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AN IMPRESSION OF A MATHEMATICIAN.

Y. WATANABE.

The Journal of Gakugei, Tokushima University (Mathematics) has been published just to its tenth Volume. It should be congratulated at any rate.

However, even a certain authority once blamed though pertinently in some respect that there are so many publications of mathematical essays, that he cannot look over one by one. But, behold how plenty mass communications of compositions not only on sciences, but much more on criticisms, literatures, pictures and musics &c.! Is not it rather a proud of the present age inspired with creative genius of the world, to have gone far above the Runaissance, especially in natural sciences? Accordingly, as mathematicians, we ought somehow to contrive to contribute to the world of knowledge following the general tendency in the academic field so as our studies rarely adapt to promote the proper course. Furthermore, as the great Cantor claims, the essence of mathematics exists in its freedom of thinkingways. What seems sometimes to be trifle or injurious conventionally, might be a germ of a good flourishing towering tree. To say the privilege of a scientist, it is merely to delight himself stealthily when he could accomplish his research by displaying his own creative instinct artistically, yet through Dante's purgatory, i. e. not without suffering under the intolerably heavy pressure of immense classical conventions, whose obstacle is only surmountable for a genius. Therefore there is little need to restrain scientific works too particularly. As A. Josano, a Japanese poetess, sings "I also will nail a small gold stud at the eternally extending palace of beauties," so also we would make some contribution if minute to the forever constructing palace of truths.

Y. Watanabe.

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ON THE COMPLETE TENSOR PRODUCT OF MODULES

By

Motoyoshi SAKUMA

(Received September 30, 1959)

In our previous paper [2], we introduced the notion of the complete tensor product of modules. Namely, let E and E' be, respectively, finite modules over an $\mathfrak{m}$- and an $\mathfrak{m}'$-adic Zariski rings A and A'. Assume A and A' contain a common subfield K. Put $G_n = E/\mathfrak{m}^n E \otimes_K E'/\mathfrak{m}'^n E'$. Then the system $\{G_n, \phi_n\}$ $(n = 1, 2, \dots)$ constitutes an inverse system of $A \otimes_K A'$-modules, where ϕ_n denotes the canonical homomorphisms $G_n \to G_{n-1}$. Its projective limit $E \hat{\otimes}_K E'$ is referred to as a complete tensor product of E and E' over K. In this note, we shall mainly investigate, following closely the recent work of Satô [4], the relation between the multiplicities $e_E(\mathfrak{q})$, $e_{E'}(\mathfrak{q}')$ and $e_{E\hat{\otimes}E'}((\mathfrak{q}, \mathfrak{q}')(A \hat{\otimes} A'))$ in the case when A and A' are, respectively, local rings, where we denote by $\mathfrak{q}$ and $\mathfrak{q}'$ primary ideals belonging to the maximal ideals of A and A' respectively.[1]

This relation was studied first, in a restricted case, by Samuel [3] and continued by Nagata [1] and Satô [4] in the case of rings.

1. General remarks on the complete tensor product of modules.

We start with the following proposition which is fundamental in this note.

PROPOSITION 1. *Let A and A' be, respectively, an $\mathfrak{m}$-adic and an $\mathfrak{m}'$-adic Zariski rings which contain a common subfield K and let E, F and G be finite A-modules such that*

$$0 \to F \to E \to G \to 0 \quad \text{(exact)}.$$

Then, for any finite A'-module E', we have the following exact sequence of finite $A \hat{\otimes}_K A'$-modules :

$$0 \to F \hat{\otimes}_K E' \to E \hat{\otimes}_K E' \to G \hat{\otimes}_K E' \to 0.$$

And we have

$$E \hat{\otimes}_K E' \approx (E \otimes_K E') \otimes_{A \otimes A'} (A \hat{\otimes} A').^{2)}$$

For the proof we refer the reader to [2].

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1) For the notations and terminology we refer the reader to [2].

2) In the following we shall omit K and $A \otimes A'$ if any confusion does not occur.

COROLLARY. (*With the same notations and assumptions*). *For any submodules F and G (resp. F' and G') of E (resp. E'), we have*

i) $(F+G)\hat{\otimes}(F'+G') = F\hat{\otimes}F'+F\hat{\otimes}G'+G\hat{\otimes}F'+G\hat{\otimes}G'$.

ii) $(E/G)\hat{\otimes}(E'/G') \approx (E\hat{\otimes}E')/(G\hat{\otimes}E'+E\hat{\otimes}G')$.

iii) $(F\cap G)\hat{\otimes}(F'\cap G') = F\hat{\otimes}F'\cap F\hat{\otimes}G'\cap G\hat{\otimes}F'\cap G\hat{\otimes}G'$.

iv) $(F:G)\hat{\otimes}A'=(F\hat{\otimes}A'):(G\hat{\otimes}A')$ *and* $A\hat{\otimes}(F':G')=(A\hat{\otimes}F'):(A\hat{\otimes}G')$.

PROOF. Since the functor $T(F, F')=(F\otimes F')\otimes_{A\otimes A'}(A\hat{\otimes}A')$ is a covariant additive exact functor in both variables, we can prove the corollary in the same way as was given in Lemma 2 in [2].

Remark: Let again F and F' be submodules of E and E' respectively, then by Proposition 1, the canonical mappings $F\hat{\otimes}E' \to E\hat{\otimes}E'$ and $F\hat{\otimes}F' \to F\hat{\otimes}E'$ are injective. Therefore the composed mapping $F\hat{\otimes}F'\to E\hat{\otimes}E'$ is also injective. Hence if we restrict our attention to submodules $M_\lambda(\lambda\in I)$ of $E\otimes E'$ such that M_λ is a finite sum of type $F\otimes F'$, the functor $T(\cdot) = \cdot\otimes_{A\otimes A'}(A\hat{\otimes}A')$ is exact.

PROPOSITION 2. *Let E and E' be finite modules over an* $\mathfrak{m}$*-adic and an* $\mathfrak{m}'$*-adic Zariski rings A and A' respectively, and assume A and A' contain a common subfield K. Then, for any submodules F of E and F' of E' and ideals* $\mathfrak{a}$ *of A and* $\mathfrak{a}'$ *of A',* we have

i) $F\otimes F'$ *is a closed submodule of* $E\otimes E'$.

ii) $(\mathfrak{a}, \mathfrak{a}')(F\hat{\otimes}F')\cap(F\otimes F') = (\mathfrak{a}, \mathfrak{a}')(F\otimes F')$.

iii) $F\otimes F' = (F\otimes E')\cap(E\otimes F')$ *and* $F\hat{\otimes}F' = (F\hat{\otimes}E')\cap(E\hat{\otimes}F')$.

iv) $(\mathfrak{a}\,\mathfrak{a}')(F\otimes F') = (\mathfrak{a}\,F)\otimes(\mathfrak{a}'F') = (\mathfrak{a}\,F\otimes E')\cap(E\otimes\mathfrak{a}'F')$ *and* $(\mathfrak{a}\,\mathfrak{a}')(F\hat{\otimes}F') = (\mathfrak{a}\,F)\hat{\otimes}(\mathfrak{a}'F') = (\mathfrak{a}\,F\hat{\otimes}E')\cap(E\hat{\otimes}\mathfrak{a}'F')$.

We assume further that $A/\mathfrak{m}\hat{\otimes}A'/\mathfrak{m}'$ *is Noetherian, then*

v) $(\mathfrak{a}, \mathfrak{a}')(F\hat{\otimes}F') = (\mathfrak{a}\,F)\hat{\otimes}F'+F\hat{\otimes}(\mathfrak{a}'F')$.

vi) $(F\hat{\otimes}F')/(\mathfrak{a}, \mathfrak{a}')(F\hat{\otimes}F') \approx (F/\mathfrak{a}F)\hat{\otimes}(F'/\mathfrak{a}'F')$.

PROOF. i) Consider the sequence of submodules of $E\otimes E'$: $E\otimes E'\supset F\otimes E'\supset F\otimes F'$. Then, to prove i) it is enough to show $F\otimes E'$ is a closed subspace of $E\otimes E'$. Since we proved in the proof of Proposition 3 in [2] that there exists an integer r such that

$$(\mathfrak{m}, \mathfrak{m}')^{2n}(F\otimes E')\subseteq(\mathfrak{m}^n, \mathfrak{m}'^n)(E\otimes E')\cap(F\otimes E')\subseteq(\mathfrak{m}^{n-r}, \mathfrak{m}'^{n-r})(F\otimes E')\subseteq(\mathfrak{m}, \mathfrak{m}')^{n-r}(F\otimes E')$$

for any $n\geqq r$, $F\otimes E'$ is a subspace of $E\otimes E'$. Therefore it remains to show $F\otimes E'$ is closed in $E\otimes E'$. To see this we remark first that

$$\bigcap_n((F+\mathfrak{m}^nE)\otimes E') = F\otimes E' \text{ and } \bigcap_n((F\otimes E')+(E\otimes\mathfrak{m}'^nE')) = F\otimes E'.$$

In fact, let ξ be an element in $\bigcap_n((F+\mathfrak{m}^nE)\otimes E')$, then it can be written as

$$\xi = y_1 \otimes y_1' + \dots + y_t \otimes y_t' \text{ with } y_i \in E \text{ and } y_i' \in E',$$

and we may assume $y_1', \dots, y_t'$ are linearly independent over K. Therefore $y_i \in F + \mathfrak{m}^n E$ for any n, hence $\xi \in F \otimes E'$ since $\bigcap_n (F + \mathfrak{m}^n E) = F$ which proves the first equality. As for the second, by passing to the residue module, we may assume $F = 0$ and by the consideration similar to the first part, we get $\bigcap_n (E \otimes \mathfrak{m}'^n E') = 0$.

Now, by virtue of these remarks, we have

closure of $F \otimes E'$ in $E \otimes E' = \bigcap_n ((F \otimes E') + (\mathfrak{m}, \mathfrak{m}')^n (E \otimes E'))$

$= \bigcap_n (F \otimes E' + (\mathfrak{m}^n, \mathfrak{m}'^n)(E \otimes E')) = \bigcap_n ((F \otimes E') + (\mathfrak{m}^n E \otimes E') + (E \otimes \mathfrak{m}'^n E'))$

$= \bigcap_n ((F + \mathfrak{m}^n E) \otimes E' + E \otimes \mathfrak{m}'^n E') \subseteq \bigcap_i \bigcap_n ((F + \mathfrak{m}^i E) \otimes E' + E \otimes \mathfrak{m}'^n E')$

$= \bigcap_i ((F + \mathfrak{m}^i E) \otimes E') = F \otimes E'$.

ii) This follows from $(\mathfrak{a}, \mathfrak{a}')(F \hat{\otimes} F') \cap (F \otimes F') =$ closure of $(\mathfrak{a}, \mathfrak{a}')(F \otimes F')$ in $F \otimes F' = (\mathfrak{a}, \mathfrak{a}')(F \otimes F')$ by i).

iii) We take a base $\{x_i\}_{i \in I}$ (resp. $\{x_i'\}_{i \in I'}$) of F (resp. F') over K and extend this base to a base $\{x_i, y_j\}_{i \in I\, j \in J}$ (resp. $\{x'_i, y'_j\}_{i \in I'\, j \in J'}$) of E (resp. E') over K. Then the set $\{y_i \otimes x_{i'}, y_j \otimes y_{j'}, x_i \otimes y_{j'}, y_j \otimes x_{i'}\}_{i \in I,\, i' \in I',\, j \in J,\, j' \in J'}$ foms a base of $E \otimes E'$ over K. By making use of this base we see easily that $(F \otimes E') \cap (E \otimes F') \subseteq F \otimes F'$. Converse inclusion is obvious. The second equality follows from

$(F \hat{\otimes} F') = (F \otimes F') \otimes_{A \otimes A'} (A \hat{\otimes} A') = ((F \otimes E') \cap (E \otimes F')) \otimes (A \hat{\otimes} A')$

$= ((F \otimes E') \otimes (A \hat{\otimes} A')) \cap ((E \otimes F') \otimes (A \hat{\otimes} A')) = (F \hat{\otimes} E') \cap (E \hat{\otimes} F')$

by the remark stated after Corollary to Proposition 1.

iv) We have $\mathfrak{a}\,\mathfrak{a}'(F \otimes F') = (\mathfrak{a} \otimes A')(A \otimes \mathfrak{a}')(F \otimes F') = (\mathfrak{a} F) \otimes (\mathfrak{a}' F')$ $= (\mathfrak{a} F \otimes E') \cap (E \otimes \mathfrak{a}' F')$ by iii), and $\mathfrak{a}\,\mathfrak{a}'(F \hat{\otimes} F') = \mathfrak{a}\,\mathfrak{a}'((F \otimes F') \otimes (A \hat{\otimes} A'))$ $= ((\mathfrak{a} F) \otimes (\mathfrak{a}' F')) \otimes (A \hat{\otimes} A') = \mathfrak{a} F \hat{\otimes} \mathfrak{a}' F' = (\mathfrak{a} F \hat{\otimes} E') \cap (E \hat{\otimes} \mathfrak{a}' F')$.

v) Clearly it is enough to show that $\mathfrak{a}(F \hat{\otimes} F') = (\mathfrak{a} F) \hat{\otimes} F'$. Since $A \hat{\otimes} A'$ is complete in an $(\mathfrak{m}, \mathfrak{m}')$-adic topology, and since $A/\mathfrak{m} \hat{\otimes} A'/\mathfrak{m}'$ is Noetherian, $A \hat{\otimes} A'$ is also Noetherian [3, Corollary to Proposition 1, p. 21], therefore a Zariski ring. Hence $\mathfrak{a}(F \hat{\otimes} F')$ is closed in $F \hat{\otimes} F'$. Whence $(\mathfrak{a} F) \hat{\otimes} F' =$ closure of $(\mathfrak{a} F) \otimes F'$ in $F \hat{\otimes} F' \subseteq \mathfrak{a}(F \hat{\otimes} F')$. The coverse inclusion is obvious.

vi) Since $(F/\mathfrak{a} F) \hat{\otimes} (F'/\mathfrak{a}' F') \approx F \hat{\otimes} F'/((\mathfrak{a} F) \hat{\otimes} F' + F \hat{\otimes} (\mathfrak{a}' F'))$, by Corollary to Proposition 1, the assetion follows by virtue of v),

2. Multiplicities.

In his paper [4], Satô studied the relations between prime divisors and primary components of ideals $\mathfrak{a}$ and $\mathfrak{a}'$ of Zariski rings A and A' and those of $(\mathfrak{a}, \mathfrak{a}')(A \hat{\otimes} A')$. For our purpose, the following lemma, due to Satô, is necessary.

LEMMA 1. *Let* $(A, \mathfrak{m})$ *and* $(A', \mathfrak{m}')$[3] *be, respectively, local rings of rank*

3) By a local ring $(A, \mathfrak{m})$ we mean a local ring A with the maximal ideal $\mathfrak{m}$.

d and d' which contain a common subfield K. Assume $A/\mathfrak{m}\otimes A'/\mathfrak{m}'$ *is an Artin ring. Then* $A\hat{\otimes}A'$ *becomes a semi-local ring*[4] *of rank* $d+d'$, $(\mathfrak{q},\ \mathfrak{q}')(A\hat{\otimes}A')$ *is a defining ideal of* $A\hat{\otimes}A'$, *any prime divisor of* $(\mathfrak{q},\ \mathfrak{q}')(A\hat{\otimes}A')$ *is isoloted and the lengths of its primary components are the same and are equal to* $l(\mathfrak{q})l(\mathfrak{q}')c$ *where* $l(\mathfrak{q})$ (*resp.* $l(\mathfrak{q}')$) *stands for the length of primary ideal* $\mathfrak{q}$ (*resp.* $\mathfrak{q}'$) *and c stands for the common length of primary components of* $(\mathfrak{m},\ \mathfrak{m}')(A\hat{\otimes}A')$.

Remark that in the case when A and A' are fields, we have $A\hat{\otimes}A'=A\otimes A'$.

LEMMA 2. *Let E and E' be, respectively, finite dimensional vector spaces over the fields L and L'. Assume that both L and L' are the extensions of a field K and* $L\otimes L'$ *is an Artin ring. Then*

$$l(E\otimes E') = \dim_L E\ \dim_{L'} E'\ l(L\otimes L')$$

where $l(E\otimes E')$ (*resp.* $l(L\otimes L')$) *means the length as the finite module* $E\otimes E'$ *over the Artin ring* $L\otimes L'$ (*resp. the length of the Artin ring* $L\otimes L'$).

PROOF. Put $s=\dim_L E$, $s'=\dim_{L'} E'$ and $l=l(L\otimes L')$. In the case when $s=1$, we can proceed by applying induction to s' as follows : Since our lemma is trivially valid in the case when $s'=1$, we may assume $s'>1$. Let E' be a subspace of E' such that $\dim_{L'} E_1'=s'-1$. Then $E'/E_1'\approx L'$ and by our induction hypothesis we have $l(L\otimes E_1')=(s'-1)l$. Therefore

$$l(L\otimes E')=l(L\otimes E_1')+l(L\otimes (E'/E_1'))=l(L\otimes E_1')+l(L\otimes L')=sl,$$

which is to be shown. General case follows from this in the same way by applying induction to s.

PROPOSITION 3. *Let* $\mathfrak{q}$ *and* $\mathfrak{q}'$ *be, respectively, primary ideals belonging to the maximal ideals of local rings* $(A,\ \mathfrak{m})$ *and* $(A',\ \mathfrak{m}')$ *which contain a common subfield K. Assume* $A/\mathfrak{m}\otimes A'/\mathfrak{m}'$ *is an Artin ring. Then, for any finite A- and A'-module E and E', we have*

$$l((E/\mathfrak{q}\ E)\hat{\otimes}(E'/\mathfrak{q}'\ E'))=l(E/\mathfrak{q}\ E)\ l(E'/\mathfrak{q}'\ E')\ l(A/\mathfrak{m}\otimes A'/\mathfrak{m}').$$

PROOF. First we consider the case when $\mathfrak{q}=\mathfrak{m}$ and proceed by applying induction to the length of $\mathfrak{q}'$. Since our proposition is true, by Lemma 2, in the case when $l(\mathfrak{q}')=1$, i. e., $\mathfrak{q}'=\mathfrak{m}'$, we may assume $l(\mathfrak{q}')>1$. Let $\mathfrak{m}'=\mathfrak{q}_1'\supset\mathfrak{q}_2'\supset\ldots\supset\mathfrak{q}_t'=\mathfrak{q}'$ be a chain of $\mathfrak{m}'$-primary ideals and asuume that each inclusion is strict and no $\mathfrak{m}'$-primary ideals can be inserted between $\mathfrak{q}_i'$ and $\mathfrak{q}_{i+1}'$ $(i=1,\ \ldots,\ t-1)$. From the exact sequence

$$0\to\mathfrak{q}_{t-1}'\ E'/\mathfrak{q}_t'\ E'\to E'/\mathfrak{q}_t'\ E'\to E'/\mathfrak{q}'_{t-1}\ E'\to 0,$$

4) In this case $A\hat{\otimes}A'$ is Noetherian as we remarked in the proof of v) of Proposition 2, hterefore semi-local [3, §1 e, p. 7].

we have, by Proposition 1, the exact sequence of $A\hat{\otimes}A'$-modules:

$$0\to(E/\mathfrak{m}E)\hat{\otimes}(\mathfrak{q}_{t-1}'E'/\mathfrak{q}_t'E')\to(E/\mathfrak{m}E)\hat{\otimes}(E'/\mathfrak{q}_t'E')\to(E/\mathfrak{m}E)\hat{\otimes}(E'/\mathfrak{q}_{t-1}E')\to 0,$$

hence
$$l((E/\mathfrak{m}E)\hat{\otimes}(E'/\mathfrak{q}_t'E'))=l((E/\mathfrak{m}E)\hat{\otimes}(\mathfrak{q}_{t-1}'E'/\mathfrak{q}_t'E'))+l((E/\mathfrak{m}E)\hat{\otimes}(E'/\mathfrak{q}_{t-1}'E')).$$

Since $\mathfrak{m}'\mathfrak{q}_{t-1}'\subset\mathfrak{q}_t'$ and $\mathfrak{q}_{t-1}'=(\mathfrak{q}_t',x)$ for any element $x\in\mathfrak{q}_{t-1}'$, $x\notin\mathfrak{q}_t'$, $\mathfrak{q}_{t-1}'/\mathfrak{q}_t'$ is isomorphic to $A'/\mathfrak{m}'$, hence $\mathfrak{q}_{t-1}'E'/\mathfrak{q}_t'E'\approx E'/\mathfrak{m}'E'$. Therefore

$$l((E/\mathfrak{m}E)\hat{\otimes}(\mathfrak{q}_{t-1}'E'/\mathfrak{q}_t'E'))=l((E/\mathfrak{m}E)\hat{\otimes}(E'/\mathfrak{m}'E'))=l(E/\mathfrak{m}E)l(E'/\mathfrak{m}'E')l$$

where $l=l(A/\mathfrak{m}\otimes A'/\mathfrak{m}')$. On the other hand, by our induction hypothesis, we have

$$l((E/\mathfrak{m}E)\hat{\otimes}(E'/\mathfrak{q}_{t-1}'E'))=l(E/\mathfrak{m}E)l(E'/\mathfrak{q}_{t-1}'E')l.$$

Therefore, by combining these relations, we get

$$l((E/\mathfrak{m}E)\hat{\otimes}(E'/\mathfrak{q}'E'))=l(E/\mathfrak{m}E)l(E'/\mathfrak{q}'E')l.$$

Now the general case follows from this relation by applying induction to the length of $\mathfrak{q}$ in the same way as above.

COROLLARY. $l((A/\mathfrak{q})\hat{\otimes}(A'/\mathfrak{q}'))=l(A/\mathfrak{q})\,l(A'/\mathfrak{q}')\,l(A/\mathfrak{m}\otimes A'/\mathfrak{m}')$.

LEMMA 3. *Let E and E' be, respectively, finite modules over an $\mathfrak{m}$-adic and an $\mathfrak{m}'$-adic Zariski rings A and A' and let $\mathfrak{a}$ and $\mathfrak{a}'$ be ideals of A and A' respectively. Assume A and A' contain a common subfield K and $A/\mathfrak{m}\hat{\otimes}A'/\mathfrak{m}'$ is Noetherian. Then we have*

$$(\mathfrak{a}^n,\ \mathfrak{a}^{n-1}\mathfrak{a}',\ \dots,\ \mathfrak{a}^{n-i+1}\mathfrak{a}'^{i-1},\ \mathfrak{a}^{n-i-1}\mathfrak{a}'^{i+1},\ \dots,\ \mathfrak{a}\,\mathfrak{a}'^{n-1},\ \mathfrak{a}'^n)(E\hat{\otimes}E')\cap(\mathfrak{a}^{n-i}\mathfrak{a}'^i)(E\hat{\otimes}E')\subseteq(\mathfrak{a}^{n-i+1}\mathfrak{a}'^i,\ \mathfrak{a}^{n-i}\mathfrak{a}'^{i+1})(E\hat{\otimes}E').$$

PROOF. We take a special base of the vector space E over K as follows: First we take a base $\{x_{n\lambda};\ \lambda\in\Lambda_n\}$ of $\mathfrak{a}^nE$ over K, and then extend this base to the base $\{x_{n\lambda},\ x_{n-1\mu};\ \lambda\in\Lambda_n,\ \mu\in\Lambda_{n-1}\}$ of $\mathfrak{a}^{n-1}E$ over K. Continuing this process we obtain a base $\{x_{n\lambda},\ x_{n-1\mu},\ x_{n-2\nu},\ \dots;\ \lambda\in\Lambda_n,\ \mu\in\Lambda_{n-1},\ \nu\in\Lambda_{n-2},\ \dots\}$ of E over K. In the same way, we construct a base $\{x'_{n\lambda'},\ x'_{n-1\mu'},\ x_{n-2\nu'},\ \dots;\ \lambda'\in\Lambda_n',\ \mu'\in\Lambda'_{n-1},\ \nu'\in\Lambda'_{n-2},\ \dots\}$ of E' over K. By making use of these bases, we see easily that

$$(\mathfrak{a}^n,\mathfrak{a}^{n-1}\mathfrak{a}',\ \dots,\ \mathfrak{a}^{n-i+1}\mathfrak{a}'^{i-1},\ \mathfrak{a}^{n-i-1}\mathfrak{a}'^{i+1},\ \dots,\ \mathfrak{a}\,\mathfrak{a}'^{n-1},\mathfrak{a}'^n)(E\otimes E')\cap(\mathfrak{a}^{n-i}\mathfrak{a}'^i)(E\otimes E')\subseteq(\mathfrak{a}^{n-i+1}\mathfrak{a}'^i,\ \mathfrak{a}^{n-i}\mathfrak{a}'^{i+i})(E\otimes E').$$

Operating $\otimes_{A\otimes A'}(A\hat{\otimes}A')$ to the both side of this relation, we get a required relation by virtue of the remark stated after Corollary to Proposition 1.

LEMMA 4. *Let E be a finite module over a Noetherian ring A and $E_1,\ \dots,\ E_n$ be submodules of E. And let $\mathfrak{a}$ be an ideal of A such that corank $\mathfrak{a}=0$. Put $F_i=\mathfrak{a}E_i(i=1,\ \dots,\ n)$. Assume $(E_1+\dots+E_{i-1}+E_{i+1}+\dots+E_n)\cap E_i\subseteq F_i$ $(i=1,\ \dots,\ n)$.* Then we have

$$l((E_1+\dots+E_n)/(F_1+\dots+F_n)) = l(E_1/F_1)+\dots+l(E_n/F_n).$$

PROOF. Consider a sequence of submodules of $E_1+\dots+E_n$: $E_1+\dots+E_n \supset E_1+\dots+E_{n-1}+F_n \supset E_1+\dots+E_{n-2}+F_{n-1}+F_n \supset \dots \supset E_1+F_2+\dots+F_n \supset F_1+\dots+F_n$. Since

$$l((E_1+\dots+E_i+F_{i+1}+\dots+F_n)/(E_1+\dots+E_{i-1}+F_i+F_{i+1}+\dots+F_n))$$
$$=l((E_1+\dots+E_{i-1}+F_i+\dots+F_n)+E_i/(E_1+\dots+E_{i-1}+F_i+\dots+F_n))$$
$$=l(E_i/E_i\cap(E_1+\dots+E_{i-1}+F_i+\dots+F_n))$$
$$=l(E_i/(E_i\cap(E_1+\dots+E_{i-1}+F_{i+1}+\dots+F_n))+F_i) = l(E_i/F_i),$$

by our assumption, therefore we have

$$l((E_1+\dots+E_n)/(F_1+\dots+F_n)) = \sum_{i=1}^{n} l((E_1+\dots+E_i+F_{i+1}+\dots+F_n)/(E_1+\dots+E_{i-1}+F_i+F_{i+1}+\dots+F_n)) = \sum_{i=1}^{n} l(E_i/F_i).$$

Now we shall prove the theorem which is the main purpose of this note.

THEOREM. *Let* $(A, \mathfrak{m})$ *and* $(A', \mathfrak{m}')$ *be local rings which contain a common subfield* K *and assume* $A/\mathfrak{m}\otimes A'/\mathfrak{m}'$ *is an Artin ring. And let* E *and* E' *be finite modules over* A *and* A' *respectively. Then, for any* $\mathfrak{m}$*-primary ideal* $\mathfrak{q}$ *and* $\mathfrak{m}'$*-primary ideal* $\mathfrak{q}'$, *we have*

$$e_{E\hat{\otimes}E'}((\mathfrak{q}, \mathfrak{q}')(A\hat{\otimes}A')) = e_E(\mathfrak{q})e_{E'}(\mathfrak{q}')\,l(A/\mathfrak{m}\otimes A'/\mathfrak{m}').$$

PROOF. We first show that $l(E\hat{\otimes}E'/(\mathfrak{q}, \mathfrak{q}')^n(E\hat{\otimes}E')) = \sum_{i+j<n} l((\mathfrak{q}^iE/\mathfrak{q}^{i+1}E)\hat{\otimes}(\mathfrak{q}'^jE'/\mathfrak{q}'^{j+1}E'))$, for any integer n. In fact, since $(E\hat{\otimes}E')/(\mathfrak{q}, \mathfrak{q}')(E\hat{\otimes}E') \approx (E/\mathfrak{q}E)\hat{\otimes}(E'/\mathfrak{q}'E')$, by Proposition 2, our assertion is true in the case when $n=1$. We assume our assertion is true in the case when $n=r$, and consider the case when $n=r+1$. Then

$$l(E\hat{\otimes}E'/(\mathfrak{q}, \mathfrak{q}')^{r+1}(E\hat{\otimes}E'))$$
$$= l(E\hat{\otimes}E'/(\mathfrak{q}, \mathfrak{q}')^r(E\hat{\otimes}E')) + l((\mathfrak{q}, \mathfrak{q}')^r(E\hat{\otimes}E')/(\mathfrak{q}, \mathfrak{q}')^{r+1}(E\hat{\otimes}E'))$$
$$= \sum_{i+j<r} l((\mathfrak{q}^iE/\mathfrak{q}^{i+1}E))\hat{\otimes}(\mathfrak{q}'^jE'/\mathfrak{q}'^{j+1}E')) + l((\mathfrak{q}, \mathfrak{q}')^r(E\hat{\otimes}E')/(\mathfrak{q}, \mathfrak{q}')^{r+1}(E\hat{\otimes}E'))$$

(by our induction hypothesis)

$$= \sum_{i+j<r} l((\mathfrak{q}^iE/\mathfrak{q}^{i+1}E)\hat{\otimes}(\mathfrak{q}'^jE'/\mathfrak{q}'^{j+1}E')) + \sum_{s+t=r} l((\mathfrak{q}^sE/\mathfrak{q}^{s+1}E)\hat{\otimes}(\mathfrak{q}'^tE'/\mathfrak{q}'^{t+1}E'))$$

(by Lemma 3 and 4)

$$= \sum_{i+j<r+1} l(\mathfrak{q}^iE/\mathfrak{q}^{i+1}E)\hat{\otimes}(\mathfrak{q}'^jE'/\mathfrak{q}'^{j+1}E')).$$

Therefore, by Proposition 3, we have

$$l(E\hat{\otimes}E'/(\mathfrak{q}, \mathfrak{q}')^n(E\hat{\otimes}E')) = \sum_{i+j<n} l(\mathfrak{q}^iE/\mathfrak{q}^{i+1}E)\,l(\mathfrak{q}'^jE'/\mathfrak{q}'^{j+1}E')\,l(A/\mathfrak{m}\otimes A'/\mathfrak{m}')$$
$$= \sum_{i+j<n} f(i)g(j)l,$$

where $f(i) = l(\mathfrak{q}^iE/\mathfrak{q}^{i+1}E)$, $g(j) = l(\mathfrak{q}'^jE'/\mathfrak{q}'^{j+1}E')$ and $l = l(A/\mathfrak{m}\otimes A'/\mathfrak{m}')$. This formula enables us to calculate the multiplicity of the defining ideal $(\mathfrak{q}, \mathfrak{q}')$

$(A \hat{\otimes} A')$ of the semi-local ring $A \hat{\otimes} A'$ in $E \hat{\otimes} E'$ in the same way as was given in [3] replacing $e(\mathfrak{q})$ and $e(\mathfrak{q}')$ by $e_E(\mathfrak{q})$ and $e_{E'}(\mathfrak{q}')$.

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SOME EXCEPTIONAL EXAMPLES TO STUDENT'S DISTRIBUTION

By

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The famous Student's test in stochastics is only applicable so far the population distribution is normal. In the present note the author intends to show that, if the universe is not properly normal, e. g. it may be truncatedly normal[1)] or Laplace's truncated distribution, the Student-like ratio of the sample mean to the sample S. D. $t = \bar{x}/s$, distributes quite differently from the ordinary t-distribution.

§ **1.** *Some Preliminary Remarks on the Simplex.* Let some sample taken from an universe with a non-negative variable be $\{x_1, x_2, \dots, x_n\}$ and its sample mean be

(1. 1) $$\bar{x} = \sum_1^n x_i/n \quad (= \text{determinate} > 0).$$

The space occupied by these sample-points forms a simplex of the $(n-1)$-th order, S_{n-1}, whose vertices $A_i (i = 1, 2, \dots, n)$ have co-ordinates such as all $x_j = 0$ except only one $x_i = n\bar{x}$. Really every point P on any side A_iA_j has two positive co-ordinates $x_i = n\bar{x}/(1+\lambda)$, $x_j = \lambda n\bar{x}/(1+\lambda)$ with $\lambda > 0$ and $P \varepsilon S_{n-1}$, while for every point Q on produced parts of $A_i A_j (\lambda < 0)$ at least one of x_i, x_j becomes negative and $Q \bar{\varepsilon} S_{n-1}$. Hence the length of one side is $a = n\bar{x}\sqrt{2}$. In general, let $S_{m-1} (0 < m < n)$ be the simplex formed by all points whose non-negative co-ordinates $x_1, \dots, x_m$ amount to $\sum_1^m x_i = n\bar{x}$, but $x_{m+1} = \dots = x_n = 0$. If a vertex $A_j (j > m)$ be adjoined to S_{m-1}, the simplex thus obtained $S_m \subset S_{n-1}$. For, P being any point of S_{m-1}, the co-ordinates of any point on the join PA_j are $X_i = x_i/(1+\lambda)(i = 1, \dots, m)$ and $X_j = \lambda n\bar{x}/(1+\lambda)$, but the remaining $X_k = 0$ $(k \neq i, j)$, so that $\sum_1^n X_i = n\bar{x}$. Moreover, X_i, X_j are non-negative so far $\lambda > 0$, while if $\lambda < 0$ at least one of X_i, X_j becomes negative. Hence all points of $S_m \varepsilon S_{n-1}$. Similarly $S_l \subset S_m$ if $l < m < n-1$. Consequently

(1. 2) $$S_0 \,(\text{vertex}) \subset S_1 \,(\text{side}) \subset S_2 \,(\text{face}) \subset S_3 (\text{tetrahedron}) \subset S_4 \subset \dots \subset S_{m-1} \subset \dots \subset S_{n-1}.$$

Clearly all simplexes $S_1, S_2, \dots, S_{n-1}$ are compact and convex. In fact, if $P_1(x_{11}, x_{12}, \dots, x_{1n})$ and $P_2(x_{21}, \dots, x_{2n})$ be any two boundary points of S_{n-1}, i. e. all of these co-ordinates be non-negative and $\sum x_{1i} = \sum x_{2i} = n\bar{x}$, but $x_{1j} = x_{2k} = 0$ $(j \neq k)$, $x_{2j} > 0$, $x_{1k} > 0$, then every point Q which lies on the join

$\overline{P_1P_2}$ also belongs to S_{n-1}, but every point Q' that is on the produced parts of $\overline{P_1P_2}$ cannot belong to S_{n-1}, because, the co-ordinates of Q' being $x'_i = \frac{x_{1i}+\lambda x_{2i}}{1+\lambda}$ with $\lambda < 0$, either $x'_j = \frac{\lambda x_{2j}}{1+\lambda}$ or $x'_k = \frac{x_{1k}}{1+\lambda}$ becomes necessarily negative.

Further G_{m-1} the centroid of S_{m-1} has as its co-ordinates $n-m$ zeros and m co-ordinates with each $n\bar{x}/m$. Specially for $G_{n-1} \equiv G$, the centroid of the whole S_{n-1}, it is $(\bar{x}, \bar{x}, \ldots, \bar{x})$.

Now, on excluding one vertex, say A_1, we obtain a subsimplex S_{n-2} formed by all points whose co-ordinates are $x_1 = 0$ and $\sum_2^n x_i = n\bar{x}$ with non-negative x_2, $\ldots$, x_n. To find the minimal distance from A_1 to S_{n-2}, we have to ask the relative minimum of the squared distance

$$y = (n\bar{x})^2 + \bar{x}_2^2 + \ldots + x_n^2$$

under condition that $\sum_2^n x_i = n\bar{x}$, or making use of the undetermined multiplier λ, the absolute minimum of

$$z = y - 2\lambda(\sum_2^n x_i - n\bar{x}).$$

Hence, on putting $\frac{\partial z}{\partial x_i} = 2x_i - 2\lambda = 0\ (i=2, \ldots, n)$, we obtain $\lambda = x_2 = \ldots = x_n = \sum_2^n x_i/(n-1) = n\bar{x}/(n-1)$. Therefore the required point is $G_{n-2}(0, \lambda, \ldots, \lambda)$, i. e. the centroid of S_{n-2}, and the minimal distance becomes

$$\sqrt{(n\bar{x})^2+(n-1)\lambda^2} = n\bar{x}\sqrt{n/(n-1)}.$$

Moreover, this line A_1G_{n-2} is really normal to S_{n-2}. For, if $P(0, x_2, \ldots, x_n)$ be any point on S_{n-2}, we have

$$\overline{PG}_{n-2}^2 = \sum_2^n (x_i-\lambda)^2 = \sum_2^n x_i^2-(n-1)\lambda^2 \quad \text{and} \quad \overline{PA_1^2} = (n\bar{x})^2 + \sum_2^n x_i^2$$

while $\quad \overline{A_1G_{n-2}^2} = (n\bar{x})^2+(n-1)\lambda^2, \quad$ so that $\quad \overline{A_1G_{n-2}^2} + \overline{PG_{n-2}^2} = \overline{PA_1^2}$.

Thus A_1G_{n-2} being perpendicular to every line PG_{n-2} drawn through G_{n-2} in the base simplex S_{n-2}, it may be called the height of S_{n-1} against the base simplex S_{n-2} and its length is

$$h_{n-1} = n\bar{x}\sqrt{n/(n-1)}. \tag{1. 3}$$

Further, if the straight line $\overline{A_1G_{n-2}}$ be divided internally in the ratio $n-1$: 1, the point of division Q has the co-ordinates

$$x_1 = (1.\ n\bar{x}+0)/n = \bar{x}, \qquad x_i = [0+(n-1)\lambda]/n = \bar{x}(i = 2, \ldots n),$$

and thus Q coincides with G. Consequently

$$\overline{GG_{n-2}} = h_{n-1}/n = \bar{x}\sqrt{n/(n-1)},$$

which should be the shostest distance between G and S_{n-2}, because GG_{n-2} is perpendicular to S_{n-2}. More generally, if we treat a subsimplex $S_{m-1}(m<n)$,

we can prove that the shortest distance from G to S_{m-1} is the central join GG_{m-1}, which is normal to S_{m-1}, and

$$\overline{GG_{m-1}} = \bar{x}\sqrt{n(n-m)/m}, \tag{14}$$

where G and G_{m-1} are the centroids of S_{n-1} and S_{m-1} respectively. Thus the centroids of each in set (1. 2): S_{n-1}, S_{n-2}, ... , S_{m-1}, ... , $S_0(=A)$ are apart away from the whole centroid

$$0,\ \bar{x}\sqrt{n/(n-1)},\ \bar{x}\sqrt{2n/(n-2)},\ \ldots\ldots,\ \bar{x}\sqrt{(n-m)n/m},\ \ldots,\ \bar{x}\sqrt{n(n-2)/2},\ \bar{x}\sqrt{n(n-1)}, \tag{1.5}$$

respectively.

Lastly let us find the volume of the simplex S_{n-1}. If we join A_1 into all points of the base simplex S_{n-2} and divide all of these joins in a same fractional ratio $r:1$, all the resulting points form again a simplex S'_{n-2}, which is parallel to S_{n-2} and accordingly its measure is $r^{n-2}\ S_{n-2}$. The height $h=\overline{A_1G_{n-2}}$ being normal to S'_{n-2}, we get as an elementary volume $r^{n-2}S_{n-2}\ \Delta\ h$. Hence the required volume shall be

$$S_{n-1} = \lim_{\Delta h\to 0}\sum r^{n-2}S_{n-2}\,\Delta h.$$

Making $r=m/N$, $0<m<N\to\infty$, $N\Delta h=h$, we obtain

$$S_{n-1} = S_{n-2}\,h\lim_{\Delta h\to 0}\frac{1}{N}\sum\left(\frac{m}{N}\right)^{n-2} = S_{n-2}h\int_0^1 r^{n-2}dr = S_{n-2}\,h_{n-1}/(n-1). \tag{1.6}$$

This formula holds equally good for $n-2$, $n-3$, ... up to 2. In fact, when $n=2$, it reduces to $S_1=S_0h_1$. But, (1. 3) renders, $h_1=n\bar{x}\sqrt{2}$, what can be translated as the height of a linear simplex S_1 with two vertices, because its length $n\bar{x}\sqrt{2}$ may be conveniently deemed as its height with one end point S_0 as base. On the other hand the zero-dimensional S_0 may be measured as $S_0=1^0=1$. Consequently $S_1=n\bar{x}\sqrt{2}=h_1S_0$ is still consistent.

Now, writing down the recurring formula thus obtained (1. 6) successively, we get

$$\begin{aligned} S_{n-1} &= S_{n-2}\,h_{n-1}/(n-1),\\ S_{n-2} &= S_{n-3}\,h_{n-2}/(n-2),\\ &\ldots\ldots\ldots\ldots\\ S_2 &= S_1\,h_2/2,\\ S_1 &= S_0\,h_1. \end{aligned}$$

Multiplying all these equations sides by sides, cancelling the same factors and applying (1. 3), we attain

$$S_{n-1} = h_{n-1}\,h_{n-2}\ldots\ h_2h_1/(n-1)! = (n\bar{x})^{n-1}\sqrt{n}/(n-1)! \tag{1.7}$$

Or, if the length of one side $a=\sqrt{2}n\bar{x}$ be substituted, we obtain

$$S_{n-1} = (a/\sqrt{2})^{n-1}\sqrt{n}/(n-1)! \tag{1.8}$$

e. g. $S_2=a^2\sqrt{3}/4$, $S_3=a^3/6\sqrt{2}$, which may be readily verified directly.

§ **2.** *The Distribution of the Sample Mean from a Positively Truncated Universe.* Let $f(x)$ be the frequency function of an universe U with a non-negative variable x, e. g. as a truncated Laplace distribution

(2. 1) $$f(x) = e^{-x/\sigma}/\sigma \qquad (x > 0)$$

whose mean and S. D. are both σ. Now take a random sample $\{x_1, \ldots, x_n\}$ from U, and form the sample mean and variance

(2. 2) $$\bar{x} = \sum_1^n x_i/n > 0,$$

(2. 3) $$s^2 = \sum_1^n (x_i - \bar{x})^2/n.$$

We are to discover the frequency function $f(\bar{x})$, the probability element being

(2. 4) $$dp = f(x_1)\, f(x_2) \ldots\ldots f(x_n)\, dx_1 \ldots\ldots dx_n.$$

Firstly we assume that the product $f(x_1) \ldots\ldots f(x_n)$ reduces to some function of $\bar{x}$ alone, $g(\bar{x})$ say, what is the case for (2. 1). Ignoring s, therefore, we may only evaluate

$$dP = f(\bar{x})\, d\bar{x} = g(\bar{x})\, dV, \qquad dV = \int dx_1 \ldots dx_n,$$

where the integration is extended over all $\{x_1, \ldots, x_n\}$ satisfying (2. 2) only, and the volume element dV has, as its base S_{n-1}, and height $d(\sqrt{n}\,\bar{x})$. Really (2. 2) being a n-dimensional hyperplane H, it may be written as

(2. 5) $$\sum_1^n x_i/\sqrt{n} = \sqrt{n}\,\bar{x},$$

so its normal from origin has direction cosines $1/\sqrt{n}$, $1/\sqrt{n}$, $\ldots$, $1/\sqrt{n}$ and the normal length $\sqrt{n}\,\bar{x}$. Naturally this normal is also perpendicular to S_{n-1}, because $S_{n-1} \subset H$. Indeed, we see that the co-ordinates of $G(\bar{x}, \ldots, \bar{x})$ satisfy (2. 5) and on joining the co-ordinate origin O into G and any point $P(x_1, \ldots, x_n)$ on (2. 5), it follows that

$$\overline{OG}^2 + \overline{GP}^2 = n\bar{x}^2 + \sum_1^n (x_i - \bar{x})^2 = \sum_1^n x_i^2 = \overline{OP}^2.$$

Furthermore, since for a second hyperplane H': $\sum_1^n x_i / \sqrt{n} = \sqrt{n}(\bar{x} + d\bar{x})$, the same holds, so H and H' are parallel to each other, and the normal from origin is common in direction, only their lengths differ by $\sqrt{n}\, d\bar{x}$. Therefore dV has its base S_{n-1} and height $d(\sqrt{n}\,\bar{x})$. Accordingly we obtain

(2. 6) $$dV = S_{n-1}\, d(\sqrt{n}\,\bar{x}) = (n\bar{x})^{n-1}\, n\, d\bar{x}/(n-1)\,!$$

in view of (1. 7). Thus, e. g. if $f(x) = e^{-x/\sigma}/\sigma$, we have

(2. 7) $$dP = e^{-n\bar{x}} \left(\frac{n\bar{x}}{\sigma}\right)^{n-1} d\left(\frac{n\bar{x}}{\sigma}\right) / (n-1)\,!$$

and

(2. 8) $$f(\bar{x}) = e^{-n\bar{x}} (n\bar{x}/\sigma)^{n-1} n/\sigma(n-1)\,! \qquad (n = 1, 2, \ldots)$$

which is a gamma distribution and gives the frequency function of the sample mean $\bar{x}$. Consequently

$$E(\bar{x}^k) = \int_0^\infty \bar{x}^k f(\bar{x})\,d\bar{x} = \frac{\Gamma(n+k)}{\Gamma(n)}\left(\frac{\sigma}{n}\right)^k, \quad (k = 0, 1, 2, \ldots).$$

In particular, $E(\bar{x}) = \sigma$, $E(\bar{x}^2) = \dfrac{n+1}{n}\sigma^2$, $D^2(\bar{x}) = \sigma^2/n$, so that $\bar{x}$ is still an unbiased estimate of the population mean. For the normal population, we know that not only the sample mean $\bar{x}$ but also the sample median $\tilde{x}$ is an unbiased estimate of the parent mean m. However, now with the truncated Laplace distribution, this would not hold, e. g. for $n = 3$, we get

$$f(\tilde{x}) = \frac{3!}{\sigma}\, e^{-\tilde{x}/\sigma}\int_0^{\tilde{x}} e^{-\bar{x}/\sigma}\,dx/\sigma \int_{\tilde{x}}^\infty e^{-x/\sigma}\,dx/\sigma, \qquad \int_0^\infty \tilde{x} f(\tilde{x})\,d\tilde{x} = \frac{5}{6}\sigma \,(\neq \sigma = m).$$

The critical lower and upper limits for a significant test would be found from

$$\int_0^{x_0} f(\bar{x})\,d\bar{x} = \alpha \quad \text{and} \quad \int_{x_1}^{\beta} f(\bar{x})\,d\bar{x} = \beta,$$

where α, β the levels of significance are e. g. 0. 05, 0. 025, 0. 01, 0. 005, &c. For the truncated Laplace distribution (2. 8) these limits may be found from Pearson's tables of the incomplete gamma function by

$$\frac{1}{\Gamma(n)}\int_0^{nx_0/\sigma} e^{-t}\, t^{n-1}\,dt = \alpha \quad \text{and} \quad \frac{1}{\Gamma(n)}\int_{nx_1/\sigma}^\infty e^{-t}\, t^{n-1}\,dt = \beta,$$

assumed the parent mean σ as known.

For a large sample, however, in virtue of the central limit theorem, the standardized variable $\xi = (\bar{x} - \sigma)\sqrt{n}/\sigma$ distributes asymtotically normally. Really, on substituting $\bar{x} = \sigma + \sigma\xi/\sqrt{n}$ in (2. 8) and approximating $\Gamma(n)$ by Stirling, it reduces to

$$f(\bar{x})\,d\bar{x} \cong \frac{1}{\sqrt{2\pi}} \exp\{-\xi^2/2\}\,d\xi \quad (-\sqrt{n} < \xi < \infty),$$

i. e. a truncated normal distribution, so that the usual normal test may be applied, at least on the upper side.

Again, let the population be a truncated normal distribution

$$f(x) = \sqrt{2/\pi\sigma^2}\, \exp\{-x^2/2\sigma^2\} \qquad (x > 0) \tag{2. 9}$$

with $E(x) = \sqrt{\dfrac{2}{\pi}}\,\sigma$ and $D^2(x) = \sigma^2\left(1 - \dfrac{2}{\pi}\right)$. Given a sample $\{x_1, \ldots, x_n\}$ from (2. 9) and formed (2. 2) and (2. 3), the probability element now becomes

$$dP = \left(\frac{1}{\sigma}\sqrt{\frac{2}{\pi}}\right)^n \exp\left\{-\frac{n}{2\sigma^2}(\bar{x}^2 + s^2)\right\} dV = f(\bar{x},\, s)\,d\bar{x}\,ds, \tag{2. 10}$$

where dV denotes the measure of the aggregate of $\{x_1, \ldots, x_n\}$ satisfying both (2. 2) and (2. 3), and it contains possibly both of $\bar{x}$ and s against the foregoing, and it shall be discussed in the following sections.

§ **3.** *The Joint Distribution of the Sample Mean and Variance taken from a Positively Truncated Universe.* Let a random sample $\{x_1, \ldots, x_n\}$ be drawn from an universe with a non-negative variable x, and the sample mean and variance be (2. 2) and (2. 3). Now in the probability element $dp = f(x_1) \ldots . f(x_n)\, dx_1 \ldots dx_n$ assuming that the product $f(x_1) \ldots f(x_n)$ reduces to some function $g(\bar{x},\ s)$ as is the case for (2. 1) or (2. 10) and consequently

$$dP = f(\bar{x},\ s)\, d\bar{x}\, ds = g(\bar{x},\ s)\, dV, \tag{3. 1}$$

and it needs to find

$$dV = \int dx_1 \ldots dx_n, \tag{3. 2}$$

where the integration is to be extended over the aggregate of points $\{x_1, \ldots, x_n\}$ which make mean (2. 2) between $\bar{x}$ and $\bar{x}+d\bar{x}$ and variance (2. 3) between s^2 and s^2+ds^2 (or approximately S. D. between s and $s+ds$). As already mentioned, dV has its height $\sqrt{n}\, d\bar{x}$. However, now its base is not the whole S_{n-1}, but only its portion whose points besides (2. 2) satisfy (2. 3). Now (2. 3) is a n-dimensional hypersphere K_n (as surface) of the radius $\sqrt{n}s$ with the center $G(\bar{x}, \bar{x}, \ldots, \bar{x})$. Hence a boundary of dV is the intersection of S_{n-1} and K_n, which is a $(n-1)$-dimensional sphere K_{n-1} still with center G and radius $\sqrt{n}s$. The second sphere K'_{n-1} of radius $\sqrt{n}s + d\sqrt{n}s$ being concentric with K_{n-1}, the required base is a $(n-1)$-dimensional spherical shell with thickness $\sqrt{n}ds$. The volume of the $(n-1)$-dimensional sphere of radius r $(=\sqrt{n}s)$ being $v = (\sqrt{\pi}r)^{n-1} \Big/ \Gamma\left(\frac{n+1}{2}\right)$, that of the spherical shell is, as differential of v,

$$\sqrt{\pi}^{n-1}(n-1)r^{n-2}\, dr \Big/ \Gamma\left(\frac{n+1}{2}\right) = \sqrt{n\pi}^{n-1}(n-1)s^{n-2} ds \Big/ \Gamma\left(\frac{n+1}{2}\right).$$

This being multiplied by the height $\sqrt{n}\, d\bar{x}$, we obtain

$$dV = \frac{\sqrt{n}^n \sqrt{\pi}^{n-1}}{\Gamma((n+1)/2)} (n-1)\, s^{n-2}\, ds d\bar{x}. \tag{3. 3}$$

The above is a simple imitation to Fisher's deduction in case when there is no limitation about $\bar{x}$. However, it will equally hold, if s be small compared to $\bar{x}$, i. e. if K_{n-1} lies wholly within S_{n-1}, or if the radius $\sqrt{n}s$ of K_{n-1} be smaller than the central distance $\overline{GG_{n-2}} = \sqrt{n}\bar{x}/\sqrt{n-1}$, that is, if

$$0 < s \leqq \bar{x}/\sqrt{n-1} \qquad \text{or} \qquad 0 < s/\bar{x} \equiv \tau \leqq 1/\sqrt{n-1}.$$

However, if $\tau > 1/\sqrt{n-1}$, so K_{n-1} protrudes partly outside of S_{n-1}, because then $GG_{n-2} < \sqrt{n}s$ and consequently GG_{n-2} produced to $\sqrt{n}s$, the points at end shall have negative co-ordinates. Therefore we must subtract these protruded parts. Indeed when s increases there occur several circumstances. If the radius of K_{n-1} be of magnitude between GG_{m-1} and GG_{m-2}, then by (1. 4), $\bar{x}\sqrt{n(n-m)/m} < \sqrt{n}s < \bar{x}\sqrt{n(n-m+1)/(m-1)}$ i. e. $\sqrt{(n-m)/m} < \tau < \sqrt{(n-m+1)/(m-1)}$, where $m = n,\ n-1,\ \ldots,\ 2, 1$. Thus there are the following n subcases:

$$0<\tau<\frac{1}{\sqrt{n-1}},\quad \frac{1}{\sqrt{n-1}}<\tau<\sqrt{\frac{2}{n-2}},\quad \ldots,\quad \sqrt{\frac{n-2}{2}}<\tau<\sqrt{n-1} \quad \text{and} \tag{3. 4}$$

$$\text{lastly } \sqrt{n-1} < \tau < \infty.$$

Of course, the last n-th case $\sqrt{n-1}\,\bar{x}<s<\infty$ means that the whole S_{n-1} lies within K_{n-1} and there is no point of intersection, that is no point of dV, so that we have nothing to consider. Among the remaining $n-1$ cases, the first case I : $0<\tau<1/\sqrt{n-1}$ was discusscd above. Next, if II : $1/\sqrt{n-1}<\tau<\sqrt{2/(n-2)}$, the protruded parts consist of n calottes (spherical segments) having no common portion. Hence, to solve this subcase, we have only to refer to the formula for the $(n-2)$-dimensional calotte :

$$C_{n-2}=\int_0^{r_0}\sec\gamma\, dK_{n-2}$$

where γ is the angle between the direction GG_{n-2} (i. e. AG_{n-2}) and GP drawn from G to any point P on the swelling spherical surface of K_{n-1}, while K_{n-2} denotes the volume of sphere with center G_{n-2} and radius $r=G_{n-2}P'$ where P' being the projection of P on S_{n-2}, $0\leqq r\leqq r_0=\sqrt{ns^2-\overline{GG}^2_{n-2}}=\sqrt{n(s^2-\bar{x}^2/(n-1))}$. Thus

$$C_{n-2}=\int_0^{r_0}\frac{\sqrt{n}s}{\sqrt{ns^2-r^2}}\frac{\sqrt{\pi}^{n-2}}{\Gamma(n/2)}(n-2)r^{n-3}\,dr=\frac{2\sqrt{n\pi s^2}^{n-2}}{\Gamma(n/2-1)}\int_0^{\theta_0}\sin^{n-3}\theta d\theta, \tag{3. 5}$$

where $\theta_0=\sin^{-1}r_0/\sqrt{n}s=\cos^{-1}\bar{x}/s\sqrt{n-1}$. Hence, e. g. $C_1=2\sqrt{3}\,s\cos\bar{x}/s\sqrt{2}$, $C_2=8\pi s^2(1-\bar{x}/s\sqrt{3})$, which is the celebrated Archimedes' theorem, and $C_3=10\sqrt{5}\pi s^3\left(\cos^{-1}\frac{\bar{x}}{2s}-\frac{\bar{x}}{2s}\sqrt{1-\frac{\bar{x}^2}{4s^2}}\right)$, which shall be verified in (3. 19) later on ; also compare the reference cited at the end of this note[2].

Therefore, in the case II : $1/\sqrt{n-1}<\tau<\sqrt{2/(n-2)}$ we obtain dV by subtracting $n^2\,C_{n-2}\,dsd\bar{x}$ from (3. 3) :

$$dV=\frac{\sqrt{n}^n\sqrt{\pi}^{n-1}(n-1)s^{n-2}}{\Gamma((n+1)/2)}\left[1-\frac{n\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2}-1\right)}\int_0^{\cos^{-1}\bar{x}/s\sqrt{n-1}}\sin^{n-3}\theta d\vartheta\right]dsd\bar{x}. \tag{3. 6}$$

To proceed similarly to the subcase III : $\sqrt{2/(n-2)}<\tau<\sqrt{3/(n-3)}$, &c., the matters become much more complex, now that the calottes have some common portions and further corrections are necessary. However, there is really a general method to find inductively the results for case $n=k+1$ with all its k subcases from those of case $n=k$ with $k-1$ subcases. But, this is preferably to be illustrated well by example. Therefore before to show it, we need to recapitulate each case $n=2$, 3, 4 separately, as it would at the same time make the facts much more clear.

Case $n=2$. In this trivial case, the linear simplex S_1 is a linear segment of length $2\sqrt{2\bar{x}}$ with centroid $G(\bar{x},\bar{x})$, while the linear sphere K_1 consists of two points, either of which are $\sqrt{2}s$ apart from the center G (Fig. 1). Hence the elementar y volume becomes

$$dV=2\,d(\sqrt{2}\bar{x})\,d(\sqrt{2}s)=4dsd\bar{x}, \tag{3. 7}$$

which agrees with what follows by putting $n=2$ in (3. 3). The series (3. 4) reduces to only one adoptable subcase; $0 < s/\bar{x} < 1$.

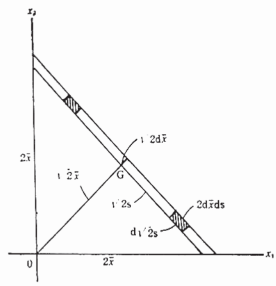

Fig. 1

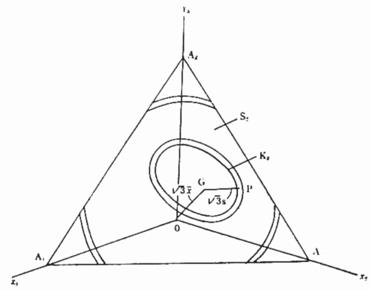

Fig. 2

Case $n = 3$. In this case the simplex S_2 is an equilateral triangle of side $3\sqrt{2}\bar{x}$ and centroid $G(\bar{x}, \bar{x}, \bar{x})$, while K_2 is a circle of radius $\sqrt{3}s$ with center G. There are only two non-trivial subcases:

I: $0 < s/\bar{x} = \tau < 1\sqrt{2}$. Observing the concentric circles in Fig. 2 directly, or by (3. 3), we get

(3. 8) $$dV = 2\pi(\sqrt{3}s)\,d(\sqrt{3}s)\,d(\sqrt{3}\bar{x}) = 6\sqrt{3}\pi s\, ds d\bar{x} = 6\sqrt{3}\pi\bar{x}^2\, d\bar{x}\ \tau d\tau.$$

II: $1\sqrt{2} < \tau < \sqrt{2}$. The circular arc of K_2 is cut into three pieces. On calculating the circular arc directly, or using (3. 6), we have

(3. 9) $$\begin{aligned} dV &= 18\sqrt{3}\, s\,(\pi/3 - \cos^{-1}\bar{x}/s\sqrt{2})\, ds d\bar{x} \\ &= 18\sqrt{3}\bar{x}^2 d\bar{x}.\ \tau(\pi/3 - \cos^{-1}1/\tau\sqrt{2})\, d\tau. \end{aligned}$$

Case $n = 4$. The simplex S_3 is a tetrahedron having as face the equilateral triangle with side $4\sqrt{2}\bar{x}$, centroid $G(\bar{x}, \bar{x}, \bar{x}, \bar{x})$, while K_3 is a sphere of radius $2s$, center G.

I: $0 < s/\bar{x} = \tau < 1/\sqrt{3}$. By (3. 3), or directly

(3. 10) $$dV = 4\pi(2s)^2\, 2ds.\ 2d\bar{x} = 64\pi s^2\, ds d\bar{x} = 64\pi\bar{x}^3\, d\bar{x}.\ \tau^2\, d\tau.$$

II: $1/\sqrt{3} < \tau < 1$. By (1. 4) $GG_2 = \bar{x}\sqrt{4/3}$, so that the height of the calotte C_2 swelled outside S_3 is $2s - 2\bar{x}/\sqrt{3}$. Its surface is after Archimedes $2\pi(2s)(2s - 2\bar{x}/\sqrt{3}) = 8\pi s(s - \bar{x}/\sqrt{3})$ coinciding with (3. 5). There are 4 faces. Hence $128\pi s(s - \bar{x}/\sqrt{3})ds d\bar{x}$ being subtracted from (3. 8), or else by (3. 6), we obtain

(3. 11) $$dV = 64\pi s(2\bar{x}/\sqrt{3} - s)\, ds d\bar{x} = 64\pi\bar{x}^3 d\bar{x}.\ (2\tau/\sqrt{3} - \tau^2)\, d\tau.$$

III: $1 < s/\bar{x} < \sqrt{3}$. Now the radius $2s$ of K_3 being between $GG_1 = 2\bar{x}$ and $GA = 2\sqrt{3}\bar{x}$ only some portions of the spherical surface S_3 contribute to integration. Let the tetrahedron S_3 be $ABCD$ with centroid G, height $AG_2 = 8\bar{x}/\sqrt{3}$ (Fig. 3). Taking conveniently G as origin, GA as ζ-axis and GE, a parallel to G_2G_1, as ξ-axis, complete $\xi\eta\zeta$-rectangular axes. Then the equation of the face ABC shall be expressed by $\xi/GE + \zeta/GA = 1$. But $GE = \frac{3}{4}\, G_2G_1 = \sqrt{6}\bar{x}/2$, $GA = 2\sqrt{3}\bar{x}$, so that the equation becomes $\zeta = 2\sqrt{3}\bar{x} - 2\sqrt{2}\xi$, or in cylindrical co-ordinates $\zeta = 2\sqrt{3}\bar{x} - 2\sqrt{2}\rho\cos\theta$. The equation of K_3 is $\rho^2 + \zeta_2 = 4s^2$. These two equations combined together, express their intersection curves

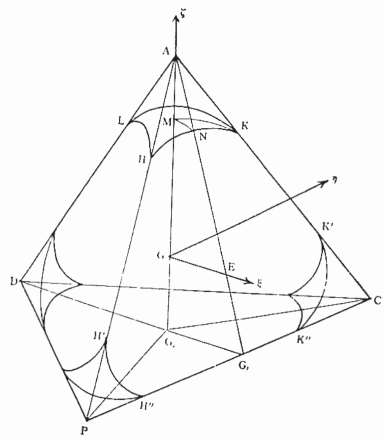

Fig. 3

HK, $H'K'$ &c. We are to evaluate 4 × surface HKL or 24 × surface MNK(= F say), where MN and MK are the intersections of the sphere K_3 with the planes AGE and AG_2C, respectively. But

$$F = F(\bar{x}, s) = \int_0^{\pi/3}\int_0^{\rho_1}\sqrt{1+\left(\frac{\partial\zeta}{\partial\rho}\right)^2}\rho d\rho d\theta,$$

where $\zeta = \sqrt{4s^2-\rho^2}$ and ρ_1 is found by solving $\rho^2+(2\sqrt{3}\bar{x}-2\sqrt{2}\rho\cos\theta)^2 = 4s^2$ to be

$$\rho_1 = 2\ [2\sqrt{6}\bar{x}\cos\theta - \sqrt{(1+8\cos^2\theta)s^2-3\bar{x}^2}]\ /\ (1+8\cos^2\theta),$$

the double signs ± being chosen to be negative, since the larger one corresponds to $H'H''K''K'$ in Fig. 3. Now the inner integral of the above double integral reduces to

$$\int_0^{\rho_1}\sqrt{1+\rho^2/(4s^2-\rho^2)}\,\rho d\rho = 2s\int_0^{\rho_1}\rho d\rho\,/\,\sqrt{4s^2-\rho^2} = 4s^2-2s\sqrt{4s^2-\rho_1^2},$$

which integrated with respect to θ, yields

$$F = \frac{4\pi s^2}{3} - 4s\int_0^{\pi/3}\frac{\sqrt{s^2(1+8\cos^2\theta)^2-\{2\sqrt{6}\bar{x}\cos\theta-\sqrt{(1+8\cos^2\theta)s^2-3\bar{x}^2}\}^2}}{1+8\cos^2\theta}\,d\theta$$

$$= \frac{4\pi s^2}{3} - 4s\int_0^{\pi/3}\frac{\sqrt{3}\bar{x}+\sqrt{8}\cos\theta\sqrt{(1+8\cos^2\theta)s^2-3\bar{x}^2}}{1+8\cos^2\theta}\,d\theta.$$

Performing the integration, we attain finally

$$F = \frac{2\pi s}{3}\left(\frac{2\bar{x}}{\sqrt{3}}-s\right) - \frac{4s\bar{x}}{\sqrt{3}}\tan^{-1}\sqrt{\frac{3(s^2-\bar{x}^2)}{2\bar{x}^2}} + 4s^2\tan^{-1}\sqrt{\frac{s^2-\bar{x}^2}{2s^2}}\,.$$

Hence

(3.12) $$dV = 24F(\bar{x}, s)2ds\,2d\bar{x} = 96F(\bar{x}, \bar{x}\tau)\,\bar{x}d\tau d\bar{x}\quad(s = \bar{x}\tau)$$

$$= 128\bar{x}^3d\bar{x}\left\{\frac{\pi}{2}\left(\frac{2}{\sqrt{3}}\tau-\tau^2\right) - \sqrt{3}\tau\tan^{-1}\sqrt{\frac{3}{2}(\tau^2-1)} + 3\tau^2\tan^{-1}\sqrt{\frac{1}{2}\left(1-\frac{1}{\tau^2}\right)}\right\}d\tau.$$

So far we have discussed the problem geometrically. However, even when $n=5$, S_4 and K_4 being four dimensional, the matter becomes less intuitional, and mucn less for cases $n>5$. Hence, we are oblidged to proceed analytically below.

In Case $n=5$, there are 4 subcases; I: $0<\tau<1/2$, II: $1/2<\tau<\sqrt{2/3}$, III $\sqrt{2/3}<\tau<\sqrt{3/2}$, IV: $\sqrt{3/2}<\tau<2$. For subcase I we get immediately by (3.3)

$$dV = 50\sqrt{5}\pi^2 s^3\, ds d\bar{x}, \tag{3.13}$$

and for subcase II (3.6) may be employed.

However, in order to explain the general method before mentiened methodologcially, let us treat this subcase II by the very general method, that can be quite similarly applied to any $n=k+1$, if the case $n=k$ were already solved.

Putting one variable x_1 aside for a while, the remaining four variables x_2, x_3, x_4, x_5 form, as their mean and variance,

$$\bar{x}' = \sum_2^5 x_i/4, \qquad s'^2 = \sum_2^5 (x^i - \bar{x}')^2/4$$

with 3 subcases: I′: $0<s'/\bar{x}'=\tau'<1/\sqrt{3}$, II′: $1/\sqrt{3}<\tau'<1$, III′: $1<\tau'<\sqrt{3}$. It is easy to show that there exist the relations

$$\bar{x}' = \frac{1}{4}(5\bar{x}-x_1), \qquad s'^2 = \frac{5}{16}\left[4s^2-(x_1-\bar{x})^2\right], \tag{3.14}$$

where $\bar{x}'$, s' being real and non-negative, and

$$0<x_1<5\bar{x}, \qquad \bar{x}-2s<x_1<\bar{x}+2s(=\gamma). \tag{3.15}$$

Given $\bar{x}$ and s so $s/\bar{x}=\tau$ also, we wish to grasp how the variable x_1 runs its course. For this purpose, we draw the graph of

$$\tau' = \frac{s'}{\bar{x}'} = \frac{\sqrt{5(4s^2-(x_1-\bar{x})^2)}}{5\bar{x}-x_1} \tag{3.16}$$

for several values of $\tau(=s/\bar{x})$ (Fig. 4). We need not consider those points outside

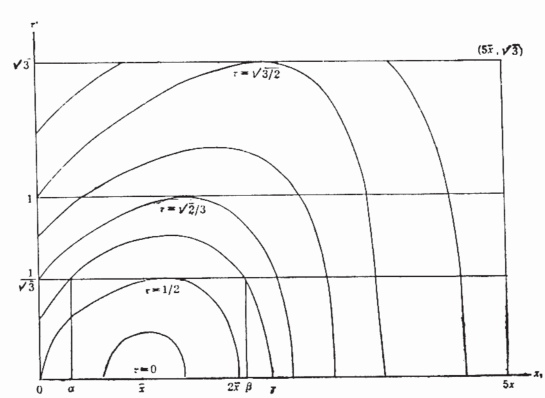

Fig. 4

the rectangle with sides $5\bar{x}$ and $\sqrt{3}$, because of (3.15) and $0<\tau'<\sqrt{3}$. The curve (3.16) has, as its τ'- and x_1-intercept, $\tau'_0=\sqrt{(4\tau^2-1)/5}$ and $\gamma=\bar{x}+2s=\bar{x}(1+2\tau)$, while its maximum arises at the point $((1+\tau^2)\bar{x},\ \sqrt{5}\tau/\sqrt{4-\tau^2})$. Hence, we obtain the following table:

τ	0	1/2	$\sqrt{2/3}$	$\sqrt{3/2}$	2
τ'-intercept $=\tau'_0$	imag.	0	$1/\sqrt{3}$	1	$\sqrt{3}$
x_1-intercept $=\gamma$	$\bar{x}$	$2\bar{x}$	$2.63\cdots\bar{x}$	$3.45\cdots\bar{x}$	$5\bar{x}$
mode m_0	$\bar{x}$	$5\bar{x}/4$	$5\bar{x}/3$	$5\bar{x}/2$	$5\bar{x}$
maximum τ_m	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞

In particular, if $\tau=0$, the τ'-curve degenerates to a single point $x_1=\bar{x}$, and it grows up larger and larger as τ increases up to $\tau=2$, in which case, however, the τ'-curve becomes needless.

Now, if e.g. $1/2<\tau<\sqrt{2/3}$, the corresponding τ'-curve lies actually between those corresponding to $\tau=1/2$ and $\tau=\sqrt{2/3}$. It starts from a point $(0,\ \tau'_0)$, such as $0<\tau'_0<1/\sqrt{3}$ and first ascending to a maximum, that lies between two parallels $\tau'=1/\sqrt{3}$, $\tau'=1$ and then descends up to $(\gamma,0)$. Its points of intersection with the parallel $\tau'=1/\sqrt{3}$ is found by solving the equation (3.16) for $\tau'=1/\sqrt{3}$ to be

$$\alpha,\ \beta=\frac{1}{4}\left[5x\mp\sqrt{15\,(4s^2-\bar{x}^2)}\right]. \tag{3.17}$$

Hence, the elementary volume in the $\bar{x}'$-s' distribution is given by (3.10) and (3.11) as

$$dV'_1=64\pi s'^2 d\,s'd\bar{x}' \quad \text{for} \quad 0<x_1<\alpha \ \text{ as well as } \ \beta<x_1<\gamma,$$

and

$$dV'_2=64\pi s'\left(\frac{2\bar{x}'}{\sqrt{3}}-s'\right)ds'd\bar{x}' \quad \text{for} \quad \alpha<x_1<\beta.$$

Transforming the variables $\bar{x}'$, s' into $\bar{x}$, s by (3.14) with the Jacobian

$$J=\frac{\partial(\bar{x}',\ s')}{\partial(\bar{x},\ s)}=\frac{25}{16}\frac{s}{s'},$$

and integrating about x_1 in the above described interals, we get

$$dV_1=\int_{x_1}dV'_1\,dx_1=100\pi\,s\,ds\,d\bar{x}\int_{x_1}s'dx_1,$$

where the integrsals are really $\int_0^\alpha+\int_\beta^\gamma$, as well as

$$dV_2=\int_{x_1}dV'_2dx_1=100\pi\ s\,ds\,d\bar{x}\int_\alpha^\beta(2\bar{x}'/\sqrt{3}-s')\,dx_1.$$

Upon substituting (3.14) in these integrals and integrating, we attain finally

(3.18) $dV = dV_1 + dV_2 = 250\sqrt{5}\pi s\left[\frac{\bar{x}}{4}\sqrt{4s^2-\bar{x}^2}+\left(\frac{\pi}{5} - \cos^{-1}\frac{\bar{x}}{2s}\right)s^2\right]dsd\bar{x}$

for the subcase II: $1/2<s/\bar{x}<\sqrt{2/3}$, and this coincides with what follows from (3.6).

Remark. If we subtract (3.18) from (3.13) which in II may comprise five 3-dimensional calottes, where some x_i's become certainly negative but still $\sum_1^5 x_i = 5\bar{x}$ and $s^2 = \sum_1^5 (x_i-\bar{x})^2/5$ hold, we obtain the superfluous volume

$$250\sqrt{5}\pi s[\cos^{-1}\bar{x}/2s - \bar{x}\sqrt{4s^2-\bar{x}^2}/4]dsd\bar{x}.$$

Therefore, if this be divided by $5d\sqrt{5}sd\sqrt{5}\bar{x}$, we shall get, as the volume of the 3-dimentional calotte

(3.19) $C_3 = 10\sqrt{5}\pi s\,[s^2\cos^{-1}\bar{x}/2s - \bar{x}\sqrt{4s^2-\bar{x}^2}/4],$

which precisely coincides with (3.5) for $n-2=3$, and thus the very formula is not a result of mere formal extension, but has an actual conformity.

§4. *Truncated Laplace Distribution.* By the results in the foregoing section it is possible to write down the volume element:

(4.1) $dV = dV_{n-1}(\bar{x}, s) = g_{n-1}(\bar{x})d\bar{x}.\ h_{n-1}(\tau)d\tau$ with $\tau = s/\bar{x}$,

where the coefficients are factorized so as to hold for every n the identity (4.6) below holds. Therefore, if the universe be e. g. a truncated Laplace distribution $f(x)=e^{-x}(x>0)$, the probability element for the joint distribution of the n-sized sample mean $\bar{x}$ and S. D. s shall be given by

(4.2) $dP = f_{n-1}(\bar{x}, s)d\bar{x}ds = e^{-n\bar{x}}g_{n-1}(\bar{x})d\bar{x}.\ h_{n-1}(\tau)d\tau,$

so that $\bar{x}$ and τ are independent. Really for $n=2, 3, 4$, we obtain the following:

(4.3) $f_1(\bar{x}, s)\,d\bar{x}ds = 4e^{-3\bar{x}}\bar{x}\cdot d\tau$

(4.4) $f_2(x, s)dxds = \frac{27}{2}e^{-3x}x^2dx\cdot\frac{4\pi}{3\sqrt{3}}\tau d\tau$ or $\frac{27}{2}e^{-3x}\bar{x}^2dx$

$$\frac{4}{\sqrt{3}}\left(\frac{\pi}{3} - \cos^{-1}\frac{1}{\tau\sqrt{2}}\right)\tau d\tau\ (0<\tau<1/\sqrt{2} \text{ or } 1/\sqrt{2}<\tau<\sqrt{2});$$

(4.5) $f_3(\bar{x}, s)d\bar{x}ds = \frac{128}{3}e^{-4\bar{x}}\bar{x}^3d\bar{x},\ \frac{3\pi}{2}\tau^2d\tau$ or $\frac{128}{3}e^{-4\bar{x}}\bar{x}^3d\bar{x},\ \frac{3\pi}{2}\left(\frac{2\tau}{\sqrt{3}}-\tau^2\right)d\tau$

$(0<\tau<1/\sqrt{3}$ or $1/\sqrt{3}<\tau<1)$, or $\frac{128}{3}e^{-4\bar{x}}\bar{x}^3d\bar{x},\ \frac{3\pi}{2}\left(\frac{2\tau}{\sqrt{3}}-\tau^2\right)-3\sqrt{3}\tau$

$$\tan^{-1}\sqrt{\frac{3}{2}(\tau^2-1)}+9\tau^2\tan^{-1}\sqrt{\frac{1}{2}\left(1-\frac{1}{\tau^2}\right)}\quad(1<\tau<\sqrt{3}).$$

And we have

(4.6) $\int_0^\infty e^{-n\bar{x}}g_{n-1}(\bar{x})d\bar{x} = 1,$ where $g_{n-1}(\bar{x}) = \frac{n^n}{\Gamma(n)}\bar{x}^{n-1},$

as well as $\int_0^{\sqrt{n-1}} h_{n-1}(\tau)d\tau = 1,$ where

$$h_1(\tau) = 1 \quad \text{in} \quad 0 < \tau < 1;$$
$$h_2(\tau) = \frac{4\pi}{3\sqrt{3}}\tau \quad \text{in} \quad 0<\tau<1/\sqrt{2}, \quad \text{but} \quad \frac{4}{\sqrt{3}}\tau\left(\frac{\pi}{3} - \cos^{-1} 1/\tau\sqrt{2}\right) \quad \text{in} \quad 1/\sqrt{2}<\tau<\sqrt{2};$$
$$h_3(\tau) = \frac{3\pi}{2}\tau^2 \quad \text{in} \quad 0<\tau<1/\sqrt{3} \quad \text{and} \quad \frac{3\pi}{2}\left(\frac{2\tau}{\sqrt{3}} - \tau^2\right) \quad \text{in } 1/\sqrt{3}<\tau<1, \text{ and lastly}$$
$$\frac{3\pi}{2}\left(\frac{2\tau}{\sqrt{3}} - \tau^2\right) - 3\sqrt{3}\tau \tan^{-1}\sqrt{\frac{3}{2}(\tau^2-1)} + 9\tau^2 \tan^{-1}\sqrt{\frac{\tau^2-1}{2\tau^2}} \quad \text{in} \quad 1<\tau<\sqrt{3}, \text{ \&c.}$$

Consequently we may utilize the quantity $\tau = s/\bar{x}$ for testing its significance, whether the universe is really $f(x) = e^{-x}(x>0)$ or not. The lower limit τ_0 of significance level $\alpha(=0.01$ or 0.05, &c.) is found by use of (3. 3) from

(4. 7)
$$\int_0^{\tau_0} h_{n-1}(\tau)\, d\tau = \frac{\Gamma(n)}{\Gamma((n+1)/2)}\sqrt{\frac{\pi}{n}}^{\,n-1} \frac{\tau_0^{n-1}}{\sqrt{n}} = \alpha$$

to be

$$\tau_0 = \frac{n}{\pi}\left[\frac{\alpha\sqrt{n}\ \Gamma((n+1)/2}{\Gamma(n)}\right]^{1/(n-1)}.$$

If however this value exceeds $1/\sqrt{n-1}$, we must refer to the second subinterval II, &c. Thus, e. g. for $n = 4$, we shall get

(4. 8) $\tau_0 = 0.1853$ or 0.4144 according as $\alpha = 0.01$ or 0.05.

As to the upper limit τ_1, we have to find it from

(4. 9)
$$\int_{\tau_1}^{\sqrt{n-1}} h_{n-1}(\tau)\, d\tau = \alpha,$$

what is a pretty intricate. We shall obtain for case $n=4$ by means of Newton's successive approximation

(4. 10) $\tau_1 = 1.4212$, or 1.2513 for $\alpha = 0.01$ or 0.05.

However, the classical Student's ratio being in fact

$$t = \frac{\bar{x}-m}{s}\sqrt{n-1}, \quad \text{or} \quad t = \frac{\bar{x}}{s}\sqrt{n-1} \quad \text{if} \quad m = 0,$$

our $\tau = s/\bar{x}$ is the reciprocal of t multiplied by $\sqrt{n-1}$

(4. 11) $\tau = \sqrt{n-1}/t$, i. e. $t = \sqrt{n-1}/\tau = \sqrt{n-1}\,\bar{x}/s$.

Hence, the previous $h_{n-1}(\tau)$ if expressed by t becomes $f_{n-1}(t)$ and in details:

(4. 12) $f_1(t) = t^{-2}$ $(1<t<\infty)$;

(4. 13) $f_2(t) = \dfrac{8\pi}{3\sqrt{3}}t^{-3}$ or $\dfrac{8}{\sqrt{3}}\left(\dfrac{\pi}{3} - \cos^{-1}\dfrac{t}{2}\right)t^{-3}$ $(2<t<\infty$, or $1<t<2)$;

(4. 14) $f_3(t) = \dfrac{9\sqrt{3}\pi}{2}t^{-4}$ or $9\sqrt{3}\pi\,(t/3 - 1/2)t^{-4}$ $(3<t<\infty$, or $\sqrt{3}<t<3)$,

$$\text{or} \quad 9\sqrt{3}\left[\pi\left(\frac{t}{3} - \frac{1}{2}\right) - t\tan^{-1}\sqrt{\frac{3}{2}\left(\frac{3}{t^2}-1\right)} + 3\tan^{-1}\sqrt{\frac{1}{2}\left(1-\frac{t^2}{3}\right)}\right]t^{-4}$$
$(1<t<\sqrt{3})$;

and generally

$$\int_0^\infty f_{n-1}(t)\,dt = \int_1^\infty f_{n-1}(t)\,dt = 1.$$

For the sake of comparison, if the figures of (4. 8), (4. 10) be expressed in t by (4. 11), we obtain as the upper limits 9. 347, 4. 180 and as the lower limits, 1.219, 1.384 corresponding to $\alpha=0.01$, 0.05 respectively, while the Student's Table for $n=4$ delivers ± 4.541 and ± 2.353 by reason of the symmetry.

We have argued on such a truncated Laplacian population as: A whole Laplace distribution $f(X)=\frac{1}{2\sigma}\exp\{-|X-m|/\sigma\}$ is truncated into half at $X=m$, only the part $X>m$ adopted, and the factor 1/2 removed in order to make the resulting expression furnish a frequency function, and finally the variable X transformed into x by $(X-m)/\sigma=x$, so that $f(x)=e^{-x}(x>0)$ holds. Hence, if the original distribution be regarded, of course, the Student-like ratio $\frac{X-m}{s}\sqrt{n-1}$ should be consulted.

§ **5.** *Truncated Normal Distribution as Universe.* Lastly we shall consider the sample distribution in case that the universe is, as in (2.9),

$$f(x) = \sqrt{\frac{2}{\pi}}\exp\left\{-\frac{x^2}{2}\right\} \quad (x>0). \tag{5. 1}$$

The probability element which yields the n-sized sample mean $\bar{x}$ and S. D. s so that $\tau=s/\bar{x}$ is

$$dP = \sqrt{\frac{2}{\pi}}^{\,n}\exp\left\{-\frac{n}{2}\bar{x}^2(1+\tau^2)\right\} g_{n-1}(\bar{x})\,h_{n-1}(\tau)\; d\bar{x}d\tau,$$

where $g_{n-1}(\bar{x})$ and $h_{n-1}(\tau)$ are those given in (4. 6). Therefore, this time, $\bar{x}$ and τ are by no means independent, However, on considering τ as fixed, and integrating about $\bar{x}$, we obtain

$$\int_0^\infty \exp\left\{\frac{n\bar{x}^2}{2}(1+\tau^2)\right\} g_{n-1}(\bar{x})\,d\bar{x} = \frac{1}{2}\,\frac{\Gamma(n/2)}{\Gamma(n)}\left(\frac{2n}{1+\tau^2}\right)^{n/2},$$

so that the frequency function of τ shall be given by

$$\psi_{n-1}(\tau) = \frac{1}{2}\,\frac{\Gamma(n/2)}{\Gamma(n)}\sqrt{\frac{4n}{\pi}}^{\,n}\,\frac{h_{n-1}(\tau)}{(1+\tau^2)^{n/2}}\,. \tag{5. 2}$$

More in details:

$$\psi_1(\tau) = 4/\pi(1+\tau^2) \qquad (0<\tau<1); \tag{5. 3}$$

$$\psi_2(\tau) = 4\tau(1+\tau^2)^{-3/2} \text{ or } 4\tau(1+\tau^2)^{-3/2}\left(1-\frac{3}{\pi}\cos^{-1}1/\tau\sqrt{2}\right) \qquad (0<\tau<1/\sqrt{2} \text{ or } 1/\sqrt{2}<\tau<\sqrt{2}); \tag{5. 4}$$

$$\begin{aligned}\psi_3(\tau) &= \frac{32\tau^2}{\pi}(1+\tau^2)^{-2} \text{ or } \frac{32}{\pi}\left(\frac{2\tau}{\sqrt{3}}-\tau^2\right)(1+\tau^2)^{-2} \qquad (0<\tau<1/\sqrt{3} \text{ or } 1/\sqrt{3}<\tau<1),\\ &= \frac{32}{\pi^2}(1+\tau^2)^{-2}\left[\pi\left(\frac{2}{\sqrt{3}}-\tau\right)\tau - 2\sqrt{3}\tau\tan^{-1}\sqrt{\frac{3}{2}(\tau^2-1)}\right.\\ &\qquad\left. + 6\tau^2\tan^{-1}\sqrt{\frac{1}{2}(1-1/\tau^2)}\,\right] \qquad (1<\tau<\sqrt{3})^{3)}.\end{aligned} \tag{5. 5}$$

Or , if $\tau=\sqrt{n-1}/t$ be adopted, we shall get, as the Student-like distribution,

(5. 6) $f_1(t)=4/\pi(1+t^2)$ $\qquad (1<t<\infty)$;

(5. 7) $f_2(t)=2\sqrt{2}\,(1+t^2/2)^{-3/2}$ or $2\sqrt{2}\left(1-\frac{3}{\pi}\cos^{-1}\frac{t}{2}\right)\left(1+\frac{t^2}{2}\right)^{-3/2}$

$(2<t<\infty$ or $1<t<2)$;

(5. 8) $f_3(t)=\dfrac{32}{\pi\sqrt{3}}(1+t^2/3)^{-2}$ or $\dfrac{32}{\pi\sqrt{3}}\left(\frac{2}{3}t-1\right)(1+t^2/3)^{-2}$

$(3<t<\infty$ or $\sqrt{3}<t<3)$,

$$=\frac{32}{\pi^2\sqrt{3}}(1+t^2/3)^{-2}\Big[\pi\left(\frac{2}{3}t-1\right)-2t\tan^{-1}\sqrt{(3-t^2)/2t^2}+6\tan^{-1}\sqrt{(3-t^2)/6}\Big] \qquad (1<t<\sqrt{3});$$

whereas the classical Student's distributions deliver

$$s_1(t)=\frac{1}{\pi}(1+t^2)^{-1},\quad s_2(t)=\frac{1}{2\sqrt{2}}(1+t^2)^{-3/2},\quad s_3(t)=\frac{2}{\sqrt{3}\pi}(1+t^2/3)^{-2},$$

and in general

$$s_{n-1}(t)=\frac{1}{\sqrt{(n-1)\pi}}\frac{\Gamma(n/2)}{\Gamma((n-1)/2)}\left(1+\frac{t^2}{n-1}\right)^{-n/2}, \qquad (-\infty<t<\infty).$$

The significant upper or lower limits with level α of our T. N. D. can be found from

$$\int_{t_1}^{\infty}f_{n-1}(t)\,dt=\alpha \quad \text{or} \quad \int_1^{t_0}f_{n-1}(t)\,dt=\alpha,$$

of which the former is readily computed, while for the latter it requires generally Newton's method of successive approximation. The following table shows a comparison between the significant lower- or upper-limit t_0, t_1 of our T. N. D. and those of Student's ordinary N. D.

		$\alpha=0.05$		$\alpha=0.025$		$\alpha=0.01$		$\alpha=0.005$	
		ours	Student	ours	Student	ours	Student	ours	Student
$n=2$	t_0	1.082	−6.314	1.040	−12.706	1.016	−31.821	1.008	−63.657
	t_1	25.452	+6.314	50.926	+12.706	127.321	+31.821	254.996	+63.657
$n=3$	t_0	1.381	−2.920	1.261	−4.303	1.161	−6.065	1.114	−9.925
	t_1	5.248	+2.920	7.471	+4.303	12.805	+6.065	18.130	+9.925

§ **6.** *Concluding Remark.* Whatever the universe may be if its mean and variance exist, and when the sample size so large that the central limit theorem holds for the distribution of sample mean, the ordinary normal test will convert to use. It is probable that our $f_{n-1}(t)$ shall also follow that theorem just as it is the case for the ordinary Student ratio. But, to prove this strictly, we have to find the general expression for $f_{n-1}(t)$, or at least to show the existence of its mean and variance, what however seems plausible by the general argument done in section 3. Since, however, with large samples the classical normal test will do at any rate, there is little need to know the exact form $f_{n-1}(t)$ for large n. On the contrary, the exact sampling distribution with small size forms certainly a subject of discussion.

Our examples in this note were rather simple. To conceive a little more copmplex case, e. g. let the universe be

(6. 1) $$f(x) = \alpha^{m+1}x^m e^{-\alpha x}/\Gamma(m+1), \qquad (x>0, \quad m=1, 2, \ldots),$$

or more practically, the χ_k^2 -distribution, i. e.

(6. 2) $$f(x) = x^{\frac{k}{2}-1} e^{-\frac{x}{2}} \Big/ 2^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right) \qquad (x>0, \quad k=1,\ 2,\ \ldots).$$

With the exact sample of size n we have the probability element for (6. 1)

(6. 3) $$dP = e^{-n\bar{x}} \prod_1^n x_i^m\, dV/\Gamma(m+1) \quad \text{where} \quad dV = \int \prod_1^n dx_i.$$

We may calculate the sample moments

(6. 4) $$\nu_k = \sum_1^n x_i^k/n \qquad k = 1, 2, \ \ldots\ , \ n,$$

or, more concretely the central moments

(6. 5) $$\mu_k = \sum_1^n (x_i-\bar{x})^k/n\,,$$

which, namely, give the sample mean $\bar{x} = \nu_1\,(\mu_1 = 0)$, variance $s^2 = \mu_2$, skewness $\alpha_3 = \mu_3/s^3$ and kurtosis $\alpha_4 = \mu_4/s^4$, &c. We should express the probability element (6. 3) so as

(6. 6) $$dP = f(\bar{x},\ s,\ \mu_3,\ \ldots\ ,\ \mu_n)\ d\bar{x}dsd\mu_3 \ldots\ d\mu_n,$$

or else, parallel to $s = \sqrt{\mu_2}$ writing $\sqrt[k]{\mu_k}$ as variables for every $k \geq 3$ also.

The expression $\prod_1^n x_i^m$ may be denoted by a combinaticn of $x, s, \mu_3, \ldots, \mu_n$, while $\int \prod_1^n dx_i$ should be expressed as $g(\bar{x}, s, \mu_3, \ldots)\, d\bar{x}ds\ d\mu_3 \ldots\ d\mu_n$, and in particular accessibly[4] for cases $n = 2,\ 3,\ 4$.

However, the present author will leave the remaining work unfinished to any investigator who is interested in this theme. Of course, if merely the frequency function for sample mean be required, it could be readily obtained by a simple application of the convolution theory to the Γ- or X^2-distribution.

References : 1) Herald Cramér, Mathematical Methods of Statistics, p. 247 (1946).

2) Émile Borel, Introduction géométrique, quelques théories physiques (1914). Using the notations in that text, pp. 63, 64, we obtain, as the area of the spherical calotte,

$$C = \frac{S}{2} - S_1 = 2\pi R^{m-1} \int_0^\theta \sin^{m-2} \varphi_1\ d\varphi_1 \int_0^\pi \sin^{m-3} \varphi_1\, d\varphi_1 \ldots \int_0^\pi \sin \varphi_{m-2}\, d\varphi_{m-2}$$

$$= \frac{2(\pi R^2)^{\frac{m-1}{2}}}{\Gamma\left(\frac{m-1}{2}\right)} \int_0^\theta \sin^{m-2} \varphi_1\, d\varphi_1\,,$$

where $\cos\theta = \alpha/R$ and α denotes the central distance of the base, and this result really coincides exactly with our (3.5) if we put $m = n-1$, $R^2 = ns^2$ and $\cos\theta = \bar{x}/s\sqrt{n-1}$.

3) The author has verified that the total probabilities always become unity for several distributions which were treated in this note. From that point of view, however, it requires for (5.5)

$$\int_0^\infty \psi_3(\tau)d\tau = \frac{96}{\pi^2}\int_1^{\sqrt{3}} \tan^{-1}\sqrt{\frac{1}{2}\left(1-\frac{1}{\tau^2}\right)}\ \frac{d\tau}{1+\tau^2} = 1,$$

what could really be ascertained by Gauss' method of selected ordinates for numerical integrations. Also he tried to prove it by means of the theory of functions, though yet without finishing completely: Y. Watanabe, Eine Integralformel, the present volume of this Journal.

4) E.g. the case $n=2$ for X_k^2-distribution may be readily treated by means of (3.7): The fr. f. for the X_k^2-distribution being $f(x) = x^{\frac{k}{2}-1} e^{-\frac{x}{2}} \Big/ 2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)$ $(x > 0)$, it follows that after (3.7)

$$dP = \frac{4}{2^k\Gamma\left(\frac{k}{2}\right)^2}\,(x_1\,x_2)^{\frac{k}{2}-1} e^{-\bar{x}} ds d\bar{x}\ \ (0 < s < \bar{x},\ \ 0 < \bar{x} < \infty),\ \text{where}\ x_1 x_2 = \bar{x}^2 - s^2.$$

Or, writing $\bar{x}\sqrt{n-1}/s = t$, $1 < t = \bar{x}/s < \infty$ for $n=2$, the fr. f. $f_2(t)$ is found to become

$$f_2(t)dt = \left(1-\frac{1}{t^2}\right)^{\frac{k}{2}-1}\frac{dt}{t^2}\int_0^\infty \bar{x}^{k-1} e^{-\bar{x}} d\bar{x}/2^{k-2}\Gamma\left(\frac{k}{2}\right)^2 = 2\Gamma\left(\frac{k+1}{2}\right)\left(1-\frac{1}{t^2}\right)^{\frac{k}{2}-1}\frac{dt}{t^2}\Big/\sqrt{\pi}\ \Gamma\left(\frac{k}{2}\right),$$

because of $\Gamma(k) = \frac{2^{k-1}}{\sqrt{\pi}}\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{k+1}{2}\right)$, a multiplication theorem of the gammafunction, and consequently $\int_1^\infty f_2(t)\,dt = 1$. Hence, the lower- and upper-limit t_0 and t_1 for the significance level 0.05 say, such that

$$\int_1^{t_0} f(t)\,dt = 0.05 \quad\text{and}\quad \int_{t_1}^\infty f_2(t)\,dt = 0.05$$

are found to be

k	1	2	3	and so on
t_0	1.003	1.053	1.072	
t_1	12.745	20.000	25.286	

while the corresponding ordinary Student's ratios are $t = \mp 6.314$.

Similarly, with the parent distribution $f(x) = x^m e^{-x}/m!$ $(x > 0,\ m = 1, 2, 3, \ldots)$ and the sample size $n=2$, it follows that $dP = \frac{4}{m!^2}(\bar{x}^2 - s^2)^m e^{-2\bar{x}}\,ds d\bar{x}$, and the fr. f. $f_2(t) = \frac{4(2m+1)!}{m!^2}\left(1-\frac{1}{t^2}\right)^m \frac{1}{t^2}$ in $1 < t = \frac{\bar{x}}{s} < \infty$.

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NOTES ON GENERAL ANALYSIS (VIII)

By

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In the preceding paper*), we proved the next theorem. *Let the family of functions* $\{f(x)\}$ *from* E_1 *to* E_2 *satisfy following conditions* : (1) *each function* $f(x)$ *is analytic in* $\|x\| < 1$ *in* E_1 *and is a one-to-one mapping to a domain* D_f *in* E_2 *and its inverse function* $f^{-1}(x)$ *is also analytic in* D_f, (2) $\{f(x)\}$ *are bounded, that is,* $\|f(x)\| \leq M$, (3) *the norms of linear parts* $\{g_1(x)\}$ *of* $\{f^{-1}(x)\}$ *are bounded, that is,* $\|g_1\| \leq K$, (4) $f(\theta) = \vartheta$, *then each domain* D_f *includes the sphere whose radius is constant.*

In this note, we discuss the case where E_1 is composed of complex numbers and E_2 is complex Banach spaces.

Lemma 1. *If* $x = f(\alpha)$ *is an* E_2*-valued function defined in the unit circle* $|\alpha| < 1$ *in complex plane and analytic there, then*

$$\|f'(\alpha)\|(1-|\alpha|^2) = \|F'(\beta)\|(1-|\beta|^2),$$

where $F(\beta) = f\left(\dfrac{\beta+\alpha_0}{1+\beta\bar{\alpha}_0}\right)$ *and* $\beta = \dfrac{\alpha-\alpha_0}{1-\alpha\bar{\alpha}_0}$.

Proof. Since $\beta = \dfrac{\alpha-\alpha_0}{1-\alpha\bar{\alpha}_0}$, we have $\dfrac{|d\alpha|}{1-|\alpha|^2} = \dfrac{|d\beta|}{1-|\beta|^2}$.

On the other hand, $F'(\beta) = f'(\alpha)\dfrac{d\alpha}{d\beta}$.

Then we have $\|F'(\beta)\| = \|f'(\alpha)\|\dfrac{|d\alpha|}{|d\beta|} = \|f'(\alpha)\|\dfrac{1-|\alpha|^2}{1-|\beta|^2}$, and we see that

$$\|F'(\beta)\|(1-|\beta|^2) = \|f'(\alpha)\|(1-|\alpha|^2).$$

Lemma 2. *If an* E_2*-valued function* $f(\alpha)$ *defined in* $|\alpha| < 1$ *in complex plane satisfies* $f(0) = \theta$, $\|f'(0)\| = 1$ *and is a one-to-one mapping to a domain* D *in* E_2 *and its inverse function* $f^{-1}(x)$ *is also analytic in* D, *then the norm of the linear part* $g_1(x)$ *of* $f^{-1}(x)$ *is* 1.

Proof. Since $f^{-1}(x)$ is analytic in D and $f(0) = \theta$, we have

$$f^{-1}(x) = \sum_1^\infty g_n(x),$$

where $g_n(x)$ is a complex valued homogeneous polynomial of degree n for $n = 1$, 2, 3, ... Let β be complex variables, then

$$\alpha = f^{-1}(\beta x) = \sum_1^\infty g_n(x)\beta^n.$$

Since $f^{-1}(\beta x)$ is one-to-one mapping and analytic in D, $\sum_1^\infty g_n(x)\beta^n$ converges uniformly for $|\beta|\leq r<1$ at least and $g_1(x)\neq 0$. This shows that a neighbourhood U of $\beta=0$ is mapped to a neighbourhood V of $\alpha=0$. If E_2 is composed of at least two elements x, x', where x and x' are linearly independent, we have on the same way

$$\alpha = \sum_1^\infty g_n(x')\beta^n.$$

By this relation, we see that a neighbourhood U' of $\beta=0$ is mapped to a neighbourhood V$'$ of $\alpha=0$. That is, a set $\beta x'$, where $\beta\in U'$, is mapped to V'. Then the intersection $V\cdot V'$ is mapped to different sets Ux and $U'x'$ simultaneously. Ux is a set of βx, where $\beta\in U$, and $U'x'$ is a set of $\beta x'$, where $\beta\in U'$. This contradicts that $f(\alpha)$ is a one-to-one mapping, and we see that E_2 is composed of one element which is linearly independent.

On the other hand, we have

$$y = f(\alpha) = \sum_1^\infty a_n\alpha^n,$$

where $a_n\in E_2$, since $f(\alpha)$ is analytic in $|\alpha|<1$. Then we have

$$\begin{aligned}\alpha = f^{-1}(x) &= \sum_1^\infty g_n(x)\\ &= \sum_1^\infty g_n\Big(\sum_1^\infty a_n\alpha^n\Big)\\ &= g_1(a_1)\alpha + \dots\dots\end{aligned}$$

Comparing the coefficients of α, we have $g_1(a_1)=1$. Since $a_1\in E_2$, E_1 is composed of points βa_1. Then, points on $\|x\|=1$ are expressed as $a_1e^{i\theta}$ (where $0\leq\theta\leq 2\pi$), so we have

$$\|g_1\| = \sup_{0\leqq\theta\leqq 2\pi}|g_1(a_1e^{i\theta})| = 1.$$

This completes the proof.

Theorem. *If the family of functions $\{f(\alpha)\}$ from $|\alpha|<1$ in complex plane to E_2 are analytic in $|\alpha|<1$ and one-to-one mapping to $\{D_f\}$ in E_2 separately and satisfy $\{\|f'(o)\|=1\}$ and their inverse functions $\{f^{-1}(x)\}$ are also analytic in $\{D_f\}$, then each domain D_f includes the sphere whose radius is constant.*

Proof. First of all, we assume that $f(\alpha)$ is analytic on $|\alpha|\leq 1$. Since $f'(\alpha)$ is analytic on $|\alpha|\leq 1$, $\|f'(\alpha)\|\cdot(1-|\alpha|^2)$ is continuons on $|\alpha|\leq 1$. Then, there exists a point α_0 such that

$$\|f'(\alpha_0)\|\cdot(1-|\alpha_0|^2) = \underset{|\alpha|\leqq 1}{Max.}\|f'(\alpha)\|\cdot(1-|\alpha|^2).$$

When $|\alpha|=1$, $\|f'(\alpha)\|(1-|\alpha|^2)=0$. Therefore, $|\alpha_0|<1$.
Put $M=\|f'(\alpha_0)\|(1-|\alpha_0|^2)$, then $M\geqq 1$, because, $\|f'(\alpha)\|(1-|\alpha|^2)=\|f'(o)\|=1$, when $\alpha=0$. By the transformation $\beta=\dfrac{\alpha-\alpha_0}{1-\alpha\bar{\alpha}_0}$, let $f(\alpha)=F(\beta)$.
By Lemma 1, we have $\|F'(\beta)\|(1-|\beta|^2)=\|f'(\alpha)\|(1-|\alpha|^2)$. Then

$$\|F'(o)\| = \|f'(\alpha_0)\|(1-|\alpha_0|^2) = M.$$

Put $\varphi(\beta)=\frac{1}{M}(F(\beta)-F(o))$, we have $\varphi(o)=0$ and $\|\varphi'(o)\|=\frac{1}{M}\|F'(o)\|=\frac{M}{M}=1$. $\varphi(\beta)$ is also analytic in $|\beta|\leq 1$ and clearly one-to-one mapping to a domain D' in E_2, since $f(\alpha)$, $\frac{\alpha-\alpha_0}{1-\alpha\bar{\alpha}_0}$ and $\frac{1}{M}(x-x_0)$ are one-to-one mappings. From the relation $\varphi'(\beta)=\frac{1}{M}F'(\beta)$, we see that $\|\varphi'(\beta)\|=\frac{1}{M}\|F'(\beta)\|$ and then

$$\begin{aligned}\|\varphi'(\beta)\| &= \frac{1}{M}\cdot\frac{\|f'(\alpha)\|\cdot(1-|\alpha|^2)}{1-|\beta|^2}\\ &\leq \frac{M}{M(1-|\beta|^2)}\\ &= \frac{1}{1-|\beta|^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\varphi(\beta)\| &= \left\|\int_0^\beta \varphi'(\beta)\,d\beta\right\|\\ &\leq \int_0^\beta \|\varphi'(\beta)\|\cdot|d\beta|\\ &\leq \int_0^\beta \frac{1}{1-|\beta|^2}\,|d\beta|\\ &= \frac{1}{2}\log\frac{1+|\beta|}{1-|\beta|}.\end{aligned}$$

Put $|\beta|\leq R<1$, we have $\|\varphi(\beta)\|\leq\log\frac{1+R}{1-R}$. Put $\log\frac{1+R}{1-R}=K$ and $\varphi_1(\beta)=\frac{1}{R}\varphi(R\beta)$, then $\varphi_1(\beta)$ is analytic in $|\beta|\leq 1$ and one-to-one mapping to D'' and $\|\varphi_1'(o)\|=\frac{1}{R}\|R\varphi'(o)\|=\|\varphi'(o)\|=1$ and $\varphi_1(o)=\frac{1}{R}\varphi(o)=0$.

Therefore, we have $\|\varphi_1(\beta)\|\leqq K$, when $|\beta|\leqq 1$. Appealing to Lemma 2, we see that the norm of the linear part $g_1(x)$ of the inverse function $\varphi_1^{-1}(x)$ is 1, since $\varphi_1^{-1}(x)$ is also analytic, one-to-one mapping, $\varphi_1(o)=0$ and $\|\varphi_1'(o)\|=1$. Thus, by the theorem written at the biginning of this paper, we see that D'' includes the sphere whose radius is constant. Therefore, D includes also the sphere whose radius is constant. It is easy as well as the usual way that the assumption $|\alpha|\leqq 1$ is removed.

References

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FINITE SEMIGROUPS IN WHICH LAGRANGE'S THEOREM HOLDS

By

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In the theory of finite groups, it is familiar as Lagrange's theorem that the order of every subgroup of a group G is a factor of the order of G. We should like to study the structure of a semigroup with such a property. A finite semigroup S is said to have $\mathfrak{S}_1$-property if the order of any subsemigroup is a divisor of the order of S. On the other hand $\mathfrak{S}$-property is defined as follows.

If a semigroup S of order n contains no proper subsemigroup of order greater than $n/2$, then S is said to have $\mathfrak{S}$-property.

Immediately $\mathfrak{S}_1$-property implies $\mathfrak{S}$-property. A finite semigroup with $\mathfrak{S}$-property and one with $\mathfrak{S}_1$-property are called $\mathfrak{S}$-semigroup and $\mathfrak{S}_1$-semigroup respectively. In the present paper we shall determine the types of $\mathfrak{S}$-semigroups, and at last the result will make the reader see that $\mathfrak{S}$-property is equivalent to $\mathfrak{S}_1$-property. We add that any semigroup of order at most 2 have $\mathfrak{S}$-property and so this case will be sometimes out of consideration.

1. Notations.

If S is a finite simple semigroup, then S is represented as a regular matrix semigroup with a ground group G and a defining matrix $P=(p_{ji})$ of type (l, m). (See [1])

If $p_{ji} \neq 0$ for all i, j, then

either $\quad S=\{(x\,;\ i\ j) \mid x \in G,\ i=1, \ldots, m\,;\ j=1, \ldots, l\}$

or $\quad S=\{(x\,;\ i\ j) \mid x \in G,\ i=1, \ldots, m\,;\ j=1, \ldots, l\} \cup \{0\}$

the multiplication of which is

$$(x\,;\ i\ j)(y\,;\ s\ t)=(x p_{js}\, y\,;\ i\ t)$$
$$0^2=0\,(x\,;\ i\ j)=(x\,;\ i\ j)\,0=0 \qquad \text{if } S \text{ has } 0.$$

If there is $p_{ji}=0$, then

$$S=\{(x\,;\ i\ j) \mid x \in G,\ i=1, \ldots, m\,;\ j=1, \ldots, l\} \cup \{0\}$$

with multiplication

$$(x\,;\ i\ j)(y\,;\ s\ t)=\begin{cases} 0 & \text{if } p_{js}=0 \\ (x p_{js}\, y\,;\ i\ t) & \text{if } p_{js} \neq 0. \end{cases}$$

Let $L=\{1, \ldots, m\}$, $R=\{1, \ldots, l\}$. R and L are regarded as a right-singular[1)] semigroup and a left-singular semigroup respectively.

---

1) R is called right-singular if $xy=y$ for every $x, y \in R$.

For the sake of convenience, we shall use the notations

$$Simp.(G\,;\ P) \text{ and } Simp.(G,\ 0\,;\ P)$$

which denote simple semigroups S with a defining matrix $P=(p_{ji})$ and a ground group G. The former denotes one without zero, whence $p_{ji}\neq 0$ for all i, j, bnt the latter denotes one with zero $\mathbf{0}$, so that if $p_{ji}\neq 0$ for all i and j, S contains no zero-divisor.

$A\times B$ denotes the direct product of two semigroups A and B.

2. Important Examples of $\mathfrak{S}$-Semigroups besides Groups.

Let G be any finite group a unit of which is denoted by e. The examples given in this paragraph will be proved to be $\mathfrak{S}_1$-semigroups and hence $\mathfrak{S}$-semigroups.

Lemma 1. *$Simp.(G\,;\ \binom{e}{e})$ is an $\mathfrak{S}$-semigroup, and any subsemigroup H is either a subgroup G' of G or $Simp.(G'\,;\ \binom{e}{e})$ isomorphic to $G'\times R$ where $R=\{1,\ 2\}$ is right singular.*

Proof. Let $S=Simp.(G\,;\ \binom{e}{e})$ and let H be a proper subsemigroup of S. Putting $H_{1j}=\{(x\,;\ 1\ j)\mid (x\,;\ 1\ j)\in H\}$,
H has one of the forms ; H_{11}, H_{12}, $H_{11}\cup H_{12}$.[2)]
Further, set $\qquad G_{1j}=\{x\mid (x\,;\ 1\ j)\in H\}$.
Since H_{1j} is a subsemigroup of H, it is shown that $x\in G_{1j}$ and $y\in G_{1j}$ imply $xy\in G_{1j}$. Hence G_{1j} is a subgroup of G. If $H=H_{1j}$ $(j=1, 2)$, H is isomorphic to the subgroup G_{1j} of G. If $H=H_{11}\cup H_{12}$, then from $H_{11}H_{12}\subseteq H_{12}$ and $H_{12}H_{11}\subseteq H_{11}$, it follows that $x\in G_{11}$ and $y\in G_{12}$ imply $xy\in G_{12}$, $yx\in G_{11}$. Since G_{1j} is a group, we get $G_{11}\subseteq G_{12}$ and $G_{12}\subseteq G_{11}$, therefore $G_{11}=G_{12}$ which we denote by G'. Thus we have

$$H=\{(x\,;\ 1\ j)\mid x\in G',\ j=1,\ 2\}$$

that is, $\qquad H=\mathrm{Simp.}(G'\,;\ \binom{e}{e})=G'\times R.$

Letting g and g' be the orders of G and G' respectively, the order of S is $2g$ and H has the order g' or $2g'$, the factor of $2g$.

Similarly we have

Corollary 1. *$Simp.(G\,;\ (ee))$ is an $\mathfrak{S}$-semigroup, and any subsemigroup H is either a subgroup G' of G or $Simp.(G'\,;\ (ee))$ isomorphic to $G'\times L$, where $L=\{1,\ 2]$ is left-singular.*

Remark. Let S_1 and S_2 be simple semigroups given by Lemma 1 and Corollary 1 respectively. When S_1 and S_2 have a ground group G in common, S_1 and S_2 are anti-isomorphic since G has always an anti-automorphism.

Lemma 2. *Let $0\neq a\in G$. $Simp.(G\,;\ \binom{ee}{ea})$ is an $\mathfrak{S}$-semigroup and any*

2) $\cup$ denotes the set union.

subsemigroup H is isomorphic to one of the following.

(α) *a subgroup G' of G,*

(β) Simp. $(G'\,;\ \binom{e}{e})$ *isomorphic to* $G' \times R$,

(γ) Simp. $(G'\,;\ (ee))$ *isomorphic to* $G' \times L$,

(δ) Simp. $(G'\,;\ \binom{ee}{ea})$.

Proof. Any subsemigroup H has one of the forms

(α') H_{ij} $(i,\ j=1,\ 2)$,

(β') $H_{11} \cup H_{12}$, $H_{21} \cup H_{22}$,

(γ') $H_{11} \cup H_{21}$, $H_{12} \cup H_{22}$,

(δ') $H_{11} \cup H_{12} \cup H_{21} \cup H_{22}$.

These are easily shown by considering all the subsemigroups of $R \times L$. Clearly G_{11}, G_{12}, G_{21} and $G_{22}a$ are subgroups of G, and H_{11}, H_{12}, H_{21} and H_{22} are isomorphic to G_{11}, G_{12}, G_{21}, and $G_{22}a$ respectively. Similarly as Lemma 1 and Corollary 1, we can prove :

If $H = H_{11} \cup H_{12}$, then $G_{11} = G_{12}\ (=G_1)$ and $H = Simp.\,(G_1\,;\ (e\ e))$,

if $H = H_{11} \cup H_{21}$, then $G_{11} = G_{21}\ (=G_2)$ and $H = Simp.\,(G_2\,;\ \binom{e}{e})$.

Let us discuss the other cases :

If $H = H_{21} \cup H_{22}$, we get $G_{21} = G_{22}a$ because $G_{21} \subseteq G_{22}a$ from $H_{21} H_{22} \subseteq H_{22}$, and $G_{22}a \subseteq G_{21}$ from $H_{22}\ H_{21} \subseteq H_{21}$; and hence

$$H = \{(x\,;\ 2\,1) \mid x \in G_{21}\} \cup \{(x\,;\ 2\,2) \mid x \in G_{21}\,a^{-1}\}.$$

It is proved that H is isomorphic to

$$Simp.\,(G_{21}\,;\ \binom{e}{e})$$

under the mapping f defined as

$$f((x\,;\ 2\,1)) = (x\,;\ 2\,1), \qquad f((x\,;\ 2\,2)) = (xa\,;\ 2\,2).$$

If $H = H_{12} \cup H_{22}$, then $G_{12} = aG_{22}$; and H is isomorphic to $Simp.$ $(G_{12}\,;\,(ee))$ under the mapping g defined as

$$g((x\,;\ 1\,2)) = (x\,;\ 1\,2), \qquad g((x\,;\ 2\,2) = (ax\,;\ 2\,2).$$

Finally if $H = H_{11} \cup H_{12} \cup H_{21} \cup H_{22}$,

we get $G_{11} = G_{12} = G_{21} = a\,G_{22} = G_{22}\,a\,(\text{put} = G')$.

Since G' is a subgroup of G, we can consider $(e\,;\ 1\,2)$, $(e\,;\ 2\,1) \in H$ and $(a\,;\ 1\,1) = (e\,;\ 1\,2)\ (e\,;\ 2\,1) \in H$ so that $a \in G'$ naturally $a^{-1} \in G'$. At last $G' = G_{11} = G_{12} = G_{21} = G_{22}$, which contains a. Therefore it follows that $H =$ Simp. $(G'\,;\ \binom{ee}{ea})$. Let g and g' be the orders of G and G' respectively. The order of S is $4g$ and the order H is g' or $2g'$ or $4g'$, the factor of $4g$.

3. General Case.

Lemma 3. *Let $S = Simp.\,(G,\ 0\,;\ (p_{ji}),\ i=1,\ \ldots,\ m\,;\ j=1,\ \ldots,\ l)$ of order >2. S has no $\mathfrak{S}$-property.*

Proof. Let g be the order of G. Then the order of S is

$$n = glm + 1.$$

When $l = m = 1$, S is a group with zero adjoined ; then $g > 1$ since we have assumed $n > 2$. But the order g of a proper subsemigroup G is not a divisor of $n = g + 1$.

When at least one of l and m is $\geqq 2$ e. g. $l \geqq 2$, there is a proper subsemigroup T of S

$$T = \{(x ; i\, j) \mid x \in G, \quad i = 1, \ldots, m ; \quad j = 1, \ldots, l-1\} \cup \{0\}$$ [2)]

whose order is $n' = gm\,(l-1) + 1$. On the other hand, we see

$$2n' - n = gm\,(l-2) + 1 > 0$$

whence n' is not a divisor of n. Therefore S has no $\mathfrak{S}$-property if $n > 2$. q. e. d.

Lemma 4. *A finite non-simple semigroup has no $\mathfrak{S}$-property.*

Proof. Let I be a maximal ideal of a finite non-simple semigroup S. Then the difference semigroup $(S : I) = D$ is a simple semigroup with zero. Set $D = \{G, 0 ; (p_{ji})\ i = 1, \ldots, m ; j = 1, \ldots, l\}$ and let g, i, d and n be the orders of G, I, D, and S respectively. Then $d = glm + 1$, $n = i + d - 1 = glm + i$. We may assume $n > 2$, $i > 1$.

First, if $lm = 1$ (i. e. $l = m = 1$), D is a group with zero adjoined, so that S is the union of I and a group G:

$$S = I \cup G, \quad I \cap G = \emptyset, \quad IG \subseteqq I, \quad GI \subseteqq I$$

where $n = i + g$. Then S contains a proper subsemigroup T of order $> n/2$. In fact

$T = I$ if $i > g$

$T = I \cup \{e\}$ where e is a unit of G if $i = g$

$T = G$ if $i < g$.

Second, if $lm > 1$, e. g. $l > 1$, then the proof of Lemma 3 shows that D contains a subsemigroup D' of order $d' = gm\,(l-1) + 1$. Let S' be the inverse image of D' under the homomorphism $S \to D$. Evidently S' is a proper subsemigroup of S such that $(S' : I) = D'$. Then the order n' of S' is greater than $n/2$, because, from $n' = gm\,(l-1) + i$

we get $$2n' - n = gm\,(l-2) + i > 0.$$

In all cases S has no $\mathfrak{S}$-property.

Lemma 5. *Let* $S = Simp.\,(G ; (p_{ji})\ i = 1, \ldots, m ; j = 1, \ldots, l)$. *If S is an $\mathfrak{S}$-semigroup, then S has one of the following structures :*

(1) *a finite group*

(2) $Simp.\,(G ; \binom{e}{e})$

(3) $Simp.\,(G ; (ee))$

(4) $Simp.\,(G ; \begin{pmatrix} e & e \\ e & a \end{pmatrix})$, $a \neq 0$.

2) T is not always simple.

Proof. Let g be the order of the ground group G of S, then the order n of S is

$$n = glm.$$

Suppose that S has $\mathfrak{S}$-property and at least one of l and m is $\geqq 3$ e. g. $l \geqq 3$, and consider

$$T = \{(x\,;\ i\ j)\,|\,x \in G,\ i = 1,\ \ldots\ ,\ m\,;\ j = 1,\ \ldots\ ,\ l-1\}.$$

T is clealy a subsemigroup of S and its order n' is

$$n' = gm\,(l-1)$$

and

$$2n' - n = gm\,(l-2) > 0$$

whence n' is not a factor of n. This contradicts $\mathfrak{S}$-property of S. Therefore we must have the following four cases of S

(i) $l = m = 1$,
(ii) $l = 2, \quad m = 1$,
(iii) $l = 1, \quad m = 2$,
(iv) $l = 2, \quad m = 2$.

If (i), S is a group. According to Rees' theory [1] the defining matrix is equivalent to $\binom{e}{e}$ in the case of (ii), equivalent to $(e\ e)$ in the case of (iii), equivalent to $\binom{e\,e}{e\,a}$ in the case of (iv).

Combining Lemmas 3, 4, and 5 with Lemmas 1, 2 and Corollary 1, we have the following theorem.

Theorem *A finite semigroup S is an $\mathfrak{S}$-semigroup of order $\geqq 2$ if and only if S has one of the following structures*:

(0) *a semigroup of order* 2[3)]
(1) *a group of order* $\geqq 2$
(2) *Simp.* $(G\,;\ \binom{e}{e})$
(3) *Simp.* $(G\,;\ (e\ e))$
(4) *Simp.* $(G\,;\ \binom{e\,e}{e\,a})$

where e is a unit of G, $0 \neq a \in G$, and the order g of G is $\geqq 1$.

A subsemigroup T of S is called proper if it is neither S itself nor a subsemigroup composed of only an idempotent. As a special case of $\mathfrak{S}$-semigroups, we have

Corollary 2. *A finite semigroup which contains no proper subsemigroup is either a semigroup of order at most 2 or a cyclic group of prime order.*

Proof. Let S be a semigroup satisfying this condition. According to Theorem, if the order g of G is $\geqq 2$, the simple semigroups (2), (3), (4) contain

3) In detail, (0) is either $\begin{array}{c|cc} & 0 & a \\ \hline 0 & 0 & 0 \\ a & 0 & 0 \end{array}$ or $\begin{array}{c|cc} & a & b \\ \hline a & a & a \\ b & a & b \end{array}$. Besides them, there are 3 types, which belong to (1) $\sim$ 3).

proper subsemigroups e. g. G. Hence, apart from the case of order 2, S must be a group. The theory of groups teaches us that an $\mathfrak{S}$-group is a cyclic group of order prime.

Added Note Corollary 2 holds even if the condition "finite" is exclnded.

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EIN ADDENDUM ZU "ZUR LAPLACESCHEN ASYMPTOTISCHEN FORMEL.", DIESES JOURN., VOL. IX (1958).

Von

Yoshikatsu WATANABE und Yoshihiro ICHIJO

(Eingegangen am 30 September, 1959)

Der Verfasser fand seines altes Manuskript in welches nochmal die folgenden weiteren Korrekturglieder geschrieben sind :

(3. 14. 2)

$$O\left(\frac{1}{n^3}\right) \cong \frac{1}{16n^3}\left\{-\frac{\varphi^{VI}}{3F''^3}+\frac{1}{3F''^4}\left[7F'''\varphi^{V}+\frac{35}{4}F^{IV}\varphi^{IV}+7F^{V}\varphi'''+\frac{7}{2}F^{VI}\varphi'' + F^{VII}\varphi'+\frac{1}{8}F^{VIII}\varphi\right]\right.$$

$$-\frac{7}{2F''^5}\left[\frac{5}{2}F'''^2\varphi^{IV}+5F'''F^{IV}\varphi'''+\left(\frac{15}{8}F^{IV2}+3F'''F^{V}\right)\varphi''+F'''F^{VI}\varphi' + \left(\frac{1}{4}F^{IV}F^{VI}+\frac{3}{20}F^{V2}+\frac{1}{7}F'''F^{VII}\right)\varphi\right]$$

$$+\frac{77}{4F''^6}\left[\frac{10}{9}F'''^3\varphi'''+\frac{5}{3}F'''^2F^{IV}\varphi''+\left(\frac{5}{4}F'''F^{IV2}+F'''^2F^{V}\right)\varphi' +\left(\frac{5}{48}F^{IV3}+\frac{1}{6}F'''^2F^{VI}+\frac{1}{2}F'''F^{IV}F^{V}\right)\varphi\right]$$

$$-\frac{5005}{72F''^7}\left[\frac{1}{2}F'''^4\varphi''+F'''^3F^{IV}\varphi'+\left(\frac{3}{8}F'''^2F^{IV2}+\frac{1}{5}F'''^3F^{V}\right)\varphi\right]$$

$$\left.+\frac{5}{36F''^8}\left[\frac{10125}{16}F'''^5\varphi'+\frac{1}{1001}F'''^4F^{IV}\varphi\right]-\frac{85085}{663552}\frac{F'''^6\varphi}{F''^9}\right\},$$

wobei die in Z. 1, S. 4, loc. cit. gemachte Aussage wieder gültig bleibt.

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EINE INTEGRALFORMEL

Von

Yoshikatsu WATANABE

(Eingegangen am 30 September 1959)

Der Verfasser sich bemühte um eine kleine Identität

$$\int_1^{\sqrt{3}} \operatorname{arctg} \sqrt{\frac{1}{2}\left(1-\frac{1}{x^2}\right)}\, \frac{dx}{1+x^2} = \frac{\pi^2}{96} = 0.102808\ldots,$$

die zwar nach einem statistischen Problem[1] gewiß bestehen soll, und tatsächlich nach der Gaußschen Methode numerischer Integration gesichert worden ist, formal zu prüfen. Obgleich es selbst trivial ist, jedoch mag die Denkensart als eine elementare Aufgabe zur Funktionentheorie für Studenten lehrreich dienen.

Wird das Integral teilweise integriert, so ergibt sich

$$\int_1^{\sqrt{3}} \operatorname{arctg} \sqrt{\frac{1}{2}\left(1-\frac{1}{x^2}\right)}\, \frac{dx}{1+x^2} = \frac{\pi^2}{18} - \sqrt{2}\int_1^{\sqrt{3}} \frac{\operatorname{arctg} x\, dx}{(3x^2-1)\sqrt{x^2-1}},$$

was wieder andere Arkstangensfunktion enthält. Beachtet man aber den letzten Integrand ins Komplexe, so ist die Funktion

$$f(z) = \operatorname{arctg}\ z/(3z^2-1)\sqrt{z^2-1}$$

eindeutig sogar regulär im Bereich B welcher von der halbkreisförmigen Kontur C wie in der Fig. 1 begrenzt wird. Wird der Einfachheit halber das Integral

$$J = \int_1^{\infty} f(x)\, dx$$

betrachtet, so kommt nach Cauchy

$$0 = \oint_C f(z)\, dz = (1) + (2) + (3) + \ldots\ldots + (15),$$

wobei die Nummern (1), (2), (3), ..., die längs jedem gleichlautenden Teilweg mit Pfilen in der Fig. 1 erstreckten Teilintegrale bedeuten.

Es seien arc $(z \mp 1) = 0$ auf Weg 1, und demnach bestehen für Teilintegrale

$$(1) = \lim_{\rho\to 0,\ R\to\infty} \int_{1+\rho}^{R} f(x)\, dx = J = (7),$$

aber $(9) + (14) = 0$ sowie $(11) + (12) = 0$ wegen je entgegengesetztes Vorzeichn.

Ferner stellt das sich in der längs die Imaginäreachse zwischen $(i,\ \infty i)$ sowie $(-i,\ -\infty i)$ gesperrten Zahlenebene E, bei $z = re^{i\theta}$

1) Y. Watanabe, Some exceptional examples to Student's distribution, gegenwärtiges Journ. S. 27, Fußnote.

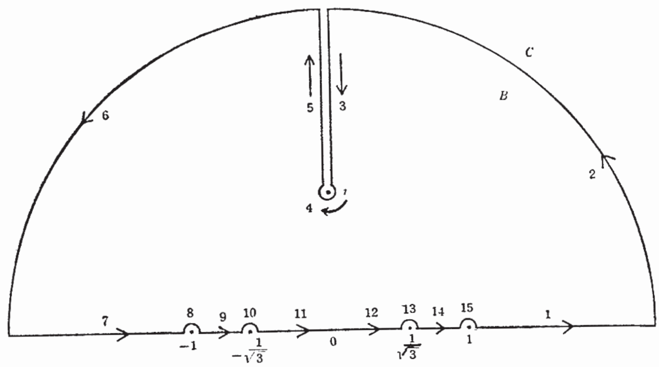

Fig. 1

$$\operatorname{arctg} z = \int_0^z \frac{dw}{1+w^2} = \frac{1}{2}\left[\operatorname{arctg}\frac{r-\sin\theta}{\cos\theta} + \operatorname{arctg}\frac{r+\sin\theta}{\cos\theta}\right] + \frac{i}{4}\log\left|\frac{r^2+2r\sin\theta+1}{r^2-2r\sin\theta+1}\right|$$

dar, und also ist arctg z eindeutig und überhaupt regulär in E, außer daß es nur für $r^2 \pm 2r \sin\theta + 1 \to 0$ (*d. h.* $z \to \pm i$), logarithmisch unendlich wird. Wenn die Radien der kleinen und großen Kreise im Integrationsweg $\rho\to 0$, $R\to\infty$ streben, so werden ersichtlich

$$(2) = (6) = O(R^{-2}),\quad (8) = (15) = O(\rho^{\frac{1}{2}}) \text{ und } (4) = O(\rho \log \rho).$$

Daher verschwinden alle diese Teilintegrale sämtlich bei $\rho \to 0$, $R \to \infty$.

Da aber die Faktoren arctg z in Integrande von (3) und (5) dieselben Imaginäreteile aber verschiedenen Reelleteil $\pm\pi/2$ bzw. besitzen, so betragen beide Teilintegrale (3) und (5) zusammen

$$-\pi\int_1^\infty \frac{dyi}{-(3y^2+1)\sqrt{y^2+1}\,e^{\pi i/2}} = \frac{\pi}{\sqrt{2}}\operatorname{arctg}(3-2\sqrt{2}) = \frac{\pi}{2\sqrt{2}}\operatorname{arctg}\frac{1}{2\sqrt{2}}.$$

Endlich gilt für (13), das Integral um den Pol $z = 1/\sqrt{3}$, asymptotisch

$$\int_\pi^0 \operatorname{arctg}\left(\frac{1}{\sqrt{3}}\right)\rho e^{i\theta}\, i d\theta \Big/ 3\rho e^{i\theta}\left(\frac{2}{\sqrt{3}}\right)\sqrt{\frac{2}{3}}\, e^{\pi i/2} = -\frac{\pi^2}{12\sqrt{2}}$$

und ganz ebenso für (10). Also schlüßt man

$$2J + \frac{\pi}{2\sqrt{2}}\operatorname{arctg}\frac{1}{2\sqrt{2}} - \frac{\pi^2}{6\sqrt{2}} = 0,$$

uud daraus

$$J = \frac{\pi}{4\sqrt{2}}\left[\frac{\pi}{3} - \operatorname{arctg}\frac{1}{2\sqrt{2}}\right] = 0.392766\ \ldots\ .$$

Man soll noch das Integral $\int_{\sqrt{3}}^\infty f(x)\,dx$ versuchen, was getan wird, falls wir das allgemeinere Integral $\int_a^\infty f(x)\, dx$ $(a > 1)$ ausfinden können. Für diese Leistung braucht man den Radius vom großen Halbkreis $R = a$ endlich und fest zu erhalten anstatt ∞ zu machen. Das neue über diesen Halbkreis erstreckte Integral ist wirklich gleich

$$\Re 2 \int_0^{\pi/2} f(Re^{i\theta})\, Re^{i\theta}\, i d\theta.$$

Oder, sonst, kann man den Halbkreis als Integrationsweg durch zwei aufwärts gezogene Lote $x = \pm \sqrt{3}$ mit entgegengesetzten Richtungen ersetzen. Jedoch werden diese fuuktionentheoretischen Abschätzungen etwas beschwerlich und weitere Untersuchungen sind als Aufgaben für Studenten übergelassen.

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SEMIGROUPS OF ORDER $\leqq$ 10 WHOSE GREATEST C-HOMOMORPHIC IMAGES ARE GROUPS

By

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In the present note we show all the isomorphically and anti-isomorphically distinct semigroups of order $\leqq 10$ whose greatest commutative homomorphic[1] images are groups. Especially a semigroup which has no proper commutative homomorphic image is called c-indecomposable. The purpose of the computation is to obtain examples by which we test and clarify the theory of finite semigroups of this kind. Here we shall only show tables of the results, the theory being discussed precisely in publication elsewhere.

1. On Tables 1 and 2. Since a finite simple semigroup is completely simple, it is represented as a regular matrix semigroup, which is determined by a ground group G (or G_0 with zero) and a defining matrix P [1].

At first we shall explain the notations of Table 1 with examples.

$\lambda.\ \mu$	the μ-th simple semigroup of order λ
3. 1 $\{\varepsilon\}$. 3—1	the first simple semigroup of order 3 with the ground group $G = \{\varepsilon\}$ and the defining matrix $P = \begin{pmatrix}\varepsilon\\\varepsilon\\\varepsilon\end{pmatrix}$.
4. 3. $\{\varepsilon,\ \alpha\}$, 2—1	the ground group : $G = \{\varepsilon,\ \alpha\}$, $\alpha^2 = \varepsilon$, the defining matrix : $P = \begin{pmatrix}\varepsilon\\\varepsilon\end{pmatrix}$.
6. 4 $\{\varepsilon, \alpha, \beta\}$, 2—1	$G = \{\varepsilon, \alpha, \beta\}$, $\beta = \alpha^2$, $\alpha^3 = \varepsilon$, $P = \begin{pmatrix}\varepsilon\\\varepsilon\end{pmatrix}$.
4. 2 $\{\varepsilon\}$, 2—2	$G = \{\varepsilon\}$, $P = \begin{pmatrix}\varepsilon & \varepsilon\\\varepsilon & \varepsilon\end{pmatrix}$.

Remark. $m—l$ symbols the matrix with m-rows l-columns, all the elements of which are ε.

5. 3 $\{0, \varepsilon\}$, $\begin{pmatrix}\varepsilon & \varepsilon\\\varepsilon & 0\end{pmatrix}$	$G_0 = \{0, \varepsilon\}$, $P = \begin{pmatrix}\varepsilon & \varepsilon\\\varepsilon & 0\end{pmatrix}$. Of course S has zero 0.
r-sing.	right-singular semigroup i. e. $xy = y$ for every x, y.
3. 1′	the dual form of the semigroup 3. 1, i. e. the multiplication $x \cdot y$ of 3. 1′ is defined as $x \cdot y = yx$ where yx is the multplication of 3. 1

1) "commutative homomorphic" will be called "c-homomorphic".

2.2 × 2.1	the direct product of the semigroups 2.2 and 2.1
c-ind.	c-indecomposable
c-dec.	c-decomposable
comm.	commutative
ind.	indecomposable i. e. having no proper homomorphism
self-dual	anti-isomorphic to itself.
	One unfilled in the column of "self-dual or not" is not self-dual.

Remark. For example, the semigroup 4. 3, $\{\varepsilon, \alpha\}$, 2—1, is composed of the elements

$$(\varepsilon,\ 11),\quad (\alpha,\ 11),\quad (\varepsilon,\ 12),\quad (\alpha,\ 12)$$

which are denoted by a, b, c, d respectively in the alphabet order ; 4. 3 represents

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	a	b	c	d
d	b	a	d	c

The semigroup 4. 2. $\{\varepsilon\}$, 2—2, is composed of

$$a=(\varepsilon,\ 11),\quad b=(\varepsilon,\ 12),\quad c=(\varepsilon,\ 21),\quad d=(\varepsilon,\ 22)$$

with the table

	a	b	c	d
a	a	b	a	b
b	a	b	a	b
c	c	d	c	d
d	c	d	c	d

The semigroup 5. 3, $\{0,\ \varepsilon\}$, $\begin{pmatrix}\varepsilon & \varepsilon\\ \varepsilon & 0\end{pmatrix}$ shows

	a	b	c	d	e
a	a	a	a	a	a
b	a	b	c	b	c
c	a	b	c	a	a
d	a	d	e	d	e
e	a	d	e	a	a

where $a=0$, $b=(\varepsilon,\ 11)$, $c=(\varepsilon,\ 12)$, $d=(\varepsilon,\ 21)$, $e=(\varepsilon,\ 22)$.

The semigroup 7. 3 $\{0,\ \varepsilon\}$, $\begin{pmatrix}\varepsilon & \varepsilon & \varepsilon\\ \varepsilon & \varepsilon & 0\end{pmatrix}$ consists of the elements
$a=0$, $b=(\varepsilon,\ 11)$, $c=(\varepsilon,\ 12)$, $d=(\varepsilon,\ 21)$, $e=(\varepsilon,\ 22)$, $f=(\varepsilon,\ 31)$, $g=(\varepsilon,\ 32)$.

The semigroup 9. 13 $\{0, \varepsilon, \alpha\}$, $\begin{pmatrix}\varepsilon & \varepsilon\\ \varepsilon & 0\end{pmatrix}$ consists of the elements $a=0$, $b=(\varepsilon,\ 11)$, $c=(\alpha,\ 11)$, $d=(\varepsilon,\ 12)$, $e=(\alpha,\ 12)$, $f=(\varepsilon,\ 21)$, $g=(\alpha,\ 21)$, $h=(\varepsilon,\ 22)$, $i=(\alpha,\ 22)$.

In Table 2, there are given automorphisms of some simple semgroups and some non-simple semigroups. The table shows that, for example, the automorphisms of the semigroup 3. 2 are

$$\begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}, \qquad \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix},$$

and those of 5. 4 are

$$\begin{pmatrix} 0 & 11 & 12 & 21 & 22 \\ 0 & 11 & 12 & 21 & 22 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 11 & 12 & 21 & 22 \\ 0 & 22 & 21 & 12 & 11 \end{pmatrix}$$

which are also denoted by $\begin{pmatrix} a & b & c & d & e \\ a & b & c & d & e \end{pmatrix}$, $\begin{pmatrix} a & b & c & d & e \\ a & e & d & c & b \end{pmatrix}$ respectively.

With respect to 7. 3, there is an automorphism

$$\begin{pmatrix} 0 & 11 & 12 & 21 & 22 & 31 & 32 \\ 0 & 21 & 22 & 11 & 12 & 31 & 32 \end{pmatrix}$$

besides the identical mapping. The automorphisms are useful for us to exclude isomorphic semigroups in our computation.

2. General Remark. An ideal I[2] of a semigroup S is called proper if I is neither S itself nor an ideal composed of only zero. A proper ideal I of S is called minimal, if I contains no proper ideal of S i. e. $\{0\} \subset J \subset I$ for no ideal J of S. If S is finite and not simple, then a minimal ideal exists. It is known that a minimal ideal of a finite semigroup is either a simple semigroup or a semigroup defined as $xy = 0$ for all x, y. [2] The latter will be called zero-semigroup. Since a homomorphic image of a c-indecomposable semigroup is also c-indecomposable, the difference semigroup $D = (S: I)$ of a c-indecomposable semigroup S modulo an ideal I is c-indecomposable. Further if I is minimal and simple, then I is also c-indecomposable.

Our computation is to find all c-indecomposable semigroups S such that $D = (S: I)$ and I is a minimal ideal of S when I (simple c-indecomposable semigroup or a zero-semigroup) and a c-indecomposable semigroup D with zero are given. In particular the Tables $3 \sim 11$ show the cases where D is moreover simple.

When I is not a zero-remigroup, S is completely determined by a system of some right translations φ of I:

$$\Phi = \{\varphi_\alpha \mid \alpha \in D, \quad \alpha \neq 0\},$$

and a system of some left translations ψ of I:

$$\Psi = \{\psi_\alpha \mid \alpha \in D, \quad \alpha \neq 0\}$$

where the correspondence $\alpha \to \varphi_\alpha$ is a ramified homorphism of D and $\alpha \to \psi_\alpha$ is a ramified anti-homomorphis of D. Let f_a and g_a be an inner right translation of I and an inner left translation of I respectively :

$$\begin{aligned} f_a(x) &= xa \\ g_a(x) &= ax. \end{aligned} \qquad a \in I \qquad x \in I$$

I is called right-regular if the correspondence $a \to \varphi_a$ is one-to-one; left-regularity

2) By an ideal we mean a two sided ideal.

is defined dually.

Especially if I is right-regular, S is completely determined by only the system Φ.

When I is a zero-semigroup, S is not always determined by Φ and Ψ : and then there is necessity for giving adequately the product of certain elements in order to determine S uniquely. We note that we may adopt $\{\varphi_\alpha \mid \alpha \in B \subset D, \alpha \neq 0\}$ instead of Φ, $\{\psi_\alpha \mid \alpha \in B \subset D, \alpha \neq 0\}$ instead of Ψ, where B is called the base of D.

Next, let S be a finite semigroup (c. d. g.) whose greatest c-homomorphic image G is a group, and let I be a minimal ideal of S. Then I is a simple semigroup without zero, and G is the greatest c-homomorphic image of I under the mapping $S \rightarrow G$, and further the difference semigroup $D = (S: I)$ is c-indecomposable. When there are given a simple semigroup (c. d. g.) I and a c-indecomposable semigroup D with zero, we must find S such that $D = (S: I)$ where I is a minimal ideal of S. The method of computation is like the case of c-indecomposable S.

3. On Contents of Tables. Generally $I - D$ symbols the type of a semigroup S such that I is a minimal ideal of S and $D = (S: I)$ is simple. $I_1 - \tilde{I}_2 - D'$ symbols that I_1 is a minimal ideal of S and the difference semigroup $D = (S: I_1)$ is not simple but have type $\tilde{I}_2 - D'$. In other words there is an ideal I_2 such that

$$(S:\ I_2) = D', \qquad (I_2:\ I_1) = \tilde{I}_2$$

where D' is simple. By $I_1 - \tilde{I}_2 - \tilde{I}_3 - D''$ we mean that $(S: I_1)$ is of type $\tilde{I}_2 - \tilde{I}_3 - D''$, namely there is a sequence of the ideals $I_1 \subset I_2 \subset I_3 \subset S$ where $(S: I_3) = D''$, $(I_3: I_2) = \tilde{I}_3$, $(I_2: I_1) = \tilde{I}_2$, and D'' is simple.

See Contents of Tables.

(r-sing.) — 5	The type in which I is right-singular and D is a simple semigroup of order 5 with zero i. e. 5.3 or 5.4.
5 (simp. 0)	A simple semigroup of order 5 with zero, 5.3 or 5.4.
5 (simp. 0) — 5	$I = 5$ (simp. 0), and D is also 5 (simp. 0)
(z) — 5	I is a zero-semigroup and D is same as the above.
3_0 — 5	I is the zero-semigroup of order 3, and D same as the above.
(sing.) — 7	I is singular, that is, right-singular or left-singular, and $D = 7$ (simp. 0) i. e. one of 7.3 ~ 7.6.
2 (sing.) — 9_ε	I is a singular semigroup of order 2 i. e. 2.1 or 2.1′, and D is one of 9.6 ~ 9.12.
2 (r-sing.) — $9_{\varepsilon,\alpha}$	$I = 2.1$, and D is either 9.13 or 9.14.
4 (r-sing. × l-sing.) — 7	I is of order 4 and the direct product of a right-singular semigroup and a left-singular semigroup.

3_0—3_0—5	I_1 is the zero-semigroup of order 3, and D has type 3_0—5; i. e. $\tilde{I}_2 = 3_0$, $D' = 5$ (simp. 0)
2 (r-sing.)—3_0—3_0—5	$I_1 = 2.1$ and D has type 3_0—3_0—5; $\tilde{I}_2 = 3_0$, $\tilde{I}_3 = 3_0$.
c. d. g.	c-decomposable and its greatest c-homomorphic image is a group.
(simp. or g.)	Either a group or the direct product of a group and a singular semigroup.
2_g	the group of order 2.

4. On Tables of the Non-simple. See Table 3, 3. 1—5. 3. We find three isomorphically distinct semigroups S denoted by (3. 1—5. 3) 1, (3. 1—5. 3) 2, (3. 1—5. 3) 3 :

(3. 1—5. 3) 1	$\varphi_{11} = (a\ a\ a)$	$\varphi_{22} = (a\ a\ a)$,
(3. 1—5. 3) 2	$\varphi_{11} = (b\ b\ b)$	$\varphi_{22} = (a\ a\ a)$,
(3. 1—5. 3) 3	$\varphi_{11} = (a\ c\ c)$	$\varphi_{22} = (a\ a\ b)$,

where 11, 22 form the base of D. The Tables show φ or ψ for the base of D.

As far as (3. 1—5. 3) 3 is concerned, we get

$$\varphi_{12} = \varphi_{11}\ \varphi_{22} = \begin{pmatrix} a & b & c \\ a & c & c \end{pmatrix} \begin{pmatrix} a & b & c \\ a & a & b \end{pmatrix} = \begin{pmatrix} a & b & c \\ a & b & b \end{pmatrix}, \quad \varphi_{21} = \varphi_{22}\ \varphi_{11} = \begin{pmatrix} a & b & c \\ a & a & c \end{pmatrix}$$

and

		a	b	c	11 d	12 e	21 f	22 g
	a	*a*	*b*	*c*	*a*	*a*	*a*	*a*
	b	*a*	*b*	*c*	*c*	*b*	*a*	*a*
	c	*a*	*b*	*c*	*c*	*b*	*c*	*b*
11	*d*	*a*	*b*	*c*	*d*	*e*	*d*	*e*
12	*e*	*a*	*b*	*c*	*d*	*e*	*a*	*a*
21	*f*	*a*	*b*	*c*	*f*	*g*	*f*	*g*
22	*g*	*a*	*b*	*c*	*f*	*g*	*a*	*a*

where, if, for example, we set $x = ef \in I$, then $\varphi_x = \varphi_{12}\ \varphi_{21} = (aaa)$ implies $x = a$ because I is right regular; the others are likewise found. Tables 3, 4, 6, 8, and 9 are seen in the same manner as this.

In Table 4, we seem that there is only one belonging type 5. 4—5. 3, but (5. 3—5. 4) 1 may be admitted into the category 5. 4—5. 3.

See Table 5. When I is 3_0 and D is 5. 3, the reguired $S = \{a, b, c, d, e, f, g\}$ is completely determined by φ_{11} and φ_{22} because we can prove that $\varphi_{11} = (a\,c\,c)$ and $\varphi_{22} = (a\,a\,b)$ imply $xy = a$ for $x = d, e, f, g$ and $y = a, b, c$, and moreover we get $g^2 = (22)^2 = a$ and hence $uv = a$ if $uv \in I$, $u \in S$, $v \in S$. Similarly we have 3_0—5. 4, 5_0—5. 4 in Table 5, and 3_0—7, 4_0—7 in Table 7. In these cases, $xy \in I$, $x \overline{\in} I$, $y \overline{\in} I$ implies $xy = a$.

On the other hand, even if φ_{11}, φ_{22}, ψ_{11}, ψ_{22} are assigned, $S = \{a, b, c, d, e, f, g, h, i\}$ of type 5_0—5. 3 is not uniquely determined, but we have (5_0—

5. 3) 1 or (5_0—5. 3) 2 according as $i^2=(22)^2=a$ or b.
We add that if i^2 is given, every $xy \in I$ $(x \notin I,\ y \notin I)$ is naturally determined :

$$gh = gi = ih = i^2 = a \qquad \text{in } (5_0\text{—}5.\,3)\ 1,$$
$$gh = e,\quad gi = c,\quad ih = d,\quad i^2 = b \qquad \text{in } (5_0\text{—}5.\,3)\ 2.$$

See Tables 10, and 11, For example, (4. 2—5. 3) 3 is completely determined by φ_e, φ_h, ψ_e, ψ_h. In fact we calculate

$$\varphi_f = \varphi_e\,\varphi_h = (aacc), \qquad \varphi_g = \varphi_h\,\varphi_e = (bbdd),$$
$$\psi_f = \psi_h\,\psi_e = (cdcd), \qquad \psi_g = \psi_e\,\psi_h = (abab),$$

and $\varphi_{xy} = \varphi_x\varphi_y$, $\psi_{xy} = \psi_y\,\psi_x$ for $x \notin I$, $y \notin I$, $xy \in I$
from which all xy are uniquely determined :

$$h^2 = a,\quad hg = b,\quad fh = c,\quad fg = d.$$

In Tables 2 ~ 11, thus, we have seen the type I—D, while there are the type I_1—$\tilde{I}_2$—D' in Tables 12 ~ 17, 23 ~ 26, and the types I_1—$\tilde{I}_2$—$\tilde{I}_3$—D'' in Tables 18 and 27.

In Tables 12 and 13, I_2 is denoted by $\{a\,b\,c\,a\,a\}$, $\{a\,a\,a\,b\,c\}$ etc., which represent

	a	b	c	d	e
a	a	b	c	a	a
b	a	b	c	a	a
c	a	b	c	a	a
d	a	b	c	a	a
e	a	b	c	a	a

,

	a	b	c	d	e
a	a	a	a	b	c
b	a	a	a	b	c
c	a	a	a	b	c
d	a	a	a	b	c
e	a	a	a	b	c

etc.

respectively. The automorphisms of I_2 have already listed in Table 2. φ and ψ shown in Tables 12, 13 are right translations and left translations of I_1. We note that I_2 cannot be prepared arbitrarily, but is somewhat restricted by I_1, φ and ψ.

In Table 14, I_2 is denoted by

$$\begin{pmatrix} \varphi_e = (aacc) \\ \varphi_f = (aacc) \\ \psi_e = (abab) \\ \psi_f = (abab) \end{pmatrix}.$$

This represents

	a	b	c	d	e	f
a	a	b	a	b	a	a
b	a	b	a	b	a	a
c	c	d	c	d	c	c
d	c	d	c	d	c	c
e	a	b	a	b	a	a
f	a	b	a	b	a	a

which is obtained like (4. 2—5. 3).

Referring to the examples which we have explained, all the tables can be understood.

References

[1] D. Rees, On semigroups, Proc. Cambridge Philos. Soc., 36, 1940 387—400.
[2] S. Schwarz, On semigroup having a kernel, Czechoslovak Math. Jour., 1 76, 1951 25—88.

Contents of Tables

Contents by Classification of the Orders

Table 1 Simple Semigroups whose greatest c-homomorphic images are groups

Order	No.	defining matrix	Remark	c-decompo-sability	self-dual or not
2	2.1	$\{\varepsilon\}$, 2-1	r-sing.	c-ind.	
	2.2		group	comm.	
3	3.1	$\{\varepsilon\}$, 3-1	r-sing.	c-ind.	
	3.2		group	comm.	
4	4.1	$\{\varepsilon\}$, 4-1	r-sing.	c-ind.	
	4.2	$\{\varepsilon\}$, 2-2	2.1×2.1′	c-ind.	self-dual
	4.3	$\{\varepsilon, \alpha\}$, 2-1	2.2×2.1	c-dec.	
	4.4		cyclic group	comm.	
	4.5		group 2.2×2.2	comm.	
5	5.1	$\{\varepsilon\}$, 5-1	r-sing.	c-ind.	
	5.2		group	comm.	
	5.3	$\{0, \varepsilon\}$, $\begin{pmatrix}\varepsilon\varepsilon\\ \varepsilon 0\end{pmatrix}$		ind.	self-dual
	5.4	$\{0, \varepsilon\}$, $\begin{pmatrix}\varepsilon 0\\ 0\varepsilon\end{pmatrix}$		ind.	self-dual
6	6.1	$\{\varepsilon\}$, 6-1	r-sing.	c-ind.	
	6.2	$\{\varepsilon\}$, 2-3	2.1×3.1′	c-ind.	
	6.3	$\{\varepsilon, \alpha\}$, 3-1	2.2×3.1	c-dec.	
	6.4	$\{\varepsilon, \alpha, \beta\}$, 2-1	3.2×2.1	c-dec.	
	6.5		cyclic group	c-dec.	
	6.6		symmetric group	c-dec.	
7	7.1	$\{\varepsilon\}$, 7-1	r-sing.	c-ind.	
	7.2		group	comm.	
	7.3	$\{0, \varepsilon\}$, $\begin{pmatrix}\varepsilon\varepsilon\varepsilon\\ \varepsilon\varepsilon 0\end{pmatrix}$		c-ind.	
	7.4	$\{0, \varepsilon\}$, $\begin{pmatrix}\varepsilon\varepsilon\varepsilon\\ \varepsilon 00\end{pmatrix}$		c-ind.	
	7.5	$\{0, \varepsilon\}$, $\begin{pmatrix}0\varepsilon 0\\ \varepsilon 0\varepsilon\end{pmatrix}$		c-ind.	
	7.6	$\{0, \varepsilon\}$, $\begin{pmatrix}\varepsilon\varepsilon 0\\ \varepsilon 0\varepsilon\end{pmatrix}$		ind.	
8	8.1	$\{\varepsilon\}$, 8-1	r-sing.	c-ind.	
	8.2	$\{\varepsilon\}$, 4-2	4.1×2.1′	c-ind.	
	8.3	$\{\varepsilon, \alpha\}$, 2-2	2.2×4.2	c-dec.	self-dual
	8.4	$\{\varepsilon, \alpha\}$, $\begin{pmatrix}\varepsilon\varepsilon\\ \varepsilon\alpha\end{pmatrix}$		c-ind.	self-dual
	8.5	$\{\varepsilon, \alpha\}$, 4-1	2.2×4.1	c-dec.	
	8.6		cyclic group	comm.	
	8.7		group 2.2×4.4	comm.	
	8.8		group 2.2×4.5	comm.	
	8.9		dihedral group	c-dec.	
	8.10		quaternion group	c-dec.	
9	9.1	$\{\varepsilon\}$, 9-1	r-sing,	c-ind.	
	9.2	$\{\varepsilon\}$, 3-3	3.1×3.1′	c-ind.	
	9.3	$\{\varepsilon, \alpha, \beta\}$, 3-1	3.2×3.1	c-dec.	
	9.4		cyclic group	comm.	
	9.5		group 3.2×3.2	comm.	
	9.6	$\{0, \varepsilon\}$, $\begin{pmatrix}\varepsilon\varepsilon\varepsilon\varepsilon\\ \varepsilon\varepsilon\varepsilon 0\end{pmatrix}$		c-ind.	
	9.7	$\{0, \varepsilon\}$, $\begin{pmatrix}\varepsilon\varepsilon\varepsilon\varepsilon\\ \varepsilon\varepsilon 00\end{pmatrix}$		c-ind.	
	9.8	$\{0, \varepsilon\}$, $\begin{pmatrix}\varepsilon\varepsilon\varepsilon 0\\ \varepsilon\varepsilon 0\varepsilon\end{pmatrix}$		c-ind.	
	9.9	$\{0, \varepsilon\}$, $\begin{pmatrix}\varepsilon\varepsilon\varepsilon\varepsilon\\ \varepsilon 000\end{pmatrix}$		c-ind.	
	9.10	$\{0, \varepsilon\}$, $\begin{pmatrix}\varepsilon\varepsilon 00\\ \varepsilon 0\varepsilon\varepsilon\end{pmatrix}$		c-ind.	
	9.11	$\{0, \varepsilon\}$, $\begin{pmatrix}\varepsilon\varepsilon\varepsilon 0\\ 000\varepsilon\end{pmatrix}$		c-ind.	
	9.12	$\{0, \varepsilon\}$, $\begin{pmatrix}\varepsilon\varepsilon 00\\ 00\varepsilon\varepsilon\end{pmatrix}$		c-ind.	
	9.13	$\{0, \varepsilon, \alpha\}$, $\begin{pmatrix}\varepsilon\varepsilon\\ \varepsilon 0\end{pmatrix}$		c-ind.	self-dual
	9.14	$\{0, \varepsilon, \alpha\}$, $\begin{pmatrix}\varepsilon 0\\ 0\varepsilon\end{pmatrix}$		c-ind.	self-dual

	10. 1	$\{\varepsilon\}$, 10-1	r-sing.	c-ind.	
	10. 2	$\{\varepsilon\}$, 2-5	2. 1×5. 1′	c-ind.	
	10. 3	$\{\varepsilon, \alpha\}$, 5-1	2. 2×5. 1	c-dec.	
	10. 4	$\{\varepsilon, \alpha, \beta, \gamma, \delta, \}$ 2-1	5. 2×2. 1	c-dec.	
	10. 5		cyclic group	comm.	
	10. 6		non-commutative group	c-dec.	
	10. 7	$\{0, \varepsilon\}$, $\begin{pmatrix} \varepsilon\varepsilon\varepsilon \\ \varepsilon\varepsilon\varepsilon \\ \varepsilon\varepsilon 0 \end{pmatrix}$		c-ind.	self-dual
	10. 8	$\{0, \varepsilon\}$, $\begin{pmatrix} \varepsilon\varepsilon\varepsilon \\ \varepsilon\varepsilon\varepsilon \\ \varepsilon 00 \end{pmatrix}$		c-ind.	
	10. 9	$\{0, \varepsilon\}$, $\begin{pmatrix} \varepsilon\varepsilon\varepsilon \\ \varepsilon\varepsilon 0 \\ \varepsilon 0\varepsilon \end{pmatrix}$		ind.	self-dual
	10. 10	$\{0, \varepsilon\}$, $\begin{pmatrix} \varepsilon\varepsilon\varepsilon \\ \varepsilon\varepsilon 0 \\ \varepsilon 00 \end{pmatrix}$		ind.	self-dual
10	10. 11	$\{0, \varepsilon\}$, $\begin{pmatrix} \varepsilon\varepsilon 0 \\ \varepsilon\varepsilon 0 \\ \varepsilon 0\varepsilon \end{pmatrix}$		ind.	
	10. 12	$\{0, \varepsilon\}$, $\begin{pmatrix} \varepsilon\varepsilon 0 \\ \varepsilon 0\varepsilon \\ 0\varepsilon\varepsilon \end{pmatrix}$		ind.	self-dual
	10. 13	$\{0, \varepsilon\}$, $\begin{pmatrix} \varepsilon\varepsilon\varepsilon \\ \varepsilon 00 \\ \varepsilon 00 \end{pmatrix}$		c-ind.	self-dual
	10. 14	$\{0, \varepsilon\}$, $\begin{pmatrix} \varepsilon\varepsilon 0 \\ \varepsilon 0\varepsilon \\ \varepsilon 00 \end{pmatrix}$		ind.	
	10. 15	$\{0, \varepsilon\}$, $\begin{pmatrix} \varepsilon\varepsilon 0 \\ \varepsilon\varepsilon 0 \\ 00\varepsilon \end{pmatrix}$		c-ind.	self-dual
	10. 16	$\{0, \varepsilon\}$, $\begin{pmatrix} \varepsilon\varepsilon 0 \\ \varepsilon 0\varepsilon \\ 0\varepsilon 0 \end{pmatrix}$		ind.	self-dual
	10. 17	$\{0, \varepsilon\}$, $\begin{pmatrix} 00\varepsilon \\ \varepsilon\varepsilon 0 \\ \varepsilon 00 \end{pmatrix}$		ind.	
	10. 18	$\{0, \varepsilon\}$, $\begin{pmatrix} \varepsilon 00 \\ \varepsilon 00 \\ 0\varepsilon\varepsilon \end{pmatrix}$		c-ind.	
	10. 19	$\{0, \varepsilon\}$, $\begin{pmatrix} \varepsilon 00 \\ 0\varepsilon 0 \\ 00\varepsilon \end{pmatrix}$		ind.	self-dual

Table 2 Automorphisms

(sing.) (singular semigroup)	all permutations
(Z) ($xy = 0$ for all x, y)	all permutations which fix 0
2. 2	*ab*
3. 2	*abc*, *acb*
4. 2	*abcd*, *badc*, *cdab*, *dcba*
4. 3	*abcd*, *cdab*
4. 4	*abcd*, *adcb*
4. 5	*abcd*, *abdc*, *acbd*, *adcb*
5. 2	*abcde*, *acebd*, *adbec*, *aedcb*
5. 3	$\begin{matrix} a & b & c & d & e \\ 0 & 11 & 12 & 21 & 22 \end{matrix}$
5. 4	$\begin{matrix} a & b & c & d & e, & a & e & d & c & b \\ 0 & 11 & 12 & 21 & 22 & 0 & 22 & 21 & 12 & 11 \end{matrix}$
6. 2	*abcdef*, *badcfe*, *abefcd*, *bafedc*, *cdabef*, *dcbafe*, *cdefab*, *dcfeba*, *efabcd*, *febadc*, *efcdab*, *fedcba*
6. 3	*abcdef*, *abefcd*, *cdabef*, *cdefab*, *efabcd*, *efcdab*

6. 4……………………*abcdef*, *acbdfe*, *defabc*, *dfeacb*

6. 5……………………*abcdef*, *afedcb*

6. 6………………… { *abcdef*, *abcefd*, *abcfde*, *acbdfe*, *acbedf*, *acbfed* }

7. 3……………………0 11 12 21 22 31 32, 0 21 22 11 12 31 32

7. 4……………………0 11 12 21 22 31 32. 0 11 12 31 32 21 22

7. 5……………………0 11 12 21 22 31 32, 0 31 32 21 22 11 12

7. 6……………………0 11 12 21 22 31 32, 0 12 11 32 31 22 21

9. 6…………………
0 11 12 21 22 31 32 41 42
0 11 12 31 32 21 22 41 42
0 21 22 11 12 31 32 41 42
0 21 22 31 32 11 12 41 42
0 31 32 11 12 21 22 41 42
0 31 32 21 22 11 12 41 42

9. 7…………………
0 11 12 21 22 31 32 41 42
0 11 12 21 22 41 42 31 32
0 21 22 11 12 31 32 41 42
0 21 22 11 12 41 42 31 32

9. 8…………………
0 11 12 21 22 31 32 41 42
0 21 22 11 12 31 32 41 42
0 12 11 22 21 42 41 32 31
0 22 21 12 11 42 41 32 31

9. 9…… ……………
0 11 12 21 22 31 32 41 42
0 11 12 21 22 41 42 31 32
0 11 12 31 32 21 22 41 42
0 11 12 31 32 41 42 21 22
0 11 12 41 42 21 22 31 32
0 11 12 41 42 31 32 21 22

9. 10 ………………
0 11 12 21 22 31 32 41 42
0 11 12 21 22 41 42 31 32

9. 11 ………………
0 11 12 21 22 31 32 41 42
0 11 12 31 32 21 22 41 42
0 21 22 11 12 31 32 41 42
0 21 22 31 32 11 12 41 42
0 31 32 11 12 21 22 41 42
0 31 32 21 22 11 12 41 42

9. 12 ………………
0 11 12 21 22 31 32 41 42
0 11 12 21 22 41 42 31 32
0 21 22 11 12 31 32 41 42
0 21 22 11 12 41 42 31 32
0 32 31 42 41 12 11 22 21
0 32 31 42 41 22 21 12 11
0 42 41 32 31 12 11 22 21
0 42 41 32 31 22 21 12 11

{*abaa*}……………………*abcd*, *abdc*

{*abab*}……………………*abcd*, *badc*

{*abcaa*} ……………………*abcde*, *abced*, *acbde*, *acbed*

{*abcab*} ……………………*abcde*, *baced*,

{*abcdaa*}……………………*axyzef*, *axyzfe* ($x, y, z = b, c, d$)

{*abcdab*}……………………*abcdef*, *abdcef*, *bacdfe*, *badcfe*

{*aaabc*} ……………………*abcde*, *acbed*

3) $\left(4.2;\ \begin{matrix}\varphi_e = (aacc), & \varphi_f = (aacc) \\ \psi_e = (abab), & \psi_f = (abab)\end{matrix}\right)$…………*abcdef*, *abcdfe*

$\left(4.2;\ \begin{matrix}\varphi_e = (aacc), & \varphi_f = (bbdd) \\ \psi_e = (abab), & \psi_f = (abab)\end{matrix}\right)$…………*abcdef*, *badcfe*

(2. 1—5. 3) 1…………………*abcdef*

(2. 1—5. 3) 2…………………*abcdef*

(2. 1—5. 4) 1…………………*abcdef*, *abfedc*

(2. 1—5. 4) 2…………………*abcdef*, *bafedc*

3) See p. 48, this paper.

{*abaaaa*}······················*abxyzu* ((*x y z u*) perm. of *c*, *d*, *e*, *f*)[4]

{*abaabb*}····················· {*abcdef*, *abcdfe*, *abdcef*, *abdcfe*, *baefcd*, *baefdc*, *bafecd*, *bafedc*

{*abaaab*}······················*abxyzf* ((*x y z*) perm. *of* *c*, *d*, *e*)

{*abaaba*}······················*abxyez* ((*x y z*) perm. *of* *c*, *d*, *f*)

{*ababab*}····················· {*abcdef*, *abcfed*, *abedcf*, *abefcd*, *badcfe*, *badefc*, *bafcde*, *bafedc*

{*abaacd*}······················*abcdef*, *abdcfe*

4) (*x y z u*) runs throughout the permutations of *c*, *d*, *e*, *f*.

Table 3

c-ind. (r-sing.) —5

D / I	5.3 $\begin{pmatrix}\varepsilon & \varepsilon\\ \varepsilon & 0\end{pmatrix}$			5.4 $\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon\end{pmatrix}$		
	No.	φ_{11}	φ_{22}	No.	φ_{12}	φ_{21}
2.1	1	*a a*	*a a*	1	*a a*	*a a*
	2	*a a*	*b b*	2	*a a*	*b b*
3.1	1	*a a a*	*a a a*	1	*a a a*	*a a a*
	2	*b b b*	*a a a*	2	*a a a*	*b b b*
	3	*a c c*	*a a b*	8	*a a b*	*a c a*
				4	*a a b*	*b c c*
4.1	1	*a a a a*	*a a a a*	1	*a a a a*	*a a a a*
	2	*b b b b*	*a a a a*	2	*a a a a*	*b b b b*
	3	*a d a d*	*a a a b*	3	*a a a b*	*a d a a*
	4	*a d d d*	*a a a b*	4	*a a a b*	*a d d a*
	5	*c d c d*	*a a a b*	5	*a a a b*	*b d b d*
	6	*a c c a*	*a a b b*	6	*a a a b*	*b d d d*
	7	*a c c c*	*a a b b*	7	*a a a b*	*c d c c*
				8	*a a a b*	*c d d d*
				9	*a a b b*	*a c a c*
				10	*a a b b*	*b c c b*
5.1	1	*a a a a a*	*a a a a a*	1	*a a a a a*	*a a a a a*
	2	*b b b b b*	*a a a a a*	2	*a a a a a*	*b b b b b*
	3	*a e a a e*	*a a a a b*	3	*a a a a b*	*a e a a a*
	4	*a e a e e*	*a a a a b*	4	*a a a a b*	*a e a e a*
	5	*a e e e e*	*a a a a b*	5	*a a a a b*	*a e e e a*
	6	*c e c c e*	*a a a a b*	6	*a a a a b*	*b e b b e*
	7	*c e c e e*	*a a a a b*	7	*a a a a b*	*b e b e e*
	8	*a d a d a*	*a a a b b*	8	*a a a a b*	*b e e e e*
	9	*a d a d d*	*a a a b b*	9	*a a a a b*	*c e c c c*
	10	*a d d d a*	*a a a b b*	10	*a a a a b*	*c e c e c*
	11	*a d d d d*	*a a a b b*	11	*a a a a b*	*c e e c e*
	12	*c d c d c*	*a a a b b*	12	*a a a a b*	*c e e e e*
	13	*c d c d d*	*a a a b b*	13	*a a a b b*	*a d a a d*
	14	*a c c a a*	*a a b b b*	14	*a a a b b*	*a d d a a*
	15	*a c c a c*	*a a b b b*	15	*a a a b b*	*a d d a d*
	16	*a c c c c*	*a a b b b*	16	*a a a b b*	*b d b d b*
	17	*a d e d e*	*a a a b c*	17	*a a a b b*	*b d b d d*
				18	*a a a b b*	*b d d d b*
				19	*a a a b b*	*c d c c d*
				20	*a a a b b*	*c d d d c*
				21	*a a b b b*	*a c a c c*
				22	*a a b b b*	*b c c b b*
				23	*a a a b c*	*a d e a a*
				24	*a a a b c*	*b d e d d*
				25	*a a a b c*	*c d e e e*
6.1	1	*a a a a a a*	*a a a a a a*	1	*a a a a a a*	*a a a a a a*
	2	*b b b b b b*	*a a a a a a*	2	*a a a a a a*	*b b b b b b*
	3	*a f a a a f*	*a a a a a b*	3	*a a a a a b*	*a f a a a a*
	4	*a f a a f f*	*a a a a a b*	4	*a a a a a b*	*a f a a f a*
	5	*a f a f f f*	*a a a a a b*	5	*a a a a a b*	*a f a f f a*
	6	*a f f f f f*	*a a a a a b*	6	*a a a a a b*	*a f f f f a*
	7	*c f c c c f*	*a a a a a b*	7	*a a a a a b*	*b f b b b f*

	8	c f c c f f	a a a a a b	8	a a a a a b	b f b b f f
	9	c f c f f f	a a a a a b	9	a a a a a b	b f b f f f
	10	a e a a e a	a a a a b b	10	a a a a a b	b f f f f f
	11	a e a e e a	a a a a b b	11	a a a a a b	c f c c c c
	12	a e e e e a	a a a a b b	12	a a a a a b	c f c c f c
	13	a e a a e e	a a a a b b	13	a a a a a b	c f c f f c
	14	a e a e e e	a a a a b b	14	a a a a a b	c f f c c f
	15	a e e e e e	a a a a b b	15	a a a a a b	c f f c f f
	16	c e c c e c	a a a a b b	16	a a a a a b	c f f f f f
	17	c e c c e e	a a a a b b	17	a a a a b b	a e a a a e
	18	c e c e e c	a a a a b b	18	a a a a b b	a e a e a a
	19	c e c e e e	a a a a b b	19	a a a a b b	a e a e a e
	20	a d a d a a	a a a b b b	20	a a a a b b	a e e e a a
	21	a d a d a d	a a a b b b	21	a a a a b b	a e e e a e
	22	a d a d d d	a a a b b b	22	a a a a b b	b e b b e b
	23	a d d d a a	a a a b b b	23	a a a a b b	b e b b e e
6.1	24	a d d d a d	a a a b b b	24	a a a a b b	b e b e e b
	25	a d d d d d	a a a b b b	25	a a a a b b	b e b e e e
	26	c d c d c c	a a a b b b	26	a a a a b b	b e e e e b
	27	c d c d c d	a a a b b b	27	a a a a b b	c e c c c e
	28	c d c d d d	a a a b b b	28	a a a a b b	c e c e c c
	29	a c c a a a	a a b b b b	29	a a a a b b	c e c e c e
	30	a c c a a c	a a b b b b	30	a a a a b b	c e e c e c
	31	a c c a c c	a a b b b b	31	a a a a b b	c e e c e e
	32	a c c c c c	a a b b b b	32	a a a a b b	c e e e e c
	33	a e f a e f	a a a a b c	33	a a a b b b	a d a a d d
	34	a e f e e f	a a a a b c	34	a a a b b b	a d d a a d
	35	a e f f e f	a a a a b c	35	a a a b b b	a d d a d d
	36	d e f d e f	a a a a b c	36	a a a b b b	b d b d b b
	37	a d f d a f	a a a b b c	37	a a a b b b	b d b d b d
	38	a d f d d f	a a a b b c	38	a a a b b b	b d d d b b
	39	a d f d f f	a a a b b c	39	a a a b b b	c d c c d d
				40	a a a b b b	c d d d c c
				41	a a b b b b	a c a c c c
				42	a a b b b b	b c c b b b
				43	a a a a b c	a e f a a a
				44	a a a a b c	a e f e a a
				45	a a a a b c	a e f f a a
				46	a a a a b c	b e f b e e
				47	a a a a b c	b e f e e e
				48	a a a a b c	b e f f e e
				49	a a a a b c	d e f d d d
				50	a a a a b c	d e f e e e
				51	a a a a b c	d e f f f f
				52	a a a b b c	a d f a d a
				53	a a a b b c	a d f a f a
				54	a a a b b c	b d f d b d
				55	a a a b b c	b d f d f d
				56	a a a b b c	c d f f c f
				57	a a a b b c	c d f f d f

Table 4 c-ind. 5 (simp. 0) — 5

D / I	5.3 $\begin{pmatrix}\varepsilon & \varepsilon\\ \varepsilon & 0\end{pmatrix}$			5.4 $\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon\end{pmatrix}$		
	No.	φ_{11}	φ_{22}	No.	φ_{12}	φ_{21}
5.3 $\begin{pmatrix}\varepsilon\varepsilon\\ \varepsilon 0\end{pmatrix}$	1	a a a a a	a a a a a	1	a a a a a	a a a a a
aaaaa	2	a b b d d	a b b d d	2	a b b d d	a b b d d
abcbc	3	a b b d d	a c c e e	3	a b b d d	a c c e e
abcaa	4	a b b d d	a b a d a	4	a b b d d	a b a d a
adede	5	a b b d d	a c a e a	5	a b b d d	a c a e a
adeaa	6	a b a d a	a b a d a	6	a b a d a	a b a d a
	7	a c c e e	a b b d d	7	a c c e e	a c c e e
	8	a c c e e	a c c e e			
	9	a b a d a	a b b d d			
5.4 $\begin{pmatrix}\varepsilon 0\\ 0\varepsilon\end{pmatrix}$		a b a d a	a b a d a	1	a a a a a	a a a a a
aaaaa				2	a b a d a	a b a d a
abcaa				3	a a b a d	a c a e a
aaabc						
adeaa						
aaade						

Table 5

c-ind. (z) — 5

D \ I	No.	φ_{11}	φ_{22}	ψ_{11}	ψ_{22}	$(22)^2$	φ_{12}	φ_{21}	ψ_{12}	ψ_{21}	$(12)^2=(21)^2$
		5.3 $\begin{pmatrix}\varepsilon\varepsilon\\ \varepsilon 0\end{pmatrix}$					5.4 $\begin{pmatrix}\varepsilon 0\\ 0\varepsilon\end{pmatrix}$				
3_0		*acc*	*aab*	*aaa*	*aaa*	*a*	*aab*	*aca*	*aaa*	*aaa*	*a*
5_0	1	*adede*	*aaabc*	*accee*	*aabad*	*a*	*aaabc*	*adeaa*	*acaea*	*aabad*	*a*
	2	*adede*	*aaabc*	*accee*	*aabad*	*b*					

Table 6

c-ind. (sing.) — 7

D \ I	No.	φ_{11}	φ_{21}	φ_{32}	No.	φ_{11}	φ_{22}	φ_{31}
		7.3 $\begin{pmatrix}\varepsilon & \varepsilon & \varepsilon\\ \varepsilon & \varepsilon & 0\end{pmatrix}$				7.4 $\begin{pmatrix}\varepsilon & \varepsilon & \varepsilon\\ \varepsilon & 0 & 0\end{pmatrix}$		
2.1	1	*a a*	*a a*	*a a*	1	*a a*	*a a*	*a a*
	2	*b b*	*b b*	*a a*	2	*b b*	*a a*	*b b*
3.1	1	*a a a*	*a a a*	*a a a*	1	*a a a*	*a a a*	*a a a*
	2	*b b b*	*b b b*	*a a a*	2	*b b b*	*a a a*	*b b b*
	3	*a c c*	*a c c*	*a a b*	3	*a c c*	*a a b*	*a a c*
4.1	1	*a a a a*	*a a a a*	*a a a a*	1	*a a a a*	*a a a a*	*a a a a*
	2	*b b b b*	*b b b b*	*a a a a*	2	*b b b b*	*a a a a*	*b b b b*
	3	*a d a d*	*a d a d*	*a a a b*	3	*a d a d*	*a a a b*	*a a a d*
	4	*a d a d*	*a d d d*	*a a a b*	4	*a d a d*	*a a a b*	*a a d d*
	5	*a d d d*	*a d d d*	*a a a b*	5	*a d d d*	*a a a b*	*a a a d*
	6	*c d c d*	*c d c d*	*a a a b*	6	*a d d d*	*a a a b*	*a a d d*
	7	*a c c a*	*a c c a*	*a a b b*	7	*c d c d*	*a a a b*	*c c c d*
	8	*a c c a*	*a c c c*	*a a b b*	8	*c d c d*	*a a a b*	*d d c d*
	9	*a c c c*	*a c c c*	*a a b b*	9	*a c c a*	*a a b b*	*a a c a*
					10	*a c c a*	*a a b b*	*a a c c*
					11	*a c c c*	*a a b b*	*a a c a*
					12	*a c c c*	*a a b b*	*a a c c*
	No.	ψ_{11}	ψ_{21}	ψ_{32}	No.	ψ_{11}	ψ_{22}	ψ_{31}
2.1′	1	*a a*	*a a*	*a a*	1	*a a*	*a a*	*a a*
	2	*a a*	*b b*	*a a*	2	*a a*	*a a*	*b b*
	3	*b b*	*b b*	*a a*	3	*b b*	*a a*	*a a*
					4	*b b*	*a a*	*b b*
3.1′	1	*a a a*	*a a a*	*a a a*	1	*a a a*	*a a a*	*a a a*
	2	*a a a*	*b b b*	*a a a*	2	*a a a*	*a a a*	*b b b*
	3	*b b b*	*b b b*	*a a a*	3	*b b b*	*a a a*	*a a a*
	4	*b b b*	*c c c*	*a a a*	4	*b b b*	*a a a*	*b b b*
	5	*a c c*	*a c c*	*a a b*	5	*b b b*	*a a a*	*c c c*
					6	*a c c*	*a a b*	*a b b*
4.1′	1	*a a a a*	*a a a a*	*a a a a*	1	*a a a a*	*a a a a*	*a a a a*
	2	*a a a a*	*b b b b*	*a a a a*	2	*a a a a*	*a a a a*	*b b b b*
	3	*b b b b*	*b b b b*	*a a a a*	3	*b b b b*	*a a a a*	*a a a a*
	4	*b b b b*	*c c c c*	*a a a a*	4	*b b b b*	*a a a a*	*b b b b*
	5	*a d a d*	*a d a d*	*a a a b*	5	*b b b b*	*a a a a*	*c c c c*
	6	*a d a d*	*c d c d*	*a a a b*	6	*a d a d*	*a a a b*	*a b a b*
	7	*a d d d*	*a d d d*	*a a a b*	7	*a d d d*	*a a a b*	*a b b b*
	8	*c d c d*	*c d c d*	*a a a b*	8	*a d d d*	*a a a b*	*a c c c*
	9	*a c c a*	*a c c a*	*a a b b*	9	*c d c d*	*a a a b*	*a b a b*
	10	*a c c c*	*a c c c*	*a a b b*	10	*c d c d*	*a a a b*	*c b c b*
					11	*a c c a*	*a a b b*	*a b b a*
					12	*a c c a*	*a a b b*	*d c c d*
					13	*a c c c*	*a a b b*	*a b b b*

D \ I	No.	φ_{11}	φ_{22}	φ_{31}	No.	φ_{12}	φ_{21}	φ_{31}
		7.5 $\begin{pmatrix}0 & \varepsilon & 0\\ \varepsilon & 0 & \varepsilon\end{pmatrix}$				7.6 $\begin{pmatrix}\varepsilon & \varepsilon & 0\\ \varepsilon & 0 & \varepsilon\end{pmatrix}$		
2.1	1	*a a*	*a a*	*a a*	1	*a a*	*a a*	*a a*
	2	*b b*	*a a*	*b b*	2	*b b*	*a a*	*a a*

3. 1	1	a a a	a a a	a a a	1	a a a	a a a	a a a
	2	b b b	a a a	b b b	2	b b b	a a a	a a a
	3	a c a	a a b	a c a	3	a c c	a b a	a a b
4. 1	1	a a a a	a a a a	a a a a	1	a a a a	a a a a	a a a a
	2	b b b b	a a a a	b b b b	2	b b b b	a a a a	a a a a
	3	a d a a	a a a b	a d a a	3	a d a d	a b a a	a a a b
	4	a d d a	a a a b	a d d a	4	a d a d	a b b a	a a a b
	5	b d b d	a a a b	b d b d	5	a d d d	a b a a	a a a b
	6	b d d d	a a a b	b d b d	6	a d d d	a b b a	a a a b
	7	c d d d	a a a b	c d d d	7	c d c d	a b a a	a a a b
	8	a c a c	a a b b	a c a c	8	c d c d	a b b b	a a a b
					9	a c c a	a b a a	a a b b
					10	a c c a	a b a b	a a b b
					11	a c c c	a b a a	a a b b
					12	a c c c	a b a b	a a b b
	No.	ψ_{11}	ψ_{22}	ψ_{31}	No.	ψ_{12}	ψ_{21}	ψ_{31}
2. 1′	1	a a	a a	a a	1	a a	a a	a a
	2	a a	a a	b b	2	a a	b b	a a
	3	b b	a a	b b	3	b b	a a	a a
					4	b b	b b	a a
3. 1′	1	a a a	a a a	a a a	1	a a a	a a a	a a a
	2	a a a	a a a	b b b	2	a a a	b b b	a a a
	3	b b b	a a a	b b b	3	b b b	a a a	a a a
	4	b b b	a a a	c c c	4	b b b	b b b	a a a
	5	a c a	a a b	a c a	5	b b b	c c c	a a a
	6	b c c	a a b	b c c	6	a c c	b b c	a a b
4. 1′	1	a a a a	a a a a	a a a a	1	a a a a	a a a a	a a a a
	2	a a a a	a a a a	b b b b	2	a a a a	b b b b	a a a a
	3	b b b b	a a a a	b b b b	3	b b b b	a a a a	a a a a
	4	b b b b	a a a a	c c c c	4	b b b b	b b b b	a a a a
	5	a d a a	a a a b	a d a a	5	b b b b	c c c c	a a a a
	6	a d d a	a a a b	a d d a	6	a d a d	b b b d	a a a b
	7	b d b d	a a a b	b d b d	7	a d d d	b b b d	a a a b
	8	b d d d	a a a b	b d d d	8	a d d d	c c c d	a a a b
	9	c d d d	a a a b	c d d d	9	c d c d	b b b d	a a a b
	10	c d c c	a a a b	c d c c	10	a c c a	a a d d	a a b b
	11	b d d d	a a a b	c d d d	11	a c c a	b b c c	a a b b
	12	c d c c	a a a b	a d a a	12	a c c c	b b c c	a a b b
	13	a c a a	a a b b	a c a a	13	a c c c	b b d d	a a b b
	14	a c a c	a a b b	a c a c				
	15	b d b d	a a b b	b d b d				
	16	b d d d	a a b b	b d d d				
	17	a d a a	a a b b	a c a a				
	18	a c a c	a a b b	b d b d				
	19	b c c c	a a b b	b d d d				

Table 7

c-ind. (z) — 7

D / I	7. 3 $\begin{pmatrix}\varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0\end{pmatrix}$							7. 4 $\begin{pmatrix}\varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & 0\end{pmatrix}$						
	No.	φ_{11}	φ_{21}	φ_{32}	ψ_{11}	ψ_{21}	ψ_{32}	No.	φ_{11}	φ_{22}	φ_{31}	ψ_{11}	ψ_{22}	ψ_{31}
3_0	1	acc	acc	aab	aaa	aaa	aaa	1	acc	aab	aac	aaa	aaa	aaa
	2	aaa	aaa	aaa	acc	acc	aab	2	aaa	aaa	aaa	acc	aab	abb
4_0		aaaa	aaaa	aaaa	addd	accc	aabb		aaaa	aaaa	aaaa	addd	aaac	abbb
D / I	7. 5 $\begin{pmatrix}0 & \varepsilon & 0 \\ \varepsilon & 0 & \varepsilon\end{pmatrix}$							7. 6 $\begin{pmatrix}\varepsilon & \varepsilon & 0 \\ \varepsilon & 0 & \varepsilon\end{pmatrix}$						
	No.	φ_{11}	φ_{22}	φ_{31}	ψ_{11}	ψ_{22}	ψ_{31}		φ_{12}	φ_{21}	φ_{31}	ψ_{12}	ψ_{21}	ψ_{31}
3_0	1	aca	aca	aab	aaa	aaa	aaa		acc	aba	aab	aaa	aaa	aaa
	2	aaa	aaa	aaa	aab	aca	aab							
4_0		aaaa	aaaa	aaaa	aada	acac	aaba		aaaa	aaaa	aaaa	adad	aacc	aabb

Table 8

c-ind. 2 (sing.) — 9_{ε}

9.6 $\begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix}$

I \ D	No.	φ_{11}	φ_{21}	φ_{31}	φ_{42}
2.1	1	a a	a a	a a	a a
	2	a a	a a	a a	b b
	No.	ψ_{11}	ψ_{21}	ψ_{31}	ψ_{42}
2.1′	1	a a	a a	a a	a a
	2	a a	a a	a a	b b
	3	a a	a a	b b	a a
	4	a a	a a	b b	b b

9.7 $\begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & 0 \end{pmatrix}$

I \ D	No.	φ_{11}	φ_{22}	φ_{32}	φ_{42}
2.1	1	a a	a a	a a	a a
	2	a a	b b	b b	b b
	No.	ψ_{11}	ψ_{22}	ψ_{32}	ψ_{42}
2.1′	1	a a	a a	a a	a a
	2	a a	b b	a a	a a
	3	b b	a a	a a	a a
	4	b b	b b	a a	a a
	5	a a	a a	a a	b b
	6	a a	b b	a a	b b
	7	b b	a a	a a	b b
	8	b b	b b	a a	b b

9.8 $\begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & 0 & \varepsilon \end{pmatrix}$

I \ D	No.	φ_{11}	φ_{21}	φ_{32}	φ_{41}
2.1	1	a a	a a	a a	a a
	2	a a	a a	b b	a a
	No.	ψ_{11}	ψ_{21}	ψ_{32}	ψ_{41}
2.1′	1	a a	a a	a a	a a
	2	a a	a a	a a	b b
	3	a a	a a	b b	a a
	4	a a	a a	b b	b b
	5	a a	b b	a a	a a
	6	a a	b b	a a	b b
	7	a a	b b	b b	a a
	8	a a	b b	b b	b b

9.9 $\begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & 0 & 0 \end{pmatrix}$

I \ D	No.	φ_{11}	φ_{22}	φ_{32}	φ_{42}
2.1	1	a a	a a	a a	a a
	2	a a	b b	b b	b b
	No.	ψ_{11}	ψ_{22}	ψ_{32}	ψ_{42}
2.1′	1	a a	a a	a a	a a
	2	b b	a a	a a	a a
	3	a a	a a	a a	b b
	4	b b	a a	a a	b b

9.10 $\begin{pmatrix} \varepsilon & \varepsilon & 0 & 0 \\ \varepsilon & 0 & \varepsilon & \varepsilon \end{pmatrix}$

I \ D	No.	φ_{12}	φ_{21}	φ_{31}	φ_{41}
2.1	1	a a	a a	a a	a a
	2	b b	a a	a a	a a
	No.	ψ_{12}	ψ_{21}	ψ_{31}	ψ_{41}
2.1′	1	a a	a a	a a	a a
	2	a a	a a	b b	a a
	3	a a	b b	a a	a a
	4	a a	b b	b b	a a
	5	b b	a a	a a	a a
	6	b b	a a	b b	a a
	7	b b	b b	a a	a a
	8	b b	b b	b b	a a

9.11 $\begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & 0 \\ 0 & 0 & 0 & \varepsilon \end{pmatrix}$

I \ D	No.	φ_{12}	φ_{22}	φ_{32}	φ_{41}
2.1	1	a a	a a	a a	a a
	2	a a	a a	a a	b b
	No.	ψ_{12}	ψ_{22}	ψ_{32}	ψ_{41}
2.1′	1	a a	a a	a a	a a
	2	a a	a a	a a	b b
	3	a a	a a	b b	a a
	4	a a	a a	b b	b b

9.12 $\begin{pmatrix} \varepsilon & \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon & \varepsilon \end{pmatrix}$

I \ D	No.	φ_{12}	φ_{21}	φ_{31}	φ_{41}
2.1	1	a a	a a	a a	a a
	2	a a	b b	b b	b b
	No.	ψ_{12}	ψ_{21}	ψ_{31}	ψ_{41}
2.1′	1	a a	a a	a a	a a
	2	a a	b b	a a	a a
	3	b b	a a	a a	a a
	4	b b	b b	a a	a a
	5	a a	a a	a a	b b
	6	a a	b b	a a	b b
	7	b b	a a	a a	b b
	8	b b	b b	a a	b b

Table 9

c-ind. 2 (r-sing.) —$9_{\varepsilon,\alpha}$

I \ D	9.13 $\{\varepsilon,\alpha\}\begin{pmatrix}\varepsilon & \varepsilon\\ \varepsilon & 0\end{pmatrix}$			9.14 $\{\varepsilon,\alpha\}\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon\end{pmatrix}$		
	No.	$\varphi_{\alpha\,11}$	$\varphi_{\alpha\,22}$	No.	$\varphi_{\alpha\,12}$	$\varphi_{\alpha\,21}$
2.1	1	*a a*	*a a*	1	*a a*	*a a*
	2	*a a*	*b b*	2	*a a*	*b b*

Table 10

c-ind. (r-sing. × *l*-sing.) — 5

I \ D	5.3 $\begin{pmatrix}\varepsilon & \varepsilon\\ \varepsilon & 0\end{pmatrix}$					5.4 $\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon\end{pmatrix}$				
	No.	φ_{11}	φ_{22}	ψ_{11}	ψ_{22}	No.	φ_{12}	φ_{21}	ψ_{12}	ψ_{21}
4.2	1	*aacc*	*aacc*	*abab*	*abab*	1	*aacc*	*aacc*	*abab*	*abab*
	2	*aacc*	*aacc*	*cdcd*	*abab*	2	*aacc*	*aacc*	*abab*	*cdcd*
	3	*bbdd*	*aacc*	*cdcd*	*abab*	3	*aacc*	*bbdd*	*abab*	*cdcd*
6.2	1	*bbddff*	*aaccee*	*cdcdcd*	*ababab*	1	*aaccee*	*bbddff*	*ababab*	*cdcdcd*
	2	*bbddff*	*aaccee*	*ababab*	*ababab*	2	*aaccee*	*bbddff*	*ababab*	*ababab*
	3	*aaccee*	*aaccee*	*ababab*	*ababab*	3	*aaccee*	*aaccee*	*ababab*	*ababab*
	4	*aaccee*	*aaccee*	*cdcdcd*	*ababab*	4	*aaccee*	*aaccee*	*ababab*	*cdcdcd*

Table 11

c-ind. 4(r-sing. × *l*-sing.) — 7

I \ D	7.3 $\begin{pmatrix}\varepsilon & \varepsilon & \varepsilon\\ \varepsilon & \varepsilon & 0\end{pmatrix}$							7.5 $\begin{pmatrix}0 & \varepsilon & 0\\ \varepsilon & 0 & \varepsilon\end{pmatrix}$						
	No.	φ_{11}	φ_{21}	φ_{32}	ψ_{11}	ψ_{22}	ψ_{32}	No.	φ_{11}	φ_{22}	φ_{31}	ψ_{11}	ψ_{22}	ψ_{31}
4.2	1	*aacc*	*aacc*	*aacc*	*abab*	*abab*	*abab*	1	*aacc*	*aacc*	*aacc*	*abab*	*abab*	*abab*
	2	*aacc*	*aacc*	*aacc*	*abab*	*cdcd*	*abab*	2	*aacc*	*aacc*	*aacc*	*abab*	*abab*	*cdcd*
	3	*aacc*	*aacc*	*aacc*	*cdcd*	*cdcd*	*abab*	3	*aacc*	*aacc*	*aacc*	*cdcd*	*abab*	*cdcd*
	4	*bbdd*	*bbdd*	*aacc*	*abab*	*abab*	*abab*	4	*bbdd*	*aacc*	*bbdd*	*abab*	*abab*	*abab*
	5	*bbdd*	*bbdd*	*aacc*	*abab*	*cdcd*	*abab*	5	*bbdd*	*aacc*	*bbdd*	*abab*	*abab*	*cdcd*
	6	*bbdd*	*bbdd*	*aacc*	*cdcd*	*cdcd*	*abab*	6	*bbdd*	*aacc*	*bbdd*	*cdcd*	*abab*	*cdcd*
	7.4 $\begin{pmatrix}\varepsilon & \varepsilon & \varepsilon\\ \varepsilon & 0 & 0\end{pmatrix}$							7.6 $\begin{pmatrix}\varepsilon & \varepsilon & 0\\ \varepsilon & 0 & \varepsilon\end{pmatrix}$						
	No.	φ_{11}	φ_{22}	φ_{31}	ψ_{11}	ψ_{22}	ψ_{31}	No.	φ_{11}	φ_{21}	φ_{31}	ψ_{11}	ψ_{21}	ψ_{31}
	1	*aacc*	*aacc*	*aacc*	*abab*	*abab*	*abab*	1	*aacc*	*aacc*	*aacc*	*abab*	*abab*	*abab*
	2	*aacc*	*aacc*	*aacc*	*abab*	*abab*	*cdcd*	2	*aacc*	*aacc*	*aacc*	*abab*	*abab*	*cdcd*
	3	*aacc*	*aacc*	*aacc*	*cdcd*	*abab*	*abab*	3	*aacc*	*aacc*	*aacc*	*abab*	*cdcd*	*abab*
	4	*aacc*	*aacc*	*aacc*	*cdcd*	*abab*	*cdcd*	4	*aacc*	*aacc*	*aacc*	*abab*	*cdcd*	*cdcd*
	5	*bbdd*	*aacc*	*bbdd*	*abab*	*abab*	*abab*	5	*aacc*	*bbdd*	*bbdd*	*abab*	*abab*	*abab*
	6	*bbdd*	*aacc*	*bbdd*	*abab*	*abab*	*cdcd*	6	*aacc*	*bbdd*	*bbdd*	*abab*	*abab*	*cdcd*
	7	*bbdd*	*aacc*	*bbdd*	*cdcd*	*abab*	*abab*	7	*aacc*	*bbdd*	*bbdd*	*abab*	*cdcd*	*abab*
	8	*bbdd*	*aacc*	*bbdd*	*cdcd*	*abab*	*cdcd*	8	*aacc*	*bbdd*	*bbdd*	*abab*	*cdcd*	*cdcd*

Table 12

c-ind. (r-sing.) —3_0 —5

I_1	I_2 \ $(S : I_1)$	3_0—5.3			3_0—5.4		
		No.	φ_{11}	φ_{22}	No.	φ_{11}	φ_{22}
2.1	{*a b a a*}	1	*a a*	*a a*	1	*a a*	*a a*
	{*a b a b*}	2	*b b*	*a a*	2	*a a*	*b b*
3.1	{*a b c a a*}	1	*a a a*	*a a a*	1	*a a a*	*a a a*
	{*a b c a b*}	2	*b b b*	*a a a*	2	*a a a*	*b b b*
4.1	{*a b c d a a*}	1	*a a a a*	*a a a a*	1	*a a a a*	*a a a a*
		2	*a b b a*	*a c a a*	2	*a c a a*	*a a b a*
		3	*a b b a*	*a c a c*	3	*a c a a*	*a a b b*
		4	*a b b b*	*a c a a*	4	*a c a c*	*a a b a*
		5	*a b b b*	*a c a c*	5	*a c a c*	*a a b b*
	{*a b c d a b*}	6	*a a a a*	*b b b b*	6	*a a a a*	*b b b b*
		7	*a a a c*	*b b d d*	7	*a a a c*	*b b d b*

Table 13

c-ind. 3_0—3_0—5

I_1	(S : I_1) / I_2	5. 3 $\begin{pmatrix}\varepsilon & \varepsilon \\ \varepsilon & 0\end{pmatrix}$					5. 4 $\begin{pmatrix}\varepsilon & 0 \\ 0 & \varepsilon\end{pmatrix}$				
		No.	φ_{11}	φ_{22}	ψ_{11}	ψ_{22}	No.	φ_{12}	φ_{21}	ψ_{12}	ψ_{21}
3	{*aaaaa*}	1	*aaaee*	*aaaad*	*accaa*	*aabaa*	1	*aaaad*	*aaaea*	*aabaa*	*acaaa*
		2	*accee*	*aabad*	*aaaaa*	*aaaaa*	2	*aabad*	*acaea*	*aaaaa*	*aaaaa*
	{*aaabc*}	3	*accee*	*aabad*	*aaaaa*	*aaaaa*	3	*aabad*	*acaea*	*aaaaa*	*aaaaa*

Table 14

c-ind. 4 (r-sing. × *l*-sing.) —3_0—5

I_1	(S : I_1) / I_2	5. 3 $\begin{pmatrix}\varepsilon & \varepsilon \\ \varepsilon & 0\end{pmatrix}$					5. 4 $\begin{pmatrix}\varepsilon & 0 \\ 0 & \varepsilon\end{pmatrix}$				
		No.	φ_{11}	φ_{22}	ψ_{11}	ψ_{22}	No.	φ_{12}	φ_{21}	ψ_{12}	ψ_{21}
4. 2	$\varphi_e=(aacc)$	1	*aaccff*	*aaccae*	*ababaa*	*ababaa*	1	*aaccae*	*aaccfa*	*ababaa*	*ababaa*
	$\varphi_f=(aacc)$	2	*aaccff*	*aaccae*	*ababaa*	*cdcdcc*	2	*aaccae*	*aaccfa*	*ababaa*	*cdcdcc*
	$\psi_e=(abab)$	3	*aaccff*	*aaccae*	*cdcdcc*	*ababaa*	3	*aaccae*	*aaccfa*	*cdcdcc*	*ababaa*
	$\psi_f=(abab)$	4	*aaccff*	*aaccae*	*cdcdcc*	*cdcdcc*	4	*aaccae*	*aaccfa*	*cdcdcc*	*cdcdcc*
	$\varphi_e=(aacc)$	1	*bbddff*	*aaccae*	*ababaa*	*ababaa*	1	*aaccae*	*bbddfb*	*ababab*	*ababab*
	$\varphi_f=(bbdd)$	2	*bbddff*	*aaccae*	*ababaa*	*cdcdcc*	2	*aaccae*	*bbddfb*	*ababab*	*cdcdcd*
	$\psi_e=(abab)$	3	*bbddff*	*aaccae*	*cdcdcc*	*ababaa*	3	*aaccae*	*bbddfb*	*cdcdcd*	*ababab*
	$\psi_f=(abab)$	4	*bbddff*	*aaccae*	*cdcdcc*	*cdcdcc*	4	*aaccae*	*bbddfb*	*cdcdcd*	*cdcdcd*

Table 15

c-ind. 2(sing.) —3_0—7

(S : I_1) / I_2	(3_0—7. 3) 1				(3_0—7. 3) 2				(3_0—7. 4) 1				(3_0—7. 4) 2			
	No.	φ_{11}	φ_{21}	φ_{32}	No.	φ_{11}	φ_{21}	φ_{32}	No.	φ_{11}	φ_{22}	φ_{31}	No.	φ_{11}	φ_{22}	φ_{31}
{*abaa*}		*a a*	*a a*	*a a*	1	*a a*	*a a*	*a a*		*a a*	*a a*	*a a*	1	*a a*	*a a*	*a a*
					2	*b b*	*b b*	*a a*					2	*b b*	*a a*	*b b*
{*abab*}		*b b*	*b b*	*a a*						*b b*	*a a*	*b b*				
		ψ_{11}	ψ_{21}	ψ_{32}		ψ_{11}	ψ_{21}	ψ_{32}		ψ_{11}	ψ_{22}	ψ_{31}		ψ_{11}	ψ_{22}	ψ_{31}
{*abaa*}′	1	*a a*	*a a*	*a a*		*a a*	*a a*	*a a*	1	*a a*	*a a*	*a a*		*a a*	*a a*	*a a*
	2	*a a*	*b b*	*a a*					2	*a a*	*a a*	*b b*				
	3	*b b*	*b b*	*a a*					3	*b b*	*a a*	*a a*				
									4	*b b*	*a a*	*b b*				
{*abab*}′						*b b*	*b b*	*a a*						*b b*	*a a*	*b b*

(S : I_1) / I_2	(3_0—7. 5) 1				(3_0—7. 5) 2				(3_0—7. 6) 1				(3_0—7. 6) 2			
	No.	φ_{11}	φ_{22}	φ_{31}	No.	φ_{11}	φ_{22}	φ_{31}	No.	φ_{12}	φ_{21}	φ_{31}	No.	φ_{12}	φ_{21}	φ_{31}
{*abaa*}		*a a*	*a a*	*a a*	1	*a a*	*a a*	*a a*		*a a*	*a a*	*a a*	1	*a a*	*a a*	*a a*
					2	*b b*	*a a*	*b b*					2	*b b*	*a a*	*a a*
{*abab*}		*b b*	*a a*	*b b*						*b b*	*a a*	*a a*				
		ψ_{11}	ψ_{22}	ψ_{31}		ψ_{11}	ψ_{22}	ψ_{31}		ψ_{12}	ψ_{21}	ψ_{31}		ψ_{12}	ψ_{21}	ψ_{31}
{*abaa*}′	1	*a a*	*a a*	*a a*		*a a*	*a a*	*a a*	1	*a a*	*a a*	*a a*		*a a*	*a a*	*a a*
	2	*a a*	*a a*	*b b*					2	*a a*	*b b*	*a a*				
	3	*b b*	*a a*	*b b*					3	*b b*	*a a*	*a a*				
									4	*b b*	*b b*	*a a*				
{*abab*}′						*b b*	*a a*	*b b*						*b b*	*a a*	*a a*

Table 16

c-ind. 2 (r-sing.) —5 (simp.0) —5

(S : I_1) = (5. 3—5. 3)

I_2 \ (S : I_1)	No.	1 φ_{11}	1 φ_{22}	2 φ_{11}	2 φ_{22}	3 φ_{11}	3 φ_{22}	4 φ_{11}	4 φ_{22}	5 φ_{11}	5 φ_{22}	6 φ_{11}	6 φ_{22}	7 φ_{11}	7 φ_{22}	8 φ_{11}	8 φ_{22}	9 φ_{11}	9 φ_{22}
(2. 1—5. 3)1	1 2 3 4	*a a* *a a* *b b* *b b*	*a a* *b b* *a a* *b b*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*
(2. 1—5. 3)2	1 2	*a a* *b b*	*a a* *a a*	*a a*	*a a*	*a a*	*b b*	*a a*	*a a*	*a a*	*b b*	*a a*	*a a*	*b b*	*a a*	*b b*	*b b*	*a a*	*a a*

(S ; I_1) = (5. 3—5. 4)

I_2 \ (S : I_1)	No.	1 φ_{12}	1 φ_{21}	2 φ_{12}	2 φ_{21}	3 φ_{12}	3 φ_{21}	4 φ_{12}	4 φ_{21}	5 φ_{12}	5 φ_{21}	6 φ_{12}	6 φ_{21}	7 φ_{12}	7 φ_{21}
(2. 1—5. 3)1	1 2 3	*a a* *a a* *b b*	*a a* *b b* *b b*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*
(2. 1—5. 3)2	1 2 3	*a a* *a a* *b b*	*a a* *b b* *b b*	*a a*	*a a*	*a a*	*b b*	*a a*	*a a*	*a a*	*b b*	*a a*	*a a*	*b b*	*b b*

(S : I_1) = (5. 4—5. 3)

I_2 \ (S : I_1)	φ_{11}	φ_{22}
(2. 1—5. 4)1	*a a*	*a a*
(2. 1—5. 4)2	*b b*	*b b*

(S : I_1) = (5. 4—5. 4)

I_2 \ (S : I_1)	No.	1 φ_{12}	1 φ_{21}	2 φ_{12}	2 φ_{21}	3 φ_{12}	3 φ_{21}
(2. 1—5. 4)1	1 2 3	*a a* *a a* *b b*	*a a* *b b* *b b*	*a a*	*a a*	*a a*	*a a*
(2. 1—5. 4)2		*a a*	*b b*	*b b*	*b b*	*b b*	*a a*

Table 17

c-ind. 2(r-sing.) —5_0—5

I_2 \ (S : I_1)	(5_0—5. 3) 1 φ_{11}	(5_0—5. 3) 1 φ_{22}	(5_0—5. 3) 2 φ_{11}	(5_0—5. 3) 2 φ_{22}	(5_0—5. 4) φ_{12}	(5_0—5. 4) φ_{21}
{*abaaaa*}	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*
{*abaabb*}	*b b*	*a a*	*b b*	*a a*	*a a*	*b b*

Table 18

c-ind. 2 (r-sing.) —3_0—3_0—5

I_2 \ (S : I_1)	(3_0—3_0—5. 3) 1 φ_{11}	(3_0—3_0—5. 3) 1 φ_{22}	(3_0—3_0—5. 3) 2 φ_{11}	(3_0—3_0—5. 3) 2 φ_{22}	(3_0—3_0—5. 3) 3 φ_{11}	(3_0—3_0—5. 3) 3 φ_{22}	(3_0—3_0—5. 4) 1 φ_{12}	(3_0—3_0—5. 4) 1 φ_{21}	(3_0—3_0—5. 4) 2 φ_{12}	(3_0—3_0—5. 4) 2 φ_{21}	(3_0—3_0—5. 4) 3 φ_{12}	(3_0—3_0—5. 4) 3 φ_{21}
{*abaaaa*}	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*	*a a*
{*abaaab*}	*b b*	*a a*					*a a*	*b b*				
{*abaaba*}	*a a*	*b b*					*b b*	*a a*				
{*abaabb*}	*b b*	*b b*					*b b*	*b b*				
{*ababab*}			*b b*	*a a*	*b b*	*a a*			*a a*	*b b*	*a a*	*b b*
{*abaacd*}	*d d*	*c c*					*c c*	*d d*				

Table 19

c.d.g. (simp. or g.) —5

I \ D		$5.3\begin{pmatrix}\varepsilon & \varepsilon\\ \varepsilon & 0\end{pmatrix}$			$5.4\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon\end{pmatrix}$		
		No.	φ_{11}	φ_{22}	No.	φ_{12}	φ_{21}
group	2. 2		*a b*	*a b*	1 2	*a b* *b a*	*a b* *b a*
	3. 2		*a b c*	*a b c*	1 2	*a b c* *b c a*	*a b c* *c a b*
	4. 4		*a b c d*	*a b c d*	1 2	*a b c d* *b c d a*	*a b c d* *d a b c*
	4. 5		*a b c d*	*a b c d*	1 2	*a b c d* *b a d c*	*a b c d* *b a d c*
	5. 2		*a b c d e*	*a b c d e*	1 2	*a b c d e* *b c d e a*	*a b c d e* *e a b c d*
	6. 5		*a b c d e f*	*a b c d e f*	1 2 3	*a b c d e f* *b c d e f a* *c d e f a b*	*a b c d e f* *f a b c d e* *e f a b c d*
	6. 6		*a b c d e f*	*a b c d e f*	1 2 3	*a d c d e f* *b c a f d e* *d e f a b c*	*a b c d e f* *c a b e f d* *d e f a b c*
non-groups	4. 3	1 2	*a b a b* *a b a b*	*a b a b* *c d c d*	1 2 3 4	*a b a b* *a b a b* *b a b a* *b a b a*	*a b a b* *c d c d* *b a b a* *d c d c*
	6. 3	1 2	*a b a b a b* *a b a b a b*	*a b a b a b* *c d c d c d*	1 2 3 4	*a b a b a b* *a b a b a b* *b d b d b d* *b d b d b d*	*a b a b a b* *c d c d c d* *b d b d b d* *d c d c d c*
	6. 4	1 2	*a b c a b c* *a b c a b c*	*a b c a b c* *d e f d e f*	1 2 3 4	*a b c a b c* *a b c a b c* *b c a b c a* *b c a b c a*	*a b c a b c* *d e f d e f* *c a b c a b* *f d e f d e*

Table 20

c.d.g. (simp. or g.)—7

I \ D		$7.3\begin{pmatrix}\varepsilon & \varepsilon & \varepsilon\\ \varepsilon & \varepsilon & 0\end{pmatrix}$				$7.4\begin{pmatrix}\varepsilon & \varepsilon & \varepsilon\\ \varepsilon & 0 & 0\end{pmatrix}$			
		No.	φ_{11}	φ_{21}	φ_{32}	No.	φ_{11}	φ_{22}	φ_{31}
group	2. 2		*a b*	*a b*	*a b*		*a b*	*a b*	*a b*
	3. 2		*a b c*	*a b c*	*a b c*		*a b c*	*a b c*	*a b c*
	4. 4		*a b c d*	*a b c d*	*a b c d*		*a b c d*	*a b c d*	*a b c d*
	4. 5		*a b c d*	*a b c d*	*a b c d*		*a b c d*	*a b c d*	*a b c d*
non-group	4. 3	1 2	*a b a b* *a b a b*	*a b a b* *a b a b*	*a d a b* *c d c d*	1 2	*a b a b* *a b a b*	*a b a b* *a b a b*	*a b a b* *c d c d*
	4. 3′		*a b c d*	*a b c d*	*a b c d*		*a b c d*	*a b c d*	*a b c d*

I \ D		$7.5\begin{pmatrix}0 & \varepsilon & 0\\ \varepsilon & 0 & \varepsilon\end{pmatrix}$				$7.6\begin{pmatrix}\varepsilon & \varepsilon & 0\\ \varepsilon & 0 & \varepsilon\end{pmatrix}$			
		No.	φ_{11}	φ_{22}	φ_{31}	No.	φ_{12}	φ_{21}	φ_{31}
group	2. 2	1 2	*a b* *b a*	*a b* *b a*	*a b* *b a*		*a b*	*a b*	*a b*
	3. 2	1 2	*a b c* *b c a*	*a b c* *c a b*	*a b c* *b c a*		*a b c*	*a b c*	*a b c*
	4. 4	1 2 3	*a b c d* *c d a b* *b c d a*	*a b c d* *c d a b* *d a b c*	*a b c d* *c d a b* *b c d a*		*a b c d*	*a b c d*	*a b c d*
	4. 5	1 2	*a b c d* *b a d c*	*a b c d* *b a d c*	*a b c d* *b a d c*		*a b c d*	*a b c d*	*a b c d*
non-group	4. 3	1 2	*a b a b* *a b a b*	*a b a b* *c d c d*	*a b a b* *a b a b*	1 2	*a b a b* *c d c d*	*a b a b* *a b a b*	*a b a b* *a b a b*
	4. 3′		*a b c d*	*a b c d*	*a b c d*		*a b c d*	*a b c d*	*a b c d*

Table 21

c. d.g. 2g—9_{ε}

I \ D	9. 6				9. 7				9. 8			
	φ_{11}	φ_{21}	φ_{31}	φ_{42}	φ_{11}	φ_{22}	φ_{32}	φ_{42}	φ_{11}	φ_{21}	φ_{32}	φ_{41}
2. 2	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*

I \ D	9. 9				9. 10				9. 11				
	φ_{11}	φ_{22}	φ_{32}	φ_{42}	φ_{12}	φ_{21}	φ_{31}	φ_{41}	No.	φ_{12}	φ_{22}	φ_{32}	φ_{41}
2. 2	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	1 2	*a b* *b a*	*a b* *b a*	*a b* *b a*	*a b* *b a*

I \ D	9. 12				
	No.	φ_{12}	φ_{21}	φ_{31}	φ_{41}
2. 2	1 2	*a b* *b a*	*a b* *a b*	*a b* *b a*	*a b* *b a*

Table 22 c.d.g. 2g—$9_{\varepsilon, \alpha}$

I \ D	9. 13			9. 14		
	No.	$\varphi_{\alpha 11}$	$\varphi_{\alpha 22}$	No.	$\varphi_{\alpha 12}$	$\varphi_{\alpha 21}$
2. 2	1 2	*a b* *b a*	*a b* *b a*	1 2	*a b* *b a*	*a b* *b a*

Table 23 c.d.g. (simp. or g.)—3_0—5

I_1		I_2 \ (S : I_1)	3_0—5. 3		3_0—5. 4	
			φ_{11}	φ_{22}	φ_{12}	φ_{21}
group	2. 2	{*a b a a*}	*a b*	*a b*	*a b*	*a b*
group	2. 2	{*a b a b*}			*b a*	*b a*
group	2. 2	{*a b b b*}	*a b*	*a b*	*a b*	*a b*
group	3. 2	{*a b c a a*}	*a b c*	*a b c*	*a b c*	*a b c*
group	3. 2	{*a b c a b*}			*c a b*	*b c a*
group	3. 2	{*a b c b b*}	*a b c*	*a b c*	*a b c*	*a b c*
group	3. 2	{*a b c b c*}			*c a b*	*b c a*
group	4. 4	{*a b c d a a*}	*a b c d*	*a b c d*	*a b c d*	*a b c d*
group	4. 4	{*a b c d b b*}	*a b c d*	*a b c d*	*a b c d*	*a b c d*
group	4. 4	{*a b c d c c*}	*a b c d*	*a b c d*	*a b c d*	*a b c d*
group	4. 4	{*a b c d a b*}			*d a b c*	*b c d a*
group	4. 4	{*a b c d a c*}			*c d a b*	*c d a b*
group	4. 4	{*a b c d b c*}			*d a b c*	*b c d a*
group	4. 4	{*a b c d b d*}			*c d a b*	*c d a b*
group	4. 5	{*a b c d a a*}	*a b c d*	*a b c d*	*a b c d*	*a b c d*
group	4. 5	{*a b c d b b*}	*a b c d*	*a b c d*	*a b c d*	*a b c d*
group	4. 5	{*a b c d a b*}			*b a d c*	*b a d c*
group	4. 5	{*a b c d c d*}			*b a d c*	*b a d c*
non-group	4. 3	{*a b c d a a*}	*a b a b*	*a b a b*	*a b a b*	*a b a b*
non-group	4. 3	{*a b c d a b*}	*b a b a*	*b a b a*	*b a b a*	*b a b a*
non-group	4. 3	{*a b c d a c*}	*c d c d*	*a b a b*	*a b a b*	*c d c d*
non-group	4. 3	{*a b c d a d*}	*d c d c*	*b a b a*	*b a b a*	*d c d c*
non-group	4. 3	{*a b c d b a*}	*b a b a*	*b a b a*		
non-group	4. 3	{*a b c d b b*}	*a b a b*	*a b a b*	*a b a b*	*a b a b*
non-group	4. 3	{*a b c d b c*}	*d c d c*	*b a b a*		
non-group	4. 3	{*a b c d b d*}	*c d c d*	*a b a b*	*a b a b*	*c d c d*

Table 24

c.d.g. 2g—3_0—7

I_1	I_2	(S : I_2)	7.3			7.4			7.5			7.6		
		No. of (3_0-7. i) $3 \leqq i \leqq 6$	φ_{11}	φ_{21}	φ_{32}	φ_{11}	φ_{22}	φ_{31}	φ_{11}	φ_{22}	φ_{31}	φ_{12}	φ_{21}	φ_{31}
2.2	{*a b a a*}	1	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*
		2 (i ≠ 6)	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*			
	{*a b a b*}	1							*b a*	*b a*	*b a*			
		2 (i ≠ 6)							*b a*	*b a*	*b a*			
	{*a b b a*}	1							*b a*	*b a*	*b a*			
		2 (i ≠ 6)							*b a*	*b a*	*b a*			
	{*a b b b*}	1	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*			
		2 (i ≠ 6)	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*

Table 25

c.d.g. 2_g—5 (simp. 0) —5

I_1	I_2 \ (S : I_1)	(5.3—5.3)			(5.3—5.4)		
		No.	φ_{11}	φ_{22}	No.	φ_{12}	φ_{21}
2.2	(2.2-5.3)	1	*a b*	*a b*	1	*a b*	*a b*
		2	*a b*	*a b*	2	*b a*	*b a*
		3	*a b*	*a b*	3	*a b*	*a b*
		4	*a b*	*a b*	4	*a b*	*a b*
		5	*a b*	*a b*	5	*a b*	*a b*
		6	*a b*	*a b*	6	*a b*	*a b*
		7	*a b*	*a b*	7	*a b*	*a b*
		8	*a b*	*a b*	8	*a b*	*a b*
		9	*a b*	*a b*			

I_1	I_2 \ (S : I_1)	(5.4—5.3)		(5.4—5.4) 1			(5.4—5.4) 2		(5.4—5.4) 3	
		φ_{11}	φ_{22}	No.	φ_{12}	φ_{21}	φ_{12}	φ_{21}	φ_{12}	φ_{21}
2.2	(2.2-5.4) 1	*a b*	*a b*	1	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*
				2	*b a*	*b a*				
	(2.2-5.4) 2	*a b*	*a b*		*b a*	*b a*	*a b*	*a b*	*b a*	*b a*

Table 26

c.d.g 2_g—5_0—5

I_2 \ (S : I_1)	(5_0—5.3) 1		(5_0—5.3) 2		(5_0—5.4)	
	φ_{11}	φ_{22}	φ_{11}	φ_{22}	φ_{12}	φ_{21}
{*a b a a a a*}	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*
{*a b a b b a*}					*b a*	*b a*
{*a b b b b b*}	*a b*	*a b*			*a b*	*a b*

Table 27

c.d.g. 2_g—3_0—3_0—5

I_3 \ (S : I_1)	(3_0—3_0—5.3) 1		(3_0—3_0—5.3) 2		(3_0—3_0—5.4) 1		(3_0—3_0—5.4) 2	
	φ_{11}	φ_{22}	φ_{11}	φ_{22}	φ_{12}	φ_{21}	φ_{12}	φ_{21}
{*a b a a a a*}	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*
{*a b a a b b*}	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*
{*a b a b a b*}					*a b*	*a b*	*b a*	*b a*
{*a b a b b a*}							*b a*	*b a*
{*a b b b b b*}	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*	*a b*

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