Journal of Gakugei, Tokushima University
Vol. XI. 1960

ON GROUPS OF SPECIAL MOTIONS IN A SUBSPACE OF A RIEMANNIAN SPACE

By

Yoshihiro ICHIJYO

(Received September 30, 1960)

§ 1. In the first place let us consider the tangential infinitesimal point transformation

$$\bar{x}^{\lambda'} = x^{\lambda} + B_i^{\cdot\lambda} v^i dt \tag{1,1}$$

in L_n with coordinates $x^i (i = 1, 2, \cdots, n)$ which is an n-dimensional subspace of an affinly connected m-dimensional space L_m with torsion and with coordinates $x^{\alpha} (\alpha = 1, 2, \cdots, m)$, where we denote by $B_i^{\cdot\lambda}$ the tangent vectors $\partial x^{\lambda}/\partial x^i$.

By Prof. K. Yano [5][1], rewriting (1, 1) as

$$\bar{x}^{\lambda} = x^{\lambda}(x^i) + \frac{\partial x^{\lambda}}{\partial x^i} v^i dt = x^{\lambda}(x^i + v^i dt),$$

we may consider this transformation as a change of parameters

$$x^i \rightarrow \bar{x}^{i'} = x^i + v^i dt. \tag{1,2}$$

Although in general the Lie differential is defined by

$$\pounds\, T^{\lambda\mu}_{\cdot\cdot\nu} dt = T^{\lambda\mu}_{\cdot\cdot\nu}(\bar{x}) - T^{\lambda'\mu'}_{\cdot\cdot\nu'}(x), \tag{1,3}$$

in our case we shall define the Lie differential of composite tensors $T^{\cdot\lambda\cdots i\cdots}_{\cdots u\cdots j\cdot}(x^i)$ as

$$\pounds\, T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot} dt = T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot}(\bar{x}) - \widetilde{T}^{\cdot\lambda\cdots i'\cdots}_{\cdots\mu\cdots j'\cdot}(x), \tag{1,4}$$

where $\widetilde{T}^{\cdot\lambda\cdots i'\cdots}_{\cdots\mu\cdots j'\cdot}(x)$ is parallelly displaced quantities of $T^{\cdot\lambda\cdots i'\cdots}_{\cdots\mu\cdots j'\cdot}(x)$ by (1, 1) with respect to the affine connection of L_m.

Calculating (1, 4), we have

$$\begin{aligned}
&\pounds\, T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot} dt \\
&= T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot}(x^i) + \partial_k T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot}(x^i) v^k dt - \widetilde{T}^{\cdot\lambda\cdots a\cdots}_{\cdots\mu\cdots b\cdot}(x^i) \cdots A^{i'}_a \cdots A^b_{j'} \cdots^{2)} \\
&= T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot}(x^i) + \partial_k T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot}(x^i) v^k dt \\
&\quad - [T^{\cdot\lambda\cdots a\cdots}_{\cdots\mu\cdots b\cdot}(x^i) \cdots - \Gamma^{\lambda}_{\rho\nu} T^{\cdot\nu\cdots a\cdots}_{\cdots\mu\cdots b\cdot}(x^i) B_k^{\cdot\rho} v^k dt - \cdots \\
&\qquad \cdots + \Gamma^{\nu}_{\rho\mu} T^{\cdot\lambda\cdots a\cdots}_{\cdots\nu\cdots b\cdot}(x^i) B_k^{\cdot\rho} v^k dt + \cdots] \\
&\qquad \cdots \times [\delta^i_a + \partial_a v^i dt] \times \cdots \times [\delta^b_j - \partial_j v^b dt] \times \cdots
\end{aligned}$$

---

1) Nunbers in brackets refer to the references at the end of this paper.

2) $A^{i'}_a$ and $A^b_{j'}$ stand for $\dfrac{\partial x^{i'}}{\partial x^a}$ and $\dfrac{\partial x^b}{\partial x^{j'}}$ respectively.

$$= [v^k T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot};_k - \cdots - T^{\cdot\lambda\cdots a\cdots}_{\cdots\mu\cdots j\cdot} v_a^{\cdot i} - \cdots \quad \cdots + T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots a\cdot} v_j^{\cdot a} + \cdots]\, dt,$$

where we put

$$\left.\begin{aligned} v_a^{\cdot i} &= \partial_a v^i + \Gamma^i_{ja} v^j = v^i_{;a} + 2\, S_{ba}^{\cdot\cdot i} v^b, \\ S_{ba}^{\cdot\cdot i} &= \frac{1}{2}\,(\Gamma_{ba}^{\cdot\cdot i} - \Gamma_{ab}^{\cdot\cdot i}). \end{aligned}\right\} \quad (1, 5)$$

Hence we obtain

$$\text{£}\, T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot} = v^k\, T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot};_k - \cdots - T^{\cdot\lambda\cdots a\cdots}_{\cdots\mu\cdots j\cdot} v_a^{\cdot i} - \cdots$$
$$\cdots + T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots a\cdot} v_j^{\cdot a} + \cdots. \quad (1, 6)$$

Similarly we obtain

$$\begin{aligned} \text{£}\, \Gamma^i_{jk} &= \partial_k \partial_j v^i + v^h \partial_h \Gamma^i_{jk} + \Gamma^i_{hk} \partial_j v^h + \Gamma^i_{jh} \partial_k v^h - \Gamma^h_{jk} \partial_h v^i \\ &= v_j^{\cdot i};_k + R^i_{\cdot jkh}\, v^h. \end{aligned} \quad (1, 7)$$

and

$$\text{£}\,(\delta\, u^k) - \delta\,(\text{£}\, u^k) = (\text{£}\, \Gamma^k_{ij})\, u^j\, dx^i. \quad (1, 8)$$

§ **2.** In this section we consider n-dimensional subspace V_n immersed in an m-dimensional Riemannian space V_m, then we have

$$S_{jk}^{\cdot\cdot i} = 0, \quad \Gamma^\lambda_{\mu\nu} = \{{}^{\lambda}_{\mu\nu}\}, \quad \Gamma^i_{jk} = \{{}^{i}_{jk}\},$$

and

$$\text{£}\, T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot} = v^k\, T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots j\cdot};_k - \cdots - T^{\cdot\lambda\cdots a\cdots}_{\cdots\mu\cdots j\cdot} v^i_{;a} - \cdots$$
$$\cdots + T^{\cdot\lambda\cdots i\cdots}_{\cdots\mu\cdots a\cdot} v^a_{;j} + \cdots, \quad (2, 1)$$

from which

$$\text{£}\, g_{ij} = v_{j;i} + v_{i;j}, \quad (2, 2)$$

$$\text{£}\, B_i^{\cdot\lambda} = v^k H_{ik}^{\cdot\cdot\lambda} + v^a_{;i}\, B_a^{\cdot\lambda}, \quad (2, 3)$$

and

$$\text{£}\, H_{ij}^{\cdot\cdot\lambda} = H_{ij}^{\cdot\cdot\lambda};_k\, v^k + H_{aj}^{\cdot\cdot\lambda}\, v^a_{;i} + H_{ia}^{\cdot\cdot\lambda}\, v^a_{;j}\ , \quad (2, 4)$$

hence, we see these results coincide with Yano's [5] and Hlavatý's [4], where we denote by $H_{ij}^{\cdot\cdot\lambda}$ Euler-Schouten's tensor, that is,

$$H_{ij}^{\cdot\cdot\lambda} = B_i^{\cdot\lambda};_j\,.$$

Now we attach to any point x^i of V_n in V_m a frame $(B_i^{\cdot\lambda}, B_P^{\cdot\lambda})$ $(P = n+1, n+2, \cdots, m)$ in such a way that $B_i^{\cdot\lambda}$ are n tangent vectors of $V_n$and $m-n$ vectors $B_P^{\cdot\lambda}$ are orthogonal to V_n and mutually orthogonal, then we have

$$H_{ij}^{\cdot\cdot\lambda} = \sum_P H_{ijP}\, B_P^{\cdot\lambda},$$

$$\text{£}\, B_P^{\cdot\lambda} = v^k B_{P;k}^{\cdot\lambda}\,.$$

By Weingarten's formula

$$B_{P}^{\cdot\lambda}{}_{;k} = -g^{ij} H_{jkP} B_i^{\cdot\lambda} + \sum_Q L_{PQ|k} B_Q^{\cdot\lambda},$$

where we put

$$L_{PQ|k} = g_{\alpha\beta} B_P^{\cdot\alpha} B_{Q,k}^{\cdot\beta} + [\gamma\delta, \beta] B_k^{\cdot\gamma} B_Q^{\cdot\delta} B_P^{\cdot\beta},$$

we obtain

$$\pounds B_P^{\cdot\lambda} = v^k [-g^{ij} H_{jkP} B_i^{\cdot\lambda} + \sum_Q L_{PQ|k} B_Q^{\cdot\lambda}], \tag{2, 5}$$

$$\pounds H_{ijP} = H_{ijP;k} v^k + H_{kjP} v^k{}_{;i} + H_{ikP} v^k{}_{;j}. \tag{2, 6}$$

As in general, we call a point transformation of V_n in V_m *a motion* if it satisfies

$$\pounds g_{ij} = 0. \tag{2, 7}$$

§ **3.** In this section we consider the case $m = n+1$, then we have

$$L_{PQ|k} = 0, \quad B_P^{\cdot\alpha} = B_{n+1}^{\cdot\alpha},$$

from which we can put

$$B_{n+1}^{\cdot\alpha} = \xi^\alpha, \quad H_{ij\,n+1} = H_{ij}.$$

Now we call a motion *an absolute motion* if it satisfies

$$\pounds H_{ij} = 0. \tag{3, 1}$$

Then the condition that a point transformation (1, 1) is an absolute motion is given as

$$\left.\begin{aligned} v_{i;j} + v_{j;i} = 0, \\ v^k H_{ij;k} + H_{kj} v^k{}_{;i} + H_{ik} v^k{}_{;j} = 0. \end{aligned}\right\} \tag{3, 2}$$

By multiplying g^{ij} to the second equation, summing up with i and j and using the first equation, we get

$$v^k \partial_k H = 0, \tag{3, 3}$$

where we put

$$H = \frac{1}{n} g^{ij} H_{ij}.$$

Thus we have

Theorem 3. 1. *If an infinitesimal tangential point transformation in a V_n immersed in a V_{n+1} is an absolute motion, then the vector $\rho_i = \partial_i H$ must be orthogonal to the Killing vector v_i.*

As a motion in V_n in V_{n+1} is, however, a motion in a Riemannian space V_n, we consider a group of motions of order r which is generated by those r Killing vectors $\underset{a}{v}^k (a=1, 2, \cdots, r)$, if V_n admits such a group of motions.

Now let us consider a group of motions which satisfy (3, 1) and call it *a* *group of absolute motions*. Therefore if V_n admits a group of absolute motions and the rank of $(\underset{a}{v^i})$ is n, then we get

$$H = \text{const.}.$$

Hence we have

Theorem 3. 2. *If a V_n in a V_{n+1} admits a group of absolute motions of order r $(r \geqq n)$ and the rank of the matrix $(\underset{a}{v^i})$ is n, then the V_n must be a hypersurface with constant mean curvature in V_{n+1}.*

§ **4.** Here we consider the order r of a group G_r of absolute motions of V_n in V_{n+1}.

Since G_r must be a group of motions in V_n, we get

$$r \leqq \frac{n(n+1)}{2}.$$

If the equality holds, V_n is a space of constant curvature. In this case it is clear that $G_{n(n+1)/2}$ is a group of absolute motions if V_n is either a completely totally umbilical or a totally geodesic hypersurface in V_{n+1}. Hence we have

Theorem 4. 1. *In the case where a V_n is a space of constant curvature and either a completely totally umbilical or a totally geodesic hypersurface in a V_{n+1}, then V_n admits a group of absolute motions of the maximum order $n(n+1)/2$.*

Now applying the analogus method of proof of Egorov's theorem [2] [7], we have

Theorem 4. 2. *The maximum order of the group of absolute motions in those V_n's which are immersed in V_{n+1} and are neither completely totally umbilical nor totally geodesic is less than or equals to $\frac{1}{2}n(n-1)+1$.*

And again we have

Theorem 4. 3. *If the V_n in V_{n+1} admits a group of absolute motions of order greater than $\frac{1}{2}n(n-1)+1$, then the V_n is either a completely totally umbilical or a totally geodesic hypersurface in V_{n+1}.*

In fact, since absolute motions are motions, if the operator $\underset{v}{£} f$ is that of an absolute motion, we have

$$\underset{v}{£}\, g_{ij} = v_{i;j} + v_{j;i} = 0, \tag{4, 1}$$

$$\underset{v}{£}\, H_{ij} = v^k H_{ij;k} + H_{kj} v^k{}_{;i} + H_{ik} v^k{}_{;j} = 0. \tag{4, 2}$$

Using (4, 1) we can write (4, 2) also in the form[1)]

1) Here we have used a new kind of indices P_α, P_β also running 1 to n which we assume that $P_\alpha \neq P_\beta$ for $\alpha \neq \beta$ and that the summation convention does not hold.

$$\underset{v}{\pounds} H_{ij} = v^k H_{ij;k} + \sum_{P_\alpha > P_\beta}^{1\cdots n} T_{ij}^{\cdot\cdot P_\alpha P_\beta} v_{P_\alpha ; P_\beta} = 0, \tag{4, 3}$$

where

$$T_{ij}^{\cdot\cdot P_\alpha P_\beta} = 4\, \delta_{(i}^{[P_\alpha} H_{j)}^{\cdot P_\beta]}, \quad H_j^{\cdot k} = g^{ki} H_{ji}. \tag{4, 4}$$

Now we proceed the calulation by such a method as Prof. Egorov's. The $(P_\alpha\, P_\beta)$ rank, or $(i\, j)$ rank, of the matrix $(T_{ij}^{\cdot\cdot P_\alpha P_\beta})$[2] $(i \leqq j,\ P_\alpha < P_\beta)$ must be less than

$$\frac{1}{2} n\, (n+1) - (\frac{1}{2}\, n\, (n-1) + 1) = n - 1,$$

and we have

$$\begin{cases} H_{P_\alpha}^{\cdot P_\beta} = 0, \\ H_1^{\cdot 1} = H_2^{\cdot 2} = \cdots = H_n^{\cdot n} \equiv \rho, \end{cases}$$

that is,

$$H_i^{\cdot j} = \delta_i^j\, \rho,$$

from which

$$H_{ij} = \rho\, g_{ij}.$$

Putting this into (4, 2) we have

$$v^k \rho_k g_{ij} + \rho\, g_{kj}\, v^k_{\ ;i} + \rho\, g_{ik}\, v^k_{\ ;j} = 0,$$

from which by using (4, 1) we have

$$v^k \rho_k = 0.$$

Since this equation must hold for all vectors $\underset{a}{v}^k$

$$(a = 1, 2, \cdots, r;\ r < \frac{1}{2}\, n\, (n-1) + 1)$$

which are generating vectors of G_r,

$$\underset{a}{v}^k \rho_k = 0.$$

The rank of $(\underset{a}{v}^k)$ being n,

$$\rho = \text{const.}.$$

Hence the V_n in V_{n+1} must be a completely totally umbilical or a totally geodesic in V_{n+1}.

Moreover from these theorems we have

Theorem 4. 4. *The necessary and sufficient condition that a V_n in V_{n+1} admits a group of absolute motions of order $\frac{1}{2}\, n\, (n+1)$ is that the V_n is a space of*

---

2) We have considered here the matrix $T_{ij}^{\cdot\cdot P_\alpha P_\beta}$ by letting the two lower indices denote the rows and the two upper indices the columns, and we have called the rank of this matrix $(P_\alpha P_\beta)$ rank or $(i\, j)$ rank.

constant curvature and either a completely totally umbilical or a totally geodesic hypersurface in V_{n+1}.

§ **5.** Again in this section we consider the order of a group of absolute motions by a similar method of Egorov's [3].

The conditions

$$\underset{v}{\pounds} H_{ij} = 0 \tag{5, 1}$$

can be written as

$$\underset{v}{\pounds} H_{ij} = v^a H_{ij;a} + T_{ija}^{\cdot\cdot\cdot b} v^a{}_{;b}\,,$$

where we put

$$T_{ija}^{\cdot\cdot\cdot b} = \delta_j^b H_{ia} + \delta_i^b H_{aj}\,. \tag{5, 2}$$

From which

$$\begin{aligned} T_{ija}^{\cdot\cdot\cdot b} v^a{}_{;b} = T_{ijab} v^{a;b} &= \sum_{p<q}^{1\cdots n} 2\, T_{ij[pq]} v^{p;q} \\ &= \sum_{p<q}^{1\cdots n} S_{ijpq} v^{p;q}, \end{aligned} \tag{5, 3}$$

where we put

$$2\, T_{ij[pq]} = S_{ijpq}. \tag{5, 4}$$

Hence (5, 1) gives

$$v^a H_{ij;a} + \sum_{p<q}^{1\cdots n} S_{ijpq} v^{p;q}. \tag{5, 5}$$

Now we consider the $(i\,j)$ rank of the matrix (S_{ijpq}), where

$$S_{ijpq} = g_{qj} H_{ip} + g_{qi} H_{pj} - g_{pj} H_{iq} - g_{pi} H_{qj}. \tag{5, 6}$$

Without loss of generality we may consider the case $i \leqq j$, since

$$S_{ijpq} = S_{jipq} \quad .$$

Here we suppose that the rank of matrix (H_{ij}) is l and determinant equation

$$| H_{ij} - \lambda g_{ij} | = 0$$

has only simple roots $\lambda = H_{(1)}, H_{(2)}, \cdots, H_{(l)}$ except $\lambda = 0$. Then for convenience we choose the coordinate system in such a way that the lines of curvatures passing through an arbiterary point in V_n in V_{n+1} are parameter curves of V_n —— if $l < n$, we can take as n parameter curves l lines of curvatures determined by the above $H_{(p)}$ $(1 \leqq p \leqq l)$ and $n-l$ lines of curvatures which are orthogonal mutually and also to the former lines of curvatures since the lines of curvatures are indeterminate in V_{n-l} where $\lambda = 0$ [1].

From our assumption

$$H_{(p)} \neq H_{(q)} \text{ for } p \neq q. \tag{5, 7}$$

Moreover we put $H_{(j)}$ as follows:

if $1 \leqq j \leqq l$, the $H_{(j)}$ are solutions of the proper equation, if $l < j \leqq n$, then $H_{(j)}$ vanish.

Then we have

$$g_{ij} \overset{*}{=} \delta_{ij}, \quad H_{ij} \overset{*}{=} \delta_{ij} H_{(j)}, \tag{5, 8}$$

from which

$$S_{ijpq} \overset{*}{=} \delta_{qj}\delta_{pi} H_{(i)} + \delta_{qi}\delta_{pj} H_{(j)} - \delta_{pj}\delta_{qi} H_{(i)} - \delta_{pi}\delta_{qj} H_{(j)}. \tag{5, 9}$$

After some calculations we obtain that

$$S_{ijij} \overset{*}{=} H_{(i)} - H_{(j)}, \quad S_{ijji} \overset{*}{=} H_{(j)} - H_{(i)} \ ,$$

and the other components of S_{ijpq} all vanish.

Then the nonvanishing columns and rows of the matrix (S_{ijpq}) $(i \leqq j)$ are

$(ij) \backslash (pq)$	(1 2)	(1 3) ⋯	(1 l)	(2 3) ⋯	(2 l)	(3 4) ⋯	($l-1$, l)
(1 2)	$H_{(1)}-H_{(2)}$	0	⋯	⋯	⋯	⋯	0
(1 3)	0	$H_{(1)}-H_{(3)}$					⋮
⋮	⋮	⋮	⋱				⋮
(1 l)	⋮	⋮	$H_{(1)}-H_{(l)}$	0			⋮
(2 3)	⋮	⋮	0	$H_{(2)}-H_{(3)}$			⋮
⋮	⋮	⋮		⋱			⋮
(2 l)	⋮	⋮			$H_{(2)}-H_{(l)}$	0	⋮
(3 4)	⋮	⋮			0	$H_{(3)}-H_{(4)}$	⋮
⋮	⋮	⋮				⋱	⋮
($l-1$, l)	0	0	⋯	⋯	⋯	⋯	$H_{(l-1)}-H_{(l)}$

Hence in our assumption $v^{a;b}$ must satisfy the conditions more than $l(l-1)/2$ adding to Killing's conditions. Thus we obtain

Theorem 5. 1. *Let a V_n in V_{n+1} admit a group of absolute motions, the rank of (H_{ij}) be l, and the equation $|H_{ij} - \lambda g_{ij}| = 0$ have only l non-vanishing simple roots in every point of V_n, then*

$$r \leqq \frac{1}{2}(n+l)(n-l+1) \quad .$$

From which we have

Theorem 5. 2. *When a V_n in V_{n+1} admits a group of absolute motions, n principal directions are determinate and the n principal curvatures $H_{(i)}$ are not equal each other in every point of V_n, then*

$$r \leqq n.$$

References

[1] L. P. Eisenharrt : Riemannian Geometry, *Princeton Univ. Press* (1949).

[2] P. Egorov : On a strengthening of Fubini's theorem on the order of the group of motions of a Riemannian space, *Doklady Aoad. Nauk SSSR* (*N. S.*) 66 (1949), 793—796.

[3] P. Egorov : A tensor characterisation of A_n of nonzero curvature with maximum mobility, *Doklady Acad, Nauk SSSR* (*N. S.*) 84 (1952), 209—212.

[4] V. Hlavatý : Deformation theory of subspaces in a Riemannian space, *Proc. Amer. Math. Soc.* 16 (1950), 600—617.

[5] K. Yano : Sur la deformation infinitesimale tangentielle d'un sous-espace. *Proc. Imp. Acad.* 21 (1945), 261—268.

[6] K. Yano : On n-dimensional Riemannian spaces admitting a group of motions of order $\frac{1}{2}n(n-1)+1$. *Trans. Amer. Math. Soc.* 74 (1953), 260—279.

[7] K. Yano : The Theory of Lie Derivatives and its Applications. *North-Holland Publishing Co.* Amsterdam (1957).