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NOTES ON THE SPACE OF REGULAR FUNCTIONS OF COMPLEX VARIABLES

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We consider a set Ω of regular functions of complex variables on $|z|<1$. It is clear that this set is considered as the vector space, which has complex numbers as operators. That is, if $f(z),\ g(z)\in\Omega,\ \alpha f(z)+\beta g(z)\in\Omega$ for arbitrary complex numbers α and β.

Now, we introduce their norms. Put $||f||=|c_n|$, where $f(z)=\sum_{n=0}^{\infty}c_nz^n$. Then we have

i) $||f+g||\leqq ||f||+||g||$,

ii) $||\alpha f||=|\alpha|\cdot||f||$,

iii) $||f||=0$, if and only if $|c_n|=0$.

If the sequence $\{f_p\}$ in Ω is the Cauchy-sequence, there exists limit point f_0. Indeed, let

$$||f_p-f_q||<\varepsilon,$$

for $p,\ q\geqq n_0$, which depends on an arbitrary positive number ε. Put $f_p(z)=\sum_{n=0}^{\infty}c_{pn}z^n$ and $f_q(z)=\sum_{n=0}^{\infty}c_{qn}z^n$, then $||f_p-f_q||=|c_{pn}-c_{qn}|<\varepsilon$. The sequence $\{c_{pn}\}$ of complex numbers is the Cauchy-sequence and so we have c_0 such that $\lim_{p\to\infty} c_{pn}=c_0$. Let $f_0(z)=c_0z^n$, then we have $||f_p-f_0||=|c_{pn}-c_0|\to 0$, when p thends to ∞.
Thus, we see that Ω is complete. Then we have next theorem.

Theorem 1. *Ω is the complex Banach space.*

Put $||f||_r=\sqrt{\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2d\theta}$ for $0<r<1$. Then Ω is the Hilbert space with respect to $|z|=r$.

Theorem 2. *If a sequence $\{f_n\}$ is a Cauchy-sequence with respect to the norm $||f||_r$ on $|z|=r$, for an arbitrary r, which satisfies $0<r<1$, then there exists $f_0(z)$ in Ω such that $f_n(z)$ converges uniformly to $f_0(z)$ on any closed circle $|z|\leqq\sigma(<1)$ in $|z|<1$, and moreover $||f_n-f_0||$ tends to 0.*

Proof. Since $||f_q-f_p||_r$ tends to 0 when p and q tend to ∞, then there exists $f_0(z)$ in the sence of L_2 on $|z|=r$ such that $||f_n-f_0||_r\to 0$ on $|z|=r$. Put $F(z)=\frac{1}{2\pi i}\int_C\frac{f_0(\zeta)}{\zeta-z}d\zeta$, where C is a circle whose radius is r. Then

$$|f_p(z)-F(z)| = \left| \frac{1}{2\pi i}\int_c \frac{f_p(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_c \frac{f_0(\zeta)}{\zeta - z}\,d\zeta \right|$$

$$= \left| \frac{1}{2\pi i}\int_c \frac{f_p(\zeta)-f_0(z)}{\zeta - z}\,d\zeta \right|$$

$$= \left| \frac{1}{2\pi}\int_0^{2\pi} \frac{f_p(re^{i\theta})-f_0(re^{i\theta})}{re^{i\theta}-\rho e^{i\varphi}}\,re^{i\theta}d\vartheta \right|, \text{ where } \zeta = re^{i\theta} \text{ and } z=\rho e^{i\varphi}.$$

$$\leqq \frac{r}{2\pi(r-\rho)}\int_0^{2\pi} |f_p(re^{i\theta})-f_0(re^{i\theta})|\,d\vartheta$$

Appealing to Holder's inequality, we have

$$\leqq \frac{r}{2\pi(r-\rho)}\sqrt{\int_0^{2\pi}|f_p(re^{i\theta})-f_0(re^{i\theta})|^2 d\theta}\sqrt{\int_0^{2\pi} d\vartheta}$$

$$= \frac{r}{\sqrt{2\pi}(r-\rho)}\,||f_p - f_0||.$$

If $\rho \leqq r_1 < r$, then $|f_p(z)-F(z)| = \frac{r}{\sqrt{2\pi}(r-\rho)}\,||f_p-f_0|| \to 0$, when $p\to\infty$. This shows that $f_p(z)$ converges to $F(z)$ uniformly on $|z|\leqq r_1$ for an arbitrary r_1 such that $r_1 < r$ and we see that $F(z)$ is regular on $|z|<r$. Since r is arbitrary in $0<r<1$, $f_p(z)$ converges uniformly to $F(z)$ in $|z|\leqq\sigma<1$ and we see that $F(z)$ is regular in $|z|<1$ and so $F(z)\in\Omega_0$.

$$||f_p - F|| = \left| \frac{1}{2\pi i}\int_c \frac{f_p(\zeta)}{\zeta^{n+1}}\,d\zeta - \frac{1}{2\pi i}\int_c \frac{f_0(\zeta)}{\zeta^{n+1}}\,d\zeta \right|$$

$$= \left| \frac{1}{2\pi i}\int_0^{2\pi} \frac{f_p(re^{i\theta})-f_0(re^{i\theta})}{r^{n+1}e^{i(n+1)\theta}}\,ire^{i\theta}\,d\vartheta \right|$$

$$\leqq \frac{1}{2\pi r^n}\int_0^{2\pi}|f_p(re^{i\theta})-f_0(re^{i\theta})|\,d\vartheta$$

$$\leqq \frac{1}{2\pi r^n}\sqrt{\int_0^{2\pi}|f_p(re^{i\theta})-f_0(re^{i\theta})|^2}\ \sqrt{\int_0^{2\pi} d\vartheta}$$

$$\leqq \frac{1}{\sqrt{2\pi}\,r^n}\,||f_n - f_0||_r.$$

This shows that $||f_p - F||\to 0$, when $||f_p - F||_r \to 0$. This completes the proof.