tau d\tau}{(9\tau^2-1)\sqrt{3\tau^2-1}}.$$

Further, on transforming the integration variable τ into z by $\sqrt{3\tau^2-1}=z$ or $z\tau$ in W_1 or W_0, respectively, we obtain

$$(11.25) \qquad W_1(t)=\frac{3c_3}{2t^2}\int_0^{\sqrt{2}}\left(\frac{1}{t^2z^2+t^2-3}+\frac{2}{(t^2z^2+t^2-3)^2}\right)\frac{dz}{3z^2+2} \qquad (t<-\sqrt{3}),$$

$$(11.26) \qquad W_0(t)=c_3\int_0^{\sqrt{2}}\left[\frac{t}{(z^2+t^2-3)^2}-\frac{1}{t^3}\right]\frac{dz}{z^2+6} \qquad (t<-\sqrt{3}).$$

These being formally integrated on assumption that $t<-\sqrt{3}$, we find

$$(11.27) \qquad W_1(t)=\frac{27}{t^2-9}\left[\frac{\pi(t^2-3)}{\sqrt{3}\,t^2(t^2-9)}-\frac{2}{3(t^2-1)(t^2-3)}-\frac{t^2-5}{(t^2-3)(t^2-9)}X\right],$$

(11.28) $$W_0(t)=\frac{18t}{t^2-9}\left[\frac{\pi}{6\sqrt{3}\,(t^2-9)}-\frac{1}{(t^2-1)(t^2-3)}-\frac{3(t^2-5)}{2(t^2-3)(t^2-9)}Y\right],$$

where

(11.29) $$X=\sqrt{\frac{2t^2}{t^2-3}}\tan^{-1}\sqrt{\frac{2t^2}{t^2-3}} \quad \text{and} \quad Y=\sqrt{\frac{2}{t^2-3}}\tan^{-1}\sqrt{\frac{2}{t^2-3}}.$$

These combined with (23), we obtain the d. f. for $-\infty<t<-\sqrt{3}$

(11.30) $$S(t)=\frac{-\sqrt{3}\,\pi t}{(t^2-9)^2}+\frac{18}{(1-t)(t^2-3)(t^2-9)}-\frac{27(t^2-5)}{(t^2-3)(t^2-9)^2}(X+tY).$$

Whence evaluated $S(t)$ for several negative values of t:

(11.31)

t	$-\sqrt{3}-0$	-2	-3	-4	-5	-6	-7	-8	-9	$-\infty$
$S(t)$	.09505	.07732	.03984	.02324	.01474	.00993	.00701	.00514	.00324	0

in which $S(-\sqrt{3})$ and $S(-3)$ shall be specially noticed: Although (30) seems apparently to be singular at $t=-\sqrt{3}$, it is simply *hebbar*. It is true that X and Y tend ∞ as $t\to-\sqrt{3}-0$. But, on putting $z=\sqrt{2/(t^2-3)}>1$, we have

$$tY=tz\tan^{-1}z=tz\left(\frac{\pi}{2}-\frac{1}{z}+\frac{1}{3z^2}-\cdots\cdots\right)=\frac{\pi}{2}zt-t+\frac{t}{3z^2}- \qquad ,$$

$$X=-tz\tan^{-1}(-tz)=-\frac{\pi}{2}zt-1+\frac{1}{3z^2t^2}-\cdots\cdots.$$

Hence

$$X+tY=-1-t+\frac{1}{3z^2}\left(t+\frac{1}{t^2}\right)+O\left(\frac{1}{z^4}\right),$$

where $1/z^2=(t^2-3)/2=o(1)$ as $t\to-\sqrt{3}$. This substituted in (30) and making $t\to-\sqrt{3}$, we can readily certify that $S(-\sqrt{3}-0)=\pi/12-1/6=0.09502$. Similarly for $S(-3)$. But, this time to examine more in detail, let us put $t=-3+\zeta$ and expand S in a power series of ζ about $t=-3$: The terms of negative powers in W disappear at all, as

$$W_1=\left(\frac{5\pi}{48\sqrt{3}}+\frac{31}{384}\right)+\left(\frac{5\pi}{27\sqrt{3}}+\frac{19}{128}\right)\zeta+\left(\frac{2263\pi}{10368\sqrt{3}}+\frac{21043}{110592}\right)\zeta^2+\cdots\cdots,$$

$$W_0=\left(\frac{5\pi}{288\sqrt{3}}-\frac{27}{192}\right)-\left(\frac{\pi}{72\sqrt{3}}+\frac{45}{128}\right)\zeta-\left(\frac{731\pi}{13824\sqrt{3}}-\frac{795}{2048}\right)\zeta^2+\cdots\cdots,$$

when $|\zeta|<3-\sqrt{3}$. So that

(11. 32) $W = W_1 + W_0$

$$= \left(\frac{35\pi}{288\sqrt{3}} - \frac{25}{192}\right) + \left(\frac{37\pi}{216\sqrt{3}} - \frac{13}{64}\right)\zeta + \left(\frac{6859\pi}{41472\sqrt{3}} - \frac{21887}{110592}\right)\zeta^2 + \cdots\cdots.$$

Also (23) yields

(11. 33) $$P - U + V = -\frac{\pi}{36\sqrt{3}} - \frac{5\pi\zeta}{108\sqrt{3}} - \frac{17\pi\zeta^2}{432\sqrt{3}} - \cdots\cdots.$$

Therefore, the d.f., as the sum of (32) (33), is given by

(11. 34) $$S(t) = \left(\frac{3\pi}{32\sqrt{3}} - \frac{25}{192}\right) + \left(\frac{\pi}{8\sqrt{3}} - \frac{13}{64}\right)\zeta + \left(\frac{5131\pi}{41472\sqrt{3}} - \frac{21887}{110592}\right)\zeta^2 + \cdots\cdots$$

$$= 0.03984 + 0.02260\zeta + 0.08837\zeta^2 + \cdots\cdots\cdots$$

in the vicinity of $t = -3$. Hence, the significant point for level $\alpha = 0.1$, i.e. $\alpha/2 = 0.05$, lies in the neighbourhood of $t = -3$. Equating the above expression to 0.05, we get

(11. 35) $$f(\zeta) = -0.01016 + 0.02260\zeta + 0.08837\zeta^2 + \cdots = 0,$$

in which $+\zeta^3\cdots$ neglected, it yields $\zeta = 0.25$, so that $t_{0.1} = \pm 2.75$ roughly. More exactly, using the whole expression (30) and by to-and-fro linear interpolations, we find $S(-2.7) = 0.0478$, $S(-2.6) = 0.0510$, $S(-2.63) = 0.0500$. Hence we have approximately $t_{0.1} = \pm 2.63$. Similarly we obtain $S(-3.995) = 0.0285$ and $S(-3.996) = 0.0247$, so that $t_{0.05} = \pm 3.996$ nearly.

To proceed more systematically, we shall make use of Newton's method of successive approximations, for which however $S'(t) = s(t)$ is required for $t < -\sqrt{3}$ at least. It shall be found by differentiating (30), or else integrating (17) (18), that

(11. 36) $$S'(t) = \frac{3\sqrt{3}\,\pi(t^2+3)}{(t^2-9)^3} + \frac{18(8t^4 - 7t^3 - 51t^2 + 39t + 27)}{(1-t)^2(t^2-3)^2(t^2-9)^2}$$
$$+ \frac{27(4t^6 - 33t^4 + 54t^2 + 135)}{(t^2-3)^2(t^2-9)^3}\left[\frac{X}{t} + Y\right].$$

Thus, starting from the first approximation $t_0 = -2.63$, $f(t_0) = S(t_0) - 0.05 = 0.000034$, $S'(t_0) = 0.0424$, we obtain the second approximation $t_1 = t_0 - f(t_0)/S'(t_0) = -2.631$ and $S(t_1) = 0.05 = 0.0000014$. Hence $t_{0.1} \pm 2.631$.

To obtain $t_{0.01}$, which is certainly > 5.841 (the classical value for N.D.), we may avail Laurent's expansion of (30) for $|t| > 3$. We expand first $X = z\tan^{-1}z$ with $z = \sqrt{2t^2/(t^2-3)} > 1$ as $|t| > 3$:

$$X = z\left[\frac{\pi}{2} - \frac{1}{z} + \frac{1}{3z^3} - \frac{1}{5z^5} + \cdots\cdots\right] = \frac{\pi}{2}z - 1 + \frac{1}{3z^2} - \frac{1}{5z^4} + \cdots,$$

where

$$z=\sqrt{2}\left(1-\frac{3}{t^2}\right)^{-1/2}=\sqrt{2}\left(1+\frac{3}{2t^2}+\frac{27}{8t^4}+\frac{135}{16t^6}+\cdots\cdots\right),\quad \frac{1}{z^2}=\frac{1}{2}\left(1-\frac{3}{t^2}\right).$$

And accordingly

$$(11.37)\qquad X=\left(\frac{\pi}{\sqrt{2}}-A\right)+\frac{3}{t^2}\left(\frac{\pi}{2\sqrt{2}}-B\right. +\frac{9}{2t^4}\left(\frac{3\pi}{4\sqrt{2}}-C\right)+\frac{9}{2t^6}\left(\frac{15\pi}{8\sqrt{2}}-D\right)+\cdots,$$

where

$$(11.38)\qquad \frac{\pi}{\sqrt{2}}=2.22144,\quad A=\sum_{n=0}^{\infty}\frac{(-1)^n n}{(2n+1)2^n}=0.87042,$$

$$B=-\sum_{n=1}^{\infty}\frac{(-1)^n n}{(2n+1)2^n}=0.10188,$$

$$C=\sum_{n=2}^{\infty}\frac{(-1)^n n(n-1)}{(2n+1)2^n}=0.04170,$$

$$D=-\sum_{n=3}^{\infty}\frac{(-1)^n n(n-1)(n-2)}{(2n+1)2^n}=0.08083,\quad \&c.$$

These constants can be found by conceiving the real function

$$(11.39)\qquad A(\zeta)=\frac{1}{2i\sqrt{\zeta}}\log\frac{1+i\sqrt{\zeta}}{1-i\sqrt{\zeta}},$$

where $i=\sqrt{-1}$ and ζ donotes a real positive variable. Really (39) does not contain $i=\sqrt{-1}$ because of $\left|\frac{1+i\sqrt{\zeta}}{1-i\sqrt{\zeta}}\right|=1$. When $\zeta=1/2$, we have

$$A=A\left(\frac{1}{2}\right)=\frac{1}{\sqrt{2}}\cos^{-1}\frac{1}{3},\qquad B=-\zeta A'(\zeta)\Big|_{1/2}=\frac{1}{2}\frac{1}{\sqrt{2}}\cos^{-1}\frac{1}{3}-\frac{1}{3},$$

$$C=\zeta^2A''(\zeta)\Big|_{1/2},\qquad D=-\zeta^3A'''(\zeta)\Big|_{1/2},\quad \&\text{c.}$$

Consequenty

$$(11.40)\qquad X=1.35102+\frac{3.63780}{t^2}+\frac{7.39971}{t^4}+\frac{13.88524}{t^6}+\cdots\cdots.$$

Next, it holds for $|t|>3>\sqrt{5}$, $z_1=\sqrt{2/(t^2-3)}<1$ and

$$(11.41)\qquad Y=z_1\tan^{-1}z_1=z_1^2-\frac{1}{3}z_1^4+\frac{1}{5}z_1^6-\cdots=\frac{2}{t^2}+\frac{14}{3t^4}+\frac{58}{5t^6}+\cdots.$$

Lastly, these being substituted in (30) and further their coefficient as well as the first two terms of (30) expanded by binomial series in powers of t^{-1}, we get after all

$$(11.42)\quad S(t)=-\frac{5.4414}{t^3}-\frac{36.4775}{t^4}-\frac{169.9452}{t^5}-\frac{699.8612}{t^6}-\frac{2546.2597}{t^7}$$

$$-\frac{9337.3073}{t^8}-\frac{31390.3168}{t^9}-\frac{109698.1690}{t^{10}}+\cdots \qquad (t<-3).$$

Equating this to $\alpha/2=0.005$ and writing $-1/t=x$, solved for x by Horner, we get $t=-8.809$. Notwithstanding, if taken only up to the term of t^{-9} and solved, we obtain a better result $t=-8.086$: For, the whole expression (30) yeields $S(-8.809)=0.00408$, while $S(-8.086)=0.00502$. So that we have $t_{0.01}=\pm 8.086$ almost. Really the series (42) being alternate $(t<0)$, the sum of the first n terms, S_n, oscilates about the true sum S Although both $S_{2n}(<S)$ and $S_{2n-1}(>S)$ tend to S as $n\to\infty$, monotonic increasing and decreasing, respectively, yet it gives rise to the above paradox, that S_{2a} converges slower than S_{2n-1}.

The expansion (42) being divergent on and inside its boundary circle $|t|=3$, it cannot be applied at all for $|t_{0.1}|<3$. Also $|t_{0.05}|$ being somewhat near by 3, the convergency there becomes slow and the result found by equating (42) to 0.025 is unpleasing; it needs to take much more terms. To evaluate $t_{0.1}$ by series, we ought to expand (30) by a Taylor-Laurent series in $\sqrt{5}<|t|<3$, which however converges not so rapid, and hence unpractical.

12. *An Inductive Method.* We have hitherto argued the matter rather geometrically, so that the calculations for several values of the size n were disconnectedly made, and it is impossible to attain the general case thereby. The author imagines that $s_n(t)$ might be deduced from $s_{n-1}(t)$, if such-like as the convolution theory could be somehow applied. Really, given a n-sized sample $\{x_1, x_2, \cdots, x_n\}$ with statistics $\bar{x}$, s, t, if e.g. x_n be put aside and the remaining $\{x_1, \cdots\cdots x_{n-1}\}$ be considered as a $(n-1)$-sized sample with statistics $\bar{x}'$, s', t', we have the following relations:

$$(12.1)\quad n\bar{x}=(n-1)\bar{x}'+x_n,$$

$$(12.2)\quad ns^2=(n-1)s'^2+\frac{n-1}{n}(\bar{x}'-x_n)^2=(n-1)s'^2+\frac{n}{n-1}(\bar{x}-x_n)^2,$$

$$(12.3)\quad t=\frac{\bar{x}-m}{s/\sqrt{n-1}}=\frac{(n-1)\bar{x}'+x_n-nm}{\sqrt{ns'^2+(\bar{x}'-x_n)^2}}.$$

We can compute every single fr. f. for each term under expressions. Yet unfortunately they are not necessarily independent of others, so that the ordinary convolution formula is hardly applicable. We ought rather to deal with their compound fr. fs., but then the determination of their limits of integrations become complicated. Here only the simple cases $n=2$, 3 are introductorily discussed, hoping to accomplish their generalization in some future.

Case $n=2$.

Let the parent fr.f. be $f(x)=1$ in $(0,1)$ and conceive a two sized sample

$\{x_1, x_2\}$ with $\bar{x}=(x_1+x_2)/2$, $s^2=(x_1-x_2)^2/4$, where $0<\bar{x}<1$, $0<s<1/2$. The Jacobian being

$$J=\frac{\partial(\bar{x}, s)}{\partial(x_1, x_2)}=\pm\frac{1}{2},$$

we get the probability element $dp=f(x_1)f(x_2)dx_1dx_2$, so that it appears that $dp=dx_1dx_2/|J|=2d\bar{x}ds$ and $f(\bar{x}, s)=2$. However, with the n-sized sample statistics usually the arguments are taken as $\sqrt{n}\,\bar{x}$ and $\sqrt{n}\,s$, instead of sole $\bar{x}$ and s, so that their differentials are also as $d(\sqrt{n}\,\bar{x})$, $d(\sqrt{n}\,s)$. Thus the elementary probability for $(\sqrt{n}\,\bar{x}\cdots\sqrt{n}\,\bar{x}+d\sqrt{n}\,\bar{x}, \sqrt{n}\,s\cdots\sqrt{n}\,s+d\sqrt{n}\,s)$ is denoted by $dP=f(\sqrt{n}\,\bar{x}, \sqrt{n}\,s)d\sqrt{n}\,\bar{x}d\sqrt{n}\,s$, which may briefly be written as $f(\bar{x}, s)nd\bar{x}ds$. Hence, the foregoing $f(\bar{x}, s)$ for $n=2$ should be 4 instead of 2. Thus

$$f(\bar{x}, s)d\bar{x}ds=4d\bar{x}ds \qquad \text{for} \quad n=2. \tag{12.4}$$

Now that $s/\bar{x}=|x_1-x_2|/(x_1+x_2)\leq 1$, we get a fundamental relation

$$s\leq\bar{x}, \tag{12.5}$$

which we call the lowest inequality when $0<\bar{x}<1/2$. However for $1/2<x<1$, the lowest inequality reduces to

$$s\leq 1-\bar{x}(<\bar{x}). \tag{12.6}$$

For, on putting $1-x_1=x_1'$, $1-x_2=x_2'$, we obtain $0<\bar{x}'=1-\bar{x}<1/2$ when $1/2<\bar{x}<1$, but $s=s'$, so that $s/(1-\bar{x})=s'/\bar{x}'<1$. The total probability$=1$ is reassured by

$$\int_0^1\int_0^1 dx_1dx_2=4\int_0^{1/2}d\bar{x}\int_0^{\bar{x}}ds+4\int_{1/2}^1 d\bar{x}\int_0^{1-\bar{x}}ds=1.$$

Now the fr. f. of $t=(\bar{x}-1/2)/s$ is given as

$$f(t)dt=\int f\left(\bar{x}, s\left(=\left(\bar{x}-\frac{1}{2}\right)\Big/t\right)d\bar{x}\cdot\left(\bar{x}-\frac{1}{2}\right)dt\Big/t^2,\right.$$

that is

$$f(t)=\frac{4}{t^2}\int_{x_0}^{x_1}\left(\bar{x}-\frac{1}{2}\right)d\bar{x}=\frac{2}{t^2}\left(\bar{x}-\frac{1}{2}\right)^2\Big|_{x_0}^{x_1} \tag{12.7}$$

When $0<\bar{x}<1/2$, $t<0$, $0<s=(\bar{x}-1/2)/t<\bar{x}$, so that $x_0=1/2(1-t)$, $x_1=1/2$. Therefore

$$f(t)=\frac{1}{2(1-t)^2} \qquad \text{for} \quad t<0,$$

as shown in [II] and similarly

$$f(t)=\frac{1}{2(1+t)^2} \qquad \text{for} \quad t>0.$$

Case $n=3$. (The grouping I, II, $\cdots$ obtained in [I] is asserted below).

Now writing $n=3$ and $x_n=x$ in (1) (2), we conceive the transformation

$$(12.8)\qquad 3\bar{x}=2\bar{x}'+x,\qquad 3s^2=2s'^2+\frac{2}{3}(\bar{x}'-x)^2\left(=2s'^2+\frac{3}{2}(\bar{x}-x)^2\right).$$

The Jacobian being

$$J=\frac{\partial(\bar{x},s)}{\partial(\bar{x}',s')}=\frac{4s'}{9s},\qquad \frac{\partial(\bar{x}',s',x)}{\partial(\bar{x},s,x)}=\frac{9s}{4s'},$$

the probability element becomes

$$(12.9)\qquad dx_1dx_2dx_3=4d\bar{x}ds'dx=\frac{9s}{s'}d\bar{x}dsdx=f(\bar{x},s,x)d\bar{x}dsdx.$$

Hence, the fr. f. of Student's ratio $t=(\bar{x}-1/2)\sqrt{2}/s$ shall be given by

$$f(t)dt=\iint f\left(\bar{x},s\left(=\left(\bar{x}-\frac{1}{2}\right)\frac{\sqrt{2}}{t}\right),x\right)d\bar{x}ds\left(=\left|\bar{x}-\frac{1}{2}\right|\frac{\sqrt{2}\,dt}{t^2}\right)dx,$$

that is to say

$$(12.10)\qquad f(t)=\frac{9}{t^2}\iint s\left(=\left(\bar{x}-\frac{1}{2}\right)\frac{\sqrt{2}}{t}\right)\left|\bar{x}-\frac{1}{2}\right|\frac{\sqrt{2}\,d\bar{x}}{s'}dx$$

$$=\frac{9}{2|t|^3}\int_{\bar{x}}(2\bar{x}-1)^2d\bar{x}\int_x\frac{dx}{s'},$$

where the inner integral becomes after (8)

$$\int\frac{dx}{s'}=\frac{d(x-\bar{x})}{\sqrt{\frac{3}{4}\left[\left(\frac{2\bar{x}-1}{t}\right)^2-(x-\bar{x})^2\right]}}=\frac{2}{\sqrt{3}}\int d\sin^{-1}\frac{x-\bar{x}}{(2\bar{x}-1)/t}.$$

Subcase I: $0<\sqrt{2}\,s<\bar{x}$. This condition for $s=(\bar{x}-1/2)\sqrt{2}/t$, $t<0$ implies

$$0<\frac{2\bar{x}-1}{t}<\bar{x},\qquad \text{i.e.}\qquad \frac{1}{2-t}<\bar{x}<\frac{1}{2}.$$

Hence the outer integral should be taken from $1/(2-t)$ to $1/2$. As to the inner integral, we notice that the above inequalities yield $0<(2\bar{x}-1)/t<1/(2-t)<\bar{x}<1/2$, so that $0<\bar{x}\pm(2x-1)/t<1$. Therefore, we ought to take x's integration-interval:

$$\bar{x}-\frac{2\bar{x}-1}{t}<x<\bar{x}+\frac{2\bar{x}-1}{t},\qquad \text{i.e.}\qquad |x-\bar{x}|<\frac{2x-1}{t},$$

so that the inner integral reduces to $2\pi/\sqrt{3}$. Thus we obtain

$$(12.11)\quad f_I(t)=\frac{3\sqrt{3}\,\pi}{|t|^3}\int_{1/(2-t)}^{1/2}(2\bar{x}-1)^2d\bar{x}=\frac{\sqrt{3}\,\pi}{2(2-t)^3}\qquad \text{for}\quad t<0,$$

which agrees with (1.1) in the previous paper [II].

II: $\bar{x}<\sqrt{2}\,s<2\bar{x}$. For the sake of simplicity, we shall here consider the interval $-\infty<t<-1$ only, so that $0<\bar{x}<1/3$ only, because the Student distribution for any universe distributed symmetrically about its mean, is also symmetrical about the origin and besides $0>t>-1$ for $1/3<\bar{x}<1/2$ in II: $\bar{x}<(2\bar{x}-1)/t<2\bar{x}$, since ,we get whence $2-1/\bar{x}<t<1-1/2\bar{x}$, or $-1<t<-1/2$ when $1/3<x<1/2$. We obtain still the same integrand as in (10)

$$f_{II}(t)=\frac{3\sqrt{3}}{|t|^3}\int_{\bar{x}}(2\bar{x}-1)^2 d\bar{x}\int_{x} d\sin^{-1}\frac{(x-\bar{x})t}{2\bar{x}-1}. \tag{12.12}$$

Now condition II for $\sqrt{2}\,s=(2\bar{x}-1)/t$, $t<-1$ $(0<\bar{x}<1/3)$ implies that

$$(a)\quad \frac{1}{2(1-t)}<\bar{x}<\frac{1}{2-t} \qquad \text{and} \qquad (b)\quad \bar{x}<\frac{2\bar{x}-1}{t}<2\bar{x}<\frac{2}{3}.$$

Hence, in order that the inner integral

$$\int\frac{d(x-\bar{x})}{\sqrt{\left(\frac{2\bar{x}-1}{t}-\bar{x}+x\right)\left(\frac{2\bar{x}-1}{t}+\bar{x}-x\right)}}=\int d\sin^{-1}\frac{(x-\bar{x})t}{2\bar{x}-1}\qquad (t<-1)$$

may be real, as the first factor in radical >0 by (b), the second factor must be so also. Hence we get

$$(c)\quad 0<x<\frac{2\bar{x}-1}{t}+\bar{x}=r \text{ say,}$$

whence $r<3\bar{x}<1$ after (b). Moreover we should have the following inequalities in view of (8) and (5)

$$s'^2=\frac{3}{2}s^2-\frac{3}{4}(\bar{x}-x)^2<\bar{x}'^2=\frac{1}{4}(3\bar{x}-x)^2, \tag{12.13}$$

and consequently for $s^2=(2\bar{x}-1)^2/2t^2$,

$$\frac{3}{4t^2}(2\bar{x}-1)^2<\frac{1}{4}(3\bar{x}-x)^2+\frac{3}{4}(\bar{x}-x)^2,$$

i. e.

$$x^2-3\bar{x}x+3x^2-3(2\bar{x}-1)^2/4t^2>0.$$

This quadratic has a positive discriminant and also positive absolute term because of (b), while its linear cofficient is negative. So that it has two real positive roots

$$\alpha,\beta=\frac{3}{2}\bar{x}\pm\frac{\sqrt{3}}{2}\sqrt{\left(\frac{2\bar{x}-1}{t}\right)^2-\bar{x}^2}.$$

Hence we should have either

(i) $x>\alpha$ or (ii) $\beta>x$.

On the other hand, we have $\alpha<r$: For, on squaring the ambiguous ine-

qualities $\alpha \leqq \gamma$, where the squaring is permissible without changing the sense of double signs in view of (b) we reach an absolute inequality $0<[(2\bar{x}-1)/2t-\bar{x}]^2$, so that only the upper sign is to be chosen. Hence we obtain as the required interval (i) $\alpha<x<\gamma<1$, for which the inner integral in (12) becomes

$$\left[\sin^{-1}\frac{(x-\bar{x})t}{2\bar{x}-1}\right]_{x=\alpha}^{\gamma}=\sin^{-1}\frac{\sqrt{3}\,\bar{x}|t|-\sqrt{(1-2\bar{x})^2-\bar{x}^2t^2}}{2(1-2\bar{x})} \qquad (t<0).$$

This being substituted in (12) and integrated by parts, we get

$$(12.14) \qquad f_{II(i)}(t)=-\frac{\pi\sqrt{3}}{6(2-t)^3}+\frac{\sqrt{3}}{2t^2}\int_{1/2(1-t)}^{1/(2-t)}\frac{(1-2x)^2dx}{\sqrt{(1-2x)^2-t^2x^2}}.$$

Next, integrating along (ii) $0<x<\beta$, we obtain

$$(12.15) \qquad f_{II(ii)}(t)=-\frac{\pi\sqrt{3}}{3(2-t)^3}+\frac{\sqrt{3}}{t^2}\int_{1/2(1-t)}^{1/(2-t)}\frac{(1-2x)^2dx}{\sqrt{(1-2x)^2-x^2t^2}}.$$

On summing up (11) (14) (15) all together, we attain finally

$$(12.16) \qquad f(t)=\frac{3\sqrt{3}}{2t^2}\int_{1/2(1-t)}^{1/(2-t)}\frac{(1-2x)^2dx}{\sqrt{(1-2x)^2-x^2t^2}},$$

which coincides with (1.8) in paper [II]. As the further treatment was done in [II] already, here it shall be not repeated.

Although the above analytical procedure may seem rather tedious than the individually before made geometrical method, yet it would suggest the possibility of the method inducing from n to $n+1$. To the present author it reveals an inkling of hope to advance on the general treatment, which however shall be prepared in some future chance.

However, to examplify the above case $n=3$ methodologically, we ought still to show that the joint fr. f. $f(\bar{x}, s)$ can be deduced without employing geometrical intuition.

We have by means of (8) and (9) to compute

$$(12.17) \qquad f(\bar{x}, s)=\int f(\bar{x}, s, x)dx=\int\frac{9s}{s'}dx=6\sqrt{3}\,s\int\frac{dx}{\sqrt{2s^2-(x-\bar{x})^2}},$$

where the limits of integration x_0, x_1 must be determined. The grouping of subcases I, II, ... described in paper [I] would be availed, since it is generally obtained.

For $n=3$ we have (confining to the case $0<\bar{x}<1/2$ by reason of symmetry)

I: $0<\sqrt{2}\,s<\bar{x}, 0<\bar{x}<1/2$. The radical in (17), $\sqrt{(x-(\bar{x}-))\sqrt{2}s(\sqrt{2}s+\bar{x}-x)}$ becomes real so far $0<\bar{x}-\sqrt{2}\,s<x<\bar{x}+\sqrt{2}\,s<2\bar{x}<1$, which follow from condition I. Hence we get $x_0=\bar{x}-\sqrt{2}\,s$ and $x_1=\bar{x}+\sqrt{2}\,s$, and

$$(12.18) \qquad f(\bar{x}, s)=6\sqrt{3}\,\pi s\,.$$

II: $\bar{x}<\sqrt{2}\,s<2\bar{x},\ 0<\bar{x}<1/3$. Now that $\bar{x}-\sqrt{2}\,s<0$ but $0<\bar{x}+\sqrt{2}\,s<3\bar{x}<1$, we see that

$$(a)\qquad 0<x<\bar{x}+\sqrt{2}\,s\equiv r<1,$$

which is necessary but not sufficient. Indeed, the assumption that $x_0=0,\ x_1=r$ leads to an illusory conclusion $f(\bar{x},s)=6\sqrt{3}\,s(\pi-\cos^{-1}\bar{x}/\sqrt{2}\,s)$. We ought to contemplate another condition besides II. Now that $3\bar{x}=2\bar{x}'+x<1$, so $\bar{x}'<1/2$ and the lowest inequaliy $s'<\bar{x}'$ holds. This condition being applied to (8), it yields just the same relation as (13):

$$s'^2=\frac{3}{2}s^2-\frac{3}{4}(\bar{x}-x)^2<\bar{x}'^2=\frac{1}{4}(3\bar{x}-x)^2,\qquad \text{i.e.}$$

$$(12.\,19)\qquad x^2-3\bar{x}x+3\bar{x}^2-\frac{3}{2}s^2>0,$$

where the quadratic has a positive discriminant $3(2s^2-\bar{x}^2)>0$ and also the absolute term $3(2\bar{x}^2-s^2)/2>0$, because of II. Hence, the quadratic has 2 real positive roots $\alpha,\beta=[3\bar{x}\pm\sqrt{3(2s^2-\bar{x}^2)}]/2$. And accordingly we shall have

$$(b)\quad x>\alpha \qquad \text{or else} \qquad (c)\quad x<\beta.$$

Further to examine $\alpha\leqq r$, which means $\sqrt{3}\sqrt{2s^2-\bar{x}^2}\leqq 2\sqrt{2}\,s-\bar{x}(>\sqrt{2}\,s-\bar{x}>0)$. On squaring both positive sides, we attain $0\leqq(s-\sqrt{2}\,\bar{x})^2$, so that $\alpha<r$ follows. Therefore, we get

$$(d)\quad 1>r>\alpha>\beta>0,$$

and the required intervals are

$$\alpha<x<r \qquad \text{as well as} \qquad 0<x<\beta.$$

Consequently

$$(12.\,20)\qquad f(\bar{x},s)=6\sqrt{3}\,s\left[\sin^{-1}\frac{x-\bar{x}}{\sqrt{2}\,s}\Bigg|_{\alpha}^{r}+\sin^{-1}\frac{x-\bar{x}}{\sqrt{2}\,s}\Bigg|_{0}^{\beta}\right]$$
$$=6\sqrt{3}\,s\left(\pi-3\cos^{-1}\frac{\bar{x}}{\sqrt{2}\,s}\right).$$

II′ $\bar{x}<\sqrt{2}\,s<1-\bar{x},\ 1/3<\bar{x}<1/2$. Here hold $\bar{x}-\sqrt{2}\,s<0$ and $\sqrt{2}\,s+\bar{x}<1$, so that again (a) $0<x<\sqrt{2}\,s+\bar{x}=r<1$ hold as necessary. Also $2s^2>\bar{x}^2$ and $2\bar{x}^2>s^2$ result. For, from $2/3>1-\bar{x}>1/2$ and $\sqrt{2}\,s<1-\bar{x}<2/3$, it follows that $s^2<2/9$(maximum variance). On the other hand $2/9<2\bar{x}^2<1/2$ and whence $s^2<2\bar{x}^2$ follows. Besides (d) $1>r>\alpha>\beta>0$ hold again. Consequently just the same formula as (20) does hold.

However, to tell the truth, presently in consequence of $1<3\bar{x}=2\bar{x}'+x<3/2$ it follows that $1-x<2\bar{x}'<3/2-x$, so that $\bar{x}'<1/2$ if $x>1/2$ and the lowest inequality is certainly $s'<\bar{x}'$, where we could avail inequality (19). We shall call the set of these points $\{x\}$ to be of the category C_1. On the contrary, for some $x<1/2$, it might occur that $\bar{x}'=(3\bar{x}-x)/2>1/2$ and the lowest inequality now becomes $s'<1-x'$. Such points form a second category C_0. The intermediate point $x_0=3\bar{x}-1$, which lies in $(0, 1/2)$ in the present case, gives rise $\bar{x}'=1/2$ correspondingly. Any point x belongs to C_1 or C_0, according as $x\gtrless x_0$, e.g. the point $(x=1)\epsilon C_1$, but $(x=0)\epsilon C_0$. Now it can be shown that $\beta>3\bar{x}-1$, so that α, β both ϵC_1. Therefore we have first, as a contribution from the interval $x_0<x<1$:

$$(12.21)\quad 6\sqrt{3}\,s\left\{\sin^{-1}\frac{x-\bar{x}}{\sqrt{2}\,s}\Big|_{\alpha}^{\gamma}+\sin^{-1}\frac{x-\bar{x}}{\sqrt{2}\,s}\Big|_{x_0}^{\beta}\right\}=6\sqrt{3}\,s\left[\cos^{-1}\frac{x_0-\bar{x}}{\sqrt{2}\,s}-2\cos\frac{\bar{x}}{\sqrt{2}\,s}\right].$$

It remains to get the further contribution from $0<x<x_0$: Now, instead of (19) we obtain the inequality

$$(12.22)\qquad x^2-(3\bar{x}-1)x+(1-3\bar{x}+3\bar{x}^2)-\frac{3}{2}s^2>0$$

with two roots $\delta, \varepsilon=[3\bar{x}-\pm\sqrt{3(2s^2-(1-\bar{x})^2)}\,]/2$ that are imaginary. Hence the quadratic becomes positive definite, so that (22) gives no limitation about x. Therefore its contribution becomes

$$(12.23)\quad 6\sqrt{3}\,s\sin^{-1}\frac{x-\bar{x}}{\sqrt{2}\,s}\Big|_{0}^{x_0}=6\sqrt{3}\,s\left[\pi-\cos^{-1}\frac{x_0-\bar{x}}{\sqrt{2}\,s}-\cos^{-1}\frac{\bar{x}}{\sqrt{2}\,s}\right].$$

This together with (21) just amounts to (20).

II'': $(\bar{x}<)1-\bar{x}<\sqrt{2}\,s<2\sqrt{\bar{x}^2-\bar{x}+1/3}\,(<2\bar{x})$, $1/3<\bar{x}<1/2$. Now that $1<\bar{x}+\sqrt{2}\,s=\gamma$, the first necessary condition (a) holds by itself and of no use. Also II'' being a partial interval of II: $\bar{x}/\sqrt{2}<s<\sqrt{2}\,\bar{x}$, of course, inequalities $\bar{x}^2<2s^2$, $s^2<2\bar{x}^2$ and $(1-x)^2<2s^2$ all hold. Hence the roots of quadratics (19) as well as (22), α, β, δ, ε, become all real positive fractions. Moreover α, $\beta\,\epsilon\, C_1$, while γ, $\delta\,\epsilon\, C_0$ and we obtain finally

$$(12.24)\quad f(\bar{x},s)=6\sqrt{3\,s}\left\{\sin^{-1}\frac{x-\bar{x}}{\sqrt{2}\,s}\Big|_{\alpha}^{1}+\quad ,,\quad\Big|_{x_0}^{\beta}+\quad ,,\quad\Big|_{\delta}^{x_0}+\quad ,,\quad\Big|_{0}^{\varepsilon}\right\}$$

$$=6\sqrt{3}\,s\left[\pi-3\cos^{-1}\frac{\bar{x}}{\sqrt{2}\,s}-3\cos^{-1}\frac{1-\bar{x}}{\sqrt{2}\,s}\right].\qquad \text{Q.E.I.}$$

Making use of the joint fr. f. $f(\bar{x}, s)$ thus obtained for $n=3$, we may further find either single fr. f. $f(\bar{x})$ or $f(s)$. Although the former was already remarked at the end of section I in the previous paper [II], its general form can be far briefly given, again owing to Cramér[1]. Namely, the fr. f. of the sample mean $\bar{x}=(x_1+x_2+\cdots\cdots+x_n)/n$ is

1) H. Cramér, loc. cit., p. 245.

(12. 25) $$f_n(\bar{x})=\frac{n}{(n-1)!}\left[(n\bar{x})^{n-1}-\binom{n}{1}(n\bar{x}-1)^{n-1}+\binom{n}{2}(n\bar{x}-2)^{n-1}-\cdots\cdots\right]$$

where the summation is continued as long as the arguments $n\bar{x}$, $n\bar{x}-1$, $n\bar{x}-2$, $\cdots\cdots$ are positive. In particular

$$f_1(\bar{x})=1 \qquad \text{in } 0<\bar{x}<1,$$
$$=0 \qquad \text{outside } (0,\ 1)$$

(12. 26) $$f_2(\bar{x})=4\bar{x} \qquad \text{in } 0<\bar{x}<\frac{1}{2},$$
$$=4(1-\bar{x}) \qquad \text{in } \frac{1}{2}<\bar{x}<1.$$
$$f_3(\bar{x})=\frac{27}{2}\bar{x}^2 \qquad \text{in } 0<\bar{x}<\frac{1}{3},$$
$$=\frac{4}{9}-27\left(\frac{1}{2}-\bar{x}\right)^2 \qquad \text{in } \frac{1}{3}<\bar{x}<\frac{2}{3}$$
$$=\frac{27}{2}(1-\bar{x})^2 \qquad \text{in } \frac{2}{3}<\bar{x}<1, \text{ and so on.}$$

As to the fr. f. of the sample S. D. s, however, it is not so easily obtainable. Really for $n=2$, $s_2=|x_1-x_2|/2$ and x_1, x_2 being independent, it is readily seen that $f(s_2)=4(1-2s_2)$ in its whole interval $0<s_2<1/2$. But, already with $n=3$ the calculation goes enough intricate. This can however be sought in a similar manner to that used above to find $f(t)$, which runs as follows.

After the joint fr. f. $f(\bar{x}, s)$, (18), (20), (24), we recapitulate

I. $0<s<\bar{x}/\sqrt{2}$ $(0<\bar{x}<1/2)$: $dP=6\sqrt{3}\,\pi s ds d\bar{x}$,

II. $\bar{x}/\sqrt{2}<s<\sqrt{2}\,\bar{x}(0<\bar{x}<13)$ $dP=6\sqrt{3}\,(\pi-3\cos^{-1}\bar{x}/\sqrt{2}\,s)s ds d\bar{x}$,

II′ $\bar{x}/\sqrt{2}<s<(1-\bar{x})/\sqrt{2}$ $dP=$,, ,,

II″ $\frac{1-\bar{x}}{\sqrt{2}}<s<\sqrt{2\left(\bar{x}^{-2}-\bar{x}+\frac{1}{3}\right)}$ $\left(\frac{1}{3}<x<\frac{1}{2}\right)$:

$$dP=18\sqrt{3}\left(\frac{\pi}{3}-\cos^{-1}\frac{\bar{x}}{s\sqrt{2}}-\cos^{-1}\frac{1-\bar{x}}{s\sqrt{2}}\right)s ds dx\,.$$

The annexed xs-diagram (Fig. 11) shows distinctly the subdomain I, II, II′, II″, each enclosed by the boundary lines $s=0$, $s=\bar{x}/\sqrt{2}$, $s=(1-\bar{x})/\sqrt{2}$ and $s=\sqrt{2(\bar{x}^2-\bar{x}+1/3)}$. The pencil of rays $s=(\bar{x}-1/2)\sqrt{2}/t$ with slope $m=s/(\bar{x}-1)/2=\sqrt{2}/t$ exhaust all values of Student's ratio $t<0$, and especially those rays corresponding to $t=-\infty$, -1, $-1/2$, 0 interssect straight lines $\bar{x}=1/3$, $1/2$ ($\bar{x}$-boundary lines) and $s=1/\sqrt{2}$, $\sqrt{2}/3$, $1/\sqrt{6}$, $1/2\sqrt{2}$, $1/3\sqrt{2}$ (s-boundary lines) at the boundary corners. Especially the point

($\bar{x}=1/3$, $s=\sqrt{2}/3$, $t=-1/2$) yields the max $s=\sqrt{2}/3$. Of course, for $t<0$, we ought to take the interval $1/2<\bar{x}<1$ and to draw the whole figure symmetrically about the straight line $\bar{x}=1/2$.

The intervals over which the integration about $\bar{x}$ should be performed to get $f(s)$ would be seen from Fig. 11. Accordingly we have for

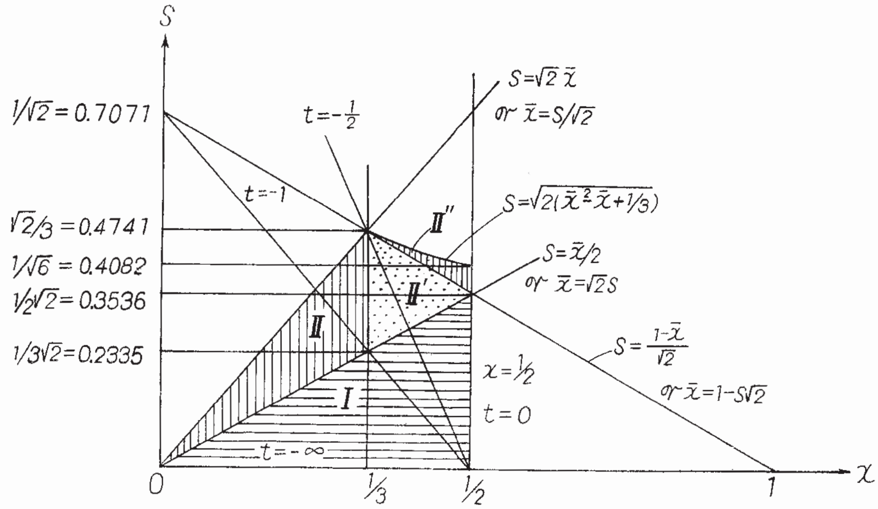

Fig. 11

(A) $0<s<1/2\sqrt{2}$

I: $6\sqrt{3}\,\pi s\int_{\sqrt{2}s}^{1/2} d\bar{x} = 6\sqrt{3}\,\pi\left(\frac{1}{2}-s\sqrt{2}\right)$

II & II′ $6\sqrt{3}\,\pi s\int_{s/\sqrt{2}}^{s\sqrt{2}}\left(\pi-3\cos\frac{\bar{x}}{s\sqrt{2}}\right)dx = 6\sqrt{3}\,s\left(\pi s\sqrt{2}-\frac{3\sqrt{3}\,s}{\sqrt{2}}\right).$

These being summed up, we get the branch of $f(s)$ corresponding to $0<x<1/2$:

(12·27) $$\frac{1}{2}f_A(s)=3s(\pi\sqrt{3}-9\sqrt{2}\,s),$$

which denotes a parabola with the vertex at ($\pi/6\sqrt{6}=0.2138$(mode), $\sqrt{2}\,\pi^2/4=3.4894$) whose double ordinate gives really $f(s)$ itself.

(B) For $1/2\sqrt{2}<s<1/\sqrt{6}$

II & II′: $6\sqrt{3}\,s\int_{s\sqrt{2}}^{1-s\sqrt{2}}(\pi-3\cos\bar{x}/s\sqrt{2})d\bar{x}$

$$=6\sqrt{3}\,s\Big[(1-\sqrt{2}\,s)(\pi-3\cos^{-1}(1/s\sqrt{2}-1)+3\sqrt{2\sqrt{2}\,s-1}-3\sqrt{\tfrac{3}{2}s}\Big)$$

$$\text{II}'':\quad 18\sqrt{3}\,s\int_{1-s\sqrt{2}}^{1/2}\left(\frac{\pi}{3}-\cos^{-1}\frac{\bar{x}}{s\sqrt{2}}-\cos^{-1}\frac{1-\bar{x}}{s\sqrt{2}}\right)d\bar{x}$$

$$=18\sqrt{3}\,s\left[\pi\left(s\sqrt{2}-\frac{1}{2}\right)+3(1-s\sqrt{2})\cos^{-1}\left(\frac{1}{s\sqrt{2}}-1\right)-3\sqrt{2\sqrt{2}\,s-1}\right],$$

so that their sum yields still the same parabola

$$(12\cdot 28)\qquad \frac{1}{2}f_B(s)=3s(\pi\sqrt{3}-9\sqrt{2}\,s).$$

(C) For $1/\sqrt{6}<s<\sqrt{2}/3$

$$\text{II \& II}':\quad 18\sqrt{3}\,s\int_{s/\sqrt{2}}^{1-s\sqrt{2}}\left(\frac{\pi}{3}-\cos^{-1}\frac{x}{s\sqrt{2}}\right)dx$$

$$=18\sqrt{3}\,s\left[\left(\frac{\pi}{3}-\cos^{-1}\left(\frac{1}{s\sqrt{2}}-1\right)\right)(1-s\sqrt{2})+\sqrt{2\sqrt{2}\,s-1}-\sqrt{\frac{3}{2}}\,s\right]$$

$$\text{II}'':\quad 18\sqrt{3}\,s\int_{1-s\sqrt{2}}^{1/2(1-\sqrt{2s^2-1/3})}\left(\frac{\pi}{3}-\cos^{-1}\frac{\bar{x}}{s\sqrt{2}}-\cos\frac{1-\bar{x}}{s\sqrt{2}}\right)d\bar{x}$$

$$=18\sqrt{3}\,\pi s\left[\frac{\pi}{6}-\left(\frac{\pi}{3}-\cos^{-1}\left(\frac{1}{s\sqrt{2}}-1\right)(1-s\sqrt{2})\right.\right.$$

$$\left.-\sqrt{2\sqrt{2}\,s-1}+\sqrt{6s^2-1}-\frac{1}{2}\sin^{-1}\frac{\sqrt{6s^2-1}}{3s^2}\right],$$

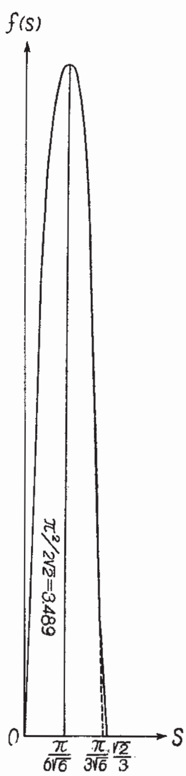

Fig. 12

$$(12\cdot 29)\qquad \frac{1}{2}f_C(s)=3s(\pi\sqrt{3}-9\sqrt{2}\,s)$$

$$+9\sqrt{3}\,s\left[2\sqrt{6s^2-1}-\sin^{-1}\frac{\sqrt{6s^2-1}}{3s^2}\right].$$

where $2\sqrt{6s^2-1}>\sin^{-1}\sqrt{6s^2-1}/3s^2$ and $1/2f_C'(s)=9\sqrt{3}\{\pi/3-2\sqrt{6}\,s+(4\sqrt{6s^2-1}-\sin^{-1}\sqrt{6s^2-1}/3s^2)\}$. Thus, the two branches $f_B(s)$ and $f_C(s)$ together with their first derivatives being coincident at the point of junction $s=1/\sqrt{6}=0.4082$, $f(s)$ and $f'(s)$ are both continuous there. But, $f_C(s)$ in the interval C deviates from the parabola and it ends at $s=\sqrt{2}/3=0.4716$, while the prolonged parabola cuts the s-axis at $s=\pi/3\sqrt{6}=0.4276$ (Fig. 12). Consequently we have

$$(12.30)\qquad f(s_3)=6s(\pi\sqrt{3}-9\sqrt{2}\,s)\quad\text{(parabola)}$$

$$\text{for}\quad 0<s<1/\sqrt{6},$$

$$=6s(\pi\sqrt{3}-9\sqrt{2}\,s)$$

$$+18\sqrt{3}\,s\left(2\sqrt{6s^2-1}-\sin^{-1}\frac{\sqrt{6s^2-1}}{3s^2}\right)$$

$$\text{for}\quad 1/\sqrt{6}<s<\sqrt{2}/3.$$