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ON GROUPS OF SPECIAL MOTIONS IN A SUBSPACE OF A RIEMANNIAN SPACE

By

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§ 1. In the first place let us consider the tangential infinitesimal point transformation

$$\bar{x}^{\lambda'} = x^{\lambda} + B_i^{\cdot\lambda} v^i dt \tag{1, 1}$$

in L_n with coordinates $x^i (i = 1, 2, \cdots, n)$ which is an n-dimensional subspace of an affinly connected m-dimensional space L_m with torsion and with coordinates $x^{\alpha} (\alpha = 1, 2, \cdots, m)$, where we denote by $B_i^{\cdot\lambda}$ the tangent vectors $\partial x^{\lambda}/\partial x^i$.

By Prof. K. Yano [5][1], rewriting (1, 1) as

$$\bar{x}^{\lambda} = x^{\lambda}(x^i) + \frac{\partial x^{\lambda}}{\partial x^i} v^i dt = x^{\lambda}(x^i + v^i dt),$$

we may consider this transformation as a change of parameters

$$x^i \to \bar{x}^{i'} = x^i + v^i dt. \tag{1, 2}$$

Although in general the Lie differential is defined by

$$\pounds\, T^{\lambda\mu}_{\cdot\cdot\nu}\, dt = T^{\lambda\mu}_{\cdot\cdot\nu}(\bar{x}) - T^{\lambda'\mu'}_{\cdot\cdot\nu'}(x), \tag{1, 3}$$

in our case we shall define the Lie differential of composite tensors $T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot}(x^i)$ as

$$\pounds\, T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot}\, dt = T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot}(\bar{x}) - \widetilde{T}^{\cdot\lambda\cdots i'\cdots}_{\cdots\mu\cdots j'\cdot}(x), \tag{1, 4}$$

where $\widetilde{T}^{\cdot\lambda\cdots i'\cdots}_{\cdots\mu\cdots j'\cdot}(x)$ is parallelly displaced quantities of $T^{\cdot\lambda\cdots i'\cdots}_{\cdots\mu\cdots j'\cdot}(x)$ by (1, 1) with respect to the affine connection of L_m.

Calculating (1, 4), we have

$$\begin{aligned}
&\pounds\, T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot}\, dt \\
&\quad = T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot}(x^i) + \partial_k T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot}(x^i) v^k dt - \widetilde{T}^{\cdot\lambda\cdots a\cdots}_{\cdots\mu\cdots b\cdot}(x^i) \cdots A_a^{i'} \cdots A_{j'}^b \cdots{}^{2)} \\
&\quad = T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot}(x^i) + \partial_k T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot}(x^i) v^k dt \\
&\qquad - [T^{\cdot\lambda\cdots a\cdots}_{\cdots\mu\cdots b\cdot}(x^i) \cdots - \Gamma^{\lambda}_{\rho\nu} T^{\cdot\nu\cdots a\cdots}_{\cdots\mu\cdots b\cdot}(x^i) B_k^{\cdot\rho} v^k dt - \cdots \\
&\qquad\qquad \cdots + \Gamma^{\nu}_{\rho\mu} T^{\cdot\lambda\cdots a\cdots}_{\cdots\nu\cdots b\cdot}(x^i) B_k^{\cdot\rho} v^k dt + \cdots] \\
&\qquad\qquad \cdots \times [\delta^i_a + \partial_a v^i dt] \times \cdots \times [\delta^b_j - \partial_j v^b dt] \times \cdots
\end{aligned}$$

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1) Nunbers in brackets refer to the references at the end of this paper.

2) $A_a^{i'}$ and $A_{j'}^b$ stand for $\frac{\partial x^{i'}}{\partial x^a}$ and $\frac{\partial x^b}{\partial x^{j'}}$ respectively.

$$= [v^k T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot\,;k} - \cdots - T^{\cdot\lambda\cdots a\cdots}_{\cdots\mu\cdots j\cdot} v^{\cdot i}_{a} - \cdots \quad \cdots + T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots a\cdot} v^{\cdot a}_{j} + \cdots]\, dt,$$

where we put

$$\left.\begin{aligned} v^{\cdot i}_{a} &= \partial_a v^i + \Gamma^i_{ja} v^j = v^i_{;a} + 2\, S^{\cdot\cdot i}_{ba} v^b, \\ S^{\cdot\cdot i}_{ba} &= \frac{1}{2}\,(\Gamma^{\cdot\cdot i}_{ba} - \Gamma^{\cdot\cdot i}_{ab}). \end{aligned}\right\} \quad (1, 5)$$

Hence we obtain

$$\pounds\, T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot} = v^k\, T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot\,;k} - \cdots - T^{\cdot\lambda\cdots a\cdots}_{\cdots\mu\cdots j\cdot} v^{\cdot i}_{a} - \cdots$$
$$\cdots + T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots a\cdot} v^{\cdot a}_{j} + \cdots. \qquad (1, 6)$$

Similarly we obtain

$$\begin{aligned} \pounds\, \Gamma^i_{jk} &= \partial_k \partial_j v^i + v^h \partial_h \Gamma^i_{jk} + \Gamma^i_{hk} \partial_j v^h + \Gamma^i_{jh} \partial_k v^h - \Gamma^h_{jk} \partial_h v^i \\ &= v^{\cdot i}_{j\;;k} + R^i_{\cdot jkh}\, v^h. \end{aligned} \qquad (1, 7)$$

and

$$\pounds\,(\delta\, u^k) - \delta\,(\pounds\, u^k) = (\pounds\, \Gamma^k_{ij})\, u^j\, dx^i. \qquad (1, 8)$$

§ **2.** In this section we consider n-dimensional subspace V_n immersed in an m-dimensional Riemannian space V_m, then we have

$$S^{\cdot\cdot i}_{jk} = 0, \quad \Gamma^\lambda_{\mu\nu} = \{{}^{\lambda}_{\mu\nu}\}, \quad \Gamma^i_{jk} = \{{}^{i}_{jk}\},$$

and

$$\pounds\, T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot} = v^k\, T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot\,;k} - \cdots - T^{\cdot\lambda\cdots a\cdots}_{\cdots\mu\cdots j\cdot} v^i_{;a} - \cdots$$
$$\cdots + T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots a\cdot} v^a_{;j} + \cdots, \qquad (2, 1)$$

from which

$$\pounds\, g_{ij} = v_{j;i} + v_{i;j}, \qquad (2, 2)$$

$$\pounds\, B^{\cdot\lambda}_{i} = v^k H^{\cdot\cdot\lambda}_{ik} + v^a_{;i}\, B^{\cdot\lambda}_{a}, \qquad (2, 3)$$

and

$$\pounds\, H^{\cdot\cdot\lambda}_{ij} = H^{\cdot\cdot\lambda}_{ij\;;k} v^k + H^{\cdot\cdot\lambda}_{aj}\, v^a_{;i} + H^{\cdot\cdot\lambda}_{ia}\, v^a_{;j}\,, \qquad (2, 4)$$

hence, we see these results coincide with Yano's [5] and Hlavatý's [4], where we denote by $H^{\cdot\cdot\lambda}_{ij}$ Euler-Schouten's tensor, that is,

$$H^{\cdot\cdot\lambda}_{ij} = B^{\cdot\lambda}_{i\;;j}\,.$$

Now we attach to any point x^i of V_n in V_m a frame $(B^{\cdot\lambda}_i, B^{\cdot\lambda}_P)$ $(P = n+1, n+2, \cdots, m)$ in such a way that $B^{\cdot\lambda}_i$ are n tangent vectors of $V_n$and $m-n$ vectors $B^{\cdot\lambda}_P$ are orthogonal to V_n and mutually orthogonal, then we have

$$H^{\cdot\cdot\lambda}_{ij} = \sum_P H_{ijP}\, B^{\cdot\lambda}_P,$$

$$\pounds\, B^{\cdot\lambda}_P = v^k B^{\cdot\lambda}_{P;\,k}\,.$$

By Weingarten's formula

$$B_{P}^{\cdot\lambda}{}_{;k} = -g^{ij} H_{jkP} B_{i}^{\cdot\lambda} + \sum_{Q} L_{PQ|k} B_{Q}^{\cdot\lambda},$$

where we put

$$L_{PQ|k} = g_{\alpha\beta} B_{P}^{\cdot\alpha} B_{Q,k}^{\cdot\beta} + [\gamma\delta, \beta] B_{k}^{\cdot\gamma} B_{Q}^{\cdot\delta} B_{P}^{\cdot\beta},$$

we obtain

$$\pounds\, B_{P}^{\cdot\lambda} = v^k [-g^{ij} H_{jkP} B_{i}^{\cdot\lambda} + \sum_{Q} L_{PQ|k} B_{Q}^{\cdot\lambda}], \tag{2, 5}$$

$$\pounds\, H_{ijP} = H_{ijP;k} v^k + H_{kjP} v^k{}_{;i} + H_{ikP} v^k{}_{;j}\,. \tag{2, 6}$$

As in general, we call a point transformation of V_n in V_m *a motion* if it satisfies

$$\pounds\, g_{ij} = 0. \tag{2, 7}$$

§ **3.** In this section we consider the case $m = n+1$, then we have

$$L_{PQ|k} = 0,\ \ B_{P}^{\cdot\alpha} = B_{n+1}^{\cdot\ \ \alpha},$$

from which we can put

$$B_{n+1}^{\cdot\ \ \alpha} = \xi^{\alpha},\ \ H_{ij\,n+1} = H_{ij}.$$

Now we call a motion *an absolute motion* if it satisfies

$$\pounds\, H_{ij} = 0. \tag{3, 1}$$

Then the condition that a point transformation (1, 1) is an absolute motion is given as

$$\left.\begin{aligned} v_{i;j} + v_{j;i} &= 0, \\ v^k H_{ij;k} + H_{kj} v^k{}_{;i} + H_{ik} v^k{}_{;j} &= 0. \end{aligned}\right\} \tag{3, 2}$$

By multiplying g^{ij} to the second equation, summing up with i and j and using the first equation, we get

$$v^k \partial_k H = 0, \tag{3, 3}$$

where we put

$$H = \frac{1}{n} g^{ij} H_{ij}.$$

Thus we have

Theorem 3. 1. *If an infinitesimal tangential point transformation in a V_n immersed in a V_{n+1} is an absolute motion, then the vector $\rho_i = \partial_i H$ must be orthogonal to the Killing vector v_i.*

As a motion in V_n in V_{n+1} is, however, a motion in a Riemannian space V_n, we consider a group of motions of order r which is generated by those r Killing vectors $\underset{a}{v}^k (a=1, 2, \cdots, r)$, if V_n admits such a group of motions.

Now let us consider a group of motions which satisfy (3, 1) and call it *a* *group of absolute motions*. Therefore if V_n admits a group of absolute motions and the rank of $(\underset{a}{v^i})$ is n, then we get

$$H = \text{const.}.$$

Hence we have

Theorem 3. 2. *If a V_n in a V_{n+1} admits a group of absolute motions of order r $(r \geqq n)$ and the rank of the matrix $(\underset{a}{v^i})$ is n, then the V_n must be a hypersurface with constant mean curvature in V_{n+1}.*

§ **4.** Here we consider the order r of a group G_r of absolute motions of V_n in V_{n+1}.

Since G_r must be a group of motions in V_n, we get

$$r \leqq \frac{n(n+1)}{2}.$$

If the equality holds, V_n is a space of constant curvature. In this case it is clear that $G_{n(n+1)/2}$ is a group of absolute motions if V_n is either a completely totally umbilical or a totally geodesic hypersurface in V_{n+1}. Hence we have

Theorem 4. 1. *In the case where a V_n is a space of constant curvature and either a completely totally umbilical or a totally geodesic hypersurface in a V_{n+1}, then V_n admits a group of absolute motions of the maximum order $n(n+1)/2$.*

Now applying the analogus method of proof of Egorov's theorem [2] [7], we have

Theorem 4. 2. *The maximum order of the group of absolute motions in those V_n's which are immersed in V_{n+1} and are neither completely totally umbilical nor totally geodesic is less than or equals to $\frac{1}{2}n(n-1)+1$.*

And again we have

Theorem 4. 3. *If the V_n in V_{n+1} admits a group of absolute motions of order greater than $\frac{1}{2}n(n-1)+1$, then the V_n is either a completely totally umbilical or a totally geodesic hypersurface in V_{n+1}.*

In fact, since absolute motions are motions, if the operator $\underset{v}{\pounds} f$ is that of an absolute motion, we have

$$\underset{v}{\pounds}\, g_{ij} = v_{i;j} + v_{j;i} = 0, \tag{4, 1}$$

$$\underset{v}{\pounds}\, H_{ij} = v^k H_{ij;k} + H_{kj} v^k{}_{;i} + H_{ik} v^k{}_{;j} = 0. \tag{4, 2}$$

Using (4, 1) we can write (4, 2) also in the form[1)]

1) Here we have used a new kind of indices P_α, P_β also running 1 to n which we assume that $P_\alpha \neq P_\beta$ for $\alpha \neq \beta$ and that the summation convention does not hold.

$$\underset{v}{\pounds} H_{ij} = v^k H_{ij\,;\,k} + \sum_{P_\alpha > P_\beta}^{1\cdots n} T_{ij}^{\cdot\cdot P_\alpha P_\beta} v_{P_\alpha\,;\,P_\beta} = 0, \tag{4, 3}$$

where

$$T_{ij}^{\cdot\cdot P_\alpha P_\beta} = 4\, \delta_{(i}^{[P_\alpha} H_{j)}^{\cdot P_\beta]}, \quad H_j^{\cdot k} = g^{ki} H_{ji}. \tag{4, 4}$$

Now we proceed the calulation by such a method as Prof. Egorov's. The $(P_\alpha\, P_\beta)$ rank, or $(i\ j)$ rank, of the matrix $(T_{ij}^{\cdot\cdot P_\alpha P_\beta})$[2] $(i \leqq j,\ P_\alpha < P_\beta)$ must be less than

$$\frac{1}{2} n\,(n+1) - (\frac{1}{2}\, n\,(n-1) + 1) = n-1,$$

and we have

$$\begin{cases} H_{P_\alpha}^{\cdot P_\beta} = 0, \\ H_1^{\cdot 1} = H_2^{\cdot 2} = \cdots = H_n^{\cdot n} \equiv \rho, \end{cases}$$

that is,

$$H_i^{\cdot j} = \delta_i^j\, \rho,$$

from which

$$H_{ij} = \rho\, g_{ij}.$$

Putting this into (4, 2) we have

$$v^k \rho_k g_{ij} + \rho\, g_{kj}\, v^k_{\ ;\,i} + \rho\, g_{ik}\, v^k_{\ ;\,j} = 0,$$

from which by using (4, 1) we have

$$v^k \rho_k = 0.$$

Since this equation must hold for all vectors $\underset{a}{v^k}$

$$(a = 1, 2, \cdots, r;\ r < \frac{1}{2}\, n\,(n-1) + 1)$$

which are generating vectors of G_r,

$$\underset{a}{v^k}\, \rho_k = 0.$$

The rank of $(\underset{a}{v^k})$ being n,

$$\rho = \text{const.}.$$

Hence the V_n in V_{n+1} must be a completely totally umbilical or a totally geodesic in V_{n+1}.

Moreover from these theorems we have

Theorem 4. 4. *The necessary and sufficient condition that a V_n in V_{n+1} admits a group of absolute motions of order $\frac{1}{2}\, n\,(n+1)$ is that the V_n is a space of*

---

2) We have considered here the matrix $T_{ij}^{\cdot\cdot P_\alpha P_\beta}$ by letting the two lower indices denote the rows and the two upper indices the columns, and we have called the rank of this matrix $(P_\alpha P_\beta)$ rank or $(i\ j)$ rank.

constant curvature and either a completely totally umbilical or a totally geodesic hypersurface in V_{n+1}.

§ **5.** Again in this section we consider the order of a group of absolute motions by a similar method of Egorov's [3].

The conditions

$$\underset{v}{\pounds} H_{ij} = 0 \tag{5, 1}$$

can be written as

$$\underset{v}{\pounds} H_{ij} = v^a H_{ij;a} + T_{ija}^{\cdot\cdot\cdot b} v^a{}_{;b}\,,$$

where we put

$$T_{ija}^{\cdot\cdot\cdot b} = \delta^b_j H_{ia} + \delta^b_i H_{aj}\,. \tag{5, 2}$$

From which

$$\begin{aligned} T_{ija}^{\cdot\cdot\cdot b}\, v^a{}_{;b} = T_{ijab}\, v^{a;b} &= \sum_{p<q}^{1\cdots n} 2\, T_{ij[pq]}\, v^{p;q} \\ &= \sum_{p<q}^{1\cdots n} S_{ijpq}\, v^{p;q}, \end{aligned} \tag{5, 3}$$

where we put

$$2\, T_{ij[pq]} = S_{ijpq}. \tag{5, 4}$$

Hence (5, 1) gives

$$v^a H_{ij;a} + \sum_{p<q}^{1\cdots n} S_{ijpq}\, v^{p;q}. \tag{5, 5}$$

Now we consider the $(i\ j)$ rank of the matrix (S_{ijpq}), where

$$S_{ijpq} = g_{qj} H_{ip} + g_{qi} H_{pj} - g_{pj} H_{iq} - g_{pi} H_{qj}. \tag{5, 6}$$

Without loss of generality we may consider the case $i \leqq j$, since

$$S_{ijpq} = S_{jipq} \quad .$$

Here we suppose that the rank of matrix (H_{ij}) is l and determinant equation

$$|\, H_{ij} - \lambda g_{ij}\,| = 0$$

has only simple roots $\lambda = H_{(1)}, H_{(2)}, \cdots, H_{(l)}$ except $\lambda = 0$. Then for convenience we choose the coordinate system in such a way that the lines of curvatures passing through an arbiterary point in V_n in V_{n+1} are parameter curvcs of V_n —— if $l < n$, we can take as n parameter curves l lines of curvatures determined by the above $H_{(p)}$ $(1 \leqq p \leqq l)$ and $n-l$ lines of curvatures which are orthogonal mutually and also to the former lines of curvatures since the lines of curvatures are indeterminate in V_{n-l} where $\lambda = 0$ [1].

From our assumption

$$H_{(p)} \neq H_{(q)} \ \text{for} \ p \neq q. \tag{5, 7}$$

Moreover we put $H_{(j)}$ as follows:

if $1 \leqq j \leqq l$, the $H_{(j)}$ are solutions of the proper equation, if $l < j \leqq n$, then $H_{(j)}$ vanish.

Then we have

$$g_{ij} \overset{*}{=} \delta_{ij}, \quad H_{ij} \overset{*}{=} \delta_{ij} H_{(j)}, \tag{5, 8}$$

from which

$$S_{ijpq} \overset{*}{=} \delta_{qj}\delta_{pi} H_{(i)} + \delta_{qi}\delta_{pj} H_{(j)} - \delta_{pj}\delta_{qi} H_{(i)} - \delta_{pi}\delta_{qj} H_{(j)}. \tag{5, 9}$$

After some calculations we obtain that

$$S_{ijij} \overset{*}{=} H_{(i)} - H_{(j)}, \quad S_{ijji} \overset{*}{=} H_{(j)} - H_{(i)} \quad ,$$

and the other components of S_{ijpq} all vanish.

Then the nonvanishing columns and rows of the matrix (S_{ijpq}) $(i \leqq j)$ are

$$\begin{array}{c|cccccccccc}
{}_{(i\,j)}\diagdown^{(p\,q)} & (1\,2) & (1\,3) & \cdots & (1\,l) & (2\,3) & \cdots & (2\,l) & (3\,4) & \cdots & (l-1,\,l) \\
\hline
(1\,2) & H_{(1)}-H_{(2)} & 0 & \cdots & & & & & & \cdots & 0 \\
(1\,3) & 0 & H_{(1)}-H_{(3)} & & & & & & & & \vdots \\
\vdots & \vdots & \vdots & \ddots & & & & & & & \\
(1\,l) & & & & H_{(1)}-H_{(l)} & 0 & & & & & \\
(2\,3) & & & & 0 & H_{(2)}-H_{(3)} & & & & & \\
\vdots & & & & & & \ddots & & & & \\
(2\,l) & & & & & & & H_{(2)}-H_{(l)} & 0 & & \\
(3\,4) & & & & & & & 0 & H_{(3)}-H_{(4)} & & \\
\vdots & \vdots & \vdots & & & & & & & \ddots & \vdots \\
(l-1,\,l) & 0 & 0 & \cdots & & & & & & \cdots & H_{(l-1)}-H_{(l)}
\end{array}$$

Hence in our assumption $v^{a;b}$ must satisfy the conditions more than $l(l-1)/2$ adding to Killing's conditions. Thus we obtain

Theorem 5. 1. *Let a V_n in V_{n+1} admit a group of absolute motions, the rank of (H_{ij}) be l, and the equation $|H_{ij} - \lambda g_{ij}| = 0$ have only l non-vanishing simple roots in every point of V_n, then*

$$r \leqq \frac{1}{2}(n+l)(n-l+1) \quad .$$

From which we have

Theorem 5. 2. *When a V_n in V_{n+1} admits a group of absolute motions, n principal directions are determinate and the n principal curvatures $H_{(i)}$ are not equal each other in every point of V_n, then*

$$r \leqq n.$$

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NOTES ON THE SPACE OF REGULAR FUNCTIONS OF COMPLEX VARIABLES

By

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We consider a set Ω of regular functions of complex variables on $|z|<1$. It is clear that this set is considered as the vector space, which has complex numbers as operators. That is, if $f(z)$, $g(z)\in\Omega$, $\alpha f(z)+\beta g(z)\in\Omega$ for arbitrary complex numbers α and β.

Now, we introduce their norms. Put $||f||=|c_n|$, where $f(z)=\sum_{n=0}^{\infty}c_nz^n$. Then we have

i) $||f+g||\leqq||f||+||g||$,

ii) $||\alpha f||=|\alpha|\cdot||f||$,

iii) $||f||=0$, if and only if $|c_n|=0$.

If the sequence $\{f_p\}$ in Ω is the Cauchy-sequence, there exists limit point f_0. Indeed, let

$$||f_p-f_q||<\varepsilon,$$

for $p,\ q\geqq n_0$, which depends on an arbitrary positive number ε. Put $f_p(z)=\sum_{n=0}^{\infty}c_{pn}z^n$ and $f_q(z)=\sum_{n=0}^{\infty}c_{qn}z^n$, then $||f_p-f_q||=|c_{pn}-c_{qn}|<\varepsilon$. The sequence $\{c_{pn}\}$ of complex numbers is the Cauchy-sequence and so we have c_0 such that $\lim_{p\to\infty} c_{pn}=c_0$. Let $f_0(z)=c_0z^n$, then we have $||f_p-f_0||=|c_{pn}-c_0|\to 0$, when p thends to ∞.

Thus, we see that Ω is complete. Then we have next theorem.

Theorem 1. *Ω is the complex Banach space.*

Put $||f||_r=\sqrt{\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2d\theta}$ for $0<r<1$. Then Ω is the Hilbert space with respect to $|z|=r$.

Theorem 2. *If a sequence $\{f_n\}$ is a Cauchy-sequence with respect to the norm $||f||_r$ on $|z|=r$, for an arbitrary r, which satisfies $0<r<1$, then there exists $f_0(z)$ in Ω such that $f_n(z)$ converges uniformly to $f_0(z)$ on any closed circle $|z|\leqq\sigma(<1)$ in $|z|<1$, and moreover $||f_n-f_0||$ tends to 0.*

Proof. Since $||f_q-f_p||_r$ tends to 0 when p and q tend to ∞, then there exists $f_0(z)$ in the sence of L_2 on $|z|=r$ such that $||f_n-f_0||_r\to 0$ on $|z|=r$. Put $F(z)=\frac{1}{2\pi i}\int_C\frac{f_0(\zeta)}{\zeta-z}d\zeta$, where C is a circle whose radius is r. Then

$$
\begin{aligned}
|f_p(z)-F(z)| &= \left| \frac{1}{2\pi i}\int_c \frac{f_p(\zeta)}{\zeta-z}\,d\zeta - \frac{1}{2\pi i}\int_c \frac{f_0(\zeta)}{\zeta-z}\,d\zeta \right| \\
&= \left| \frac{1}{2\pi i}\int_c \frac{f_p(\zeta)-f_0(z)}{\zeta-z}\,d\zeta \right| \\
&= \left| \frac{1}{2\pi}\int_0^{2\pi} \frac{f_p(re^{i\theta})-f_0(re^{i\theta})}{re^{i\theta}-\rho e^{i\varphi}}\,re^{i\theta}d\vartheta \right|, \text{ where } \zeta=re^{i\theta} \text{ and } z=\rho e^{i\varphi}. \\
&\leqq \frac{r}{2\pi(r-\rho)}\int_0^{2\pi} |f_p(re^{i\theta})-f_0(re^{i\theta})|\,d\vartheta
\end{aligned}
$$

Appealing to Holder's inequality, we have

$$
\begin{aligned}
&\leqq \frac{r}{2\pi(r-\rho)}\sqrt{\int_0^{2\pi}|f_p(re^{i\theta})-f_0(re^{i\theta})|^2 d\theta}\sqrt{\int_0^{2\pi} d\vartheta} \\
&= \frac{r}{\sqrt{2\pi}(r-\rho)}\,\|f_p-f_0\|.
\end{aligned}
$$

If $\rho\leqq r_1<r$, then $|f_p(z)-F(z)| = \dfrac{r}{\sqrt{2\pi}(r-\rho)}\,\|f_p-f_0\| \to 0$, when $p\to\infty$. This shows that $f_p(z)$ converges to $F(z)$ uniformly on $|z|\leqq r_1$ for an arbitrary r_1 such that $r_1<r$ and we see that $F(z)$ is regular on $|z|<r$. Since r is arbitrary in $0<r<1$, $f_p(z)$ converges uniformly to $F(z)$ in $|z|\leqq\sigma<1$ and we see that $F(z)$ is regular in $|z|<1$ and so $F(z)\in\Omega_0$.

$$
\begin{aligned}
\|f_p-F\| &= \left| \frac{1}{2\pi i}\int_c \frac{f_p(\zeta)}{\zeta^{n+1}}\,d\zeta - \frac{1}{2\pi i}\int_c \frac{f_0(\zeta)}{\zeta^{n+1}}\,d\zeta \right| \\
&= \left| \frac{1}{2\pi i}\int_0^{2\pi} \frac{f_p(re^{i\theta})-f_0(re^{i\theta})}{r^{n+1}e^{i(n+1)\theta}}\,ire^{i\theta}\,d\vartheta \right| \\
&\leqq \frac{1}{2\pi r^n}\int_0^{2\pi}|f_p(re^{i\theta})-f_0(re^{i\theta})|\,d\vartheta \\
&\leqq \frac{1}{2\pi r^n}\sqrt{\int_0^{2\pi}|f_p(re^{i\theta})-f_0(re^{i\theta})|^2}\;\sqrt{\int_0^{2\pi} d\vartheta} \\
&\leqq \frac{1}{\sqrt{2\pi}\,r^n}\,\|f_n-f_0\|_r.
\end{aligned}
$$

This shows that $\|f_p-F\|\to 0$, when $\|f_p-F\|_r\to 0$. This completes the proof.

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THE STUDENT'S DISTRIBUTION FOR A UNIVERSE BOUNDED AT ONE OR BOTH SIDES

By

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The author's previons paper[1] was somewhat imperfect, because the ratio considered there $x\sqrt{n-1}/s$, as simple it appears, yet differs from the ordinary Student ratio $(\bar{x}-m)\sqrt{n-1}/s$. In the present note, he would deal with the properly called Student ratio and also the case of an upperly truncated non-negative variable, e. g. as the rectangular distribution[2].

1. *A Rectangular Distribution as Universe.* If the universe be $f(x)=1$ in $0 \leq x \leq 1$ with its true mean $m=\frac{1}{2}$, Student's distribution is readily obtainable from the volume element discussed in the previous paper: If the sample be $\{x_1, x_2, \cdots\cdots\}$ with a sample mean $\bar{x}$ and a S. D. s, its probability is simply $f(x_1) f(x_2) \cdots\cdots dx_1 dx_2 \cdots = dv$. Hence, if s be transformed into Student's ratio $t=(x-m)\sqrt{n-1}/s$, and the joint probability $g(t, \bar{x})\,|J|\,d\bar{x}\,dt$ with Jacobian $|J|=|x-m|\sqrt{n-1}/t^2$, then the fr. f. would be given by

$$s_n(t)=\frac{\sqrt{n-1}}{t^2}\int|\bar{x}-m|\,g(t,\bar{x})\,d\bar{x},$$

where the integration is extended over the whole domain of variables, such that $\sum x_i = n\bar{x}$, $\sum(x_i-\bar{x})^2=ns^2$ under the condition that all x_i's remain within the n-dimensional cube of side 1. Thus

Case $n=2$. We have $dV=4\,dsd\bar{x}$, however now confined insides the square: $0\leq x_1 \leq 1$, $0 \leq x_2 \leq 1$. As the sample mean $\bar{x}$ must lie in the interval (0, 1), so the Student ratio $t=\left(\bar{x}-\frac{1}{2}\right)\Big/s$ becomes $\lesseqgtr 0$ according as $0\leq\bar{x}<\frac{1}{2}$ or $\frac{1}{2}<\bar{x}\leq 1$. First, let $0<\bar{x}<\frac{1}{2}$, $t<0$. Then the Student's fr. f. is $s_2(t)=4\int|x-\frac{1}{2}|\,dx/t^2$, where the integration must be taken insides the rectangular triangle 011 (Fig. 1) with sides $0\leq x_1\leq 1$, $0\leq x_2 \leq 1$ under condition $\sqrt{2}s\leq\sqrt{2}\,\bar{x}$. Consequently $\left(\bar{x}-\frac{1}{2}\right)\Big/t\leq\bar{x}$, if $\bar{x}<\frac{1}{2}$, $t<0$, so that $1/2(1-t)<\bar{x}<1/2$. Therefore

$$s_2(t)=\frac{2}{t^2}\int_{1/2(1-t)}^{1/2}(1-2x)\,dx=\frac{1}{2(1-t)^2}\ (t<0).$$

1) Y. Watanabe, Some Exceptional Examples to Student's Distribution, Journal of Tokushima Univ., Vol. X, 1959, p. 11.

2) H. Cramér, Mathematical Methods of Statistics, p. 244.

Next, if $1/2<\bar{x}<1$, we should consider only those points insides unit square, such as $\sqrt{2}\,s<\sqrt{2}\,(1-\bar{x})$ (Fig. 1). Hence $\left(\bar{x}-\frac{1}{2}\right)\Big/t<1-\bar{x}$ with $\bar{x}>\frac{1}{2}$, $t>0$, so that $\frac{1}{2}<\bar{x}<(1+2t)/2(1+t)$ and we obtain

$$s_2(t)=\frac{2}{t^2}\int_{1/2}^{(1+2t)/2(1+t)}(2x-1)\,dx=\frac{1}{2\,(1+t)^2}\;(t>0).$$

As usual Student $s_2(t)$ yields Cauchy distribution, so also the present fr. f. has no mean, and besides its first derivative is discontinuous at the origin, yet still distributes symmetrically (Fig. 2).

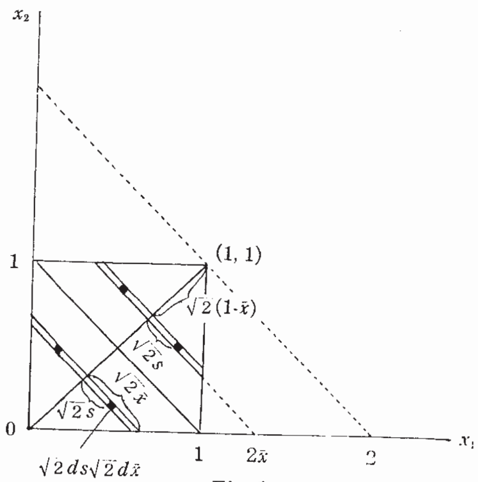

Fig. 1

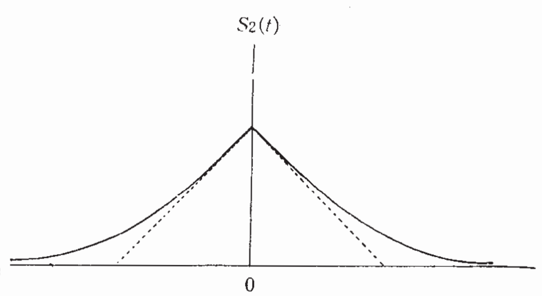

Fig. 2

The significant limits with level α are found from

$$\int_{t_1}^{\infty}s_2(t)\,dt=\frac{1}{2}\,\frac{1}{1+t_1}=\frac{\alpha}{2}\quad\text{as}\quad t_1=\frac{1}{\alpha}-1,$$

and thus e. g. for $\alpha=0.1$, 0.05, 0.01 to be $t_1=\pm 9$, ± 19, ± 99, respectively, which are of larger magnitude than those of the classical Student's ratio: ± 6.314, ± 12.706, ± 63.657.

Cace $n=3$. We shall only consider the lower half $0<\bar{x}<\frac{1}{2}$, $t<0$, since the other half $\frac{1}{2}<\bar{x}<1$, $t>0$ may be immediately obtained by symmetry.

I. Subcase $\sqrt{2}\,s<\bar{x}$. In this subcase the whole s-circle can be adopted and consequently $dV=6\sqrt{3}\,\pi\,s ds\,d\bar{x}$. Transforming s into $t=\left(\bar{x}-\frac{1}{2}\right)\sqrt{2}/s$, we get $dP=3\sqrt{3}\,\pi(2\bar{x}-1)^2 d\bar{x}\,dt/\,|\,t\,|^3$, so that, denoting the partial contribution to the Student's fr. f. $s_3(t)$ by $s_I(t)$, we have

$$s_I(t)=\frac{3\sqrt{3}\,\pi}{|\,t\,|^3}\int_{x_0}^{x_1}(2x-1)^2dx,$$

where the limits of integration are found to be $x_0=1/(2-t)$ and $x_1=1/2$ from conditions $\sqrt{2}\,s<\bar{x}$, $0<\bar{x}<1/2$ and consequently

$$s_I(t)=\frac{\sqrt{3}}{2}\frac{\pi}{(2-t)^3} \quad (t<0) \tag{1.1}$$

II. Subcase $\dfrac{\bar{x}}{\sqrt{2}}<s<\sqrt{2}\,\bar{x}$. We shall further subdivide two cases $0<\bar{x}<1/3$ and $1/3<\bar{x}<1/2$ still with $t<0$.

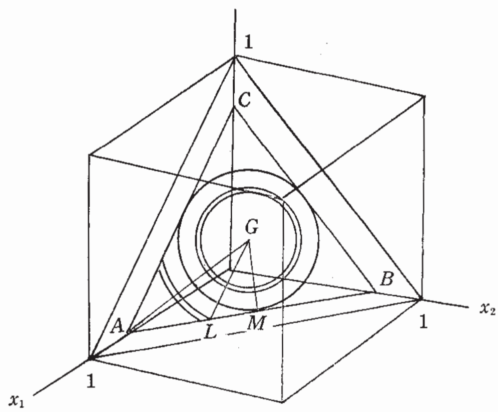

Fig. 3

II: $0<\bar{x}<1/3$. In this case the radius of s-circle $GL=\sqrt{3}\,s$ (Fig. 3) is between $GM=\sqrt{\dfrac{3}{2}}\,\bar{x}$ and $GA=\sqrt{6}\,\bar{x}$, so that $\bar{x}<\sqrt{2}\,s=(2\bar{x}-1)/t<2\bar{x}$ and thus $1/(2-t)>\bar{x}>1/2(1-t)$. But, now assumed $0<\bar{x}<1/3$, these inequalities will hold if $1/3>1/(2-t)$, i. e. $t<-1$, while, if on the contrary $1/(2-t)>1/3$, i. e. $0>t>-1$, we must take $1/3$ as the upper limit, in which case however it should be the lower limit $1/2(1-t)<1/3$, i. e. $-1/2>t>-1$. On the other hand the volume element, as described in the previous paper loc. cit., being given by $dV_{II}=18\sqrt{3}\,s\left(\dfrac{\pi}{3}-\cos^{-1}\dfrac{\bar{x}}{s\sqrt{2}}\right)ds\,d\bar{x}$, the contribution from this region is

$$s_{II}(t)=\frac{9\sqrt{3}}{|t|^3}\int_{x_0}^{x_1}(2x-1)^2\left(\frac{\pi}{3}-\cos^{-1}\frac{xt}{2x-1}\right)dx,$$

where $x_0=1/2(1-t)$, while $x_1=1/(2-t)$ if $t<-1$, but $x_1=1/3$ if $-1/2>t>-1$. Consequently we obtain if $t<-1$

$$s_{II}(t)=\frac{-\sqrt{3}\,\pi}{2(2-t)^3}+\frac{3\sqrt{3}}{2t^2}\int_{1/2(1-t)}^{1/(2-t)}h(x,t)\,dx, \tag{1.2}$$

where x is merely an integration variable and the integrand $h(x,t)$ denotes the function $(1-2x)^2/\sqrt{(2x-1)^2-x^2t^2}$ and alike in the below. However, if $-1/2>t>-1$, we have

$$s_{II}(t)=\frac{\sqrt{3}}{18|t|^3}\left(\frac{\pi}{3}-\cos^{-1}(-t)\right)+\frac{3\sqrt{3}}{2t^2}\int_{1/2(1-t)}^{1/3} h(x,t)\,dt. \tag{1.3}$$

II′. $\frac{1}{3}<\bar{x}<\frac{1}{2}$. In this subcase also the portion of equilateral triangle ABC

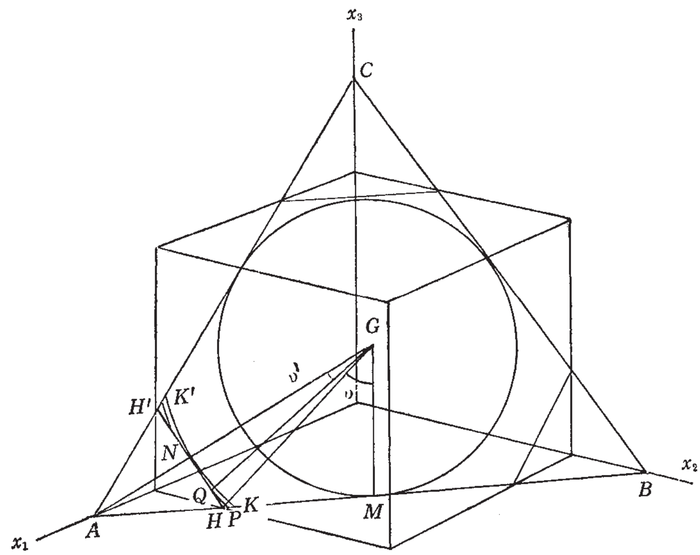

Fig. 4

that lies outsides the cube with side 1 should be rejected, so that the radius $\sqrt{3}\,s$ must be taken between $GM=\sqrt{\frac{3}{2}}\,\bar{x}$ and $GN=\sqrt{\frac{3}{2}}(1-\bar{x})$ (Fig. 4). Accordingly $\bar{x}<\sqrt{2}\,s<1-\bar{x}$. Or, substituting $s=\left(\bar{x}-\frac{1}{2}\right)\sqrt{2}/t$, we obtain now inequalities $\frac{1}{2-t}>\bar{x}>\frac{1+t}{2+t}$. However, for $(1+t)/(2+t)\gtrless 1/3$, when $t\gtrless -1/2$, the lower limit shall be $1/3$ or $(1+t)/(2+t)$ according as $-1<t<-1/2$ or $0>t>-1/2$. Hence, we get for $-1<t<-1/2$

$$s_{II'}(t)=-\frac{\sqrt{3}\,\pi}{2(2-t)^3}-\frac{\sqrt{3}}{18|t|^3}\left(\frac{\pi}{3}-\cos^{-1}(-t)\right)+\frac{3\sqrt{3}}{2t^2}\int_{1/3}^{1/(2-t)} h(x,t)\,dx, \tag{1.4}$$

while for $0>t>-1/2$

$$s_{II'}(t)=-\frac{\sqrt{3}\,\pi}{2(2-t)^3}+\frac{3\sqrt{3}}{2(2+t)^3}\left(\frac{\pi}{3}-\cos^{-1}(1+t)\right)+\frac{3\sqrt{3}}{2t^2}\int_{(1+t)/(2+t)}^{1/(2-t)} h(x,t)\,dx. \tag{1.5}$$

II″. It remains still to calculate the contribution from 6 remaining arc pieces, such as PQ (Fig. 4), which lies between the segment HH', and arc KNK', such that $GN=\sqrt{\frac{3}{2}}(1-\bar{x})<\sqrt{3}\,s< GH=\sqrt{6\bar{x}^2-6\bar{x}+2}$, or $1-\bar{x}<\sqrt{2}\,s=(1-2\bar{x})/(-t)<2\sqrt{\bar{x}^2-\bar{x}+1/3}$ with $1/3<\bar{x}<1/2$ and whence it follows that $0>t>-1/2$. Now, if $\vartheta=\angle MGP$ and $\vartheta'=\angle NGQ$ (Fig. 4),

$$PQ=\sqrt{3}\,s\left(\frac{\pi}{3}-\vartheta-\vartheta'\right)=\sqrt{3}\,s\left(\frac{\pi}{3}-\cos^{-1}\bar{x}/\sqrt{2}\,s-\cos^{-1}(1-\bar{x})/\sqrt{2}\,s\right)$$

and consequently

$$dV=18\sqrt{3}\,s\left[\frac{\pi}{3}-\cos^{-1}\frac{\bar{x}}{\sqrt{2}\,s}-\cos^{-1}\frac{1-\bar{x}}{\sqrt{2}\,s}\right]d\bar{x}\,ds \tag{1.6}$$
$$=9\sqrt{3}\left[\frac{\pi}{3}-\cos^{-1}\frac{\bar{x}t}{2\bar{x}-1}-\cos^{-1}\frac{(1-\bar{x})t}{2\bar{x}-1}\right](1-2\bar{x})^2 d\bar{x}\,dt/|t|^3,$$

so that

$$s_{II''}(t)=\frac{3\sqrt{3}}{2|t|^3}\int_{x_0}^{x_1}\frac{d}{dx}(2x-1)^3\left[\frac{\pi}{3}-\cos^{-1}\frac{xt}{2x-1}-\cos^{-1}\frac{(1-x)t}{2x-1}\right]dx,$$

where the limits of integration are found from the foregoing inequalities to be $x_1=\frac{1+t}{2+t}$ and $x_0=\frac{1}{2}\left(1+\frac{t}{\sqrt{3(1+t^2)}}\right)$, both of which lie between $\frac{1}{3}$ and $\frac{1}{2}$, so far $\frac{1}{3}<\bar{x}<\frac{1}{2}$. Hence, on integrating by parts, we get

$$s_{II''}(t)=-\frac{3\sqrt{3}}{2(2+t)^3}\left(\frac{\pi}{3}-\cos^{-1}(1+t)\right) \tag{1.7}$$
$$+\frac{3\sqrt{3}}{2t^2}\left[\int_{x_0}^{x_1}h(x,t)\,dx+\int_{1-x_1}^{1-x_0}h(x,t)\,dx\right],$$

in consequence of the equality $\cos^{-1}\frac{1}{2}\left(\sqrt{3(1-t^2)}+t\right)+\cos^{-1}\frac{1}{2}\left(\sqrt{3(1-t^2)}-t\right)=\frac{\pi}{3}$ when $|t|\leqq\frac{1}{2}$, and the limits of integration are $x_0=\frac{1}{2}\left(1+\frac{t}{\sqrt{3(1-t^2)}}\right)$, $1-x_0=\frac{1}{2}\left(1-\frac{t}{\sqrt{3(1-t^2)}}\right)$, and $x_1=\frac{1+t}{2+t}$, $1-x_1=\frac{1}{2+t}$.

On summing up all the above (1)—(7), we have for $-\infty<t<-1/2$

$$s_3(t)=\frac{3\sqrt{3}}{2t^2}\int_{1/2(1-t)}^{1/(2-t)}\frac{(2x-1)^2\,dx}{\sqrt{(2x-1)^2-x^2t^2}} \tag{1.8}$$

and for $-1/2<t<0$

$$s_3(t)=\frac{3\sqrt{3}}{2t^2}\int_{x_0}^{1/(2-t)}\frac{(2x-1)^2\,dx}{\sqrt{(2x-1)^2-x^2t^2}}+\frac{3\sqrt{3}}{2t^2}\int_{1/(2+t)}^{1-x_0}\frac{(2x-1)^2\,dx}{\sqrt{(2x-1)^2-x^2t^2}}, \tag{1.9}$$

where x_0 and $1-x_0$ are $\frac{1}{2}\left[1\pm t/\sqrt{3(1-t^2)}\right]$ respectively.

We should further integrate (8) and (9). First, from (8) we find for $-\infty < t < -2$

(1.10) $$s_3(t)=\frac{9(4+3t-4t^2)}{4(t^2-4)^2(1-t)^2}+\frac{3\sqrt{3}(2+t^2)}{2\sqrt{t^2-4}^5}\left(\frac{\pi}{2}-\sin^{-1}\frac{4-t}{2(1-t)}\right.$$
$$\left(=\sin^{-1}\frac{\sqrt{3(t^2-4)}}{2(1-t)}\right).$$

Secondly for $-2<t<-1/2$

(1.11) $$s_3(t)=\frac{9(4+3t-4t^2)}{4(4-t^2)^2(1-t)^2}+\frac{3\sqrt{3}(2+t^2)}{2\sqrt{4-t^2}^5}\log\left|\frac{4-t+\sqrt{3(4-t^2)}}{2(1-t)}\right|$$
$$\left(=\sinh^{-1}\frac{\sqrt{3(4-t^2)}}{2(1-t)}\right).$$

And when $t=-2$, we get directly from (8)

(1.12) $$s_3(-2)=\frac{3\sqrt{3}}{8}\int_{1/6}^{1/4}\frac{(2x-1)^2\,dx}{\sqrt{1-4x}}=\frac{7}{120}=0.0583.$$

It can be shown that both (10) and (11) do approach to (12) for $t\to -2\mp 0$, respectively.

Lastly, on integrating (9) and simplifying it, we obtain for $-1/2<t<0$

(1.13) $$s_3(t)=\frac{\sqrt{3}(2-5t^2)}{\sqrt{1-t^2}\,(4-t^2)^2}+\frac{3\sqrt{3}(2+t^2)}{2(4-t^2)^{5/2}}\log\frac{\sqrt{4-t^2}+\sqrt{1-t^2}}{\sqrt{4-t^2}-\sqrt{1-t^2}}.$$

Or since $t^2<1/4$ here, expanding in a power series, we have

$$s_3(t)=\frac{\sqrt{3}}{8}\left(1+\frac{3}{4}\log 3\right)-\frac{3\sqrt{3}}{128}\left(10-\frac{9}{2}\log 3\right)t^2+\cdots\cdots$$
$$=0.3949\cdots-0.2052\cdots t^2+\cdots\cdots,$$

and $s_3(0)=0.3949$. Also, as a check, it may be shown that (13) and (11) yield

$$s_3\left(-\frac{1}{2}\pm 0\right)=\frac{8}{75}+\frac{12}{25\sqrt{5}}\log\frac{2}{3-\sqrt{5}}=0.3133$$

coincidently. The fr. f. for $t>0$ shall be given by changing only the sign of t in (10), (11) and (13) itself as it stands, and the distribution is quite symmetrical with respect to the origin. In general $s_3(t)$ is regular in the whole interval, except that the derivatives become discontinuous at $t=\pm 1/2$, where however $s_3(t)$ itself still remains continuous (Fig. 5).

Furthermore, it is noteworthy that really in $-\frac{1}{2}<t<\frac{1}{2}$ $s''(t)$ is negative and the fr. f. is convex upwards, while in $|t|>\frac{1}{2}$ it is positive and concave upwards, so that there are points of inflexion at $t=\pm\frac{1}{2}$.

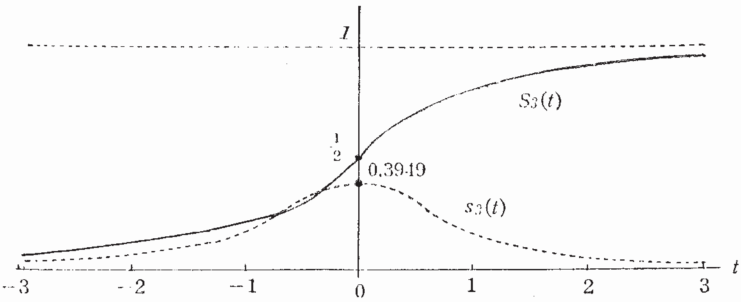

Fig. 5

Now we are to find the distribution function

$$S_3(t)=\int_{-\infty}^{t} s_3(t)\,dt,$$

for which it needs also to be treated piecewise. Firstly, integrating (10) in $-\infty<t<-2$, we get

(1.14) $S_3(t)=$

$$\frac{9}{4(t^2-4)}\left[\frac{-t}{\sqrt{3(t^2-4)}}\sin^{-1}\frac{\sqrt{3(t^2-4)}}{2(1-t)}-\frac{1}{1-t}\right]$$

with $S_3(-2-0)=5/48=0.1042$. Secondly, for $-2<t<-1/2$, on integrating (11) and taking $S_3(-2)$ into account, we obtain

t	$s_3(t)$	$S_3(t)$
0	0.3949	0.5
− 0.5	0.3133	0.2889
− 1	0.1590	0.2017
− 1.5	0.0922	0.1409
− 2	0.0583	0.1042
− 3	0.0243	0.0613
− 4	0.0153	0.0413
− 5	0.0094	0.0297
− 6	0.0061	0.0233
−10	0.0017	0.0100
−15	0.0006	0.0049

$$\text{(1.15)}\qquad S_3(t)=\frac{9}{4(4-t^2)}\left[\frac{t}{\sqrt{3(4-t^2)}}\log\left|\frac{4-t+\sqrt{3(4-t^2)}}{2(1-t)}\right|+\frac{1}{1-t}\right]$$

still with $S_3(-2+0)=5/48$. Also $S_3\left(\frac{-1}{2}\right)=\frac{2}{5}-\frac{1}{5\sqrt{5}}\log\frac{3+\sqrt{5}}{2}=0.2889.$

Thirdly, for $-1/2<t<0$ integrating (13) and using $S_3(-1/2)$ we obtain

$$\text{(1.16)}\qquad S_3(t)=\frac{1}{2}+\frac{\sqrt{3}\,t\sqrt{1-t^2}}{2(4-t^2)}+\frac{3\sqrt{3}\,t}{4\sqrt{4-t^2}}\log\frac{\sqrt{4-t^2}+\sqrt{1-t^2}}{\sqrt{4-t^2}-\sqrt{1-t^2}},$$

We are now able to determine the critical limits $\pm t_\alpha$ for a significant level α. Since $S_3(-2)=0.1042$, we may take (14) for $\alpha<0.2$ and put

$$\text{(1.17)}\quad S_3(t)=\int_{-\infty}^{t} s_3(t)\,dt=\frac{9}{4(t^2-4)}\left[\frac{-t}{\sqrt{3(t^2-4)}}\sin^{-1}\frac{\sqrt{3(t^2-4)}}{2(1-t)}-\frac{1}{1-t}\right]=\frac{\alpha}{2}.$$

We obtain by successive interpolations $t_\alpha=\pm3.59$, ±5.74 and ±14.85 nearly for $\alpha=0.1$, 0.05, 0.01 respectively. These are again greater in absolute values than the corresponding classical Student ratios ±2.920, ±4.303, ±9.925. However, comparing Cases $n=2$ and $n=3$ it is plausible that their differences shall become gradually smaller, when n is greater than 3.

Provided that the original universe has the fr. f. $f(x)=1/(b-a)$ in $a<X<b$ with the parent mean $M=(a+b)/2$, we may standardize it so as $x=(X-a)/(b-a)$ and $f(x)=1$ in $0<x<1$ with mean $m=1/2$. Let the original sample be $\{X_i\}$ with a mean $\overline{X}$, a S. D. S and Student ratio $T=(\overline{X}-M)\sqrt{n-1}/S$. From these data, we get $\bar{x}=\sum x_i/n=\sum(X_i-a)/n(b-a)=(\overline{X}-a)/(b-a)$, $\bar{x}-m=(\overline{X}-M)/(b-a)$, $s^2=\sum(x_i-\bar{x})^2/n=S^2/(b-a)^2$ and $t=(\bar{x}-m)\sqrt{n-1}/s=(\overline{X}-M)\sqrt{n-1}/S=T$. Hence, we may apply all the results obtained above about t and $\bar{x}$ directly to T and $\overline{X}$, as it stands.

Remark. Incidentally the fr. f. $f(\bar{x})$ may easily be found from $dV=dP$. In fact for $n=2$, $0<\bar{x}<1/2$, integrating $dP=4ds\,d\bar{x}$ about s under condition $s<\bar{x}$, we obtain $f(\bar{x})=4\int_0^{\bar{x}}ds=4\bar{x}$ $\left(0<\bar{x}<\frac{1}{2}\right)$; or else $4\int_0^{1-\bar{x}}ds=4(1-\bar{x})$ for $1/2<\bar{x}<1$, $s<1-\bar{x}$. Thus, we get a symmetrical triangular distribution with discontinuos derivative at $\bar{x}=1/2$.

Again, if $n=3$, for $0<\bar{x}<1/2$, integrating I: $dP=6\sqrt{3}\pi s\,ds\,d\bar{x}$ about s under condition $\sqrt{2}s<\bar{x}$, we get firstly $f_I(\bar{x})=3\sqrt{3}\pi x^2/2$. Also, if $0<\bar{x}<1/3$, from II $dP=6\sqrt{3}(\pi-3\cos^{-1}\bar{x}/s\sqrt{2})s\,ds\,d\bar{x}$ with $\bar{x}/\sqrt{2}<s<\bar{x}\sqrt{2}$ and secondly $f_{II}(\bar{x})=\int_{\bar{x}/\sqrt{2}}^{\bar{x}\sqrt{2}}6\sqrt{3}(\pi-3\cos^{-1}\bar{x}/s\sqrt{2})s\,ds=\frac{-3\sqrt{3}\pi}{2}x^2+\frac{27}{2}\bar{x}^{-2}$. Therefore

$$f(\bar{x})=f_I+f_{II}=27x^2/2 \quad \text{for} \quad 0<\bar{x}<1/3. \tag{1.18}$$

However, if $1/3<\bar{x}<1/2$, the portion swelling out of the unit cube must be rejected, so that as II′ we shall take $f_{II'}=\int_{\bar{x}/\sqrt{2}}^{(1-\bar{x})/\sqrt{2}}18\sqrt{3}s\left(\frac{\pi}{3}-\cos^{-1}\frac{\bar{x}}{2\sqrt{s}}\right)ds=\frac{9\sqrt{3}}{2}\left[(1-2\bar{x})\frac{\pi}{3}-(1-\bar{x})^2\cos^{-1}\frac{\bar{x}}{1-\bar{x}}+\bar{x}\sqrt{1-2\bar{x}}\right]$. Besides it requires, corresponding to (6), $f_{II''}=18\sqrt{3}\int_{x_0}^{x_1}\left(\frac{\pi}{3}-\cos^{-1}\frac{\bar{x}}{s\sqrt{2}}-\cos^{-1}\frac{1-\bar{x}}{s\sqrt{2}}\right)ds$, where $x_0=\frac{1}{\sqrt{2}}(1-\bar{x})$ and $x_1=\sqrt{2\bar{x}^2-\bar{x}+1/3}$. Consequently, we attain $f_{II''}=\frac{9\sqrt{3}}{2}\left[(1-x)^2\cos^{-1}\frac{\bar{x}}{1-\bar{x}}-\bar{x}\sqrt{1-\bar{x}^2}-\frac{\pi}{3}(\bar{x}-1)^2-\frac{1}{\sqrt{3}}(6\bar{x}^2-5\bar{x}+1)\right.$. Hence

$$f(\bar{x})=f_I+f_{II'}+f_{II''}=9(3\bar{x}-3\bar{x}^2-1/2) \text{ for } 1/3<\bar{x}<1/2. \tag{1.19}$$

Thus the two parabolic branches of $f(\bar{x})$ for $n=3$ at $\bar{x}=1/3$, the point of conjuction, actually possess the same values $f\left(\frac{1}{3}\right)=\frac{3}{2}, f'\left(\frac{1}{3}\right)=9$, while $f''\left(\frac{1}{3}\right)$ are yet different, as $f''\left(\frac{1}{3}-0\right)=27\neq f''\left(\frac{1}{3}+0\right)=-54$ (Cramér loc. cit. p. 245). The curve is symmetrical with respect to the mode $\bar{x}=1/2$ with $f(0)=f(1)=f'(0)=f'(1)=0$ and $f\left(\frac{1}{2}\right)=9/4$, $f'\left(\frac{1}{2}\right)=0$. Evidently the central moments are $\mu_1=\mu_3=0$, while $\mu_2=49/2880$ and $\mu_4=2639/254016$, so that $\mu_4/\mu_2^2=3.92$; thus it reveals already somewhat normal-like appearance.

Those cases with $n\geqq 4$ may be similarly argued by aid of the previous paper, although the computations shall become more cumbersome. However, as n increases, $f_n(\bar{x})$ rapidly approaches the N. D. and the corresponding Student function also approaches the classical one, it is of little value to study $s_n(t)$ for large n.

2. *Truncated Laplace Distribution as Universe.* Let the universe be $f(x)=e^{-x}(x>0)$ with the true mean $E(x)=1$, and the sample be $\{x_1, x_2, \cdots\}$ with mean $\bar{x}$, S. D. s, Student ratio $t=(\bar{x}-1)\sqrt{n-1}/s$. The probability being $e^{-n\bar{x}}\,dV$, it is required to find out the fr. f. $s_n(t)$.

Case $n=2$. Here $dV=4dsd\bar{x}$ and $dP=4e^{-2\bar{x}}dsd\bar{x}$. Or, transforming s into t, $|J|=|\bar{x}-1|/t^2$ and we get

$$s_2(t)=\frac{1}{t^2}\int 4e^{-2\bar{x}}\,|1-\bar{x}|\,d\bar{x}.$$

By the condition $0<s<\bar{x}$, $0<(\bar{x}-1)/t<\bar{x}$. First, when $t<0$, $0<\bar{x}<1$, we have $\frac{1}{1-t}<\bar{x}<1$. Hence

$$(2.1)\qquad s_2(t)=\frac{1}{t^2}\int_{1/(1-t)}^{1}4e^{-2x}(1-x)\,dx=\frac{1}{t^2}\left[e^{-2}-\frac{1+t}{1-t}e^{-2/(1-t)}\right]\;(-\infty<t<0)$$

with $s_2(-0)=2e^{-2}$, $s_2(-1)=e^{-2}$. Next, when $0<t<1$, the condition gives $\bar{x}-1<t\bar{x}$, so that $1<\bar{x}<1/(1-t)$ and accordingly still the same expression as (1) holds

$$(2.2)\qquad s_2(t)=\frac{1}{t^2}\left[e^{-2}-\frac{1+t}{1-t}e^{-2/(1-t)}\right]\quad(0<t<1)$$

with $s_2(+0)=2e^{-2}$. Lastly, when $1<t<\infty$, the condition $s=(\bar{x}-1)/t<\bar{x}$ is satisfied by itself. Hence, we are to integrate about all values of $\bar{x}>1$:

$$(2.3)\qquad s_2(t)=\frac{1}{t^2}\int_1^\infty 4e^{-2x}(x-1)\,dx=\frac{e^{-2}}{t^2}\quad(1<t<\infty).$$

Thus $s_2(t)$ is continuous and continuously derivable throughout the whole interval and even at the point of conjunction $t=1$, the two branches have the same derivatives, since $\lim\limits_{t\to1-0}\exp(-2/(1-t))/(1-t)^n$ vanish for $n=0, 1, 2, \cdots\cdots$.

However, since $ts_2(t)\cong O(1/t)$ as $t\to\pm\infty$, it cannot be integrated there and has no mean, similarly as the classical Student's distribution for case $n=2$. Of course, in the contrary to the classical case, our $s_2(t)$ has no symmetry. To show it, we shall investigate its mode by putting

$$(2.4)\qquad s_2'(t)=\frac{2e^{-2}}{t^3}\left[\frac{1-t+t^2+t^3}{(1-t)^2}e^{-2t/(1-t)}-1\right]=0.$$

Solving this equation by Newton's successive approximation, we find that thc mode is about $m_0=0.4234$ with a maximal value $s_2(m_0)=0.3258$.

Further, the distribution function $S_2(t)$ being given by $\int_{-\infty}^{t}s_2(t)\,dt$ and thus for $t<0$ as well as $0<t<1$, we get

$$(2.5)\qquad S_2(t)=1-\frac{e^{-2}}{t}+\frac{1-t}{t}e^{-2/(1-t)}$$

with $S_2(\pm0)=0.5940$, $S(1-0)=1-e^{-2}=0.8647$, $S_2(-10)=0.09644$, &c. Or, if $|t|$ be tolerably large, expanding in a power series of t^{-1}

$$(2.6)\qquad S_2(t)=1-\frac{e^{-2}}{t}+\frac{1-t}{t}\left(1-\frac{2}{1-t}+\frac{2}{(1-t)^2}-\cdots\right)=\frac{1+e^{-2}}{-t}-\frac{2}{t^2}\text{ nearly.}$$

Naturally, if $t>1$, we have by (3)

$$S_2(t)=S(1)+\int_1^t \frac{e^{-2}}{t^2}\,dt=1-\frac{e^{-2}}{t} \tag{2.7}$$

and $S_2(\infty)=1$ as expected. The median is found by equating (5) to 0.5, and solving that equation again by Newton's successive approximation to be $m_l=-0.3974$, with Max $s_2(t)=0.3258$. Thus the asymmetrical feature is remarkably manifested (Fig. 6).

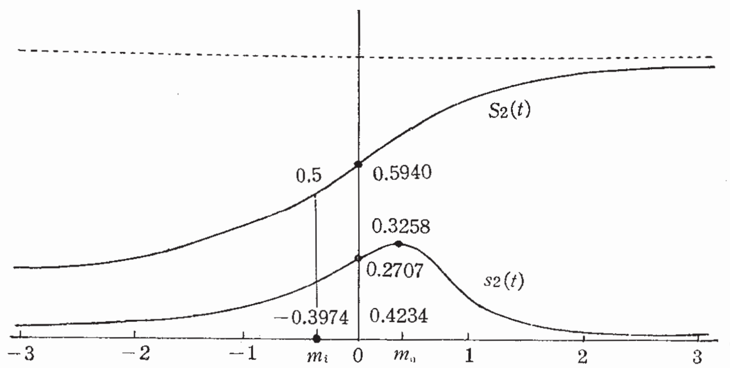

Fig. 6

Student's Distribution for Truncated Laplace Distribution.

t	$s_2(t)$	$S_2(t)$	t	$s_2(t)$	$S_2(t)$
−300	0.0000	0.0038	+0	.2707	.5940
−231.1	low. 1% pt.	.0050	+0.4232	.3258	.7283
−200	0.0000	.0056	(mode)	(maximum)	
−100	.0001	.0111	+0.5	.3215	.7477
−50	.0004	.0219	1	.1353	.8647
−43.68	low. 5% pt.	.0250	1.5	.0601	.9098
−40	.0007	.0277	2	.0338	.9317
−30	.0014	.0358	2.707	up. 10% pt.	.9500
−20.78	low. 10% pt.	.0500	3	.0150	.9549
−20	.0024	.0519	4	.0085	.9622
−10	.0082	.0964	5	.0054	.9729
−6	.0187	.1007	5.41	up. 5% pt.	.9750
−5	.0245	.1673	6	.0038	.9774
−4	.0336	.1959	10	.0014	.9865
−3	.0487	.2364	20	.0003	.9932
−2	.0723	.2849	27.07	up. 1% pt.	.9950
−1.5	.1001	.3413	30	.0002	.9959
−1	.1353	.3996	50	.0001	.9973
−0.5	.1899	.4799	100	.0000	.9986
−0.3974	(median)	.5	200	.0000	.9993
−0	.2707	.5940	300	.0000	.9995

The upper significant points $t_1(>1)$ are readily found by (7) on putting

(2. 8) $$\int_{t_1}^{\infty}\frac{e^{-2}dt}{t^2}=\frac{e^{-2}}{t_1}=\frac{\alpha}{2}$$

to be 2.707, 5.413, 27.067 for significant levels $\alpha=0.1$, 0.05, 0.01, respectively. As to the lower significant points t_0, in virtue of (6)

(2, 9) $$S_2(t)\cong\frac{1+e^{-2}}{-t}-\frac{2}{t^2}=\frac{\alpha}{2}\text{ approximately}$$

and solving the quadratic equation about t and taking the larger one, because for the smaller the expansion (9) becomes less exact, we obtain $t_0=-20.782$, -43.578, -231.12 for $\alpha=0.1$, 0.05, 0.01, respectively. Comparing all these results with the classical 10-, 5-, 1-% points, i. e. ± 6.314, ± 12.706, ± 63.657 for $n=2$, it is noticeable that the critical values are extraordinarily enlarged in magnitude at the truncated side, while on the contrary at the side of the reserved part they are rather lessened.

Case $n=3$.

I. When $\sqrt{2}\,s<\bar{x}$, i. e. $2(\bar{x}-1)/t<\bar{x}$, the whole s-circle can be adopted and $dV=12\sqrt{3}\,\pi(\bar{x}-1)^2 d\bar{x}\,ds/|t|^3$, so that, denoting by $s_I(t)$ the contribution to $s_3(t)$ from this region,

$$s_I(t)=\frac{12\sqrt{3}\,\pi}{|t|^3}\int_{x_0}^{x_1}e^{-3\bar{x}}(x-1)^2dx,$$

where the limits of integration shall be determined from the condition $2(\bar{x}-1)/t<\bar{x}$. Now, for the sake of later convenience, writing the integral

(2. 10) $$\int_{\infty}^{x}(x-1)^2e^{-3x}dx=\frac{-1}{27}e^{-3x}[9(x-1)^2+6(x-1)+2]=G(x),$$

so that $G'(x)=(x-1)^2e^{-3x}$. First, if $t<0$, $0<\bar{x}<1$, the previous condition yields $2(\bar{x}-1)>t\bar{x}$, so that $2/(2-t)<\bar{x}<1$ and we get

(2. 11) $$s_I(t)=\frac{12\sqrt{3}\,\pi}{|t|^3}\left[G(1)-G\left(\frac{2}{2-t}\right)\right],\quad(-\infty<t<0)$$

where $G(1)=-\dfrac{2}{27}e^{-3}=-0.00368793$ ($e^{-3}=0.04978707\cdots$). Next, if $0<t<2$, the condition gives $1<\bar{x}<2/(2-t)$ and consequently

(2. 12) $$s_I(t)=\frac{12\sqrt{3}\,\pi}{t^3}\left[G\left(\frac{2}{2-t}\right)-G(1)\right]\quad(0<t<2).$$

Lastly, if $t>2$, the condition $2(\bar{x}-1)<\bar{x}t$ is satisfied by itself, so that

(2. 13) $$s_I(t)=\frac{12\sqrt{3}\,\pi}{t^3}[G(\infty)-G(1)]=\frac{c}{t^3},\quad(2<t<\infty)$$

where $c=-12\sqrt{3}\,\pi\,G(1)=8\sqrt{3}\,\pi e^{-3}/9=0.2408100$. The auxiliary function $G(x)$

is negative throughout the whole interval $0<x<\infty$ and non-decreasing with $G(0)=-5/27$, $G(\infty)=-0$.

II. $\bar{x}<\sqrt{2}\,s=2(\bar{x}-1)/t<2\bar{x}$. Now the volume element being

$$dV=\frac{12\sqrt{3}\,(\bar{x}-1)^2}{|t|^3}\left(\pi-3\cos^{-1}\frac{\bar{x}t}{2(\bar{x}-1)}\right)d\bar{x}\,dt,$$

their contribution to $s_3(t)$ is

$$s_{II}(t)=\frac{12\sqrt{3}\,\pi}{|t|^3}\int_{x_0}^{x_1}G'(x)\left[1-\frac{3}{\pi}\cos^{-1}\frac{xt}{2(x-1)}\right]dx,$$

where the limits of integration are e. g. for $t<0$, $x_0=\dfrac{1}{1-t}$ and $x_1=\dfrac{2}{2-t}$. Integrating by parts, we get

$$(2.11)'\quad s_{II}(t)=\frac{12\sqrt{3}\,\pi}{|t|^3}G\left(\frac{2}{2-t}\right)+\frac{36\sqrt{3}}{t^2}\int_{1/1-t}^{2/2-t}\frac{G(x)\,dx}{(x-1)\sqrt{4(1-x)^2-x^2t^2}}\ (t<0)$$

with a positive integrand because of $G(x)<0$ and $x<1$. Next, if $0<t<1$, $x>1$ and after conditions $\bar{x}t<2(\bar{x}-1)<2\bar{x}t$, we have

$$(2.12)'\quad s_{II}(t)=-\frac{12\sqrt{3}\,\pi}{t^3}G\left(\frac{2}{2-t}\right)+\frac{36\sqrt{3}}{t^2}\int_{2/2-t}^{1/1-t}\frac{G(x)\,dx}{(x-1)\sqrt{4(x-1)^2-x^2t^2}},\quad (0<t<1)$$

where the integral really becomes negative. Further, if $1<t<2$, the condition $\bar{x}(1-t)<1$ holds by itself, but by the remaining condition $2<(2-t)\bar{x}$, so we obtain

$$(2.12)''\ s_{II}(t)=-\frac{12\sqrt{3}\,\pi}{t^3}G\left(\frac{2}{2-t}\right)+\frac{36\sqrt{3}}{t^2}\int_{2/2-t}^{\infty}\frac{G(x)\,dx}{(x-1)\sqrt{4(x-1)^2-x^2t^2}}\quad (1<t<2).$$

Lastly, when $t>2$, a part of conditions $2<\bar{x}(2-t)$ becomes impossible, so that the value $t>2$ is inadmissible in II.

To sum up I and II: In view of (10), using the notation

$$(2.14)\quad \frac{36\sqrt{3}}{t^2}\,\frac{-G(x)}{(x-1)\sqrt{4(x-1)^2-x^2t^2}}=\frac{4}{\sqrt{3}\,t^2}\,\frac{e^{-3x}[1+(2-3x)^2]}{(x-1)\sqrt{4(x-1)^2-x^2t^2}}\equiv H(x,\,t),$$

we obtain, as the fr. f. for Student's ratio in respect to the truncated Laplace population,

$$(2.15)\qquad s_3(t)=s_I(t)+s_{II}(t)=c/t^3-\int_{1/1-t}^{2/2-t}H(x,t)\,dx\qquad(-\infty<t<0)$$

$$(2.16)\qquad =c/t^3-\int_{2/2-t}^{1/1-t}H(x,t)\,dx\qquad(0<t<1)$$

$$(2.17)\qquad =c/t^3-\int_{2/2-t}^{\infty}H(x,t)\,dx\qquad(1<t<2)$$

$$(2.18)\qquad =c/t^3\qquad \text{with } c=0.2408100.\qquad(2<t<\infty)$$

Since the above integrals seem not to be expressible in any finite combination of elementary functions, so we give up to execute the integrations. However, the distribution function

$$\text{(2. 19)} \qquad S_3(t)=\int_{-\infty}^{t} s_3(t)\,dt=-\frac{c}{2t^2}-\int_{-\infty}^{t} dt\int_{x0(t)}^{x1(t)} H(x,t)\,dx$$

can be computed by interchanging the order of integrations, so as $\int dx \int H(x,t)\,dt$, where the limits of integrations should be adequately determined for each case.

Subdividing 3 cases: $1° -\infty<t<0$, $2°\ 0<t<1$ and $3°\ 1<t<2$, we shall investigate them separately. We utilize an evident indefinite integral

$$\int\frac{dt}{t^2\sqrt{a^2-b^2t^2}}=\frac{\sqrt{a^2-b^2t^2}}{-a^2t}.$$

1° $-\infty<t<0$. Using (15),

$$\text{(2. 20)}\ S_3(t_0)=\int_{-\infty}^{t_0} s_3(t)\,dt=-\frac{c}{2t_0^2}+\frac{4}{\sqrt{3}}\int_{-\infty}^{t_0}\frac{dt}{t^2}\int_{1/1-t}^{2/2-t}\frac{e^{-3x}(1+(2-3x)^2)\,dx}{(1-x)\sqrt{4(1-x)^2-x^2t^2}},\ (t<0).$$

By changing the order of repeated integrations, we get

$$\text{(2. 21)}\quad \int_{-\infty}^{t_0} dt\int_{1/1-t}^{2/2-t} dx=\int_0^{1/1-t_0} dx\int_{2-2/x}^{1-1/x} dt+\int_{1/1-t_0}^{2/2-t_0} dx\int_{2-2/x}^{t_0} dt=\text{(i)}+\text{(ii)}.$$

Since the lower limit of the new inner integrals makes the integrand vanish, it needs only to substitute the upper limit. Thus

$$\begin{aligned}\text{(i)}&=\frac{4}{\sqrt{3}}\int_0^{1/1-t_0} e^{-3x}(1+(2-3x)^2)\frac{dx}{1-x}\left[\frac{\sqrt{4(1-x)^2-x^2t^2}}{-4(1-x)^2t}\right]_{t=1-1/x}\\&=\int_0^{1/1-t_0} e^{-3x}(1+(2-3x)^2)\frac{x\,dx}{(1-x)^3},\end{aligned}$$

which, on integrating by parts, yields

$$\text{(i)}=1-\left(1-\frac{3}{t_0}-\frac{1}{t_0^2}\right)\exp\frac{-3}{1-t_0}.$$

Hence,

$$\text{(i)}'\cong\frac{e^{-3}}{t_0^2}+1-\frac{17}{2}e^{-3}+O(t_0),\quad\text{as}\quad t_0\cong 0,$$

while

$$\text{(i)}''\cong\frac{5}{2t_0^2}+\frac{9}{t_0^3}+O\left(\frac{1}{t_0^4}\right),\quad\text{as}\quad\frac{1}{t_0}\cong 0.$$

Similarly

$$\text{(ii)}=\frac{-1}{\sqrt{3}\,t_0}\int_{1/1-t_0}^{2/2-t_0} e^{-3x}(1+(2-3x)^2)\sqrt{4(1-x)^2-x^2t_0^2}\,\frac{dx}{(1-x)^3},$$

which is not so effectively integrable as in (i). However, by taking a new variable θ so as $x=1/(1-\theta t_0)$, it yields

$$(\text{ii})=\frac{1}{\sqrt{3}\,t_0^2}\int_{1/2}^{1}\frac{\sqrt{4\theta^2-1}}{\theta^3}d\theta.\left[\exp\left(\frac{-3}{1-\theta t_0}\right)\right].\left[1+\left(2-\frac{3}{1-\theta t_0}\right)^2\right].$$

When $|t_0|$ is small, we may expand the expressions under brackets in power series of θt_0 and then integrate about θ and obtain

$$(\text{ii})'\cong\frac{2e^{-3}}{\sqrt{3}\,t_0^2}\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)+O(t_0)\cong\frac{c}{2t_0^2}-\frac{e^{-3}}{t_0^2},$$

while, if $|1/t_0|$ is small,

$$(\text{ii})''\cong\frac{5}{\sqrt{3}\,t_0^2}\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)+\frac{27}{t_0^3}+\frac{1}{t_0^4}\left(\frac{189}{8}+2/\pi\sqrt{3}\right).$$

Hence, when $t_0\cong-0$, (i)$'$ and (ii)$'$ together with the first term $(0)'=-c/2t_0^2$, yields

$$S_3(t_0\cong 0)\cong 1-17e^{-3}/2=0.5768>0.5. \tag{2.22}$$

Thus, the area under 1° is greater than 1/2, which shows that the distribution is never symmetrical.

Also, (0)$'$ (i)$''$ (ii)$''$ combined all together, we can determine the lower significant limits, so as

$$\frac{\alpha}{2}=\frac{1}{t_0^2}\left(-\frac{c}{2}+\frac{10\pi}{3\sqrt{3}}\right)+\frac{27}{t_0^3}+\frac{1}{t_0^4}\left(\frac{189}{8}+2/\pi\sqrt{3}\right)\quad\text{nearly.} \tag{2.23}$$

Thus, for $\alpha=0.1,\ 0.05,\ 0.01$, we obtain $t_0=-8.75,\ -13.08,\ -32.45$, respectively. These are again larger in the absolute value compared with the corresponding classical Student ratios for case $n=3$: ± 2.920, ± 4.303, ± 9.925, while the upper limits are smaller, as will be described later on.

2° $0<t<1$. Since in this domain by (16)

$$s_3(t)=\frac{c}{t^3}-\frac{4\sqrt{3}}{t^2}\int_{2/2-t}^{1/1-t}\frac{e^{-3x}[1+(3x-2)^2]}{(x-1)\sqrt{4(x-1)^2-x^2t^2}}dx,$$

letting $0<\tau_0<\tau_1<1$, we have

$$\begin{aligned}S(\tau_1)=S(-0)+\lim_{\tau_0\to 0}\int_{\tau_0}^{\tau_1}s_3(t)dt=0.5768+\lim_{\tau_0\to 0}\Bigg[&\left(\frac{c}{2\tau_0^2}-\frac{c}{2\tau_1^2}\right)(=(0))\\&+\int_{2/2-\tau_0}^{1/1-\tau_0}dx\int_{\tau_0}^{2-2/x}dt+\int_{1-\tau_0}^{2/2-\tau_1}dx\int_{1-1/x}^{2-2/x}dt+\int_{2/2-\tau_1}^{1/1-\tau_1}dx\int_{1-1/x}^{\tau_1}dt\\&(=(\text{i})+(\text{ii})+(\text{iii}))\Bigg].\end{aligned} \tag{2.24}$$

Because the integrands in (i) as well as (ii) vanish for the inner upper limit, we get

$$(\mathrm{i})=\frac{-1}{\sqrt{3}\,\tau_0}\int_{2/2-\tau_0}^{1/1-\tau_0}e^{-3x}(1+(3x-2)^2)\sqrt{4(x-1)^2-x^2\tau_0^2}\,\frac{dx}{(x-1)^3}.$$

Or, on putting $x=1/(1-\theta\tau_0)$, it becomes, as in (ii) of 1°,

$$(\mathrm{i})=\frac{-2e^{-3}}{\sqrt{3}\,\tau_0^2}\int_{1/2}^{1}\frac{\sqrt{4\theta^2-1}}{\theta^3}d\theta.\ \exp\left(-\frac{3\theta\tau_0}{1-\theta\tau_0}\right).\frac{1+\theta\tau_0+5\theta^2\tau_0^2/2}{(1-\theta\tau_0)^2}.$$

When $\tau_0\cong 0$, expanding the integrand in a power series of $\theta\tau_0$, it yields

$$(\mathrm{i})'=-\frac{2e^{-3}}{\sqrt{3}\,\tau_0^2}\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)+O(\tau_0)=-\frac{c}{2\tau_0^2}+\frac{e^{-3}}{\tau_0^2}+O(\tau_0).$$

Therefore, together with the first term: $\quad (0)=\dfrac{c}{2\tau_0^2}-\dfrac{c}{2\tau_1^2}$

$$(0)+(\mathrm{i})'\cong\frac{e^{-3}}{\tau_0^2}-\frac{c}{2\tau_1^2}+O(\tau_0).$$

Similarly

$$(\mathrm{ii})=-\frac{1}{\sqrt{3}}\int_{1/1-\tau_1}^{2/2-\tau_1}\frac{e^{-3x}(1+(3x-2)^2)}{(x-1)^3}\left[\frac{\sqrt{4(x-1)^2-x^2t^2}}{t}\right]_{t=1-1/x}.$$

The expression between the square brackets becomes simply $\sqrt{3}\,x$. Hence, on writing $x=1-y$ and integrating by parts, we get

$$\begin{aligned}(\mathrm{ii})&=-e^{-3}\int_{\tau_0/1-\tau_0}^{\tau_1/2-\tau_1}e^{-3y}(2+8y+15y^2+9y^3)\,\frac{dy}{y^3}\\&=\exp(-3(1+y)).\left[\frac{1}{y^2}+\frac{5}{y}+3\right]_{\tau_0/(1-\tau_0)}^{\tau_1/(2-\tau_1)}\\&=\left[\left(\frac{2-\tau_1}{\tau_1}\right)^2+\frac{5(2-\tau_1)}{\tau_1}+3\right]\exp\left(\frac{-6}{2-\tau_1}\right)-\left[\left(\frac{1-\tau_0}{\tau_0}\right)^2+\frac{5(1-\tau_0)}{\tau_0}+3\right]\exp\left(\frac{-3}{1-\tau_0}\right),\end{aligned}$$

where the second square brackets become for $\tau_0\cong 0$, $(\mathrm{ii})_0=-\dfrac{e^{-3}}{\tau_0^2}+\dfrac{17}{2}e^{-3}+O(\tau_0)$.

Hence, as $\tau_0\to 0$,

$$(0)+(\mathrm{i})'+(\mathrm{ii})_0\cong-\frac{c}{2\tau_1^2}+\frac{17}{2}e^{-3}+O(\tau_0).$$

Thus, making $\tau_0\to 0$ in (24) it is of no affect. Lastly

$$(\mathrm{iii})=\frac{1}{\sqrt{3}}\int_{2/2-\tau_1}^{1/1-\tau_1}\frac{e^{-3x}(1+(3x-2)^2)}{(x-1)^3}\left[\frac{4(x-1)^2-x^2t^2}{t}\right]_{1-1/x}^{\tau_1}.$$

For its lower inner limit, we have, as in (ii),

$$\begin{aligned}(\mathrm{iii})_0=&\left[\left(\frac{1-\tau_1}{\tau_1}\right)^2+5\left(\frac{1-\tau_1}{\tau_1}\right)+3\right]\exp\left(\frac{-3}{1-\tau_1}\right)\\&-\left[\left(\frac{2-\tau_1}{\tau_1}\right)^2+5\left(\frac{2-\tau_1}{\tau_1}\right)+3\right]\exp\left(\frac{-6}{2-\tau_1}\right),\end{aligned}$$

of which the latter half just cancels out with the first brackets in (ii), while the former half reduces to $\frac{e^{-3}}{\tau_1^2}-\frac{17}{2}e^{-3}+O(\tau_1)$ as $\tau_1\to 0$, but to naught as $\tau_1\to 1-0$. As to the upper inner limit in (iii), again putting $x=1/(1-\theta\tau_1)$ similarly as in (i), it becomes

$$\text{(iii)}_1=\frac{3e^{-3}}{\sqrt{3}\,\tau_1^2}\int_{1/2}^{1}\frac{\sqrt{4\eta^2-1}}{\theta^3}\exp\left(\frac{-3\eta\tau_1}{1-\theta\tau_1}\right)\left[1+\theta\tau_1+\frac{5}{2}\theta^2\tau_1^2\right]\frac{d\eta}{(1-\theta\tau_1)^2},$$

which approaches

$$\simeq\frac{2e^{-3}}{\sqrt{3}\,\tau_1^2}\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)+O(\tau_1)\simeq\frac{c}{2\tau_1^2}-\frac{e^{-3}}{\tau_1^2},\ \text{as}\ \tau_1\to 0.$$

Consequently the full sum (0) + (i) + (ii) + (iii) approaches 0(1) as $\tau_1\to 0$. But, when $\tau_1\to 1$, the sum (0) + (i) + (ii) + (iii)_0 amounts to

$$\frac{17e^{-3}}{2}-\frac{c}{2}=0.3028.$$

On the other hand, on computing (iii)_1 for $\tau_1=1$ by means of Gauss' method of numerical integration, we obtain

$$J\equiv\frac{2e^{-3}}{\sqrt{3}}\int_{1/2}^{1}\exp\left(\frac{-3\eta}{1-\theta}\right)\cdot\left[1+\theta+\frac{5}{2}\theta^2\right]\sqrt{4\eta^2-1}\,\frac{d\eta}{\theta^3(1-\theta)^2}=0.0058\cdots. \tag{2.25}$$

Therefore, all the above taken into account, (24) amounts to

$$S_3(1)=0.5768+0.3028+0.0058=0.8854, \tag{2.26}$$

Consequently the area under the fr. f. $s_3(t)$ in 2° $0<t<1$ is 0.3086 by (22).

3° Further, for $1<t<2$, we have

$$\int_1^{t_1}s_3(t)dt=\frac{c}{2}-\frac{c}{2t_1^2}-\frac{4}{\sqrt{3}}\int_1^{t_1}\frac{dt}{t^2}\int_{2/2-t}^{\infty}\frac{e^{-3x}(1+(3x-2)^2)}{(x-1)\sqrt{4(x-1)^2-x^2t^2}}dx.$$

The repeated integral becomes

$$\frac{1}{\sqrt{3}}\int_2^{2/2-t_1}\frac{e^{-3x}[1+(3x-2)^2)}{(x-1)^3}\left[\frac{\sqrt{4(x-1)^2-x^2t^2}}{t}\right]_{t=1}^{t=2-2/x}dx$$
$$+\frac{1}{\sqrt{3}}\int_{2/2-t_1}^{\infty}\Big[\text{the same integrand}\Big]_1^{t_1}dx=J_1+J_2.$$

In particular, for $t_1=2$, the second integral J_2 reduces to naught, while the first J_1 coincides with J of (25)—Really, either when $\theta=1/(y+1)$ in J or when $x-1=1/y$ in J_1, both become a same integral:

$$\frac{2e^{-3}}{\sqrt{3}}\int_0^1\exp\left(\frac{-3}{y}\right)\cdot\left[(y+1)^2+(y+1)+\frac{5}{2}\right]\sqrt{4-(y+1)^2}\,\frac{dy}{y^2}.$$

And thus $J_1=0.0058$ when $t_1=2$. Therefore, the area of the whole domain 3° becomes $\frac{c}{2}-\frac{c}{8}-0.0058=0.0845$. Consequently

$$S_3(2) = S_3(1) + 0.0845 = 0.9699; \tag{2.27}$$

and finally

$$S_3(\infty) = S_3(2) + \int_2^\infty \frac{c}{t^3}\,dt = 0.9699 + 0.0301 = 1. \tag{2.28}$$

Lastly we shall find the upper critical limits t_1. In fact, it was already by (18) evident that

$$\int_{t_1}^\infty \frac{0.2408}{t^3}\,dt = \frac{0.1204}{t_\alpha^2} = \frac{\alpha}{2}. \tag{2.29}$$

And accordingly $t_1 = \sqrt{0.2408/\alpha}$. E. g. for $\alpha=0.1$, 0.05, 0.01, we get immediately $t_1=1.552$, 2.195, 4.907, respectively. Here the latter two $\alpha/2=0.025$, 0.005 being less than $\int_2^\infty \frac{c}{t^3}\,dt=0.0301$, the determination is legitimate, but it is not so for $\alpha/2=0.05$, the first one. Hence it seems to require some correction: namely in view of (17) we should subtract the following integral resembling to J of (25)

$$\begin{aligned} J_1(t_1) &= \frac{4}{\sqrt{3}}\int_{t_1}^2 \frac{dt}{t^2}\int_{2/2-t}^\infty \frac{e^{-3x}(1+(3x-2)^2)}{(x-1)\sqrt{4(x-1)^2-x^2t^2}}\,dx \\ &= \frac{1}{\sqrt{3}\,t_1}\int_{2/2-t_1}^\infty \frac{e^{-3x}(1+(3x-2)^2)}{(x-1)^3}\sqrt{4(x-1)^2-x^2t^2}\,dx. \end{aligned}$$

However, on computing numerically the above correction $J_1(1.552)$ actually, we get $J_1=0.00000\,005$, which is practically immaterial. Hence the upper limit for $\alpha=0.1$ will do still with 1.552.

3. *Truncated Normal Distribution as Universe.*

If the parent distribution has the fr. f. such that for $x>0$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma D}\exp\left\{-\frac{1}{2\sigma^2}(x-a)^2\right\} = \frac{1}{\sigma D}\varphi\left(\frac{x-a}{\sigma}\right), \tag{3.1}$$

where

$$D=\int_0^\infty \varphi\left(\frac{x-a}{\sigma}\right)\frac{dx}{\sigma} = 1-\int_{-\infty}^0 \varphi\left(\frac{x-a}{\sigma}\right)\frac{dx}{\sigma} = 1-\Phi\left(\frac{-a}{\sigma}\right) = \Phi\left(\frac{a}{\sigma}\right)>0,$$

then the d. f. becomes

$$F(x)=\int_0^x f(x)dx = \frac{1}{D}\int_0^x \varphi\left(\frac{x-a}{\sigma}\right)\frac{dx}{\sigma} = \frac{1}{D}\int_{-a/\sigma}^{(x-a)/\sigma}\varphi(t)dt = \left[\Phi\left(\frac{x-a}{\sigma}\right)-\Phi\left(\frac{-a}{\sigma}\right)\right]\Big/D$$

and $F(0)=0$, $F(\infty)=1$. This is the so-called truncated normal distribution[1]. Its first moment, the parent mean m is positive and given by

$$m=\int_0^\infty xf(x)dx = \frac{1}{\sqrt{2\pi}\,D}\int_0^\infty x\exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}\frac{dx}{\sigma} = \frac{\sigma}{D}\varphi\left(\frac{-a}{\sigma}\right)+a = a+\lambda\sigma, \tag{3.2}$$

1) H. Cramér, loc. cit., p. 248.

where $\lambda = \Phi'(-a/\sigma)/D > 0$. However, for the sake of simplicity, we shall write simply, as $\sigma = 1$, below. Thus the parent fr. f. is

$$(3.1)' \qquad f(x) = \varphi(x-a)/D \quad (x > 0) \quad \text{with} \quad D = 1-\Phi(-a)$$

and the parent mean

$$(3.2)' \qquad m = a+\lambda(>0) \quad \text{with} \quad \lambda = \Phi'(-a)/D(>0).$$

Now, from the universe $f(x)$ a sample $\{x_1, x_2, \cdots\cdots\}$ being drawn with a sample mean $\bar{x}$ and a S. D. s, it is again required to find the distribution function of Student's ratio $t = (\bar{x}-m)\sqrt{n-1}/s$. The golden identity $\Sigma x_i^2 = ns^2+n\bar{x}^2$ would be always of use.

Case $n = 2$.

$$dp = f(x_1)f(x_2)dx_1 dx_2 = \frac{1}{2\pi D^2} \exp\{-s^2-(\bar{x}-a)^2\}dx_1dx_2 \text{ and } dV = 4d\bar{x}ds,$$

so that $dP = \dfrac{4}{2\pi D^2} \exp\{-s^2-(\bar{x}-a)^2\}\, d\bar{x}\, ds = \dfrac{2}{\pi D^2} \exp\left\{-\dfrac{(\bar{x}-m)^2}{t^2} - (\bar{x}-a)^2\right\}$ $\dfrac{|\bar{x}-m|}{t^2}\, d\bar{x}dt$. Hence, we get the fr. f.

$$s_2(t) = \frac{2}{\pi D^2 t^2}\int_{x_0}^{x_1} |x-m| \exp\left\{-\frac{(x-m)^2}{t^2} - (x-a)^2\right\} dx,$$

where the limits of integration should be determined from the fundamental condition for the positive argument : $s<\bar{x}$. First, if $t<0$, $\bar{x}<m$ and we have by condition $(\bar{x}-m)/t<\bar{x}$, so $m/(1-t)<\bar{x}<m$. Therefore

$$(3.3) \qquad s_2(t) = \frac{2}{\pi D^2 t^2}\int_{m/1-t}^{m} (m-x) \exp\left\{-\frac{(x-m)^2}{t^2} - (x-a)^2\right\} dx \qquad (-\infty<t<0).$$

Next, if $0<t<1$, $\bar{x}>m$ and the condition gives $\bar{x}-m<\bar{x}t$ so that $m<\bar{x}<m/(1-t)$ and

$$(3.4) \qquad s_2(t) = \frac{2}{\pi D^2 t^2}\int_{m}^{m/1-t} (x-m) \exp\left\{-\frac{(x-m)^2}{t^2} - (x-a)^2\right\} dx \quad (0<t<1).$$

However, if $1<t<\infty$, the condition $s = \dfrac{\bar{x}-m}{t} < \bar{x}$ is satisfied by itself, so that $m<\bar{x}<\infty$ and

$$(3.5) \qquad s_2(t) = \frac{2}{\pi D^2 t^2}\int_{m}^{\infty} (x-m) \exp\left\{-\frac{(x-m)^2}{t^2} - (x-a)^2\right\} dx \qquad (1<t<\infty).$$

Omitting other details, we proceed straightforwardly to obtain the lower and upper significant limits only. For this purpose it requires asymptotic formulas of the d. f. $S_2(t)$ for large $-t_0(>0)$ and $t_1(>0)$. We shall content ourselves with some approximations taking few terms in expansions by power series of t^{-1}.

Firstly, for $t<0$, taking (3. 3) and integrating by parts, we get

$$y \equiv D^2\pi s_2(t) = \frac{1}{t^2}\int_{m/1-t}^{m} -2(x-a-\lambda)e^{-(x-a)2} \cdot \exp\left\{-\frac{(x_1-m)^2}{t^2}\right\} dx \quad (a = m-\lambda)$$

$$= \frac{1}{t^2}\left\{e^{-\lambda^2} - e^{-a^2}\exp\left(\frac{2am}{1-t} - \frac{2m^2}{(1-t)^2}\right) + \int_{m/1-t}^{m} e^{-(x-a)^2}\frac{2(x-m)}{t^2}\exp\left(-\frac{(x-m)^2}{t^2}\right)dx\right.$$
$$\left. + 2\lambda\int_{m/1-t}^{m}\exp\left[-(x-a)^2 - \frac{(x-m)^2}{t^2}\right]dx \quad (\equiv z, \text{ say})\right\}$$

$$= \frac{1}{t^2}\left[e^{-\lambda^2} - e^{-a^2}\operatorname{Exp}_2(t) - y + 2\lambda z\right],$$

where

$$(3.6) \quad \operatorname{Exp}_2(t) \equiv \exp\left[\frac{2am}{1-t} - \frac{2m^2}{(1-t)^2}\right] = 1 - \frac{2am}{t} - \frac{2am + 2m^2 - 2a^2m^2}{t^2} - \cdots$$

for large $|t|$. Hence, we obtain

$$(3.7) \qquad y \equiv D^2\pi s_2(t) = \frac{e^{-\lambda^2} - e^{-a^2}\operatorname{Exp}_2(t) + 2\lambda z}{1+t^2},$$

where

$$z \equiv \int_{m/1-t}^{m}\exp\left[-(x-a)^2 - \frac{(x-m)^2}{t^2}\right]dx$$
$$= \int_{m/1-t}^{m}\exp\left[-\left(1+\frac{1}{t^2}\right)x^2 + 2\left(a+\frac{m}{t^2}\right)x - \left(a^2+\frac{m^2}{t^2}\right)\right]dx$$
$$= \exp\left(\frac{-\lambda^2}{1+t^2}\right)\int_{m/1-t}^{m}\exp\left[-\left(1+\frac{1}{t^2}\right)\left(x - \frac{m+at^2}{1+t^2}\right)^2\right]dx.$$

Or, writing

$$w = \sqrt{2\left(1+\frac{1}{t^2}\right)}\left[x - \frac{m+at^2}{1+t^2}\right],$$

$$w_1 = \frac{\sqrt{2}\,\lambda}{\sqrt{1+1/t^2}} \quad\text{and}\quad w_0 = -\sqrt{\frac{2}{1+1/t^2}}\left[a + \frac{m}{t}\,\frac{1+1/t}{1-1/t}\right],$$

z is expressible as follows

$$(3.8) \quad z = \frac{\sqrt{\pi}}{\sqrt{1+1/t^2}}\exp\left(\frac{-\lambda^2}{1+t^2}\right)\int_{w_0}^{w_1} e^{-\frac{1}{2}w^2}\frac{dw}{\sqrt{2\pi}} = \frac{\sqrt{\pi}}{\sqrt{1+1/t^2}}\exp\left(\frac{-\lambda^2}{1+t^2}\right)[\Phi(w_1) - \Phi(w_0)].$$

Furthermore we expand every factor by Taylor to some power of t^{-1} say, to t^{-3}, and we obtain

$$(3.9) \quad s_2(t) \equiv \frac{y}{D^2\pi}$$
$$\cong \frac{1}{D^2\pi}\left\{\frac{1}{t^2}\left[e^{-\lambda^2} - e^{-a^2} + 2\lambda\sqrt{\pi}(\Phi(\sqrt{2}\lambda) + \Phi(\sqrt{2}a) - 1)\right] + \frac{2m^2}{t^3}e^{-a^2} + O\left(\frac{1}{t^4}\right)\right\}.$$

Therefore, the distribution function $S_2(t)$ is, if $t<0$ and $|t|$ be large,

$$(3.10) \quad S_2(t) \cong \frac{1}{D^2\pi}\left\{-\frac{1}{t}\left[e^{-\lambda^2} - e^{-a^2} + 2\lambda\sqrt{\pi}(\Phi(\sqrt{2}\,\lambda) + \Phi(\sqrt{2}a) - 1)\right] - \frac{m^2}{t^2}e^{-a^2}\right\}$$
$$+ O\left(\frac{1}{t^3}\right).$$

Accordingly this equated to $\alpha/2$:

$$S_2(t_0)=\int_{-\infty}^{t_0}s_2(t)\,dt=\frac{\alpha}{2}, \tag{3.11}$$

and solved for t_0, we can determine the lower significant limits.

For examples, if the N. D. be truncated at centroid, then $a=0$, $D=1/2$, $\lambda=m=\sqrt{2/\pi}=0.79788$. If the term of t^{-1} only taken, we get a linear equation, which yields $t_0=-14.69$, -29.38, -146.9 for $\alpha=0.1$, 0.05, 0.01. Yet, if we take up to the term of t^{-2}, the quadratic equation shall give a pair of negative roots -13.48, -2.31 and -28.23, -1.14 and -145.6, -1.12, corresponding to three values of α. Howewer, those of greater magnitude only must be selected, since the approximation is the less exact with the smaller $|t|$. Again, if the N. D. be truncated at its left quartile, then $a=0.6745$, $D=3/4$, $\lambda=0.42369$, $m=1.09819$. Solving the corresponding quadratic equations and taking larger $|t|$, we get $t_0=-10.92$, -22.67, -116.4. Further, if the N. D. be truncated at its right quartile, we have $a=-0.6745$, $D=1/4$, $\lambda=1.27108$, $m=0.59658$, and $t_0=-23.71$, -56.72, -186.6. Thus, the more truncated, the enlargement of lower limits becomes the more manifest.

Secondly, to obtain the upper significant limits, it requires another asymptotic formula, this time, for $S(t_1)=\int_{t_1}^{\infty}s_2(t)\,dt$, in which however a pretty more number of terms should be taken up, since now the magnitude of t_1 is not so large: indeed it becomes even less than that of the untruncated case. Taking (5) for $1<t<\infty$ and integrating by parts, we get

$$y\equiv D^2\pi s_2(t)=\int_m^{\infty}\frac{2(x-m)}{t^2}e^{-(x-a)2}\exp\left(-\frac{(x-m)^2}{t^2}\right)dx \qquad (m=a+\lambda)$$

$$=\frac{1}{t^2}\int_m^{\infty}(-e^{-(x-a)2})'.\ \exp\left(-\frac{(x-m)^2}{t^2}\right)dx-\frac{2\lambda}{t^2}\int_m^{\infty}\exp\left[-(x-a)^2-\frac{(x-m)^2}{t^2}\right]dx$$

$$(\equiv z,\ \text{say})$$

$$=\frac{1}{t^2}\,[e^{-\lambda 2}-y-2\lambda z].$$

Therefore

$$y\equiv D^2\pi s_2(x)=(e^{-\lambda^2}-2\lambda z)/(1+t^2), \tag{3.12}$$

where z is just similarly to (8)

$$z\equiv\int_m^{\infty}\exp\left[-(x-a)^2-\frac{(x-m)^2}{t^2}\right]dx=\exp\left(\frac{-\lambda^2}{1+t^2}\right)\int_m^{\infty}\exp\left[-\left(1+\frac{1}{t^2}\right)\left(x-\frac{at^2+m}{t^2+1}\right)^2\right]dx.$$

Or, on writing

$$w=\sqrt{2\left(1+\frac{1}{t^2}\right)}\left[x-\frac{at^2+m}{t^2+1}\right],\qquad w_0=\lambda\sqrt{\frac{2}{1+1/t^2}},$$

it follows

$$z=\frac{\sqrt{\pi}}{\sqrt{1+1/t^2}}\exp\left(\frac{-\lambda^2}{1+t^2}\right)\int_{w_0}^{\infty}e^{-\frac{1}{2}w^2}\frac{dw}{\sqrt{2\pi}}=\frac{\sqrt{\pi}}{\sqrt{1+1/t^2}}\exp\left(\frac{-\lambda^2}{1+t^2}\right)\left[1-\Phi\left(\lambda\sqrt{\frac{2}{1+1/t^2}}\right)\right].$$

Expanding again every factor and argument into power series of t^{-2}, but now taking possibly a pretty many number of terms,

$$z=\sqrt{\pi}(1-\Phi(\sqrt{2}\lambda))\left[1-\frac{1}{t^2}\left(\lambda^2+\frac{1}{2}\right)+\frac{1}{t^4}\left(\frac{\lambda^4}{2}+\frac{3}{2}\lambda^2+\frac{3}{8}\right)-\frac{1}{t^6}\left(\frac{\lambda^6}{6}+\frac{5}{4}\lambda^4+\frac{15}{8}\lambda^2+\frac{5}{16}\right)\right]$$
$$+\lambda e^{-\lambda^2}\left[\frac{1}{2t^2}-\frac{2\lambda^2+5}{8t^4}+\frac{1}{t^6}\left(\frac{\lambda^4}{12}-\frac{19}{48}\lambda^2+\frac{19}{32}\right)\right]+O\left(\frac{1}{t^8}\right). \tag{3.13}$$

This expression being substituted in (12), we attain finally

$$s_2(t)\cong\frac{1}{D^2\pi}\left\{\frac{1}{t^2}[e^{-\lambda^2}-2\lambda\sqrt{\pi}(1-\Phi(\sqrt{2}\lambda))]\right.$$
$$-\frac{1}{t^4}[(1+\lambda^2)e^{-\lambda^2}-(2\lambda^2+3)\lambda\sqrt{\pi}(1-\Phi(\sqrt{2}\lambda))]$$
$$\left.+\frac{1}{t^6}\left[\left(1+\frac{3}{2}\lambda^2+\frac{5}{4}\lambda^4\right)e^{-\lambda^2}-\left(\lambda^4+5\lambda^2+\frac{15}{4}\right)\lambda\sqrt{\pi}(1-\Phi(\sqrt{2}\lambda))\right]\right\}$$
$$\left(=\frac{A}{t^2}-\frac{B}{t^4}+\frac{C}{t^6},\quad\text{say}\right). \tag{3.14}$$

Hence we have

$$S(t_1)=\int_{t_1}^{\infty}s_2(t)dt=\frac{A}{t_1}-\frac{B}{3t_1^3}+\frac{C}{5t_1^5}-\cdots\cdots \tag{3.15}$$

from which the upper significant limit can be determined. We obtain the following upper significant limits t_1:

	$\alpha=0$	$\alpha=0.05$	$\alpha=0.01$
$a=0$	4.11	8.26	41.40
$a=0.67450$	4.74	9.58	48.00
$a=-0.67450$	3.52	7.35	36.63
N. D.	6.31	12.71	63.66.

Thus, again, the more truncated, the more departure from the complete N. D. results.

Case $n=3$.

I. $0<s<\bar{x}/\sqrt{2}$, i. e. $2(\bar{x}-m)/t<\bar{x}$. In this subcase, the whole s-circle can be adopted:

$$dV=12\sqrt{3}\pi(\bar{x}-m)^2d\bar{x}dt/|t|^3 \text{ and } dP=(\sqrt{2\pi}D)^{-3}\exp\left[-\frac{3}{2}(s^2+\overline{x-a}^2)\right]dV.$$

Hence, denoting by $s_I(t)$ the contribution to $s_3(t)$ from this portion, (3.16) $$s_I(t) = \frac{c}{|t|^3}\int_{x_0}^{x_1}(x-m)^2 e^{-Q}dx,$$

where

$$Q = \frac{3(x-m)^2}{t^2} + \frac{3}{2}(x-a)^2 \quad \text{and} \quad c = \frac{6}{D^3}\sqrt{\frac{3}{2\pi}},$$

and the limits of integration are determined by condition I. First, fot $t<0$, $0<\bar{x}<m$ by condition $2(\bar{x}-m)>\bar{x}t$ and we have $x_0=\dfrac{2m}{2-t}$, $x_1=m$. Next, for $0<t<2$, $\bar{x}>m$ and condition gives $x_0=m$, $x_1=2m/(2-t)$. Lastly, for $t>2$, the condition holds by itself, and we have simply $x_0=m$, $x_1=\infty$. Or, to simplify the writing, let us put

(3.17) $$\int_0^x (y-m)^2 e^{-Q}dy \equiv G(x),\ 0<x<\infty, \quad \text{so that } G'(x)=(x-m)^2e^{-Q}.$$

Then

$$s_I(t) = c|t|^{-3}(G(x_1)-G(x_0))$$

and consequently

(3.18) $$\begin{cases} s_I(t) = \dfrac{c}{|t|^3}\left[G(m)-G\left(\dfrac{2m}{2-t}\right)\right] & \text{for } -\infty<t<0, \\ \quad\ = \dfrac{c}{t^3}\left[G\left(\dfrac{2m}{2-t}\right)-G(m)\right] & \text{for } \quad 0<t<2, \\ \quad\ = \dfrac{c}{t^3}\left[G(\infty)-G(m)\right] & \text{for } \quad 2<t<\infty. \end{cases}$$

Evidently $G(x)$ is non-negative and monotonic increasing about x with $G(0)=0$, yet it contains t as parameter and indeed an even function of t. In particular, the value of $G(m)$ as well as $G(\infty)$ shall be actually computed later on.

II. $\bar{x}<\sqrt{2}s=2(\bar{x}-m)/t<2\bar{x}$.

The volume element being now

$$dV = 6\sqrt{3}\pi s\left(1-\frac{3}{\pi}\cos^{-1}\bar{x}/\sqrt{2}s\right)d\bar{x}ds,$$

the contribution from this portion to $s_3(t)$ is

(3.19) $$s_{II}(t) = \frac{c}{|t|^3}\int_{x_0}^{x_1}(x-m)^2e^{-Q}\left[1-\frac{3}{\pi}\cos^{-1}\frac{xt}{2(x-m)}\right]dx$$

$$= \frac{c}{|t|^3}\int_{x_0}^{x_1}G'(x)\left[1-\frac{3}{\pi}\cos^{-1}\frac{xt}{2(x-m)}\right]dx,$$

where the limits of integration are e.g. for $t<0$, $x_0=\dfrac{m}{1-t}$ and $x_1=\dfrac{2m}{2-t}$. Integrating by parts, we obtain

$$s_{II}(t) = \frac{c}{|t|^3}G\left(\frac{2m}{2-t}\right) + \frac{3cm}{\pi t^2}\int_{m/1-t}^{2m/2-t}\frac{G(x)dx}{|x-m|\sqrt{4(x-m)^2-x^2t^2}}.$$

Similarly, for $0<t<1$, we have

$$s_{II}(t) = -\frac{c}{t^3} G\left(\frac{2m}{2-t}\right) + \frac{3cm}{\pi t^2}\int_{2m/2-t}^{m/1-t} \frac{G(x)dx}{(x-m)\sqrt{4(x-m)^2-x^2t^2}}.$$

And for $1<t<2$

$$s_{II}(t) = \frac{c}{t^3}\left[G(\infty) - G\left(\frac{2m}{2-t}\right)\right] + \frac{3cm}{\pi t^2}\int_{2m/2-t}^{\infty} \frac{G(x)dx}{(x-m)\sqrt{4(x-m)^2-x^2t^2}}.$$

However, for $2<t<\infty$, the inequalities $\bar{x}/\sqrt{2}<s<\bar{x}\sqrt{2}$ yield $\bar{x}/2<(\bar{x}-m)/t<\bar{x}$ and consequently the left half becomes impossible because of $0<\bar{x}-m<\bar{x}$, $t>2$. There is no portion for II and $s_{II}(t)=0$.

Summing up all the above we obtain

$$s_3(t) = s_I(t)+s_{II}(t) = \frac{c}{|t|^3} G(m) + \int_{m/1-t}^{2m/2-t} H(x,t)\,dx \qquad (t<0), \tag{3.20}$$

$$s_3(t) = -\frac{c}{t^3} G(m) + \int_{2m/2-t}^{m/1-t} H(x,t)\,dx \qquad (0<t<1), \tag{3.21}$$

$$s_3(t) = \frac{c}{t^3}[G(\infty)-G(m)] + \int_{2m/2-t}^{\infty} H(x,t)\,dt \qquad (1<t<2), \tag{3.22}$$

$$s_3(t) = \frac{c}{t^3}[G(\infty)-G(m)] \qquad (2<t<\infty), \tag{3.23}$$

where

$$H(x,t) = \frac{3cm}{\pi t^2}\,\frac{G(x)}{|x-m|\sqrt{4(x-m)^2-x^2t^2}}. \tag{3.24}$$

We ought now to compute $G(m)$ and $G(\infty)$ for $t \gtrless 0$. However, before computing

$$G(m) = \int_0^m (x-m)^2 e^{-Q} dx \quad \text{with} \quad Q = \frac{3(x-m)^2}{t^2} + \frac{3}{2}(x-a)^2, \tag{3.25}$$

we notice an evident lemma :

$$U \equiv -\int_0^m e^{-Q} Q'(x)\,dx = \int_0^m \left[-\frac{6}{t^2}(x-m) - 3(x-a)\right] e^{-Q} dx = e^{-Q}\Big|_0^m = e^{-\frac{3}{2}\lambda^2} - \mathrm{Exp}_1(t),$$

where

$$\mathrm{Exp}_1(t) = \exp\left[-\frac{3m^2}{t^2} - \frac{3}{2}a^2\right].$$

Also, in view of $a=m-\lambda$,

$$U = -\left(\frac{6}{t^2}+3\right)\int_0^m (x-m)e^{-Q}dx - 3\lambda\int_0^m e^{-Q}dx \equiv -\frac{3(2+t^2)}{t^2}X - 3\lambda Z \text{ say.}$$

Consequently

$$X = [\mathrm{Exp}_1(t) - e^{-\frac{3}{2}\lambda^2} - 3\lambda Z]/3(1+2/t^2).$$

Now we can easily compute $G(m)$: Putting $G(m)\equiv t^2\, Y$ and integrating by parts, we obtain

$$Y=\int_0^m\left[-\frac{1}{6}\exp\left(-\frac{3(x-m)^2}{t^2}\right)\right]'(x-m)\exp\left(-\frac{3}{2}(x-a)^2\right)dx$$
$$=-\frac{m}{6}\exp\left[-\frac{3m^2}{t^2}-\frac{3}{2}a^2\right]+\frac{1}{6}\int_0^m e^{-Q}dx-\frac{1}{2}\int_0^m(x-m)(x-a)e^{-Q}dx,$$

what equals by the previous abbreviations

$$-\frac{m}{6}\mathrm{Exp}_1(t)+\frac{1}{6}Z-\frac{t^2}{2}Y-\frac{\lambda}{2}X.$$

Hence, we get

$$\left(1+\frac{t^2}{2}\right)Y=\frac{1}{6}Z-\frac{m}{6}\mathrm{Exp}_1(t)-\frac{\lambda}{2}X,$$

in which X being substituted and solved for Y, we attain at length

$$G(m)=\frac{1}{3(1+2/t^2)^2}\left\{\lambda e^{-\frac{3}{2}\lambda^2}-\left(\lambda+m+\frac{2m}{t^2}\right)\mathrm{Exp}_1(t)+\left(3\lambda^2+1+\frac{2}{t^2}\right)Z\right\}, \tag{3.26}$$

where

$$Z=\frac{1}{\sqrt{1+2/t^2}}\exp\left(\frac{-3\lambda^2}{2+t^2}\right)\sqrt{\frac{2\pi}{3}}\left\{\Phi\left(\frac{\sqrt{3}\lambda}{\sqrt{1+2/t^2}}\right)+\Phi\left(\frac{\sqrt{3}(a+2m/t^2)}{\sqrt{1+2/t^2}}\right)-1\right\}$$

and

$$\mathrm{Exp}_1(t)=e^{-\frac{3}{2}a^2}\exp\left(-\frac{3m^2}{t^2}\right)=e^{-\frac{3}{2}a^2}\left[1-\frac{3m^2}{t^2}+\frac{9m^4}{2t^4}-\cdots\right].$$

All these being expanded in power series of t^{-2},

$$G(m)\cong\frac{\lambda}{3}e^{-\frac{3}{2}\lambda^2}-\frac{1}{3}(\lambda+m)e^{-\frac{3}{2}a^2}+\left(\lambda^2+\frac{1}{3}\right)\sqrt{\frac{2\pi}{3}}\left[\Phi(\sqrt{3}\lambda)+\Phi(\sqrt{3}a)-1\right] \tag{3.27}$$
$$-\frac{1}{t^2}\left\{\left(\lambda^2+\frac{5}{3}\right)\lambda e^{-\frac{3}{2}\lambda^2}-\left[(\lambda+m)(\lambda^2+m^2+1)+\frac{2}{3}\lambda\right]e^{-\frac{3}{2}a^2}\right.$$
$$\left.+(3\lambda^4+6\lambda^2+1)\sqrt{\frac{2\pi}{3}}\left[\Phi(\sqrt{3}\lambda)+\Phi(\sqrt{3}a)-1\right]\right\}+O\left(\frac{1}{t^4}\right).$$

Therefore, if $|t|$ great, $G(m)$ may be considered as a constant, whereas, if $|t|$ not so large, some further terms of t^{-2} should be supplemented by (26) or (27).

Quite similarly

$$G(\infty)=\frac{1}{3(1+2/t^2)^2}\left\{-\left(\lambda+m+\frac{3m}{t^2}\right)\mathrm{Exp}_1(t)+\left(3\lambda^2+1+\frac{2}{t^2}\right)Z\right\}, \tag{3.28}$$

where

$$Z=\frac{1}{\sqrt{1+2/t^2}}\exp\left(\frac{-3\lambda^2}{2+t^2}\right)\sqrt{\frac{2\pi}{3}}\,\Phi\left(\frac{\sqrt{3}(a+2m/t^2)}{\sqrt{1+2/t^2}}\right).$$

Rather it is more desirable to obtain the expansion of $G(\infty)-G(m)$ more in details. In fact it is

$$(3.29)\quad G(\infty)-G(m)=\left(\lambda^2+\frac{1}{3}\right)\sqrt{\frac{2\pi}{3}}\,(1-\Phi(\sqrt{3}\lambda))-\frac{\lambda}{3}e^{-\frac{3}{2}\lambda^2}$$

$$-\frac{1}{t^2}\left[(3\lambda^4+6\lambda^2+1)\sqrt{\frac{2\pi}{3}}(1-\Phi(\sqrt{3}\lambda))-\left(\lambda^2+\frac{5}{3}\right)\lambda e^{-\frac{3}{2}\lambda^2}\right]$$

$$+\frac{1}{2t^4}\left[(9\lambda^6+45\lambda^4+45\lambda^2+5)\sqrt{\frac{2\pi}{3}}\,(1-\Phi(\sqrt{3}\lambda))-(3\lambda^4+14\lambda^2+11)\lambda e^{-\frac{3}{2}\lambda^2}\right]$$

$$-\frac{1}{2t^6}\left\{\left(9\lambda^8+84\lambda^6+210\lambda^4+140\lambda^2+\frac{35}{3}\right)\sqrt{\frac{2\pi}{3}}\,(1-\Phi(\sqrt{3}\lambda))\right.$$

$$\left.-\left(3\lambda^6+27\lambda^4+\frac{185}{3}\lambda^2+31\right)\lambda e^{-\frac{3}{2}\lambda^2}\right\}+O\left(\frac{1}{t^8}\right)$$

$$=A-\frac{B}{t^2}+\frac{C}{2t^4}-\frac{D}{2t^6},\ \text{say.}$$

In order to compute upper significant limits t_1, we have to substitute the above expression in (23). Thus we get

$$(3.30)\quad S_3(t_1)=\int_{t1}^{\infty}s_3(t)dt=c\int_{t1}^{\infty}\left[\frac{A}{t^3}-\frac{B}{t^5}+\frac{C}{2t^7}-\frac{D}{2t^9}\right]dt$$

$$=c\left[\frac{A}{2t_1^2}-\frac{B}{4t_1^4}+\frac{C}{12t_1^6}-\frac{D}{16t_1^8}\right].$$

This being equated to $\alpha/2$, we have to solve by Horner the equation for t_1^{-2}:

$$(3.31)\quad c\left(\frac{A}{t_1^2}-\frac{B}{2t_0^4}+\frac{C}{6t_1^6}-\frac{D}{8t_1^8}\right)=\alpha(=0.1,\ 0.05,\ 0.01\ \&\text{c.}).$$

By this procedure we get the following Table regarding upper limits t_1 for $n=3$:

species \ level	$\alpha=0.1$	$\alpha=0.05$	$\alpha=0.01$
$a=0$ (truncated at centroid)	2.09	3.04	6.97
$a=0.6745$ (〃 the left quartile)	2.25	3.38	7.86
$a=-0.6745$ (〃 the right quartile)	1.89	2.75	6.30
the whole untruncated N. D.	2.92	4.30	9.925

On the contrary the lower significant limits being large in absolute value, it would be more legitimately treated rather with few expansion. For this purpose making use of (20), we have for $t_0<0$

$$(3.32)\quad S_3(t_0)=\int_{-\infty}^{t_0}s_3(t)\,dt$$

$$=\int_{-\infty}^{t_0}\frac{cG(m)}{-t^3}dt+\frac{3cm}{\pi}\int_{-\infty}^{t_0}\frac{dt}{t^2}\int_{m/1-t}^{2m/2-t}\frac{G(x)\,dx}{(m-x)\sqrt{4(x-m)^2-x^2t^2}}=\frac{\alpha}{2},$$

where

$$G(x)=G(x,t)=\int_0^x(y-m)^2e^{-Q}dy\quad\text{with}\quad Q=\frac{3(y-m)^2}{t^2}+\frac{3}{2}(y-a)^2.$$

For a large $|t_0|$, $G(m)=G(m, t)$ might be assumed to be constant and with the first term of (32) only it may serve as a rough estimation of $S_3(t_0)$. Thus we get $t_0=-\sqrt{cG/\alpha}$. However, more elaborately we should take the second double integral into account. So we must estimate $G(x)$ in the integrand of (32). Since its integrand contains already t^{-2} as factor and besides the integration variable x is small because $m/(1-t)<x<2m/(2-t)$, so that $x=O\left(\frac{m}{t}\right)$ in $-\infty<t<t_0$ with large $|t_0|$, we may put approximately

$$G(x)\cong\int_0^x(y-m)^2\left[1-\frac{3(y-m)^2}{t^2}\right]\exp\left(-\frac{3}{2}(y-a)^2\right)dy. \tag{3. 33}$$

Moreover, neglecting those terms of smaller magnitude than t^{-2} as well as x^2 in order, we have

$$G(x)\cong\left(1-\frac{3m^2}{t^2}\right)e^{-\frac{3}{2}a2}\left[m^2x-\left(m-\frac{3}{2}a\right)x^2\right] \tag{3. 34}$$

and consequently $-3m^2/t^2$ may be also neglected.

Now, changing the order of the repeated integrations in (32), we obtain

$$\int_{-\infty}^{t_0}dt\int_{m/1-t}^{2m/2-t}dx=\int_{m/1-t_0}^{2m/2-t_0}dx\int_{2-2m/x}^{t_0}dt+\int_0^{m/1-t_0}dx\int_{2-m/x}^{1-m/x}dt=\text{(i)}+\text{(ii)}.$$

We utilize again the formula

$$\int\frac{dt}{t^2\sqrt{4(x-m)^2-x^2t^2}}=\frac{\sqrt{4(x-m)^2-x^2t^2}}{-4(x-m)^2t}.$$

Since the new integrands vanish at their lower limits, we have

$$\text{(i)}=\int_{m/1-t_0}^{2m/2-t_0}\left[m^2x-\left(m-\frac{3}{2}a\right)x^2\right]\frac{\sqrt{4(x-m)^2-x^2t_0^2}}{-4(m-x)^3t_0}dx.$$

Putting $x=m/(1-\theta t_0)$ and integrating about θ from $1/2$ to 1, we get

$$\text{(i)}=-\frac{3m^2}{8t_0^3}-m\left(m-\frac{3}{2}a\right)\frac{\sqrt{3}}{4t_0^4},$$

on neglecting those terms whose powers are higher than t_0^{-4}. Also

$$\text{(ii)}=\int_0^{m/1-t_0}\left[m^2x-\left(m-\frac{3}{2}a\right)x^2\right]\left[\frac{\sqrt{4(x-m)^2-x^2t^2}}{-4(m-x)^3t}\right]_{t=(x-m)/x}dx\cong-\frac{\sqrt{3}m^2}{12t_0^3}-\frac{\sqrt{3}}{8t_0^4}m\left(m-\frac{3}{4}a\right).$$

Therefore, we obtain after all

$$S_3(t_0)\cong\frac{cG(m)}{2t_0^2}-\frac{3cm^3}{4\pi}e^{-\frac{3}{2}a2}\left(\frac{3}{2}+\frac{1}{\sqrt{3}}\right)\frac{1}{t_0^3}-\frac{9c\sqrt{3}m^2}{16\pi}\left(m-\frac{5}{4}a\right)\frac{1}{t_0^4}, \tag{3. 35}$$

whose last term however may be omitted, unless the term of t_0^{-2} in $G(m)$ by (27) be taken into account. Adopting the first one or two terms and equating it to $\alpha/2$, we may compute the lower limit t_0 approximately.

For the sake of comparison, recapitulating all the significant limits above obtained, we get the following Table :

The critical values of Student's ratios for several truncated N. D.

species \ level / size		$\alpha=0.1$ $n=2$	$\alpha=0.1$ $n=3$	$\alpha=0.05$ $n=2$	$\alpha=0.05$ $n=3$	$\alpha=0.01$ $n=2$	$\alpha=0.01$ $n=3$
1° (truncated at the centroid)	t_0	-13.48	-8.26	-28.23	-11.48	-145.6	-24.15
	t_1	$+4.11$	$+2.09$	$+8.26$	$+3.04$	$+41.40$	$+6.97$
2° (truncated at the left quartile)	t_0	-10.92	-6.38	-22.67	-8.69	-116.4	-18.77
	t_1	$+4.74$	$+2.25$	$+9.58$	$+3.38$	$+48.00$	$+7.86$
3° (truncated at the right quartile)	t_0	-23.71	-9.90	-56.72	-13.45	-186.6	-28.19
	t_1	$+3.52$	$+1.89$	$+7.35$	$+2.75$	$+36.63$	$+6.30$
4° (untruncated N. D.)	$t_0,\ t_1$	∓ 6.31	∓ 2.92	∓ 12.71	∓ 4.30	∓ 63.66	∓ 9.925

Among three species, the case 2°, the truncated one at the left quartile and thus reserving almost the original figure, behaves as nearly as the whole N. D. : The enlargement of the lower limits t_0, as well as the lessening of the upper limits t_1, both are rather moderate. However, in the case 3°, the truncated one at the right qurtile, so that the original figure is almost erased away, the departure from the ordinary N. D. 4° is very striking.

4. *A N. D. Truncated at Both Ends.* If a N. D.

$$\varphi\left(\frac{x-a}{\sigma}\right) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\} (a>0) \text{ with d. f. } \Phi(z) = \int_{-\infty}^{z}\varphi(z)\,dz$$

being not only at the left negative side truncated, but also at right the part $x>2a$ erased out, the remaining portion $0<x<2a$ be considered as universe, its fr. f. becomes

$$(4.1)\qquad f(x) = \frac{1}{D\sigma}\varphi\left(\frac{x-a}{\sigma}\right) = \frac{1}{\sqrt{2\pi}\,\sigma D}\exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}, \qquad (0<x<2a)$$

where

$$(4.2)\qquad D = \int_0^{2a}\varphi\left(\frac{x-a}{\sigma}\right)\frac{dx}{\sigma} = \Phi\left(\frac{a}{\sigma}\right) - \Phi\left(-\frac{a}{\sigma}\right) = 2\Phi\left(\frac{a}{\sigma}\right) - 1$$

and the d. f.

$$(4.3)\qquad F(x) = \int_0^x f(x)\,dx = \frac{1}{D}\left[\Phi\left(\frac{x-a}{\sigma}\right) - \Phi\left(-\frac{a}{\sigma}\right)\right]$$

with $F(0)=0$, $F(2a)=1$. Writing $\lambda = 2\varphi\left(\frac{a}{\sigma}\right)/D(>0)$, the first four moments are given by

$$\nu_1 = a\,(= \text{parent mean } m),\ \nu_2 = a^2 - a\sigma\lambda + \sigma^2,\ \nu_3 = a^3 - 3a^2\sigma\lambda + 3a\sigma^2,$$
$$\nu_4 = a^4 - 7a^3\sigma\lambda + 6a^2\sigma^2 - 3a\sigma^3\lambda + 3\sigma^4$$

and the central moments

$$\mu_1 = 0,\ \mu_2 = \sigma^2 - a\sigma\lambda,\ \mu_3 = 0,\ \mu_4 = 3\sigma^4 - 3a\sigma^3\lambda - a^3\sigma\lambda,$$

so that

$$\frac{\mu_3^2}{\mu_2^3} = 0 \quad \text{and} \quad \frac{\mu_4}{\mu_2^2} = 3 - \frac{r\lambda(3r\lambda - 3 + r^2)}{(1-r\lambda)^2},$$

where $r = a/\sigma > 0$. Hence the kurtosis is $\gtreqless 3$ according as

$$\lambda \lesseqgtr (3-r^2)/3r = (3\sigma^2 - a^2)/3a\sigma.$$

If the original S. D. σ be taken as unit, then $2r = 2a/\sigma$ expresses its range. The new variable $\xi = x/\sigma$ has the fr. f.

$$f(\xi) = \frac{1}{\sqrt{2\pi}\,D} \exp\left\{-\frac{1}{2}(\xi - r)^2\right\} \qquad 0 < \xi < 2r \tag{4.4}$$

where $D = 2\Phi(r) - 1$. When r becomes sufficiently large, this distribution exhausts almost the original whole N. D. Hence it seems that the critical values of Student ratio for (4) would approach towards the classical values, as r increases. This approximation shall be the more nearer, the greater r becomes, as will be shown below.

Suppose we have drawn from universe (1) a sample with mean $\bar{x}$ and S. D. s. If the size be e. g. $n = 2$, all $x_1,\ x_2,\ \bar{x}$ are between $(0, 2a)$ with the probability

$$dp = f(x_1)f(x_2)\,dx_1 dx_2 = \frac{1}{2\pi D^2\sigma^2} \exp\left[-\frac{(x_1-a)^2 + (x_2-a)^2}{2\sigma^2}\right] dx_1 dx_2,$$
$$dP = \frac{4d\bar{x}ds}{2\pi D^2\sigma^2} \exp\left[-\frac{(\bar{x}-a)^2 + s^2}{\sigma^2}\right].$$

Or, if s be transformed into Student's t with Jacobian $|J| = |\bar{x} - a|/t^2$, we obtain, as the fr. f. for Student's ratio $t = (\bar{x} - a)/s$,

$$s(t) = \frac{2}{\pi D^2\sigma^2} \int_{x_0}^{x_1} \frac{|x-a|}{t^2} \exp\left\{-\left(1 + \frac{1}{t^2}\right)\frac{(x-a)^2}{\sigma^2}\right\} dx, \tag{4.5}$$

where the limits of integration are determined from the condition that $0 < x_1,\ x_2 < 2a$. If $0 < \bar{x} < a$, $t < 0$, the condition $s < \bar{x}$ yields $x_0 = \dfrac{a}{1-t}$, $x_1 = a$. Hence, on writing $\dfrac{\sqrt{2}}{\sigma}\sqrt{1 + \dfrac{1}{t^2}}\,(a - x) = z\,(>0)$, the integral of (5) reduces to

$$s(t) = \frac{1}{\pi D^2(1+t^2)} \int_0^{z_1} z e^{-\frac{z^2}{2}} dz = \frac{1}{\pi D^2(1+t^2)} [1 - e^{-z_1^2/2}], \tag{4.6}$$

where

$$z_1 = \frac{a}{\sigma} \frac{\sqrt{2(1+t^2)}}{1-t} \qquad \left(-\infty < t < 0,\ \frac{a}{\sigma} < z_1 < \frac{\sqrt{2}a}{\sigma}\right). \tag{4.7}$$

In fact z_1 becomes $\sqrt{2}a/\sigma$ either when $t \to -\infty$ or $t \to -0$, while it becomes a minimum a/σ when $t = -1$.

Similarly, if $a < \bar{x} < 2a$, $t > 0$ and by the condition that (x_1, x_2) lies insides the square formed by two sides $0 < x_1 < 2a$, $0 < x_2 < 2a$, we see that $x_0 = a < \bar{x} < \frac{a(1+2t)}{1+t} = x_1$ in (5). We get again on writing $\frac{\sqrt{2}}{\sigma}\sqrt{1+\frac{1}{t^2}}\,(x-a) = z\,(>0)$, just the same expression for $s(t)$ as (6), but now instead of (7), only its sign t being changed,

$$(4.8) \qquad z_1 = \frac{a}{\sigma}\frac{\sqrt{2(1+t^2)}}{1+t} \qquad \left(0 < t < \infty,\ \frac{a}{\sigma} < z_1 < \frac{\sqrt{2}a}{\sigma}\right).$$

Therefore, we have $s(-t) = s(t)$ and the fr. f. $s(t)$ is symmetrical with respect to the origin. However we obtain e. g. for $t < 0$

$$(4.9) \qquad s'(t) = \frac{1}{\pi D^2(1+t^2)}\left[\frac{-t}{1+t^2}(1-e^{-z_1^2/2}) + \frac{2a^2}{\sigma^2}\frac{1+t}{(1-t)^3}e^{-z_1^2/2}\right]$$

which is evidently positive, if $0 > t > -1$. Also, if $-1 > t > -\infty$, rewriting the above expression as

$$s'(t) = \frac{(-t)e^{-z_1^2/2}}{\pi D^2(1+t^2)^2}\left[e^{z_1^2/2} - \left(1+\frac{a^2}{\sigma^2}\frac{1+t^2}{(1-t)^2}\frac{2(1+t)}{(1-t)t}\right)\right],$$

we see its positivity, because

$$e^{z_1^2/2} > 1+\frac{z_1^2}{2} = 1+\frac{a^2}{\sigma^2}\frac{1+t^2}{(1-t)^2} > 1+\frac{a^2}{\sigma^2}\frac{1+t^2}{(1-t)^2}\frac{2(1+t)}{(1-t)t},$$

since $y = \frac{2(1+t)}{(1-t)t}$ is positive in $-\infty < t < -1$, but its maximum is only $0.34\cdots$.

Thus $s(t)$ is monotonic increasing in $-\infty < t < 0$, whereas it decreasing in $0 < t < \infty$. Also

$$s'(-0) = -s'(+0) = \frac{2a^2}{\pi D^2\sigma^2}e^{-a^2/\sigma^2} > 0,$$

and the two branches make a cusp at $t = 0$.

Further the d. f. is given in view of (6) (8) for $t_1 > 0$ by

$$(4.10) \qquad S(-t_1) = \int_{-\infty}^{-t_1} s(t)\,dt = \int_{t_1}^{\infty} s(t)\,dt$$
$$= \frac{1}{\pi D^2}\int_{t_1}^{\infty}\left[\frac{1}{1+t^2} - \frac{1}{1+t^2}\exp\left(-\frac{a^2}{\sigma^2}\frac{(1+t^2)}{(1+t)^2}\right)\right]dt$$
$$= \frac{1}{\pi D^2}\left[\frac{\pi}{2} - \tan^{-1}t_1 - \int_{t_1}^{\infty}\frac{1}{1+t^2}\exp\left(-\frac{r^2(1+t^2)}{(1+t)^2}\right)\right]dt,$$

where $r = a/\sigma$. Expanding all in powers of t^{-1}, we have

$$\frac{\pi}{2} - \tan^{-1}t_1 = \frac{\pi}{2} - \cot^{-1}\frac{1}{t_1} = \frac{1}{t_1} - \frac{1}{3t_1^3} + \frac{1}{5t_1^5} - \cdots\cdots,$$

$$\frac{1}{1+t^2}=\frac{1}{t^2}-\frac{1}{t^4}+\frac{1}{t^6}-\cdots,\quad \frac{1+t^2}{(1+t)^2}=1-\frac{2}{t}+\frac{4}{t^2}-\frac{6}{t^3}+\cdots\cdots,$$

so that

$$\exp\left[-\frac{r^2(1+t^2)}{(1+t)^2}\right]=e^{-r^2}\left[1+\frac{2r^2}{t}+\frac{1}{t^2}(2r^4-4r^2)+\frac{1}{t^3}\left(\frac{4}{3}r^6-8r^4+6r^2\right)\right.$$
$$\left.+\frac{1}{t^4}\left(\frac{2}{3}r^8-8r^6+20r^4-8r^2\right)+\cdots\cdots\right].$$

Substituting all these in (10) and integrating we get

$$(4.11)\quad \int_{t_1}^{\infty}s(t)dt=\frac{1}{\pi D^2}\left\{\frac{1}{t_1}(1-e^{-r^2})-\frac{1}{t_1^2}r^2e^{-r^2}-\frac{1}{3t_1^3}\left[1+(2r^4-4r^2-1)e^{-r^2}\right]\right.$$
$$\left.-\frac{r^2}{t_1^4}\left(\frac{1}{3}r^4-2r^2+1\right)e^{-r^2}+\frac{1}{5t_1^5}\left[1-\left(\frac{2}{3}r^8-8r^6+18r^4-4r^2+1\right)e^{-r^2}\right]\right\}$$
$$=\frac{1}{\pi D^2}\left[\frac{A}{t_1}-\frac{B}{t_1^2}-\frac{C}{t_1^3}-\frac{D}{t_1^4}+\frac{E}{t_1^5}\right],\qquad \text{say.}$$

Assumed e. g. $r=a/\sigma=0.5,\ 0.6745,\ 1,\ 2,\ 3,\ \cdots,$ all coefficients are known. The above expression being equated to $\alpha/2$ ($\alpha=0.1,\ 0.05,\ 0.01$), or what is the same thing as the last square brackets being equated to

$$\frac{\alpha}{2}\pi D^2\equiv\alpha\,\frac{\pi}{2}\,(2\Phi(r)-1)^2$$

and solved for $1/t_1=x_1$ by Horner, we can determine the critical values $t_\alpha=\pm 1/x_\alpha$. The results are obtained as the following

Table of the critical values t_α for T. N. D. (case $n=2$)

$r=a/\sigma$	$\alpha=0.1$	$\alpha=0.05$	$\alpha=0.01$
0.5	10.71	20.22	96.95
0.6745	9.93	19.16	93.54
1	9.28	17.89	86.94
1.5	7.88	15.48	76.14
2	6.97	13.81	68.67
3	6.45	12.83	64.00
4	6.41	12.76	63.67
∞ (untruncated N. D.)	6.314	12.706	63.657

The last classical values are readily obtained by putting $r=\infty$ in (10). In fact, since for $r=\infty$ the second term of (10) reduces to naught, it holds

$$S(t_1)=\frac{1}{\pi D^2}\left(\frac{\pi}{2}-\tan^{-1}t_1\right)=\frac{\alpha}{2}.$$

Also $D=2\Phi(\infty)-1=1$, so that

$$\tan^{-1}t_1=\frac{\pi}{2}(1-\alpha),\qquad \text{i. e.}\qquad t_1=\tan\frac{\pi}{2}(1-\alpha).$$

Putting here $\alpha = 0.1,\ 0.05,\ 0.01$, we obtain the classical values

$$t_{0\,1} = \tan 81^\circ = 6.314, \quad t_{0\,05} = \tan 85^\circ 30' = 12.706, \quad t_{0\,01} = \tan 89^\circ 6' = 63.657.$$

To obtain similar values for $5 < r < \infty$, we ought to take a number of terms of negative powers much more than 5. But the case $r = 4$ being already enough near to $r = \infty$ we omit the further trouble.

In practice, the universe is not necessarily the complete whole N. D. Rather it is very probable that materially it is a truncated one. Notwithstanding, referring to the classical Table for the untruncated N. D., to speak particularly the adoption or rejection of the null-bypothesis by a subtle difference of the decimal figures, it is quite of nonsense, in case that the truncation is suspicious.

5. *A U-shaped Distribution as Universe.*

The fr. f. that interested the author, as a little peculiar one, is

$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}} \quad (0 < x < 1)$$

with $\int_0^1 f(x)\,dx = 1$, $E(x)$ (= mean = antimode = median) = $\frac{1}{2}$, $D^2(x) = \frac{1}{8}$ (Fig. 7). Supposing a sample $\{x_1, x_2\}$ drawn from this universe with mean $\bar{x}$ and S. D. s, its probability is

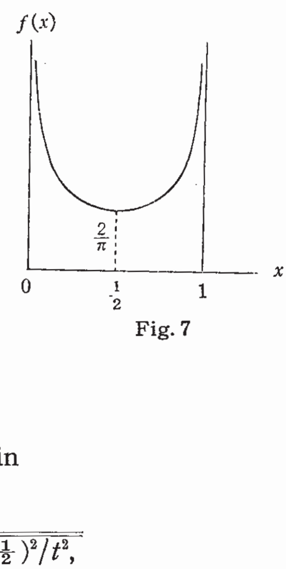

Fig. 7

$$dp = \frac{dx_1\,dx_2}{\pi^2\sqrt{x_1x_2(1-(x_1+x_2)+x_1x_2)}},$$

so that

$$dP = \frac{4\,d\bar{x}\,ds}{\pi^2\sqrt{\bar{x}^2 - s^2}\sqrt{(1-\bar{x})^2 - s^2}}. \tag{5.1}$$

Or, replacing s by Student's ratio $t = \left(\bar{x} - \frac{1}{2}\right)/s$, we obtain

$$dP = \frac{4}{\pi^2}\frac{|x - \frac{1}{2}|\,dx\,dt/t^2}{\sqrt{x^2 - (x-\frac{1}{2})^2/t^2}\sqrt{(1-x)^2 - (x-\frac{1}{2})^2/t^2}},$$

and the corresponding Student's fr. f.

$$s(t) = \frac{4}{\pi^2}\int_{x_0}^{x_1}\frac{|x - \frac{1}{2}|\,dx}{\sqrt{x^2t^2 - (x-\frac{1}{2})^2}\sqrt{(1-x)^2t^2 - (x-\frac{1}{2})^2}}. \tag{5.2}$$

The limits of integration are determined, as stated before, by conditions of boundaries (cf. Fig. 1): For $-\infty < t < 0$, $0 < x < \frac{1}{2}$ and the condition $\sqrt{2}s < \sqrt{2}x$ yields $x_0 = 1/2(1-t)$, $x_1 = \frac{1}{2}$, while, for $0 < t < \infty$, $\frac{1}{2} < x < 1$ and the condition $\sqrt{2}s < \sqrt{2}(1-x)$ gives $x_0 = \frac{1}{2}$, $x_1 = 1 - \frac{1}{2(1+t)} = \frac{1+2t}{2(1+t)}$. Thus

$$s(t)=\frac{4}{\pi^2}\int_{1/2(1-t)}^{\frac{1}{2}}\frac{(\frac{1}{2}-x)dx}{\sqrt{x^2t^2-(x-\frac{1}{2})^2}\sqrt{(1-x)^2t^2-(x-\frac{1}{2})^2}},\qquad t<0, \tag{5.3}$$

as well as

$$s(t)=\frac{4}{\pi^2}\int_{\frac{1}{2}}^{\frac{1+2t}{2+2t}}\frac{(x-\frac{1}{2})dx}{\sqrt{(1-x)^2t^2-(x-\frac{1}{2})^2}\sqrt{x^2t^2-(x-\frac{1}{2})^2}},\qquad t>0. \tag{5.4}$$

Also, on writing $x=1/2(1-\vartheta t)$, (3) reduces to

$$s(t)=\frac{4}{\pi^2}\int_0^1\frac{\vartheta\, d\vartheta}{\sqrt{1-\vartheta^2}\sqrt{(1-2\vartheta t)^2-\vartheta^2}},\qquad t<0, \tag{5.5}$$

whereas (4) becomes on putting $x=1-1/2(1+\vartheta t)$

$$s(t)=\frac{4}{\pi^2}\int_0^1\frac{\vartheta\, d\vartheta}{\sqrt{1-\vartheta^2}\sqrt{(1+2\vartheta t)^2-\vartheta^2}},\qquad t>0, \tag{5.6}$$

so that $s(-t)=s(t)$, and the fr. f. is symmetrical about the origin. Further, from (5) we obtain

$$s'(t)=\frac{4}{\pi^2}\int_0^1\frac{4\vartheta^2(1-2\vartheta t)dt}{\sqrt{1-\vartheta^2}\sqrt{(1-2\vartheta t)^2-\vartheta^2}^{\,3}},\qquad t<0, \tag{5.7}$$

which informs that $s'(t)>0$ and $s(t)$ increasing for $t<0$, and by symmetry $s'(t)<0$, $s(t)$ decreasing for $t>0$.

Moreover, if we make $t\to 0$ in (5) and (6) respectively, both denominators approach to $1-\vartheta^2$ and consequently $s(\pm 0)\to\infty$ and the fr. f. $s(t)$ behaves singularly at the origin (Fig. 8). Nevertheless we have

$$\int_{-\infty}^{\infty}s(t)dt=1. \tag{5.8}$$

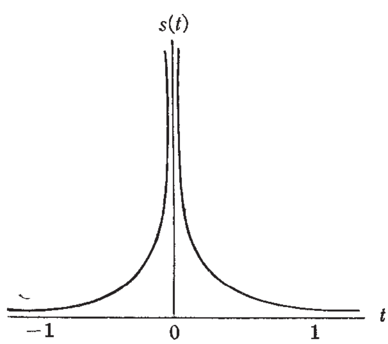

Fig. 8

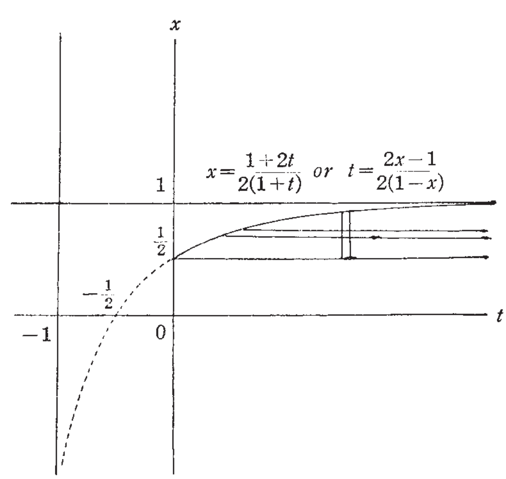

Fig. 9

This fact is evident, since (1) being a joint probability about $\bar{x}$, s, certainly (3) and (4) multiplied by dt denote the probabilities that Student's ratio lies between t and $t+dt$ and accordingly the total probability should become just unity.

However, it is rather desirable to ascertain this fact analytically. Now, the integral in (8) becomes, in view of (4) and by changing the order of integrations (Fig. 9),

$$\int_{-\infty}^{\infty} s(t)dt = 2\int_0^{\infty} s(t)dt = \frac{8}{\pi^2}J, \tag{5.9}$$

where

$$J=\int_0^{\infty}\frac{dt}{t^2}\int_{\frac{1}{2}}^{\frac{1+2t}{2+2t}}\frac{(x-\frac{1}{2})dx}{\sqrt{x^2-(x-\frac{1}{2})^2/t^2}\sqrt{(1-x)^2-(x-\frac{1}{2})^2/t^2}}$$
$$=\int_{\frac{1}{2}}^{1} dx\int_{\frac{2x-1}{2-2x}}^{\infty}\frac{(x-\frac{1}{2})dt/t^2}{x\sqrt{1-(x-\frac{1}{2})^2/x^2t^2}\sqrt{(1-x)^2-(x-\frac{1}{2})^2/t^2}}.$$

Further, on writing $\left(x-\dfrac{1}{2}\right)/(1-x)t=\zeta$ and $\dfrac{1-x}{x}=k$, we get

$$J=\int_0^1\frac{dk}{1+k}\int_0^1\frac{d\zeta}{\sqrt{1-\zeta^2}\sqrt{1-k^2\zeta^2}}=\int_0^1\frac{dk}{1+k}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}\quad(\zeta=\sin\varphi). \tag{5.10}$$

Thus the inner integral is nothing but an elliptic integral of the first kind $F\left(k, \dfrac{\pi}{2}\right)$, which becomes ∞ for $k=1$. Really, by the binomial expansion and Wallis' formula we obtain

$$F\left(k,\frac{\pi}{2}\right)=\sum_{\nu}\frac{2\nu!}{2^{2\nu}\nu!^2}\int_0^{\frac{\pi}{2}}k^{2\nu}\sin^{2\nu}\varphi d\varphi=\sum_{\nu}\frac{\pi}{2}\frac{(2\nu!)^2}{(2^{\nu}\nu!)^4}k^{2\nu}=\sum_{\nu}u_{\nu}k^{2\nu},\ \text{say.}$$

This series is convergent so far $0\leqq k<1$, because the coefficient becomes by Stirling

$$u_{\nu}\cong\frac{1}{2\nu}e^{-1/4\nu},$$

and consequently

$$\frac{u_{\nu+1}}{u_{\nu}}\cong\frac{\nu}{\nu+1}\exp\left(\frac{1}{4\nu(\nu+1)}\right)\to 1$$

as $\nu\to\infty$. Hence, the series $\sum u_{\nu}k^{2\nu}$ converges when $k^2<1$, whereas for $k^2=1$ Raabe's test yields

$$\nu\left(\frac{u_{\nu+1}}{u^{\nu}}-1\right)\cong-1,$$

so that $\lim\limits_{k\to1}F\left(k,\dfrac{\pi}{2}\right)=\sum u_r=\infty$. However, when $k^2<1$, we obtain

$$\int_0^k F\left(k,\frac{\pi}{2}\right)dk=\sum_{\nu}\frac{u_{\nu}}{2\nu+1}k^{2\nu+1}\equiv\sum_{\nu}v_{\nu}k^{2\nu+1}.$$

And this time, as $\nu \to \infty$,

$$v_\nu \cong \frac{1}{4\nu^2}$$

holds, so that even $\sum v_\nu$ converges, since $\sum \frac{1}{\nu^2} = \frac{\pi^2}{6}$. Thus $y = F\left(k, \frac{\pi}{2}\right)$ and accordingly (10) is surely integrable in $0 \leqq k \leqq 1$. Hence, we may compute it e. g. by making use of Simpson's parabolic formula with n (even$=18$, say) subsections. Of course, it occurs that in the last subsection the approximating parabola becomes impossible, because the last ordinate is $y_n = \infty$. However, this may be easily amended by taking in the last subsection another approximating curve, e. g. a cubic hyperbola $y = a/\sqrt{b-k}$, which passes through (k_ν, y_ν) : $\nu = n-2$, $n-1$ and $a > 0$, $b > 1$. Really we obtain $y = 1.30/\sqrt{1.06-k}$ and consequently $y_n = 4.4$ as a substitute of ∞. In this way it was ascertained that in fact (8) is valid.

Now we proceed to find the critical limits $\pm t_\alpha$, such that

$$S(t_\alpha) = \int_{-\infty}^{-t_\alpha} s(t)dt = \int_{t_\alpha}^{\infty} s(t)dt = \frac{\alpha}{2} \quad (\alpha = 0.1,\ 0.05,\ 0.01). \tag{5.11}$$

In fact t_α being tolerably large, we may seek an asymptotic expansion for (4) by taking up to a term of the first negative power, t_α^{-1}. Rewriting (4)

$$s(t) = \frac{4}{\pi^2}\int_{\frac{1}{2}}^{x_1} \frac{1}{\sqrt{(1-x)^2t^2 - (x-\frac{1}{2})^2}} \left(\frac{x-\frac{1}{2}}{xt}\right)\left[1-\left(\frac{x-\frac{1}{2}}{xt}\right)^2\right]^{-1/2} dx,$$

where $x_1 = \frac{1+2t}{2+2t}$. When $t > t_\alpha > 1$, $\frac{1}{2} < x < 1$, it holds that $0 < \left(x - \frac{1}{2}\right)/xt < 1/t < 1/t_\alpha < 1$. Therefore, we may expand the second square root into a binomial series and obtain

$$s(t) = \frac{4}{\pi^2}\int_{\frac{1}{2}}^{x_1}\left[\left(\frac{x-\frac{1}{2}}{xt}\right) + \frac{1}{2}\left(\frac{x-\frac{1}{2}}{xt}\right)^3 + \frac{3}{8}\left(\frac{x-\frac{1}{2}}{xt}\right)^5 + \cdots\right]\frac{dx}{\sqrt{(1-x)^2t^2 - (x-\frac{1}{2})^2}} \tag{5.12}$$

for $1 < t_\alpha < t$. Now that it is easy to verify

$$\int_{\frac{1}{2}}^{x_1} \frac{dx}{\sqrt{(1-x)^2t^2 - (x-\frac{1}{2})^2}} \cong \frac{2\log t + \log 2}{t}, \tag{5.13}$$

we may simply pick up the first term only in the large brackets of (12), and thus

$$s(t) \cong \frac{4}{\pi^2}\int_{\frac{1}{2}}^{x_1} \frac{x-\frac{1}{2}}{xt} \frac{dx}{\sqrt{(1-x)^2t^2 - (x-\frac{1}{2})^2}}. \tag{5.14}$$

But, we have also

$$\int_{\frac{1}{2}}^{x_1} \frac{dx}{x\sqrt{(1-x)^2t^2 - (x-\frac{1}{2})^2}} \cong \frac{\log t + 2\log 2}{t}, \tag{5.15}$$

which together with (13), (14) yields

$$s(t) \cong \frac{6\log t}{\pi^2 t^2} + 0\left(\frac{1}{t^2}\right). \tag{5.16}$$

Therefore, an approximate form for (11) is

$$S(t_\alpha)=\int_{t_\alpha}^{\infty}s(t)dt\simeq\frac{6}{\pi^2}\int_{t_\alpha}^{\infty}\frac{\log t}{t^2}dt\simeq\frac{6}{\pi^2}\frac{\log t_\alpha+1}{t_\alpha}. \tag{5.17}$$

This expression being equated to $\alpha/2$ and solved by Newton's method of successive approximations, the value of t_α would be obtained. Or, putting $x=1/t_\alpha$, we have to find the zero-point of

$$f(x)\equiv\frac{12}{\pi^2}(1-\log x)-\alpha=1.21585x(1-2.302585\log_{10}x)-\alpha. \tag{5.18}$$

Here $f'(x)=-1.21585\log x=-2.79961\log_{10}x$ and $f(0)=-\alpha<0$, $f'(0)=+\infty$ and $f(0.1)=0.1216-\alpha>0$ if $\alpha<0.1216$. Hence, there exists certainly a positive root of x between 0 and 0.1. We obtain thus $t_{0.1}=\pm62.5$, $t_{0.05}=\pm145.4$, $t_{0.01}=\pm955.1$, and further $t_{0.001}=\pm12705$ indeed! These figures might be seen as almost ∞ in practical statistics, so that the Student's test becomes here little worthy. However, it might be applicable in case of the U-shaped distribution, such as Pearson's example about the frequency distribution of degrees of cloudness. However, to suit this Pearson's unsymmetrical case, we must further investigate about the more general universe (the so-called Beta-distribution):

$$f(x)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}\qquad(a,b>0,\ a\neq b,\ 0<x<1),$$

whose investigation however is deferred for future.

6. *The Second Kind of Student's Ratio.*

Although the ratio the author had used in the previous paper:

$$\tau=\bar{x}/s\qquad\text{or}\qquad\tau=\bar{x}\sqrt{n-1}/s \tag{6.1}$$

differs from the proper Student ratio $t=(\bar{x}-m)\sqrt{n-1}/s$, yet both ratios have some similarity. So to speak, (1) being a short form of the latter, it may be called a second kind of Student's ratio. When the parent mean is unknown or ignored or even does not exist at all, so that the proper Student ratio becomes impossible, the Student-like ratio (1) can be still defined and may serve, as a first approximate test, to decide whether the sample does belong or not to an assigned specified universe, such as N. D. or Laplace-, Cauchy- distribution &c. The nature is vast: there may be a good many of universes. Even Cauchy's distribution is never an isolate exceptional one, but may be generalized, e. g. so as

$$f(x)=c/(1+|x-a|^b)$$

with $1<b<2$. Since this kind of functions behaves as $O(|x|^{-b})$ when $x\to\pm\infty$, it is integrable in $(-\infty,+\infty)$, so that the constants can be chosen so as to satisfy

$$\int_{-\infty}^{\infty}f(x)dx=1. \tag{6.2}$$

And accordingly it may define a certain fr. f., e. g. as $f(x)=\sqrt{3}/8\pi(1+|x|^{3/2})$. It is immaterial whether the integral

$$\int_{-\infty}^{\infty} xf(x)dx = E(x)$$

i. e. the mean exists or not. Really $E(x)$ itself is rather artificially though skilfully defined. Notwithstanding, it seems that we are liable to lay stress too much on this ficticious, conventionally made artificial flower, but not natural one. Probably there remain those fr. fs. with no mean still unawared. Therefore, as one example of its counterplan, we shall below consider the Student's ratio of the second kind applied to Cauchy's distribution, that is a Student-like distribution regarding the proper Student distribution for 2-sized sample.

Let $\bar{x}$ and s be the sample mean and S. D. of the 2-sized sample, which is drawn from a universe of Student's distribution

(6. 3) $$s_2(t) = 1/\pi(1+x^2).$$

Its probability is

$$dP=\int\frac{dx_1dx_2}{\pi^2(1+x_1^2)(1+x_2^2)}=\frac{1}{\pi^2}\frac{4d\bar{x}ds}{1+2(\bar{x}^2+s^2)+(\bar{x}^2-s^2)^2}.$$

Or transforming s into $\tau=\bar{x}/s$, we obtain

(6. 4) $$dP=\frac{4|x|\tau^2d\bar{x}d\tau}{\pi^2[\tau^4+2\tau^2(1+\tau^2)\bar{x}^2+(1-\tau^2)^2\bar{x}^4]},$$

where $\tau\gtrless 0$ according as $x\gtrless 0$. Hence the fr. f. of τ is given by

$$f(\tau)=\frac{4\tau^2}{\pi^2}\int_0^{\infty}\frac{xdx}{\tau^4+2\tau^2(1+\tau^2)x^2+(1-\tau^2)^2x^4}.$$

Writing $x^2=u$ and integrating, we get

(6. 5) $$f(\tau)=\frac{1}{\pi^2\tau}\log\left|\frac{1+|\tau|}{1-|\tau|}\right|.$$

Accordingly the fr. f. is an even function, which becomes logarithmically (hence integrably) $+\infty$ at $\tau=\pm 1$, but at the origin it behaves regularly and becomes minimum with $f(0)=\dfrac{2}{\pi^2}=0.20264$ (Fig. 10).

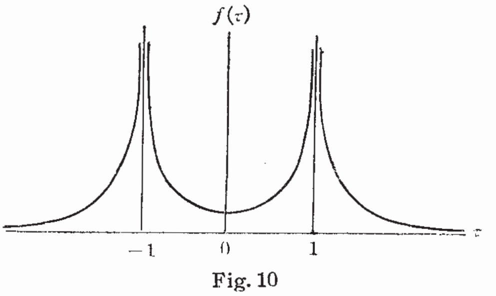

Fig. 10

Moreover, it can be readily shown that

$$(6.6)\qquad \int_{-\infty}^{1} f(\tau)d\tau=\int_{-1}^{0} f(\tau)d\tau=\int_{0}^{1} f(\tau)d\tau=\int_{1}^{\infty} f(\tau)d\tau=\frac{1}{4}.$$

In fact, we have by successive applications of integrations by parts

$$\int_0^1 f(\tau)d\tau=\frac{1}{\pi^2}\int_0^1\frac{1}{\tau}\log\frac{1+\tau}{1-\tau}d\tau=\frac{-2}{\pi^2}\int_0^1\frac{\log\tau}{1-\tau^2}d\tau=\frac{-2}{\pi^2}\int_0^1\sum_\nu \tau^{2\nu}\log\tau d\tau$$

$$=\frac{2}{\pi_2}\sum_\nu\int_0^1\frac{\tau^{2\nu}}{2\nu+1}d\tau=\frac{2}{\pi^2}\sum_\nu\frac{1}{(2\nu+1)^2}=\frac{2}{\pi^2}\frac{\pi^2}{8}=\frac{1}{4}.$$

And also

$$\int_1^\infty f(\tau)d\tau=\frac{1}{\pi^2}\int_1^\infty\frac{1}{\tau}\log\frac{\tau+1}{\tau-1}d\tau=\frac{1}{\pi^2}\int_0^1\frac{1}{\tau'}\log\frac{1+\tau'}{1-\tau'}d\tau'\left(\tau'=\frac{1}{\tau}\right)=\frac{1}{4},\ \text{\&c.}$$

Thus $\tau=\pm1$ are the upper- and lower-quartile, and

$$(6.7)\qquad \int_{-\infty}^{\infty} f(\tau)d\tau=1.$$

The d. f. may be found as follows: If $\tau_1>1$

$$(6.8)\qquad F(-\tau_1)=\int_{-\infty}^{-\tau_1} f(\tau)d\tau=\int_{\tau_1}^{\infty} f(\tau)d\tau=\frac{1}{\pi^2}\int_{\tau_1}^{\infty}\frac{1}{\tau}\left[\log\left(1+\frac{1}{\tau}\right)-\log\left(1-\frac{1}{\tau}\right)\right]d\tau$$

$$=\frac{2}{\pi^2}\left[\frac{1}{\tau_1}+\frac{1}{3^2\tau_1^3}+\frac{1}{5^2\tau_1^5}+\cdots\cdots\right].$$

Therefore, the significant limits $\pm\tau_\alpha$ are obtained from

$$(6.9)\qquad \frac{2}{\pi^2}\left[\frac{1}{\tau_\alpha}+\frac{1}{9\tau_\alpha^3}+\frac{1}{25\tau_\alpha^5}+\cdots\cdots\right]=\frac{\alpha}{2}\qquad (\alpha=0.1,\ 0.05,\ 0.01\ \text{\&c.}).$$

. They are found, by putting $1/\tau_\alpha=x$, as the positive roots of

$$(6.10)\qquad x+0.1111\cdots x^3+0.4\,x^5+\cdots\cdots=\frac{\pi^2}{4}\alpha=2.4674\,\alpha.$$

Putting $\alpha=0.1$, 0.05, 0.01 and solving the resulting equations by Horner, we get $\tau_{0.1}=\pm4.08$, $\tau_{0.5}=\pm8.12$ and $\tau_{0.01}=\pm40.53$. Also for $\alpha=\frac{0.1}{4},\ \frac{0.05}{4},\ \frac{0.01}{4}$ we get $\tau_\alpha=16.22$, 32.44, 162.1, which shall be of use later on.

We have assumed the domain of universe to be $-\infty<x<\infty$. However, if it be truncated positively as $0<x<\infty$, so its fr. f. becomes twice (3)

$$(6.11)\qquad f(x)=2/\pi(1+x^2).$$

Consequently (4) shall be multiplied by 2^2 and the fr. f. becomes

$$(6.12)\qquad f(\tau)=\frac{4}{\pi^2\tau}\log\frac{\tau+1}{\tau-1},$$

where $\tau>1$, because of $s<\bar{x}$. Hence its graph is given by the last right branch in Fig. 10, with ordinates multiplied by 4. Accordingly the upper significant limit

τ_1 for α is given by those obtained above for $\alpha/4$ about the non-truncated one: i. e. 16.22, 32.44, 162.1 for $\alpha = 0.1$, 0.05, 0.01.

But, now the distribution being unsymmetrical, we need else to find the lower significant limit τ_0. This is found from a similar expression to (9):

$$F(\tau_0) = \int_1^{\tau_0} f(\tau)\,d\tau = \frac{4}{\pi^2}\int_1^{\tau_0}\frac{1}{\tau}\left[\log\left(1+\frac{1}{\tau}\right) - \log\left(1-\frac{1}{\tau}\right)\right]d\tau$$

$$= 1 - \frac{8}{\pi^2}\left[\frac{1}{\tau_0} + \frac{1}{9\tau_0^3} + \frac{1}{25\tau_0^5} + \cdots\right] = \frac{\alpha}{2}.$$

On writing $1/\tau_0 = x$, we solve the equation

$$x + x^3/9 + x^5/25 + x^7/49 + \cdots\cdots = \frac{\pi^2}{8}\left(1-\frac{\alpha}{2}\right) = 1.23370\left(1-\frac{\alpha}{2}\right),$$

and whence roughly $\tau_0 = 1.07$, 1.05, 1.04 for $\alpha = 0.1$, 0.05, 0.01 respectively.

In general, if the domain of universe be restricted to be $a < x < b$, so also $a < \bar{x} < b$. Besides, in regard to s, we must have either $0 < \sqrt{2}s < \sqrt{2}(\bar{x}-a)$ if $a < \bar{x} < \frac{a+b}{2}$, or $0 < \sqrt{2}s < \sqrt{2}(b-\bar{x})$ if $\frac{1}{2}(a+b) < \bar{x} < b$ (Fig. 11). Or, in terms of τ, either $0 < \bar{x}/\tau < \bar{x}-a$, i. e. $\frac{a\tau}{\tau-1} < \bar{x} < \frac{a+b}{2}$, or $\bar{x}/\tau < b-\bar{x}$, i. e. $\frac{a+b}{2} < \bar{x} < \frac{b\tau}{\tau+1}$. E. g. if $a = 0$, $b = 1$, we have $0 < \bar{x} < \frac{\tau}{\tau+1}$. And in that case the fr. f. of universe shall be

(6. 13) $$f(x) = 4/\pi(1+x^2), \qquad (0 < x < 1)$$

because $x = 1$ is the right quartile of (3). Consequently, instead of (4) and (5), we should now take as the fr. f.

$$f(\tau) = \frac{64}{\pi^2}\int_0^{\frac{\tau}{1+\tau}} \frac{\tau^2 x\,dx}{\tau^4 + 2\tau^2(1+\tau^2)x^2 + (1-\tau^2)^2 x^4},$$

which, after computations, yields (Fig. 12)

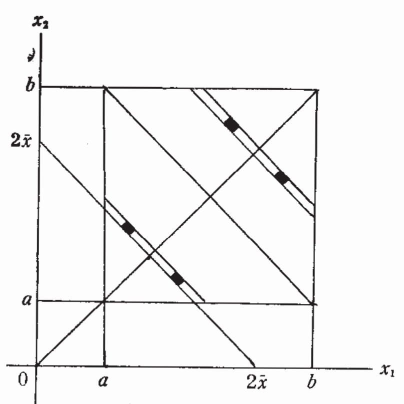

Fig. 11

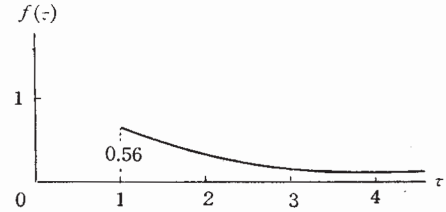

Fig. 12

$$f(\tau) = \frac{8}{\pi^2 \tau} \log \frac{(\tau+1)^2}{\tau^2+1} \quad (\tau > 1). \tag{6.14}$$

This being arranged in a series, we can compute the upper and lower significant limits, as the roots of

$$\int_{\tau_1}^{\infty} f(\tau) d\tau = \frac{16}{\pi^2}\left[\frac{1}{\tau_1} - \frac{1}{2\tau_1^2} + \frac{1}{9\tau_1^3} + \frac{1}{25\tau_1^5} + \cdots\cdots\right] = \frac{\alpha}{2},$$

and

$$\int_{1}^{\tau_0} f(\tau) d\tau = \int_{1}^{\infty} f(\tau) d\tau - \int_{\tau_0}^{\infty} f(\tau) d\tau$$
$$= 1 - \frac{16}{\pi^2}\left[\frac{1}{\tau_0} - \frac{1}{2\tau_0^2} + \frac{1}{9\tau_0^3} + \frac{1}{25\tau_0^5} + \cdots\cdots\right] = \frac{\alpha}{2},$$

respectively, and obtain $\tau_1 = 31.92$, 64.40, 323.7 and $\tau_0 = 1.159$, 10.50, 1.020.

Of course, this Cauchy distribution truncated in $0 < x < 1$ has its mean $m = (2 \log 2)/\pi$, so that there exists its ordinary Student function. In fact, that fr. f. is defined by

$$f(t) = \frac{64t^2}{\pi^2} \times \tag{6.15}$$
$$\int_{x_0}^{x_1} \frac{|m-x|\, dx}{(t^2-1)^2 x^4 + 4m(t^2-1)x^3 + 2[t^4 - t^2(m^2-1) + 3m^2]x^2 - 4m(t^2+m^2)x + (t^2+m^2)^2},$$

which however is too cumbersome to be computed. Comparing (14) and (15), it is obvious that the Student's second form is far simpler and easier to deal with than the ordinary Student ratio itself.

7. *The General Case.*

Lastly we shall outline about the general case with n-sized sample drawn from a positively truncated universe $(x > 0)$. Denote the parent mean if exists by m and the sample mean and S. D. by $\bar{x}$ and s, where, besides $s > 0$ we have also $\bar{x} > 0$, yet $\bar{x} \gtrless m$, so that $\tau = \bar{x}\sqrt{n-1}/s > 0$, but $t = (\bar{x}-m)\sqrt{n-1}/s \gtrless 0$. To speak particularly, there occur n subcases, as described in the previous paper, I: $0 < s/\bar{x} < 1/\sqrt{n-1}$, II: $1/\sqrt{n-1} < s/\bar{x} < \sqrt{2/(n-2)}$, ……, the $(n-1)$-th: $\sqrt{(n-2)/2} < s/\bar{x} < \sqrt{n-1}$, the n-th: $\sqrt{n-1} < s/\bar{x} < \infty$, where, the last one being abandoned since there is no point of the simplex S_{n-1} that lies on the sphere of radius $\sqrt{n}s > \sqrt{n(n-1)}\bar{x}$. Or, writing $\tau = \sqrt{n-1}\bar{x}/s = \bar{x}t/(\bar{x}-m)$, we have the following $n-1$ subcases:

$$\text{I}: \ \infty > \tau > n-1, \quad \text{II}: \ n-1 > \tau > \sqrt{(n-1)(n-2)/2}, \cdots\cdots, \tag{7.1}$$
$$\text{the } (n-1)\text{-th}: \ \sqrt{(n-1)2/(n-2)} > \tau > 1.$$

However, it seems apparently that the predominant contribution comes from the first portion I, because it fills almost the whole space occupied by the sample point $(\bar{x},\ s)$. Now in the portion I we have Fisher's elementary volume:

(7.2) $$dV = c_n s^{n-2} ds d\bar{x}, \quad \text{where} \quad c_n = \sqrt{n}^n \sqrt{\pi}^{n-1}(n-1)/\Gamma\left(\frac{n+1}{2}\right).$$

Or, if s be replaced by τ,

(7.3) $$dV = b_n \bar{x}^{n-1} d\bar{x} d\tau / \tau^n, \quad \text{with} \quad b_n = c_n \sqrt{n-1}^{n-1}.$$

Or, if replaced by t

(7.4) $$dV = b_n |\bar{x} - m|^{n-1} d\bar{x} dt / |t|^n.$$

In particular, if the universe be $f(x) = 1$ in $0 < x < 1$, the parent mean is $m = \frac{1}{2}$. In this case, the above elementary volumes denote at the same time the probability-elements $dP(\bar{x}, s)$, $dP(\bar{x}, \tau)$, $dP(\bar{x}, t)$, respectively. Hence speaking roughly, Student's fr. f. $s_n(t)$ for the rectangular universe shall be given by

(7.5) $$s_I(t) = \frac{b_n}{|t|^n} \int_{x_0}^{x_1} |m - \bar{x}|^{n-1} d\bar{x}.$$

To determine the limits of integration, e. g. let t be negative and given as an inner point of $(-\infty, 0)$. We must exhaust all points $\bar{x}$, which satisfy

(7.6) $$s = \frac{\bar{x} - m}{t} \sqrt{n-1} \qquad (t < 0, \ \bar{x} < m).$$

Hence, the upper limit x_1 is clearly the parent mean $m\left(=\frac{1}{2}\right)$. However, the lower limit cannot be zero. For, if $\bar{x} = 0$, $n\bar{x} = \sum x_i = 0$, so that all $x_i = 0$, because they are non-negative. Consequently $ns^2 = \sum (x_i - \bar{x})^2 = 0$, which however makes (6) impossible. Therefore, there must exist a positive lower bound for $\bar{x}$. Now, for a given $\bar{x}$, the maximal s in portion I is determined by the radius of s-sphere inscribed in the simplex S_{n-1}, namely $\sqrt{n}s = \sqrt{n}\bar{x}/\sqrt{n-1}$; thus max $s = \bar{x}/\sqrt{n-1}$. If this max s be less than (6), so also it must be for all s in I, and such $\bar{x}$ should be rejected. Hence the lower bound of $\bar{x}$ is that which makes

(7.7) $$\frac{\bar{x} - m}{t} \sqrt{n-1} = \frac{\bar{x}}{\sqrt{n-1}}, \quad \text{i. e.} \quad \lim \bar{x} = x_0 = \frac{(n-1)m}{n-1-t}.$$

Accordingly, for $f(x) = 1$, we have in view of (5)

(7.8) $$s_I(t) = \frac{b_n}{n|t|^n}\left(1 - \frac{n-1}{n-1-t}\right)^n m^n = \frac{a_n}{(n-1-t)^n}, \qquad (t < 0)$$

where

(7.9) $$a_n = b_n / n2^n = \sqrt{n}^{n-2} \sqrt{\pi}^{n-1} \sqrt{n-1}^{n+1} / \Gamma\left(\tfrac{1}{2}(n+1)\right) 2^n.$$

And hence the lower significant limit $t_\alpha (< 0)$ might be determined by

(7.10) $$S_I(t_\alpha) = \int_{-\infty}^{t_\alpha} s_I(t)\, dt = \frac{a_n}{(n-1)(n-1-t_\alpha)^{n-1}} = \frac{\alpha}{2}.$$

For example, it becomes, if $n=2$, $a_2=\dfrac{1}{2}$ and $s_I=1/2(1-t)^2$, which exactly coincides with the result obtained in section 1. Or, if $n=3$, we get $a_3=\sqrt{3}\pi/2$ and

$$(7.11) \qquad s_I(t)=\frac{\sqrt{3}\pi}{2(2-t)^3}, \qquad S_I(t_\alpha)=\int_{-\infty}^{t_\alpha} s_I(t)\,dt=\frac{\sqrt{3}\pi}{4(2-t_\alpha)^2}.$$

Whence the significant limits t_α are found from

$$(7.12) \qquad \frac{\sqrt{3}\pi}{4(2-t_\alpha)^2}=\frac{\alpha}{2}, \quad \text{i.e.} \quad -t_\alpha=\sqrt{\frac{\sqrt{3}\pi}{2\alpha}}-2,$$

which yields $t_\alpha=\pm 3.216$, ± 5.377, ± 14.495 for $\alpha=0.1$, 0.05, 0.01, respectively. The true expression of $S_3(t)$ $(-\infty<t<-2)$ being given by (1. 14), the true values of t_α can be computed by Newton's method of successive approximations. In fact it is found that

$$t_{0.1}=\pm 3.5894, \qquad t_{0.05}=\pm 5.7418, \qquad t_{0.01}=\pm 14.8496.$$

Thus the short formula (10) for the rectangular universe seems to yield tolerablly approximate values for t_α. so far the sample size is small. However, if the sample size be a little large, the resulting figures computed from (10) as in (12), $t_{0.1}$, $t_{0.05}$ for $n>6$ and $t_{0.01}$ for $n<8$, all become positive against the presupposition that $t_\alpha<0$. Therefore, some more contrivance to correct (8), e. g. by taking the portion II &c., are anyhow necessary. But this paper being already too lengthy, further discussions are delayed for a future research.

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SEMIGROUPS OF ORDER 5, 6, 7, 8 WHOSE GREATEST C-HOMOMORPHIC IMAGES ARE UNIPOTENT SEMIGROUPS WITH GROUPS[1)]

By

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In the previous paper [1] we obtained all the semigroups of order $\leqq 10$ whose greatest c-homomorphic images are groups. In the present paper we are going to list all the types of semigroups of order 5, 6, 7, 8 whose greatest c-homomorphic images are unipotent semigroups containing proper groups, without precise explanation. We should like to discuss the fundamental theorems used in this computation in another paper.

Let S be a finite semigroup whose greatest c-homomorphic image U is a unipotent semigroup without zero, a finite unipotent semigroup having a non-trivial group G as the kernel. Let I be the minimal ideal of S. Naturally I is a finite simple semigroup. Then I is mapped to G under the homomorphism $S \rightarrow U$ and the homomorphism $I \rightarrow G$ gives the greatest c-decomposition of I. Consider the difference semigroup of S module I and denote it by $D=(S:I)$. D is clearly a finite unipotent semigroup with zero. In order to construct the desired finite semigroup for given I and Z we may find S which satisfies $D=(S:I)$. Especially if the order of S is limited to 5, 6, 7, 8, then we may consider only the following cases

I	D	No. of Table
4. 3	2_2	1
	2_3, 3_3	2
	$3_4 \sim 12_4$	3
	$4_5 \sim 96_5$	4
6. 3	2_2	5
	2_3, 3_3	6
6. 4	2_2	7
	2_3, 3_3	8

1) This computation was performed in July 1960 by the four students of Tokushima University, and was revised by Tamura at the University of California in September 1960. Though the students computed all but a part of the types of order 9, 10 at the same time, the results will be published at a later date since the check is not yet finished.

where 4.3, 6.3, 6.4 are numbers seen in [1] and 2_3, 3_4, 4_5, $\cdots$, denote respectively the second semigroup of order 3, the third of order 4, the fourth of order 5, etc., which are seen in the list [2].

For the sake of the explanation below, let

$$I = \{a_1, \cdots, a_m\}, \qquad S = I \cup \{a_{m+1}, \cdots, a_n\},$$
$$D = \{0, a_{m+1}, \cdots, a_n\}$$

in which the bases of D shall be denoted $a_{k_1}, \cdots, a_{k_l}$. There exist two kinds of the systems of relations between $a_{m+1}, \cdots, a_n$, the basic relations (1) and the defining relations (2), in which the former is the relations between the bases, the latter between the bases and the other elements:

$$(1) \quad f_i(a_{k_1}, \cdots, a_{k_l}) = g_i(a_{k_1}, \cdots, a_{k_l}) \qquad i = 1, \cdots, t$$

$$(2) \quad a_j = h_j(a_{k_1}, \cdots, a_{k_l}), \qquad a_j \neq a_{k_1}, \cdots, a_{k_l}.$$

Let $\varphi_{a_1}, \cdots, \varphi_{a_m}, \varphi_{a_{m+1}}, \cdots, \varphi_{a_n}$ be the mappings of I into itself defined as

$$(x)\varphi_{a_i} = xa_i \qquad x \in I, \quad xa_i \in I$$

where φ_{a_i} $(i = 1, \cdots, m)$ are already given, but φ_{a_j} $(j = m+1, \cdots, n)$ are unknown. Φ denotes the semigroup which consists of φ_a $(a \in S)$, and Φ_0 the semigroup of $\{\varphi_a \mid a \in I\}$. Clearly S is homomorphic to Φ and so Φ_0 is an ideal of Φ.

$$(3) \quad \begin{cases} \varphi_{a_{j_1}} \varphi_{a_{j_2}} \in \Phi_0 & \text{if } \ a_{j_1}a_{j_2} = 0 \ \text{ in } D \\ \varphi_{a_{j_1}} \varphi_{a_{j_2}} = \varphi_{a_{j_3}} & \text{if } \ a_{j_1}a_{j_2} = a_{j_3} \in D, \ \neq 0 \end{cases}$$

and also for (1) and (2) we can assign the relations $(\bar{1})$ and $(\bar{2})$ under the homomorphism $a \to \varphi_a$.

$$(\bar{1}) \quad F_i(\varphi_{a_{k_1}}, \cdots, \varphi_{a_{k_l}}) = G_i(\varphi_{a_{k_1}}, \cdots, \varphi_{a_{k_l}}) \qquad i = 1, \cdots, t$$

$$(\bar{2}) \quad \varphi_{a_j} = H_j(\varphi_{a_{k_1}}, \cdots, \varphi_{a_{k_l}}), \qquad a_j \neq a_{k_1}, \cdots, a_{k_l}$$

where $(\bar{1})$ is obtained from the image of (1) by left cancellation, if necessary. Furthermore, we can easily see that $\varphi_{a_i}(i = m+1, \cdots, n)$ must be inner right translations[2)] of I if I is 4.3, and right translations of I if I is either 6.3 or 6.4.

If we find only (inner) right translations $\varphi_{a_{k_1}}, \cdots, \varphi_{a_{k_l}}$ of (4.3) 6.3 or 6.4 which satisfy $(\bar{1})$ and (3), then the other φ_{a_j} can be found by $(\bar{2})$. After $\varphi_{a_j}(j = m+1, \cdots, n)$ are determined, the remaining products in S are completely determined; in detail, since $I \ni a_i \to \varphi_{a_i}$ is one to one, if

$$\varphi_{a_{j_1}}\varphi_{a_{j_2}} = \varphi_{a_{j_3}} \in \Phi_0, \ \text{ we define } \ a_{j_1}a_{j_2} = a_{j_3} \in I.$$

By the way, we want to remark the following fact. In order that Φ_0 is an

2) With respect to right translation, see [3].

ideal of Φ, we must choose φ_{a_j} in some special subset of the set $\overline{\Phi}$ of all right translations of I, because Φ_0 may not be an ideal of $\overline{\Phi}$.

We shall add explanation of the notations in the tables below. In the tables 1, 2, 3, 4, since I, 4. 3, is

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	a	b	c	d
d	b	a	d	c

$S=I\cup\{e\}$, $D=\{0, e\}$, if D is of order 2,
$S=I\cup\{e, f\}$, $D=\{0, e, f\}$, if D is of order 3,
$S=I\cup\{e, f, g\}$, $D=\{0, e, f, g\}$, if D is of order 4,
$S=I\cup\{e, f, g, h\}$, $D=\{0, e, f, g, h\}$, if D is of order 5,

and the multiplication table of D should be understood by replacing 0, e, f, g, h by a, b, c, d, e in the list [2]. The inner right translations

$$\begin{pmatrix} a & b & c & d \\ a & b & a & b \end{pmatrix}, \quad \begin{pmatrix} a & b & c & d \\ b & a & b & a \end{pmatrix}, \quad \begin{pmatrix} a & b & c & d \\ c & d & c & d \end{pmatrix}, \quad \begin{pmatrix} a & b & c & d \\ d & c & d & c \end{pmatrix}$$

are denoted by a, b, c, d respectively for simplicity.

In the tables 5, 6, the ideal I, 6. 3, is a finite simple semigroup[3)],

$$I=\{(\xi\,;1j)\,|\,\xi=\varepsilon, \alpha\,;\,j=1, 2, 3\}$$

or $(\varepsilon\,;11)=a$, $(\alpha\,;11)=b$, $(\varepsilon\,;12)=c$, $(\alpha\,;12)=d$,
$(\varepsilon\,;13)=e$, $(\alpha\,;13)=f$,

where $\{\varepsilon, \alpha\}$ is a group of order 2 with the unit ε, and

$S=I\cup\{g\}$, $D=\{0, g\}$, if D is of order 2,
$S=I\cup\{g, h\}$, $D=\{0, g, h\}$, if D is of order 3.

The right translation φ_g or φ_h, denoted by $(ijk)\xi$, means the mapping of I into I

$$\begin{pmatrix} (\varepsilon\;;11) & (\alpha\;;11) & (\varepsilon\;;12) & (\alpha\;;12) & (\varepsilon\;;13) & (\alpha\;;13) \\ (\varepsilon\xi\,;1i) & (\alpha\xi\,;1i) & (\varepsilon\xi\,;1j) & (\alpha\xi\,;1j) & (\varepsilon\xi\,;1k) & (\alpha\xi\,;1k) \end{pmatrix}$$

where $\xi=\varepsilon$ or α ; i, j, $k=1$ or 2 or 3, and $\varepsilon\xi$ and $\alpha\xi$ are the products in the ground group $\{\varepsilon, \alpha\}$.

In the tables 7, 8, the ideal I or 6. 4 is

$$I=\{(\xi\,;1j)\,|\,\xi=\varepsilon, \alpha, \beta\,;\,j=1, 2\}$$

or $(\varepsilon\,;11)=a$, $(\alpha\,;11)=b$, $(\beta\,;11)=c$, $(\varepsilon\,;12)=d$,
$(\alpha\,;12)=e$, $(\beta\,;12)=f$,

where $\{\varepsilon, \alpha, \beta\}$ is a group of order 3, $\beta=\alpha^2$, $\varepsilon=\alpha^3$. The other notations are the same as the tables 5, 6.

3) The product is defined as $(\xi\,;1j)(\eta\,;1k)=(\xi\eta\,;1k)$.

All the tables show the system of the right translations of I which determines uniquely S, and the semigroups S constructed by the system of the right translations are neither mutually isomorphic nor anti-isomorphic.

We remark that the isomorphism of S is determined by the automorphisms of I and D. By the way, 4. 3 has an automorphism $\begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix}$ except the identical mapping and the semigroup 6. 3 has the automorphisms with the form

$$(\varepsilon\,;1i) \to (\varepsilon\,;1k_i)$$
$$(\alpha\,;1j) \to (\alpha\,;1k_j)$$

where the mapping $i \to k_i$ is a permutation of (1, 2, 3) and 6. 4 has the automorphism with the form

$$(\varepsilon\,;1\,i) \to (\varepsilon\,;1\,l_i)$$
$$(\alpha\,;1\,j) \to (\alpha'\,;1\,l_j)$$
$$(\beta\,;1\,k) \to (\beta'\,;1\,l_k)$$

where the mapping $i \to l_i$ is a permutation of (1, 2) and the mapping $\begin{pmatrix} \varepsilon & \alpha & \beta \\ \varepsilon & \alpha' & \beta' \end{pmatrix}$ is an automorphism of the group $\{\varepsilon, \alpha, \beta\}$.

Table 1 (I=4. 3)

Division Number	D	Base	Base Relations	defining Relations	φ_e	Individual Number
1. 1	2_2	e	*free*		a b	1. 1. 1 1. 1. 2

Table 2 (I=4. 3)

Division Number	D	Base	Base Relations	defining Relations	φ_e φ_f	Individual Number
2. 1	2_3	e, f	*free*		a a a b a c a d b b b d	2. 1. 1 2. 1. 2 2. 1. 3 2. 1. 4 2. 1. 5 2. 1. 6
2. 2	3_3	f		$\varphi_e=\varphi_f^2$	a a a b	2. 2. 1 2. 2. 2

Table 3 $(I=4.3)$

Division Number	D	Base	Base Relations	defining Relations	$\varphi_e\ \varphi_f\ \varphi_g$	Individual Number
3. 1	3_4	e, f, g	*free*		$a\ a\ a$ $a\ a\ b$ $a\ a\ c$ $a\ a\ d$ $a\ b\ b$ $a\ b\ c$ $a\ b\ d$ $b\ b\ b$ $b\ b\ d$	3. 1. 1 3. 1. 2 3. 1. 3 3. 1. 4 3. 1. 5 3. 1. 6 3. 1. 7 3. 1. 8 3. 1. 9
3. 2	4_4	f, g	*free*	$\varphi_e=\varphi_g^2$	$a\ a\ a$ $a\ b\ a$ $a\ c\ a$ $a\ d\ a$ $a\ a\ b$ $a\ b\ b$ $a\ c\ b$ $a\ d\ b$	3. 2. 1 3. 2. 2 3. 2. 3 3. 2. 4 3. 2. 5 3. 2. 6 3. 2. 7 3. 2. 8
3. 3	5_4	f, g	*free*	$\varphi_e=\varphi_g\varphi_f$	$a\ a\ a$ $b\ b\ a$ $c\ c\ a$ $d\ d\ a$ $b\ a\ b$ $a\ b\ b$ $d\ c\ b$ $c\ d\ b$	3. 3. 1 3. 3. 2 3. 3. 3 3. 3. 4 3. 3. 5 3. 3. 6 3. 3. 7 3. 3. 8
3. 4	6_4	f, g	$\varphi_f=\varphi_g$	$\varphi_e=\varphi_g^2$	$a\ a\ a$ $a\ b\ b$	3. 4. 1 3. 4. 2
3. 5	7_4	f, g	$\varphi_f\varphi_g=\varphi_g\varphi_f$	$\varphi_e=\varphi_f\varphi_g$	$a\ a\ a$ $b\ a\ b$ $a\ b\ b$	3. 5. 1 3. 5. 2 3. 5. 3
3. 6	8_4	f, g	$\varphi_f=\varphi_g$	$\varphi_e=\varphi_g^2$	$a\ a\ a$ $a\ b\ b$	3. 6. 1 3. 6. 2
3. 7	9_4	f, g	$\varphi_f^2=\varphi_g^2$	$\varphi_e=\varphi_g^2$	$a\ a\ a$ $a\ a\ b$ $a\ b\ b$	3. 7. 1 3. 7. 2 3. 7. 3
3. 8	10_4	f, g	$\varphi_f=\varphi_g$	$\varphi_e=\varphi_g^2$	$a\ a\ a$ $a\ b\ b$	3. 8. 1 3. 8. 2
3. 9	11_4	f, g	$\varphi_f=\varphi_g$	$\varphi_e=\varphi_g^2$	$a\ a\ a$ $a\ b\ b$	3. 9. 1 3. 9. 2
3. 10	12_4	g	*free*	$\varphi_e=\varphi_g^3$ $\varphi_f=\varphi_g^2$	$a\ a\ a$ $b\ a\ b$	3. 10. 1 3. 10. 2

Table 4 $(I=4.3)$

Division Number	D	Base	Base Relations	defining Relations	φ_e φ_f φ_g φ_h	Individual Number
4. 1	4_5	e, f, g, h	*free*		a a a a	4. 1. 1
					a a a b	4. 1. 2
					a a a c	4. 1. 3
					a a a d	4. 1. 4
					a a b b	4. 1. 5
					a a b c	4. 1. 6
					a a b d	4. 1. 7
					a a c c	4. 1. 8
					a a c d	4. 1. 9
					a a d d	4. 1. 10
					a b b b	4. 1. 11
					a b b c	4. 1. 12
					a b b d	4. 1. 13
					a b c d	4. 1. 14
					a b d d	4. 1. 15
					a d d d	4. 1. 16
					b b b b	4. 1. 17
					b b b d	4. 1. 18
					b b d d	4. 1. 19
4. 2	5_5	g, h	*free*	$\varphi_e=\varphi_h\varphi_g$ $\varphi_f=\varphi_h^2$	a a a a	4. 2. 1
					b a a b	4. 2. 2
					a c a c	4. 2. 3
					b c a d	4. 2. 4
					b a b a	4. 2. 5
					a a b b	4. 2. 6
					b c b c	4. 2. 7
					a c b d	4. 2. 8
4. 3	6_5	g, h	*free*	$\varphi_e=\varphi_g^2$ $\varphi_f=\varphi_h^2$	a a a a	4. 3. 1
					a a a b	4. 3. 2
					a c a c	4. 3. 3
					a c a d	4. 3. 4
					a a b a	4. 3. 5
					a a b b	4. 3. 6
					a c b c	4. 3. 7
					a c b d	4. 3. 8
4. 4	7_5	g, h	*free*	$\varphi_e=\varphi_g\varphi_h$ $\varphi_f=\varphi_h\varphi_g$	a a a a	4. 4. 1
					b b a b	4. 4. 2
					c a a c	4. 4. 3
					d b a d	4. 4. 4
					b b b a	4. 4. 5
					a a b b	4. 4. 6
					d b b c	4. 4. 7
					c a b d	4. 4. 8
4. 5	8_5	g, h	$\varphi_g=\varphi_h$	$\varphi_e=\varphi_f=\varphi_h^2$	a a a a	4. 5. 1
					a a b b	4. 5. 2
4. 6	9_5	g, h	$\varphi_g=\varphi_h$	$\varphi_e=\varphi_f=\varphi_h^2$	a a a a	4. 6. 1
					a a b b	4. 6. 2
4. 7	10_5	g, h	$\varphi_g=\varphi_h$	$\varphi_e=\varphi_f=\varphi_h^2$	a a a a	4. 7. 1
					a a b b	4. 7. 2
4. 8	11_5	g, h	$\varphi_g=\varphi_h$	$\varphi_e=\varphi_f=\varphi_h^2$	a a a a	4. 8. 1
					a a b b	4. 8. 2
4. 9	12_5	g, h	$\varphi_g\varphi_h=\varphi_h\varphi_g$	$\varphi_e=\varphi_g^2$ $\varphi_f=\varphi_g\varphi_h$	a a a a	4. 9. 1
					a b a b	4. 9. 2
					a a b b	4. 9. 3
					a b b a	4. 9. 4

4. 10	13_5	g, h	$\varphi_g = \varphi_h$	$\varphi_e = \varphi_f = \varphi_h^2$	$a\ a\ a\ a$ $a\ a\ b\ b$	4. 10. 1 4. 10. 2
4. 11	14_5	g, h	$\varphi_g^2 = \varphi_h^2$	$\varphi_e = \varphi_g \varphi_h$ $\varphi_f = \varphi_g^2$	$a\ a\ a\ a$ $b\ a\ a\ b$ $b\ a\ b\ a$ $a\ a\ b\ b$	4. 11. 1 4. 11. 2 4. 11. 3 4. 11. 4
4. 12	15_5	g, h	$\varphi_g^2 = \varphi_h^2$ $\varphi_g \varphi_h = \varphi_h \varphi_g$	$\varphi_e = \varphi_g^2$ $\varphi_f = \varphi_g \varphi_h$	$a\ a\ a\ a$ $a\ b\ a\ b$ $a\ a\ b\ b$	4. 12. 1 4. 12. 2 4. 12. 3
4. 13	16_5	f, h	*free*	$\varphi_e = \varphi_h^3$ $\varphi_g = \varphi_h^2$	$a\ a\ a\ a$ $a\ b\ a\ a$ $a\ c\ a\ a$ $a\ d\ a\ a$ $b\ a\ a\ b$ $b\ b\ a\ b$ $b\ c\ a\ b$ $b\ d\ a\ b$	4. 13. 1 4. 13. 2 4. 13. 3 4. 13. 4 4. 13. 5 4. 13. 6 4. 13. 7 4. 13. 8
4. 14	17_5	f, g, h	*free*	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ b\ a\ a$ $a\ c\ a\ a$ $a\ d\ a\ a$ $a\ b\ b\ a$ $a\ c\ b\ a$ $a\ d\ b\ a$ $a\ c\ c\ a$ $a\ d\ c\ a$ $a\ d\ d\ a$ $a\ a\ a\ b$ $a\ b\ a\ b$ $a\ c\ a\ b$ $a\ d\ a\ b$ $a\ b\ b\ b$ $a\ c\ b\ b$ $a\ d\ b\ b$ $a\ c\ c\ b$ $a\ d\ c\ b$ $a\ d\ d\ b$	4. 14. 1 4. 14. 2 4. 14. 3 4. 14. 4 4. 14. 5 4. 14. 6 4. 14, 7 4. 14. 8 4. 14. 9 4. 14. 10 4. 14. 11 4. 14. 12 4. 14. 13 4. 14. 14 4. 14. 15 4. 14. 16 4. 14. 17 4. 14. 18 4. 14. 19 4. 14. 20
4. 15	18_5	f, g, h	*free*	$\varphi_e = \varphi_h \varphi_g$	$a\ a\ a\ a$ $a\ b\ a\ a$ $a\ c\ a\ a$ $a\ d\ a\ a$ $b\ a\ b\ a$ $b\ b\ b\ a$ $b\ c\ b\ a$ $b\ d\ b\ a$ $c\ a\ c\ a$ $c\ b\ c\ a$ $c\ c\ c\ a$ $c\ d\ c\ a$ $d\ a\ d\ a$ $d\ b\ d\ a$ $d\ c\ d\ a$ $d\ d\ d\ a$ $b\ a\ a\ b$ $b\ b\ a\ b$ $b\ c\ a\ b$ $b\ d\ a\ b$ $a\ a\ b\ b$ $a\ b\ b\ b$ $a\ c\ b\ b$ $a\ d\ b\ b$ $d\ a\ c\ b$ $d\ b\ c\ b$ $d\ c\ c\ b$ $d\ d\ c\ b$	4. 15. 1 4. 15. 2 4. 15. 3 4. 15. 4 4. 15. 5 4. 15. 6 4. 15. 7 4. 15. 8 4. 15. 9 4. 15. 10 4. 15. 11 4. 15. 12 4. 15. 13 4. 15. 14 4. 15. 15 4. 15. 16 4. 15. 17 4. 15. 18 4. 15. 19 4. 15. 20 4. 15. 21 4. 15. 22 4. 15. 23 4. 15. 24 4. 15. 25 4. 15. 26 4. 15. 27 4. 15. 28

4.16	19_5	f, g, h	$\varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ b\ a\ a$ $a\ c\ a\ a$ $a\ d\ a\ a$ $a\ a\ b\ b$ $a\ b\ b\ b$ $a\ c\ b\ b$ $a\ d\ b\ b$	4.16. 1 4.16. 2 4.16. 3 4.16. 4 4.16. 5 4.16. 6 4.16. 7 4.16. 8
4.17	20_5	f, g, h	$\varphi_f = \varphi_g$	$\varphi_e = \varphi_h \varphi_g$	$a\ a\ a\ a$ $b\ a\ a\ b$ $a\ a\ a\ c$ $b\ a\ a\ d$ $b\ b\ b\ a$ $a\ b\ b\ b$ $b\ b\ b\ c$ $a\ b\ b\ d$	4.17. 1 4.17. 2 4.17. 3 4.17. 4 4.17. 5 4.17. 6 4.17. 7 4.17. 8
4.18	21_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4.18. 1 4.18. 2
4.19	22_5	g, h	$\varphi_g = \varphi_h^2$	$\varphi_e = \varphi_h^3$ $\varphi_f = \varphi_g$	$a\ a\ a\ a$ $b\ a\ a\ b$	4.19. 1 4.19. 2
4.20	23_5	g, h	$\varphi_g = \varphi_h^2$	$\varphi_e = \varphi_h^3$ $\varphi_f = \varphi_g$	$a\ a\ a\ a$ $b\ a\ a\ b$	4.20. 1 4.20. 2
4.21	24_5	g, h	$\varphi_g^2 = \varphi_h^3$	$\varphi_e = \varphi_h^3$ $\varphi_f = \varphi_h^2$	$a\ a\ a\ a$ $a\ a\ b\ a$	4.21. 1 4.21. 2
4.22	25_5	g, h	$\varphi_g = \varphi_h = \varphi_h^2$	$\varphi_e = \varphi_f = \varphi_g$	$a\ a\ a\ a$	4.22. 1
4.23	26_5	g, h	$\varphi_g = \varphi_h = \varphi_h^2$	$\varphi_e = \varphi_f = \varphi_g$	$a\ a\ a\ a$	4.23. 1
4.24	27_5	g, h	$\varphi_g = \varphi_h$	$\varphi_e = \varphi_h^3$ $\varphi_f = \varphi_h^2$	$a\ a\ a\ a$ $b\ a\ b\ b$	4.24. 1 4.24. 2
4.25	28_5	f, g, h	$\varphi_g \varphi_h = \varphi_h \varphi_g$	$\varphi_e = \varphi_g \varphi_h$	$a\ a\ a\ a$ $a\ b\ a\ a$ $a\ c\ a\ a$ $a\ d\ a\ a$ $b\ a\ a\ b$ $b\ b\ a\ b$ $b\ c\ a\ b$ $b\ d\ a\ b$ $a\ a\ b\ b$ $a\ b\ b\ b$ $a\ c\ b\ b$ $a\ d\ b\ b$	4.25. 1 4.25. 2 4.25. 3 4.25. 4 4.25. 5 4.25. 6 4.25. 7 4.25. 8 4.25. 9 4.25. 10 4.25. 11 4.25. 12
4.26	29_5	f, g, h	$\varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ b\ a\ a$ $a\ c\ a\ a$ $a\ d\ a\ a$ $a\ a\ b\ b$ $a\ b\ b\ b$ $a\ c\ b\ b$ $a\ d\ b\ b$	4.26. 1 4.26. 2 4.26. 3 4.26. 4 4.26. 5 4.26. 6 4.26. 7 4.26. 8
4.27	30_5	f, g, h	$\varphi_g^2 = \varphi_h^2$	$\varphi_e = \varphi_g^2$	$a\ a\ a\ a$ $a\ b\ a\ a$ $a\ c\ a\ a$ $a\ d\ a\ a$ $a\ a\ a\ b$ $a\ b\ a\ b$ $a\ c\ a\ b$ $a\ d\ a\ b$ $a\ a\ b\ b$ $a\ b\ b\ b$ $a\ c\ b\ b$ $a\ d\ b\ b$	4.27. 1 4.27. 2 4.27. 3 4.27. 4 4.27. 5 4.27. 6 4.27. 7 4.27. 8 4.27. 9 4.27. 10 4.27. 11 4.27. 12

4. 28	31_5	f, g, h	$\varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ b\ a\ a$ $a\ c\ a\ a$ $a\ d\ a\ a$ $a\ a\ b\ b$ $a\ b\ b\ b$ $a\ c\ b\ b$ $a\ d\ b\ b$	4. 28. 1 4. 28. 2 4. 28. 3 4. 28. 4 4. 28. 5 4. 28. 6 4. 28. 7 4. 28. 8
4. 29	32_5	f, g, h	$\varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ b\ a\ a$ $a\ c\ a\ a$ $a\ d\ a\ a$ $a\ a\ b\ b$ $a\ b\ b\ b$ $a\ c\ b\ b$ $a\ d\ b\ b$	4. 29. 1 4. 29. 2 4. 29. 3 4. 29. 4 4. 29. 5 4. 29. 6 4. 29. 7 4. 29. 8
4. 30	33_5	f, g, h	$\varphi_g\varphi_h = \varphi_h\varphi_f$	$\varphi_e = \varphi_g\varphi_h$	$a\ a\ a\ a$ $b\ a\ a\ b$ $a\ a\ c\ a$ $b\ a\ c\ b$ $b\ b\ b\ a$ $a\ b\ b\ b$ $b\ b\ d\ a$ $a\ b\ d\ b$	4. 30. 1 4. 30. 2 4. 30. 3 4. 30. 4 4. 30. 5 4. 30. 6 4. 30. 7 4. 30. 8
4. 31	34_5	f, g, h	$\varphi_f = \varphi_h$ $\varphi_g\varphi_h = \varphi_h^2$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ a\ c\ a$ $a\ b\ b\ b$ $a\ b\ d\ b$	4. 31. 1 4. 31. 2 4. 31. 3 4. 31. 4
4. 32	35_5	f, g, h	$\varphi_f = \varphi_h$ $\varphi_g\varphi_h = \varphi_h^2$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ a\ c\ a$ $a\ b\ b\ b$ $a\ b\ d\ b$	4. 32. 1 4. 32. 2 4. 32. 3 4. 32. 4
4. 33	36_5	f, g, h	$\varphi_f = \varphi_h$ $\varphi_g\varphi_h = \varphi_h^2$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ a\ c\ a$ $a\ b\ b\ b$ $a\ b\ d\ b$	4. 33. 1 4. 33. 2 4. 33. 3 4. 33. 4
4. 34	37_5	f, g, h	$\varphi_h^2 = \varphi_g\varphi_f$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ a\ c\ a$ $a\ b\ b\ a$ $a\ b\ d\ a$ $a\ a\ a\ b$ $a\ a\ c\ b$ $a\ b\ b\ b$ $a\ b\ d\ b$	4. 34. 1 4. 34. 2 4. 34. 3 4. 34. 4 4. 34. 5 4. 34. 6 4. 34. 7 4. 34. 8
4. 35	38_5	f, g, h	$\varphi_f = \varphi_h$ $\varphi_g\varphi_h = \varphi_h^2$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ a\ c\ a$ $a\ b\ b\ b$ $a\ b\ d\ b$	4. 35. 1 4. 35. 2 4. 35. 3 4. 35. 4
4. 36	39_5	f, g, h	$\varphi_f = \varphi_g$ $\varphi_g\varphi_h = \varphi_h\varphi_g$	$\varphi_e = \varphi_g\varphi_h$	$a\ a\ a\ a$ $b\ a\ a\ b$ $b\ b\ b\ a$ $a\ b\ b\ b$	4. 36. 1 4. 36. 2 4. 36. 3 4. 36. 4
4. 37	40_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4. 37. 1 4. 37. 2
4. 38	41_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4. 38. 1 4. 38. 2
4. 39	42_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4. 39. 1 4. 39. 2

4.40	43_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.40. 1 4.40. 2
4.41	44_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.41. 1 4.41. 2
4.42	45_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.42. 1 4.42. 2
4.43	46_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.43. 1 4.43. 2
4.44	47_5	f, g, h	$\varphi_f = \varphi_g$ $\varphi_h^2 = \varphi_g^2$	$\varphi_e = \varphi_h^2$	a a a a a a a b a b b b a b b a	4.44. 1 4.44. 2 4.44. 3 4.44. 4
4.45	48_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.45. 1 4.45. 2
4.46	49_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.46. 1 4.46. 2
4.47	50_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.47. 1 4.47. 2
4.48	51_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.48. 1 4.48. 2
4.49	52_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.49. 1 4.49. 2
4.50	53_5	f, g, h	$\varphi_f\varphi_g = \varphi_g\varphi_h = \varphi_h\varphi_f$	$\varphi_e = \varphi_g\varphi_h$	a a a a a b b b	4.50. 1 4.50. 2
4.51	54_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.51. 1 4.51. 2
4.52	55_5	f, g, h	$\varphi_f = \varphi_h$ $\varphi_g\varphi_h = \varphi_h\varphi_g$	$\varphi_e = \varphi_g\varphi_h$	a a a a a b b b b a b a b b a b	4.52. 1 4.52. 2 4.52. 3 4.52. 4
4.53	56_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.53. 1 4.53. 2
4.54	57_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.54. 1 4.54. 2
4.55	58_5	f, g, h	$\varphi_f^2 = \varphi_g\varphi_h = \varphi_h\varphi_g$	$\varphi_e = \varphi_f^2$	a a a a a a b b a b a a a b b b	4.55. 1 4.55. 2 4.55. 3 4.55. 4
4.56	59_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.56. 1 4.56. 2
4.57	60_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.57. 1 4.57. 2
4.58	61_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.58. 1 4.58. 2
4.59	62_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.59. 1 4.59. 2
4.60	63_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.60. 1 4.60. 2

4.61	64_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.61. 1 4.61. 2
4.62	65_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.62. 1 4.62. 2
4.63	66_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.63. 1 4.63. 2
4.64	67_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.64. 1 4.64. 2
4.65	68_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.65. 1 4.65. 2
4.66	69_5	f, g, h	$\varphi_g = \varphi_h$ $\varphi_f^2 = \varphi_h^2$	$\varphi_e = \varphi_h^2$	a a a a a b a a a a b b a b b b	4.66. 1 4.66. 2 4.66. 3 4.66. 4
4.67	70_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.67. 1 4.67. 2
4.68	71_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.68. 1 4.68. 2
4.69	72_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.69. 1 4.69. 2
4.70	73_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.70. 1 4.70. 2
4.71	74_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.71. 1 4.71. 2
4.72	75_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.72. 1 4.72. 2
4.73	76_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.73. 1 4.73. 2
4.74	77_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.74. 1 4.74. 2
4.75	78_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.75. 1 4.75. 2
4.76	79_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.76. 1 4.76. 2
4.77	80_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.77. 1 4.77. 2
4.78	81_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.78. 1 4.78. 2
4.79	82_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.79. 1 4.79. 2
4.80	83_5	f, g, h	$\varphi_f^2 = \varphi_g^2 = \varphi_h^2$	$\varphi_e = \varphi_h^2$	a a a a a a a b a a b b a b b b	4.80. 1 4.80. 2 4.80. 3 4.80. 4
4.81	84_5	f, g, h	$\varphi_g = \varphi_h$ $\varphi_f^2 = \varphi_h^2$	$\varphi_e = \varphi_h^2$	a a a a a a b b a b a a a b b b	4.81. 1 4.81. 2 4.81. 3 4.81. 4

4. 82	85_5	f, g, h	$\varphi_g=\varphi_h$ $\varphi_f^2=\varphi_h^2$	$\varphi_e=\varphi_h^2$	$a\ a\ a\ a$ $a\ a\ b\ b$ $a\ b\ a\ a$ $a\ b\ b\ b$	4. 82. 1 4. 82. 2 4. 82. 3 4. 82. 4
4. 83	86_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4. 83. 1 4. 83. 2
4. 84	87_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4. 84. 1 4. 84. 2
4. 85	88_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4. 85. 1 4. 85. 2
4. 86	89_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4. 86. 1 4. 86. 2
4. 87	90_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4. 87. 1 4. 87. 2
4. 88	91_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4. 88. 1 4. 88. 2
4. 89	92_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4. 89. 1 4. 89. 2
4. 90	93_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4. 90. 1 4. 90. 2
4. 91	94_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4. 91. 1 4. 91. 2
4. 92	95_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4. 92. 1 4. 92. 2
4. 93	96_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4. 93. 1 4. 93. 2

Table 5 $(I=6.3)$

Division Number	D	Base	Base Relations	defining Relations	φ_g	Individual Number
5. 1	2_2	g	*free*		(111) ε (111) α (112) ε (112) α	5. 1. 1 5. 1. 2 5. 1. 3 5. 1. 4

Table 6 ($I=6.3$)

Division Number	D	Base	Base Relations	defining Relations	φ_g	φ_h	Individual Number
6. 1	2_3	g, h	*free*		(111) ε	(111) ε	6. 1. 1
					(111) ε	(111) α	6. 1. 2
					(111) α	(111) α	6. 1. 3
					(111) ε	(222) ε	6. 1. 4
					(111) ε	(222) α	6. 1. 5
					(111) α	(222) ε	6. 1. 6
					(111) α	(222) α	6. 1. 7
					(111) ε	(112) ε	6. 1. 8
					(111) ε	(112) α	6. 1. 9
					(111) α	(112) ε	6. 1. 10
					(111) α	(112) α	6. 1. 11
					(111) ε	(221) ε	6. 1. 12
					(111) ε	(221) α	6. 1. 13
					(111) α	(221) ε	6. 1. 14
					(111) α	(221) α	6. 1. 15
					(111) ε	(322) ε	6. 1. 16
					(111) ε	(322) α	6. 1. 17
					(111) α	(322) ε	6. 1. 18
					(111) α	(322) α	6. 1. 19
					(112) ε	(222) ε	6. 1. 20
					(112) ε	(222) α	6. 1. 21
					(112) α	(222) ε	6. 1. 22
					(112) α	(222) α	6. 1. 23
					(112) ε	(333) ε	6. 1. 24
					(112) ε	(333) α	6. 1. 25
					(112) α	(333) ε	6. 1. 26
					(112) α	(333) α	6. 1. 27
					(112) ε	(112) ε	6. 1. 28
					(112) ε	(112) α	6. 1. 29
					(112) α	(112) α	6. 1. 30
					(112) ε	(221) ε	6. 1. 31
					(112) ε	(221) α	6. 1. 32
					(112) α	(221) ε	6. 1. 33
					(112) α	(221) α	6. 1. 34
6. 2	3_3	h	*free*	$\varphi_g=\varphi_h^2$	(111) ε	(111) ε	6. 2. 1
					(111) ε	(111) α	6. 2. 2
					(111) ε	(112) ε	6. 2. 3
					(111) ε	(112) α	6. 2. 4

Table 7 ($I=6.4$)

Division Number	D	Base	Base Relations	defining Relations	φ_g	Individual Number
7. 1	2_2	g	*free*		(11) ε	7. 1. 1
					(11) α	7. 1. 2

Table 8 ($I=6.4$)

Division Number	D	Base	Base Relations	defining Relations	φ_g	φ_h	Individual Number
8. 1	2_3	g, h	*free*		(11) ε	(11) ε	8. 1. 1
					(11) ε	(11) α	8. 1. 2
					(11) ε	(22) ε	8. 1. 3
					(11) ε	(22) α	8. 1. 4
					(11) α	(11) α	8. 1. 5
					(11) α	(11) β	8. 1. 6
					(11) α	(22) α	8. 1. 7
					(11) α	(22) β	8. 1. 8
8. 2	3_3	h	*free*	$\varphi_g=\varphi_h^2$	(11) ε	(11) ε	8. 2. 1
					(11) β	(11) α	8. 2. 2

Table 9 (Automorphisms of D)

2_3	(e,f)	17_5	(f, g)
3_4	permutations of e, f, g	20_5	(f, g)
7_4	(f, g)		
9_4	(f, g)		
11_4	(f, g)	28_5	(g, h)
4_5	permutations of e, f, g, h	30_5	(g, h)
15_5	(g, h)	32_5	(g, h)
		55_5	(f, h)

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