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SEMIGROUPS OF ORDER 5, 6, 7, 8 WHOSE GREATEST C-HOMOMORPHIC IMAGES ARE UNIPOTENT SEMIGROUPS WITH GROUPS[1)]

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In the previous paper [1] we obtained all the semigroups of order $\leqq 10$ whose greatest c-homomorphic images are groups. In the present paper we are going to list all the types of semigroups of order 5, 6, 7, 8 whose greatest c-homomorphic images are unipotent semigroups containing proper groups, without precise explanation. We should like to discuss the fundamental theorems used in this computation in another paper.

Let S be a finite semigroup whose greatest c-homomorphic image U is a unipotent semigroup without zero, a finite unipotent semigroup having a non-trivial group G as the kernel. Let I be the minimal ideal of S. Naturally I is a finite simple semigroup. Then I is mapped to G under the homomorphism $S \rightarrow U$ and the homomorphism $I \rightarrow G$ gives the greatest c-decomposition of I. Consider the difference semigroup of S module I and denote it by $D=(S:I)$. D is clearly a finite unipotent semigroup with zero. In order to construct the desired finite semigroup for given I and Z we may find S which satisfies $D=(S:I)$. Especially if the order of S is limited to 5, 6, 7, 8, then we may consider only the following cases

I	D	No. of Table
4. 3	2_2	1
	2_3, 3_3	2
	$3_4 \sim 12_4$	3
	$4_5 \sim 96_5$	4
6. 3	2_2	5
	2_3, 3_3	6
6. 4	2_2	7
	2_3, 3_3	8

1) This computation was performed in July 1960 by the four students of Tokushima University, and was revised by Tamura at the University of California in September 1960. Though the students computed all but a part of the types of order 9, 10 at the same time, the results will be published at a later date since the check is not yet finished.

where 4.3, 6.3, 6.4 are numbers seen in [1] and 2_3, 3_4, 4_5, $\cdots$, denote respectively the second semigroup of order 3, the third of order 4, the fourth of order 5, etc., which are seen in the list [2].

For the sake of the explanation below, let

$$I = \{a_1, \cdots, a_m\}, \qquad S = I \cup \{a_{m+1}, \cdots, a_n\},$$
$$D = \{0, a_{m+1}, \cdots, a_n\}$$

in which the bases of D shall be denoted $a_{k_1}, \cdots, a_{k_l}$. There exist two kinds of the systems of relations between $a_{m+1}, \cdots, a_n$, the basic relations (1) and the defining relations (2), in which the former is the relations between the bases, the latter between the bases and the other elements:

$$(1) \quad f_i(a_{k_1}, \cdots, a_{k_l}) = g_i(a_{k_1}, \cdots, a_{k_l}) \qquad i = 1, \cdots, t$$

$$(2) \quad a_j = h_j(a_{k_1}, \cdots, a_{k_l}), \qquad a_j \neq a_{k_1}, \cdots, a_{k_l}.$$

Let $\varphi_{a_1}, \cdots, \varphi_{a_m}, \varphi_{a_{m+1}}, \cdots, \varphi_{a_n}$ be the mappings of I into itself defined as

$$(x)\varphi_{a_i} = xa_i \qquad x \in I, \quad xa_i \in I$$

where φ_{a_i} $(i = 1, \cdots, m)$ are already given, but φ_{a_j} $(j = m+1, \cdots, n)$ are unknown. Φ denotes the semigroup which consists of φ_a $(a \in S)$, and Φ_0 the semigroup of $\{\varphi_a \mid a \in I\}$. Clearly S is homomorphic to Φ and so Φ_0 is an ideal of Φ.

$$(3) \quad \begin{cases} \varphi_{a_{j_1}} \varphi_{a_{j_2}} \in \Phi_0 & \text{if} \quad a_{j_1}a_{j_2} = 0 \quad \text{in } D \\ \varphi_{a_{j_1}} \varphi_{a_{j_2}} = \varphi_{a_{j_3}} & \text{if} \quad a_{j_1}a_{j_2} = a_{j_3} \in D, \ \neq 0 \end{cases}$$

and also for (1) and (2) we can assign the relations $(\bar{1})$ and $(\bar{2})$ under the homomorphism $a \to \varphi_a$.

$$(\bar{1}) \quad F_i(\varphi_{a_{k_1}}, \cdots, \varphi_{a_{k_l}}) = G_i(\varphi_{a_{k_1}}, \cdots, \varphi_{a_{k_l}}) \qquad i = 1, \cdots, t$$

$$(\bar{2}) \quad \varphi_{a_j} = H_j(\varphi_{a_{k_1}}, \cdots, \varphi_{a_{k_l}}), \qquad a_j \neq a_{k_1}, \cdots, a_{k_l}$$

where $(\bar{1})$ is obtained from the image of (1) by left cancellation, if necessary. Furthermore, we can easily see that $\varphi_{a_i}(i = m+1, \cdots, n)$ must be inner right translations[2)] of I if I is 4.3, and right translations of I if I is either 6.3 or 6.4.

If we find only (inner) right translations $\varphi_{a_{k_1}}, \cdots, \varphi_{a_{k_l}}$ of (4.3) 6.3 or 6.4 which satisfy $(\bar{1})$ and (3), then the other φ_{a_j} can be found by $(\bar{2})$. After $\varphi_{a_j}(j = m+1, \cdots, n)$ are determined, the remaining products in S are completely determined; in detail, since $I \ni a_i \to \varphi_{a_i}$ is one to one, if

$$\varphi_{a_{j_1}}\varphi_{a_{j_2}} = \varphi_{a_{j_3}} \in \Phi_0, \ \text{ we define } a_{j_1}a_{j_2} = a_{j_3} \in I.$$

By the way, we want to remark the following fact. In order that Φ_0 is an

2) With respect to right translation, see [3].

ideal of Φ, we must choose φ_{a_j} in some special subset of the set $\bar{\Phi}$ of all right translations of I, because Φ_0 may not be an ideal of $\bar{\Phi}$.

We shall add explanation of the notations in the tables below. In the tables 1, 2, 3, 4, since I, 4. 3, is

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	a	b	c	d
d	b	a	d	c

$S = I \cup \{e\}$, $D = \{0, e\}$, if D is of order 2,
$S = I \cup \{e, f\}$, $D = \{0, e, f\}$, if D is of order 3,
$S = I \cup \{e, f, g\}$, $D = \{0, e, f, g\}$, if D is of order 4,
$S = I \cup \{e, f, g, h\}$, $D = \{0, e, f, g, h\}$, if D is of order 5,

and the multiplication table of D should be understood by replacing 0, e, f, g, h by a, b, c, d, e in the list [2]. The inner right translations

$$\begin{pmatrix} a & b & c & d \\ a & b & a & b \end{pmatrix}, \quad \begin{pmatrix} a & b & c & d \\ b & a & b & a \end{pmatrix}, \quad \begin{pmatrix} a & b & c & d \\ c & d & c & d \end{pmatrix}, \quad \begin{pmatrix} a & b & c & d \\ d & c & d & c \end{pmatrix}$$

are denoted by a, b, c, d respectively for simplicity.

In the tables 5, 6, the ideal I, 6. 3, is a finite simple semigroup[3],

$$I = \{(\xi\,;1j) \mid \xi = \varepsilon, \alpha\,;\, j = 1, 2, 3\}$$

or $(\varepsilon\,;11) = a$, $(\alpha\,;11) = b$, $(\varepsilon\,;12) = c$, $(\alpha\,;12) = d$,
$(\varepsilon\,;13) = e$, $(\alpha\,;13) = f$,

where $\{\varepsilon, \alpha\}$ is a group of order 2 with the unit ε, and

$S = I \cup \{g\}$, $D = \{0, g\}$, if D is of order 2,
$S = I \cup \{g, h\}$, $D = \{0, g, h\}$, if D is of order 3.

The right translation φ_g or φ_h, denoted by $(ijk)\xi$, means the mapping of I into I

$$\begin{pmatrix} (\varepsilon\,;11) & (\alpha\,;11) & (\varepsilon\,;12) & (\alpha\,;12) & (\varepsilon\,;13) & (\alpha\,;13) \\ (\varepsilon\xi\,;1i) & (\alpha\xi\,;1i) & (\varepsilon\xi\,;1j) & (\alpha\xi\,;1j) & (\varepsilon\xi\,;1k) & (\alpha\xi\,;1k) \end{pmatrix}$$

where $\xi = \varepsilon$ or α ; i, j, $k = 1$ or 2 or 3, and $\varepsilon\xi$ and $\alpha\xi$ are the products in the ground group $\{\varepsilon, \alpha\}$.

In the tables 7, 8, the ideal I or 6. 4 is

$$I = \{(\xi\,;1j) \mid \xi = \varepsilon, \alpha, \beta\,;\, j = 1, 2\}$$

or $(\varepsilon\,;11) = a$, $(\alpha\,;11) = b$, $(\beta\,;11) = c$, $(\varepsilon\,;12) = d$,
$(\alpha\,;12) = e$, $(\beta\,;12) = f$,

where $\{\varepsilon, \alpha, \beta\}$ is a group of order 3, $\beta = \alpha^2$, $\varepsilon = \alpha^3$. The other notations are the same as the tables 5, 6.

3) The product is defined as $(\xi\,;1j)(\eta\,;1k) = (\xi\eta\,;1k)$.

All the tables show the system of the right translations of I which determines uniquely S, and the semigroups S constructed by the system of the right translations are neither mutually isomorphic nor anti-isomorphic.

We remark that the isomorphism of S is determined by the automorphisms of I and D. By the way, 4. 3 has an automorphism $\begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix}$ except the identical mapping and the semigroup 6. 3 has the automorphisms with the form

$$(\varepsilon\ ;1i) \to (\varepsilon\ ;1k_i)$$
$$(\alpha\ ;1j) \to (\alpha\ ;1k_j)$$

where the mapping $i \to k_i$ is a permutation of (1, 2, 3) and 6. 4 has the automorphism with the form

$$(\varepsilon\ ;1\ i) \to (\varepsilon\ ;1\ l_i)$$
$$(\alpha\ ;1\ j) \to (\alpha'\ ;1\ l_j)$$
$$(\beta\ ;1\ k) \to (\beta'\ ;1\ l_k)$$

where the mapping $i \to l_i$ is a permutation of (1, 2) and the mapping $\begin{pmatrix} \varepsilon & \alpha & \beta \\ \varepsilon & \alpha' & \beta' \end{pmatrix}$ is an automorphism of the group $\{\varepsilon,\ \alpha,\ \beta\}$.

Table 1 (I=4. 3)

Division Number	D	Base	Base Relations	defining Relations	φ_e	Individual Number
1. 1	2_2	e	*free*		a b	1. 1. 1 1. 1. 2

Table 2 (I=4. 3)

Division Number	D	Base	Base Relations	defining Relations	φ_e φ_f	Individual Number
2. 1	2_3	e, f	*free*		a a a b a c a d b b b d	2. 1. 1 2. 1. 2 2. 1. 3 2. 1. 4 2. 1. 5 2. 1. 6
2. 2	3_3	f		$\varphi_e=\varphi_f^2$	a a a b	2. 2. 1 2. 2. 2

Table 3 ($I=4.3$)

Division Number	D	Base	Base Relations	defining Relations	φ_e φ_f φ_g	Individual Number
3. 1	3_4	e, f, g	*free*		a a a a a b a a c a a d a b b a b c a b d b b b b b d	3. 1. 1 3. 1. 2 3. 1. 3 3. 1. 4 3. 1. 5 3. 1. 6 3. 1. 7 3. 1. 8 3. 1. 9
3. 2	4_4	f, g	*free*	$\varphi_e=\varphi_g^2$	a a a a b a a c a a d a a a b a b b a c b a d b	3. 2. 1 3. 2. 2 3. 2. 3 3. 2. 4 3. 2. 5 3. 2. 6 3. 2. 7 3. 2. 8
3. 3	5_4	f, g	*free*	$\varphi_e=\varphi_g\varphi_f$	a a a b b a c c a d d a b a b a b b d c b c d b	3. 3. 1 3. 3. 2 3. 3. 3 3. 3. 4 3. 3. 5 3. 3. 6 3. 3. 7 3. 3. 8
3. 4	6_4	f, g	$\varphi_f=\varphi_g$	$\varphi_e=\varphi_g^2$	a a a a b b	3. 4. 1 3. 4. 2
3. 5	7_4	f, g	$\varphi_f\varphi_g=\varphi_g\varphi_f$	$\varphi_e=\varphi_f\varphi_g$	a a a b a b a b b	3. 5. 1 3. 5. 2 3. 5. 3
3. 6	8_4	f, g	$\varphi_f=\varphi_g$	$\varphi_e=\varphi_g^2$	a a a a b b	3. 6. 1 3. 6. 2
3. 7	9_4	f, g	$\varphi_f^2=\varphi_g^2$	$\varphi_e=\varphi_g^2$	a a a a a b a b b	3. 7. 1 3. 7. 2 3. 7. 3
3. 8	10_4	f, g	$\varphi_f=\varphi_g$	$\varphi_e=\varphi_g^2$	a a a a b b	3. 8. 1 3. 8. 2
3. 9	11_4	f, g	$\varphi_f=\varphi_g$	$\varphi_e=\varphi_g^2$	a a a a b b	3. 9. 1 3. 9. 2
3. 10	12_4	g	*free*	$\varphi_e=\varphi_g^3$ $\varphi_f=\varphi_g^2$	a a a b a b	3. 10. 1 3. 10. 2

Table 4 $(I=4.3)$

Division Number	D	Base	Base Relations	defining Relations	φ_e	φ_f	φ_g	φ_h	Individual Number
4. 1	4_5	e, f, g, h	*free*		a	a	a	a	4. 1. 1
					a	a	a	b	4. 1. 2
					a	a	a	c	4. 1. 3
					a	a	a	d	4. 1. 4
					a	a	b	b	4. 1. 5
					a	a	b	c	4. 1. 6
					a	a	b	d	4. 1. 7
					a	a	c	c	4. 1. 8
					a	a	c	d	4. 1. 9
					a	a	d	d	4. 1. 10
					a	b	b	b	4. 1. 11
					a	b	b	c	4. 1. 12
					a	b	b	d	4. 1. 13
					a	b	c	d	4. 1. 14
					a	b	d	d	4. 1. 15
					a	d	d	d	4. 1. 16
					b	b	b	b	4. 1. 17
					b	b	b	d	4. 1. 18
					b	b	d	d	4. 1. 19
4. 2	5_5	g, h	*free*	$\varphi_e=\varphi_h\varphi_g$ $\varphi_f=\varphi_h^2$	a	a	a	a	4. 2. 1
					b	a	a	b	4. 2. 2
					a	c	a	c	4. 2. 3
					b	c	a	d	4. 2. 4
					b	a	b	a	4. 2. 5
					a	a	b	b	4. 2. 6
					b	c	b	c	4. 2. 7
					a	c	b	d	4. 2. 8
4. 3	6_5	g, h	*free*	$\varphi_e=\varphi_g^2$ $\varphi_f=\varphi_h^2$	a	a	a	a	4. 3. 1
					a	a	a	b	4. 3. 2
					a	c	a	c	4. 3. 3
					a	c	a	d	4. 3. 4
					a	a	b	a	4. 3. 5
					a	a	b	b	4. 3. 6
					a	c	b	c	4. 3. 7
					a	c	b	d	4. 3. 8
4. 4	7_5	g, h	*free*	$\varphi_e=\varphi_g\varphi_h$ $\varphi_f=\varphi_h\varphi_g$	a	a	a	a	4. 4. 1
					b	b	a	b	4. 4. 2
					c	a	a	c	4. 4. 3
					d	b	a	d	4. 4. 4
					b	b	b	a	4. 4. 5
					a	a	b	b	4. 4. 6
					d	b	b	c	4. 4. 7
					c	a	b	d	4. 4. 8
4. 5	8_5	g, h	$\varphi_g=\varphi_h$	$\varphi_e=\varphi_f=\varphi_h^2$	a	a	a	a	4. 5. 1
					a	a	b	b	4. 5. 2
4. 6	9_5	g, h	$\varphi_g=\varphi_h$	$\varphi_e=\varphi_f=\varphi_h^2$	a	a	a	a	4. 6. 1
					a	a	b	b	4. 6. 2
4. 7	10_5	g, h	$\varphi_g=\varphi_h$	$\varphi_e=\varphi_f=\varphi_h^2$	a	a	a	a	4. 7. 1
					a	a	b	b	4. 7. 2
4. 8	11_5	g, h	$\varphi_g=\varphi_h$	$\varphi_e=\varphi_f=\varphi_h^2$	a	a	a	a	4. 8. 1
					a	a	b	b	4. 8. 2
4. 9	12_5	g, h	$\varphi_g\varphi_h=\varphi_h\varphi_g$	$\varphi_e=\varphi_g^2$ $\varphi_f=\varphi_g\varphi_h$	a	a	a	a	4. 9. 1
					a	b	a	b	4. 9. 2
					a	a	b	b	4. 9. 3
					a	b	b	a	4. 9. 4

4. 10	13_5	g, h	$\varphi_g = \varphi_h$	$\varphi_e = \varphi_f = \varphi_h^2$	a a a a a a b b	4. 10. 1 4. 10. 2
4. 11	14_5	g, h	$\varphi_g^2 = \varphi_h^2$	$\varphi_e = \varphi_g \varphi_h$ $\varphi_f = \varphi_g^2$	a a a a b a a b b a b a a a b b	4. 11. 1 4. 11. 2 4. 11. 3 4. 11. 4
4. 12	15_5	g, h	$\varphi_g^2 = \varphi_h^2$ $\varphi_g \varphi_h = \varphi_h \varphi_g$	$\varphi_e = \varphi_g^2$ $\varphi_f = \varphi_g \varphi_h$	a a a a a b a b a a b b	4. 12. 1 4. 12. 2 4. 12. 3
4. 13	16_5	f, h	*free*	$\varphi_e = \varphi_h^3$ $\varphi_g = \varphi_h^2$	a a a a a b a a a c a a a d a a b a a b b b a b b c a b b d a b	4. 13. 1 4. 13. 2 4. 13. 3 4. 13. 4 4. 13. 5 4. 13. 6 4. 13. 7 4. 13. 8
4. 14	17_5	f, g, h	*free*	$\varphi_e = \varphi_h^2$	a a a a a b a a a c a a a d a a a b b a a c b a a d b a a c c a a d c a a d d a a a a b a b a b a c a b a d a b a b b b a c b b a d b b a c c b a d c b a d d b	4. 14. 1 4. 14. 2 4. 14. 3 4. 14. 4 4. 14. 5 4. 14. 6 4. 14, 7 4. 14. 8 4. 14. 9 4. 14. 10 4. 14. 11 4. 14. 12 4. 14. 13 4. 14. 14 4. 14. 15 4. 14. 16 4. 14. 17 4. 14. 18 4. 14. 19 4. 14. 20
4. 15	18_5	f, g, h	*free*	$\varphi_e = \varphi_h \varphi_g$	a a a a a b a a a c a a a d a a b a b a b b b a b c b a b d b a c a c a c b c a c c c a c d c a d a d a d b d a d c d a d d d a b a a b b b a b b c a b b d a b a a b b a b b b a c b b a d b b d a c b d b c b d c c b d d c b	4. 15. 1 4. 15. 2 4. 15. 3 4. 15. 4 4. 15. 5 4. 15. 6 4. 15. 7 4. 15. 8 4. 15. 9 4. 15. 10 4. 15. 11 4. 15. 12 4. 15. 13 4. 15. 14 4. 15. 15 4. 15. 16 4. 15. 17 4. 15. 18 4. 15. 19 4. 15. 20 4. 15. 21 4. 15. 22 4. 15. 23 4. 15. 24 4. 15. 25 4. 15. 26 4. 15. 27 4. 15. 28

4.16	19_5	f, g, h	$\varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	*a a a a* *a b a a* *a c a a* *a d a a* *a a b b* *a b b b* *a c b b* *a d b b*	4.16. 1 4.16. 2 4.16. 3 4.16. 4 4.16. 5 4.16. 6 4.16. 7 4.16. 8
4.17	20_5	f, g, h	$\varphi_f = \varphi_g$	$\varphi_e = \varphi_h \varphi_g$	*a a a a* *b a a b* *a a a c* *b a a d* *b b b a* *a b b b* *b b b c* *a b b d*	4.17. 1 4.17. 2 4.17. 3 4.17. 4 4.17. 5 4.17. 6 4.17. 7 4.17. 8
4.18	21_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	*a a a a* *a b b b*	4.18. 1 4.18. 2
4.19	22_5	g, h	$\varphi_g = \varphi_h^2$	$\varphi_e = \varphi_h^3$ $\varphi_f = \varphi_g$	*a a a a* *b a a b*	4.19. 1 4.19. 2
4.20	23_5	g, h	$\varphi_g = \varphi_h^2$	$\varphi_e = \varphi_h^3$ $\varphi_f = \varphi_g$	*a a a a* *b a a b*	4.20. 1 4.20. 2
4.21	24_5	g, h	$\varphi_g^2 = \varphi_h^3$	$\varphi_e = \varphi_h^3$ $\varphi_f = \varphi_h^2$	*a a a a* *a a b a*	4.21. 1 4.21. 2
4.22	25_5	g, h	$\varphi_g = \varphi_h = \varphi_h^2$	$\varphi_e = \varphi_f = \varphi_g$	*a a a a*	4.22. 1
4.23	26_5	g, h	$\varphi_g = \varphi_h = \varphi_h^2$	$\varphi_e = \varphi_f = \varphi_g$	*a a a a*	4.23. 1
4.24	27_5	g, h	$\varphi_g = \varphi_h$	$\varphi_e = \varphi_h^3$ $\varphi_f = \varphi_h^2$	*a a a a* *b a b b*	4.24. 1 4.24. 2
4.25	28_5	f, g, h	$\varphi_g \varphi_h = \varphi_h \varphi_g$	$\varphi_e = \varphi_g \varphi_h$	*a a a a* *a b a a* *a c a a* *a d a a* *b a a b* *b b a b* *b c a b* *b d a b* *a a b b* *a b b b* *a c b b* *a d b b*	4.25. 1 4.25. 2 4.25. 3 4.25. 4 4.25. 5 4.25. 6 4.25. 7 4.25. 8 4.25. 9 4.25. 10 4.25. 11 4.25. 12
4.26	29_5	f, g, h	$\varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	*a a a a* *a b a a* *a c a a* *a d a a* *a a b b* *a b b b* *a c b b* *a d b b*	4.26. 1 4.26. 2 4.26. 3 4.26. 4 4.26. 5 4.26. 6 4.26. 7 4.26. 8
4.27	30_5	f, g, h	$\varphi_g^2 = \varphi_h^2$	$\varphi_e = \varphi_g^2$	*a a a a* *a b a a* *a c a a* *a d a a* *a a a b* *a b a b* *a c a b* *a d a b* *a a b b* *a b b b* *a c b b* *a d b b*	4.27. 1 4.27. 2 4.27. 3 4.27. 4 4.27. 5 4.27. 6 4.27. 7 4.27. 8 4.27. 9 4.27. 10 4.27. 11 4.27. 12

4.28	31_5	f, g, h	$\varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b a a a c a a a d a a a a b b a b b b a c b b a d b b	4.28. 1 4.28. 2 4.28. 3 4.28. 4 4.28. 5 4.28. 6 4.28. 7 4.28. 8
4.29	32_5	f, g, h	$\varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b a a a c a a a d a a a a b b a b b b a c b b a d b b	4.29. 1 4.29. 2 4.29. 3 4.29. 4 4.29. 5 4.29. 6 4.29. 7 4.29. 8
4.30	33_5	f, g, h	$\varphi_g\varphi_h = \varphi_h\varphi_f$	$\varphi_e = \varphi_g\varphi_h$	a a a a b a a b a a c a b a c b b b b a a b b b b b d a a b d b	4.30. 1 4.30. 2 4.30. 3 4.30. 4 4.30. 5 4.30. 6 4.30. 7 4.30. 8
4.31	34_5	f, g, h	$\varphi_f = \varphi_h$ $\varphi_g\varphi_h = \varphi_h^2$	$\varphi_e = \varphi_h^2$	a a a a a a c a a b b b a b d b	4.31. 1 4.31. 2 4.31. 3 4.31. 4
4.32	35_5	f, g, h	$\varphi_f = \varphi_h$ $\varphi_g\varphi_h = \varphi_h^2$	$\varphi_e = \varphi_h^2$	a a a a a a c a a b b b a b d b	4.32. 1 4.32. 2 4.32. 3 4.32. 4
4.33	36_5	f, g, h	$\varphi_f = \varphi_h$ $\varphi_g\varphi_h = \varphi_h^2$	$\varphi_e = \varphi_h^2$	a a a a a a c a a b b b a b d b	4.33. 1 4.33. 2 4.33. 3 4.33. 4
4.34	37_5	f, g, h	$\varphi_h^2 = \varphi_g\varphi_f$	$\varphi_e = \varphi_h^2$	a a a a a a c a a b b a a b d a a a a b a a c b a b b b a b d b	4.34. 1 4.34. 2 4.34. 3 4.34. 4 4.34. 5 4.34. 6 4.34. 7 4.34. 8
4.35	38_5	f, g, h	$\varphi_f = \varphi_h$ $\varphi_g\varphi_h = \varphi_h^2$	$\varphi_e = \varphi_h^2$	a a a a a a c a a b b b a b d b	4.35. 1 4.35. 2 4.35. 3 4.35. 4
4.36	39_5	f, g, h	$\varphi_f = \varphi_g$ $\varphi_g\varphi_h = \varphi_h\varphi_g$	$\varphi_e = \varphi_g\varphi_h$	a a a a b a a b b b b a a b b b	4.36. 1 4.36. 2 4.36. 3 4.36. 4
4.37	40_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.37. 1 4.37. 2
4.38	41_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.38. 1 4.38. 2
4.39	42_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.39. 1 4.39. 2

4.40	43_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	a a a a a b b b	4.40. 1 4.40. 2
4.41	44_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	a a a a a b b b	4.41. 1 4.41. 2
4.42	45_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	a a a a a b b b	4.42. 1 4.42. 2
4.43	46_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	a a a a a b b b	4.43. 1 4.43. 2
4.44	47_5	f, g, h	$\varphi_f=\varphi_g$ $\varphi_h^2=\varphi_g^2$	$\varphi_e=\varphi_h^2$	a a a a a a a b a b b b a b b a	4.44. 1 4.44. 2 4.44. 3 4.44. 4
4.45	48_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	a a a a a b b b	4.45. 1 4.45. 2
4.46	49_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	a a a a a b b b	4.46. 1 4.46. 2
4.47	50_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	a a a a a b b b	4.47. 1 4.47. 2
4.48	51_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	a a a a a b b b	4.48. 1 4.48. 2
4.49	52_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	a a a a a b b b	4.49. 1 4.49. 2
4.50	53_5	f, g, h	$\varphi_f\varphi_g=\varphi_g\varphi_h=\varphi_h\varphi_f$	$\varphi_e=\varphi_g\varphi_h$	a a a a a b b b	4.50. 1 4.50. 2
4.51	54_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	a a a a a b b b	4.51. 1 4.51. 2
4.52	55_5	f, g, h	$\varphi_f=\varphi_h$ $\varphi_g\varphi_h=\varphi_h\varphi_g$	$\varphi_e=\varphi_g\varphi_h$	a a a a a b b b b a b a b b a b	4.52. 1 4.52. 2 4.52. 3 4.52. 4
4.53	56_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	a a a a a b b b	4.53. 1 4.53. 2
4.54	57_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	a a a a a b b b	4.54. 1 4.54. 2
4.55	58_5	f, g, h	$\varphi_f^2=\varphi_g\varphi_h=\varphi_h\varphi_g$	$\varphi_e=\varphi_f^2$	a a a a a a b b a b a a a b b b	4.55. 1 4.55. 2 4.55. 3 4.55. 4
4.56	59_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	a a a a a b b b	4.56. 1 4.56. 2
4.57	60_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	a a a a a b b b	4.57. 1 4.57. 2
4.58	61_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	a a a a a b b b	4.58. 1 4.58. 2
4.59	62_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	a a a a a b b b	4.59. 1 4.59. 2
4.60	63_5	f, g, h	$\varphi_f=\varphi_g=\varphi_h$	$\varphi_e=\varphi_h^2$	a a a a a b b b	4.60. 1 4.60. 2

4.61	64_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.61. 1 4.61. 2
4.62	65_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.62. 1 4.62. 2
4.63	66_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.63. 1 4.63. 2
4.64	67_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.64. 1 4.64. 2
4.65	68_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.65. 1 4.65. 2
4.66	69_5	f, g, h	$\varphi_g = \varphi_h$ $\varphi_f^2 = \varphi_h^2$	$\varphi_e = \varphi_h^2$	a a a a a b a a a a b b a b b b	4.66. 1 4.66. 2 4.66. 3 4.66. 4
4.67	70_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.67. 1 4.67. 2
4.68	71_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.68. 1 4.68. 2
4.69	72_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.69. 1 4.69. 2
4.70	73_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.70. 1 4.70. 2
4.71	74_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.71. 1 4.71. 2
4.72	75_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.72. 1 4.72. 2
4.73	76_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.73. 1 4.73. 2
4.74	77_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.74. 1 4.74. 2
4.75	78_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.75. 1 4.75. 2
4.76	79_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.76. 1 4.76. 2
4.77	80_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.77. 1 4.77. 2
4.78	81_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.78. 1 4.78. 2
4.79	82_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	a a a a a b b b	4.79. 1 4.79. 2
4.80	83_5	f, g, h	$\varphi_f^2 = \varphi_g^2 = \varphi_h^2$	$\varphi_e = \varphi_h^2$	a a a a a a a b a a b b a b b b	4.80. 1 4.80. 2 4.80. 3 4.80. 4
4.81	84_5	f, g, h	$\varphi_g = \varphi_h$ $\varphi_f^2 = \varphi_h^2$	$\varphi_e = \varphi_h^2$	a a a a a a b b a b a a a b b b	4.81. 1 4.81. 2 4.81. 3 4.81. 4

4.82	85_5	f, g, h	$\varphi_g = \varphi_h$ $\varphi_f^2 = \varphi_h^2$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ a\ b\ b$ $a\ b\ a\ a$ $a\ b\ b\ b$	4.82. 1 4.82. 2 4.82. 3 4.82. 4
4.83	86_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4.83. 1 4.83. 2
4.84	87_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4.84. 1 4.84. 2
4.85	88_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4.85. 1 4.85. 2
4.86	89_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4.86. 1 4.86. 2
4.87	90_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4.87. 1 4.87. 2
4.88	91_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4.88. 1 4.88. 2
4.89	92_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4.89. 1 4.89. 2
4.90	93_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4.90. 1 4.90. 2
4.91	94_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4.91. 1 4.91. 2
4.92	95_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4.92. 1 4.92. 2
4.93	96_5	f, g, h	$\varphi_f = \varphi_g = \varphi_h$	$\varphi_e = \varphi_h^2$	$a\ a\ a\ a$ $a\ b\ b\ b$	4.93. 1 4.93. 2

Table 5 ($I=6.3$)

Division Number	D	Base	Base Relations	defining Relations	φ_g	Individual Number
5. 1	2_2	g	*free*		(111) ε (111) α (112) ε (112) α	5. 1. 1 5. 1. 2 5. 1. 3 5. 1. 4

Table 6 $(I=6.3)$

Division Number	D	Base	Base Relations	defining Relations	φ_g	φ_h	Individual Number
6. 1	2_3	g, h	*free*		(111) ε	(111) ε	6. 1. 1
					(111) ε	(111) α	6. 1. 2
					(111) α	(111) α	6. 1. 3
					(111) ε	(222) ε	6. 1. 4
					(111) ε	(222) α	6. 1. 5
					(111) α	(222) ε	6. 1. 6
					(111) α	(222) α	6. 1. 7
					(111) ε	(112) ε	6. 1. 8
					(111) ε	(112) α	6. 1. 9
					(111) α	(112) ε	6. 1. 10
					(111) α	(112) α	6. 1. 11
					(111) ε	(221) ε	6. 1. 12
					(111) ε	(221) α	6. 1. 13
					(111) α	(221) ε	6. 1. 14
					(111) α	(221) α	6. 1. 15
					(111) ε	(322) ε	6. 1. 16
					(111) ε	(322) α	6. 1. 17
					(111) α	(322) ε	6. 1. 18
					(111) α	(322) α	6. 1. 19
					(112) ε	(222) ε	6. 1. 20
					(112) ε	(222) α	6. 1. 21
					(112) α	(222) ε	6. 1. 22
					(112) α	(222) α	6. 1. 23
					(112) ε	(333) ε	6. 1. 24
					(112) ε	(333) α	6. 1. 25
					(112) α	(333) ε	6. 1. 26
					(112) α	(333) α	6. 1. 27
					(112) ε	(112) ε	6. 1. 28
					(112) ε	(112) α	6. 1. 29
					(112) α	(112) α	6. 1. 30
					(112) ε	(221) ε	6. 1. 31
					(112) ε	(221) α	6. 1. 32
					(112) α	(221) ε	6. 1. 33
					(112) α	(221) α	6. 1. 34
6. 2	3_3	h	*free*	$\varphi_g=\varphi_h^2$	(111) ε	(111) ε	6. 2. 1
					(111) ε	(111) α	6. 2. 2
					(111) ε	(112) ε	6. 2. 3
					(111) ε	(112) α	6. 2. 4

Table 7 $(I=6.4)$

Division Number	D	Base	Base Relations	defining Relations	φ_g	Individual Number
7. 1	2_2	g	*free*		(11) ε	7. 1. 1
					(11) α	7. 1. 2

Table 8 $(I=6.4)$

Division Number	D	Base	Base Relations	defining Relations	φ_g	φ_h	Individual Number
8. 1	2_3	g, h	*free*		(11) ε	(11) ε	8. 1. 1
					(11) ε	(11) α	8. 1. 2
					(11) ε	(22) ε	8. 1. 3
					(11) ε	(22) α	8. 1. 4
					(11) α	(11) α	8. 1. 5
					(11) α	(11) β	8. 1. 6
					(11) α	(22) α	8. 1. 7
					(11) α	(22) β	8. 1. 8
8. 2	3_3	h	*free*	$\varphi_g=\varphi_h^2$	(11) ε	(11) ε	8. 2. 1
					(11) β	(11) α	8. 2. 2

Table 9 (Automorphisms of D)

2_3	(e, f)	17_5	(f, g)
3_4	permutations of e, f, g	20_5	(f, g)
7_4	(f, g)		
9_4	(f, g)		
11_4	(f, g)	28_5	(g, h)
4_5	permutations of e, f, g, h	30_5	(g, h)
15_5	(g, h)	32_5	(g, h)
		55_5	(f, h)

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