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ON SEMIGROUP WHOSE SUBSEMIGROUP SEMILATTICE IS THE BOOLEAN ALGEBRA OF ALL SUBSETS OF A SET

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Let S be a zero-semigroup by which we mean a semigroup defined as $xy=e$ for all $x, y \in S$. S satisfies the following conditions:

(1) Any subsemigroup contains a definite element e.

(2) Any subset of S which contains e is a subsemigroup of S.

(2) is equivalent to

(2′) $xy=x$ or y or e for every $x, y \in S$.

However these conditions do not characterize zero-semigroups, for the following counter example is given:

	e	a	b
e	e	a	e
a	a	e	a
b	e	a	e

In this note we shall determine all the types of semigroups which satisfy (1) and (2) simultaneously. Such semigroups are called β-semigroups.

Lemma 1. *A subsemigroup T of a β-semigroup S is a β-semigroup.*

Proof. Since a subsemigroup U of T is a subsemigroup of S, U contains e because of (1). If a subset V of T contains e, then V is a subsemigroup of S because of (2), and hence a subsemigroup of T.

Lemma 2. *A homomorphic image of a β-semigroup is a β-semigroup.*

Proof. Let S' be a homomorphic image of a β-semigroup S under a mapping f: $f(S)=S'$, and let U' be a subsemigroup of S' and U be the inverse image of U' under f. Since U is a subsemigroup of S, it contains e and hence U' contains $e'=f(e)$. Next, letting M' be a subset of S' which contains e', since the inverse image M of M' contains e, M is a subsemigroup of S and so $M'=f(M)$ is also a subsemigroup of S'.

Lemma 3. *$x^2=e$ for every $x \in S$, and hence S is a unipotent inversible semigroup* [1].

Proof. By the definition, $x^2=x$ or e. Suppose that there is an $x \neq e$ such that $x^2=x$. Then we have a subsemigroup $\{x\}$ of x alone outside which e

lies. This contradicts (1). Hence $x^2=e$ for every $x\epsilon S$.

Accordingly the element e is a unique idempotent. By [1] a unipotent inversible semigroup S contains a group G as a least ideal (i. e. kernel).

Lemma 4. *A subsemigroup H of G is a group.*

Proof. Clearly e is an identity of H and the existence of inverse is assured by Lemma 3.

Lemma 5. *G consists of at most 2 elements.*

Proof. Suppose that G consists of 3 elements or more, and take e, x, $y\epsilon G$ such that $x\neq e$, $y\neq e$, $x\neq y$. Let $X=\{e, x\}$ and $Y=\{e, x, y\}$. X and Y ars both subsemigroups of S or subsemigroups of G, and hence, by Lemmr 4, subgroups of G; in particular X is a subgroup of order 2 of a group Y of order 3. This contradicts the familiar theorem of groups. This lemma has been proved.

In consequence of Lemma 2, the difference semigroup of S modulo G, in Rees' sense [2], is a β-semigroup and a z-semigroup at the same time. We call it a β-z-semigroup. By a z-semigroup we mean a unipotent semigroup whose unique idempotent is a two-sided zero.

Lemma 6. *S is a β-z-semigroup, if and only if S is a zero-semigroup defined by $xy=e$ for all x, $y\epsilon S$.*

Proof. Suppose there is an $a\neq e$ such that $az=a$ for some $z\epsilon S$. Then the subset $Z=\{z;\ az=a\}$ is a subsemigroup which does not contain a. This contradicts the condition (1) of β-semigroups. Therefore we have proved that $xy\neq x$ for every non-zero x, $y\epsilon S$. Similarly we can prove $xy\neq y$ for every non-zero x, $y\epsilon S$. Consequently we have $xy=e$ for every x, $y\epsilon S$.

Next we shall determine a β-semigroup which is not a zero-semigroup. By Lemma 5, we may assume that G is of order 2, that is, $G=\{e, a\}$, $e^2=a^2=e$, $ae=ea=a$. By [1], $G=eS=Se$ and the difference semigroup of S modulo G is a zero-semigroup by Lemma 6. From this fact we see easily that

$$xe=ex=e \quad \text{for} \quad x\neq a;$$

accordingly we get $xa=ax=a$ for $x\neq a$.

Thus we see that a β-semigroup S is given as follows:

for every $x\neq a$ $\quad ax=xa=a$

otherwise $\quad xy=e$

By the theory of [1], it is assured that such a system S is a semigroup, and clearly S is a β-seigroup.

Theorem 1. *A semigroup S is a β-semigroup if and only if S is either* (3) *a zero-semigroup* $xy=e$ *for all* x, $y\epsilon S$

or (4) *a semigroup which contains $a \neq e$ and which is defined by*

$$ax = xa = a, \qquad \text{if} \quad x \neq a$$

$$xy = a^2 = e, \qquad \text{if} \quad x \neq a, \quad y \neq a.$$

It is interesting that each of (3) and (4) is uniquely determined within isomorphism by the cardinal number.

Now we shall replace e in (1) and (2) by subset E:

(5) any subsemigroup of S contains a subset E of S.

(6) any subset of S which contains E is a subsemigroup of S.

Let us consider a semigroup S satisfying the above (5) and (6). As is easily seen, E is a subsemigroup of S, and by the definition E contains no proper subsemigroup, hence no idempotent. It is impossible that E is of order $\geqq 2$, because a semigroup of infinite or finite order $\geqq 2$ contains either an idempotent or a proper subsemigroup. Hence E must be of order 1, and thus we have

Lemma 7. *The conditions (5) and (6) are equivalent to (1) and (2).*

Let $L(S)$ be the subsemigroup semilattice of a semigroup S, that is, the system of all the non-void subsemigroups of S. This forms a semilattice with respect to inclusion relation. If S is a β-semigroup, then $L(S)$ is the Boolean algebra of all the subsets including e. This property will characterize β-semigroups.

Theorem 2. *$L(S)$ is a Boolean algebra of all subsets of S containing a non-empty subset of S if and only if S is a β-semigroup.*

Proof. Let E be the least subsemigroup of S. We can see easily that S is a semigroup satisfying (5) and (6). Therefore, by Lemma 7, we conclude that S is a β-semigroup. The converse has been shown already.

According to Theorem 6, p. 159 in [3], every Boolean algebra of finite length n is isomorphic onto the system of all subsets of a set of n elements, we have easily

Theorem 3. *$L(S)$ is a Boolean algebra of finite length n as is given in Theorem 2 if and only if S is a β-semigroup of order $n+1$.*

References

[1] T. Tamura: Note on unipotent inversible semigroups, Kōdai Math. Semi. Rep., No. 3 (1954) 93—95.

[2] D. Rees: On semigroups, Proc. Cambridge Philos. Soc. 365 (1940) 387—400.

[3] G. Birkhoff: Lattice theory, Amer. Math. Soc. Coll. Publ. 25, Revised Edition (1948).

Errata

In the paper "Semigroups of order $\leqq 10$ whose greatest c-homomorphic images are groups" this Journal Vol. X (1959) p. 51, we add the following list to Table 1

Order	No.	defining matrix	Remark	c-decomposability	self-dual or not
8	8.11	4.4, 2—1		c-dec	
	8.12	4.5, 2—1		c-dec	