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FINITE SEMIGROUPS IN WHICH LAGRANGE'S THEOREM HOLDS

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In the theory of finite groups, it is familiar as Lagrange's theorem that the order of every subgroup of a group G is a factor of the order of G. We should like to study the structure of a semigroup with such a property. A finite semigroup S is said to have $\mathfrak{S}_1$-property if the order of any subsemigroup is a divisor of the order of S. On the other hand $\mathfrak{S}$-property is defined as follows.

If a semigroup S of order n contains no proper subsemigroup of order greater than $n/2$, then S is said to have $\mathfrak{S}$-property.

Immediately $\mathfrak{S}_1$-property implies $\mathfrak{S}$-property. A finite semigroup with $\mathfrak{S}$-property and one with $\mathfrak{S}_1$-property are called $\mathfrak{S}$-semigroup and $\mathfrak{S}_1$-semigroup respectively. In the present paper we shall determine the types of $\mathfrak{S}$-semigroups, and at last the result will make the reader see that $\mathfrak{S}$-property is equivalent to $\mathfrak{S}_1$-property. We add that any semigroup of order at most 2 have $\mathfrak{S}$-property and so this case will be sometimes out of consideration.

1. Notations.

If S is a finite simple semigroup, then S is represented as a regular matrix semigroup with a ground group G and a defining matrix $P=(p_{ji})$ of type (l, m). (See [1])

If $p_{ji} \neq 0$ for all i, j, then

either $\quad S=\{(x\,;\ i\ j) \mid x \in G,\ i=1, \ldots, m\,;\ j=1, \ldots, l\}$

or $\quad S=\{(x\,;\ i\ j) \mid x \in G,\ i=1, \ldots, m\,;\ j=1, \ldots, l\} \cup \{0\}$

the multiplication of which is

$$(x\,;\ i\ j)(y\,;\ s\ t)=(x p_{js}\, y\,;\ i\ t)$$
$$0^2=0\,(x\,;\ i\ j)=(x\,;\ i\ j)\,0=0 \qquad \text{if } S \text{ has } 0.$$

If there is $p_{ji}=0$, then

$$S=\{(x\,;\ i\ j) \mid x \in G,\ i=1, \ldots, m\,;\ j=1, \ldots, l\} \cup \{0\}$$

with multiplication

$$(x\,;\ i\ j)(y\,;\ s\ t)=\begin{cases}0 & \text{if } p_{js}=0 \\ (x p_{js}\, y\,;\ i\ t) & \text{if } p_{js} \neq 0.\end{cases}$$

Let $L=\{1, \ldots, m\}$, $R=\{1, \ldots, l\}$. R and L are regarded as a right-singular[1)] semigroup and a left-singular semigroup respectively.

1) R is called right-singular if $xy=y$ for every $x,\ y \in R$.

For the sake of convenience, we shall use the notations

$$Simp.(G;\ P) \text{ and } Simp.(G,\ 0;\ P)$$

which denote simple semigroups S with a defining matrix $P=(p_{ji})$ and a ground group G. The former denotes one without zero, whence $p_{ji}\neq 0$ for all i, j, bnt the latter denotes one with zero $\mathbf{0}$, so that if $p_{ji}\neq 0$ for all i and j, S contains no zero-divisor.

$A\times B$ denotes the direct product of two semigroups A and B.

2. Important Examples of $\mathfrak{S}$-Semigroups besides Groups.

Let G be any finite group a unit of which is denoted by e. The examples given in this paragraph will be proved to be $\mathfrak{S}_1$-semigroups and hence $\mathfrak{S}$-semigroups.

Lemma 1. *$Simp.(G;\ \binom{e}{e})$ is an $\mathfrak{S}$-semigroup, and any subsemigroup H is either a subgroup G' of G or $Simp.(G';\ \binom{e}{e})$ isomorphic to $G'\times R$ where $R=\{1,\ 2\}$ is right singular.*

Proof. Let $S=Simp.(G;\ \binom{e}{e})$ and let H be a proper subsemigroup of S. Putting $H_{1j}=\{(x;\ 1\ j)\mid (x;\ 1\ j)\in H\}$,
H has one of the forms ; H_{11}, H_{12}, $H_{11}\cup H_{12}$.[2)]
Further, set $\qquad G_{1j}=\{x\mid (x;\ 1\ j)\in H\}$.
Since H_{1j} is a subsemigroup of H, it is shown that $x\in G_{1j}$ and $y\in G_{1j}$ imply $xy\in G_{1j}$. Hence G_{1j} is a subgroup of G. If $H=H_{1j}$ $(j=1, 2)$, H is isomorphic to the subgroup G_{1j} of G. If $H=H_{11}\cup H_{12}$, then from $H_{11}H_{12}\subseteq H_{12}$ and $H_{12}H_{11}\subseteq H_{11}$, it follows that $x\in G_{11}$ and $y\in G_{12}$ imply $xy\in G_{12}$, $yx\in G_{11}$. Since G_{1j} is a group, we get $G_{11}\subseteq G_{12}$ and $G_{12}\subseteq G_{11}$, therefore $G_{11}=G_{12}$ which we denote by G'. Thus we have

$$H=\{(x;\ 1\ j)\mid x\in G',\ j=1,\ 2\}$$

that is, $$H=\text{Simp.}(G';\ \tbinom{e}{e})=G'\times R.$$

Letting g and g' be the orders of G and G' respectively, the order of S is $2g$ and H has the order g' or $2g'$, the factor of $2g$.

Similarly we have

Corollary 1. *$Simp.(G;\ (ee))$ is an $\mathfrak{S}$-semigroup, and any subsemigroup H is either a subgroup G' of G or $Simp.(G';\ (ee))$ isomorphic to $G'\times L$, where $L=\{1,\ 2]$ is left-singular.*

Remark. Let S_1 and S_2 be simple semigroups given by Lemma 1 and Corollary 1 respectively. When S_1 and S_2 have a ground group G in common, S_1 and S_2 are anti-isomorphic since G has always an anti-automorphism.

Lemma 2. *Let $0\neq a\in G$. $Simp.(G;\ \binom{ee}{ea})$ is an $\mathfrak{S}$-semigroup and any*

2) $\cup$ denotes the set union.

subsemigroup H is isomorphic to one of the following.

(α) *a subgroup G' of G,*

(β) Simp. $(G'\,;\ \binom{e}{e})$ *isomorphic to* $G' \times R$,

(γ) Simp. $(G'\,;\ (ee))$ *isomorphic to* $G' \times L$,

(δ) Simp. $(G'\,;\ \binom{ee}{ea})$.

Proof. Any subsemigroup H has one of the forms

(α') H_{ij} $(i,\ j=1,\ 2)$,

(β') $H_{11} \cup H_{12}$, $H_{21} \cup H_{22}$,

(γ') $H_{11} \cup H_{21}$, $H_{12} \cup H_{22}$,

(δ') $H_{11} \cup H_{12} \cup H_{21} \cup H_{22}$.

These are easily shown by considering all the subsemigroups of $R \times L$. Clearly G_{11}, G_{12}, G_{21} and $G_{22}a$ are subgroups of G, and H_{11}, H_{12}, H_{21} and H_{22} are isomorphic to G_{11}, G_{12}, G_{21}, and $G_{22}a$ respectively. Similarly as Lemma 1 and Corollary 1, we can prove :

If $H = H_{11} \cup H_{12}$, then $G_{11} = G_{12}\ (= G_1)$ and $H = Simp.\,(G_1\,;\ (e\ e))$,

if $H = H_{11} \cup H_{21}$, then $G_{11} = G_{21}\ (= G_2)$ and $H = Simp.\,(G_2\,;\ \binom{e}{e})$.

Let us discuss the other cases :

If $H = H_{21} \cup H_{22}$, we get $G_{21} = G_{22}a$ because $G_{21} \subseteq G_{22}a$ from $H_{21}H_{22} \subseteq H_{22}$, and $G_{22}a \subseteq G_{21}$ from $H_{22}\ H_{21} \subseteq H_{21}$; and hence

$$H = \{(x\,;\ 2\,1) \mid x \in G_{21}\} \cup \{(x\,;\ 2\,2) \mid x \in G_{21}\,a^{-1}\}.$$

It is proved that H is isomorphic to

$$Simp.\,(G_{21}\,;\ \binom{e}{e})$$

under the mapping f defined as

$$f((x\,;\ 2\,1)) = (x\,;\ 2\,1), \qquad f((x\,;\ 2\,2)) = (xa\,;\ 2\,2).$$

If $H = H_{12} \cup H_{22}$, then $G_{12} = aG_{22}$; and H is isomorphic to $Simp.\ (G_{12}\,;\ (ee))$ under the mapping g defined as

$$g((x\,;\ 1\,2)) = (x\,;\ 1\,2), \qquad g((x\,;\ 2\,2) = (ax\,;\ 2\,2).$$

Finally if $H = H_{11} \cup H_{12} \cup H_{21} \cup H_{22}$,

we get $\quad G_{11} = G_{12} = G_{21} = a\,G_{22} = G_{22}\,a\,(\text{put} = G')$.

Since G' is a subgroup of G, we can consider $(e\,;\ 1\,2)$, $(e\,;\ 2\,1) \in H$ and $(a\,;\ 1\,1) = (e\,;\ 1\,2)\ (e\,;\ 2\,1) \in H$ so that $a \in G'$ naturally $a^{-1} \in G'$. At last $G' = G_{11} = G_{12} = G_{21} = G_{22}$, which contains a. Therefore it follows that $H =$ Simp. $(G'\,;\ \binom{ee}{ea})$. Let g and g' be the orders of G and G' respectively. The order of S is $4g$ and the order H is g' or $2g'$ or $4g'$, the factor of $4g$.

3. General Case.

Lemma 3. *Let* $S = Simp.\,(G,\ 0\,;\ (p_{ji}),\ i = 1,\ \dots,\ m\,;\ j = 1,\ \dots,\ l)$ *of order* > 2. *S has no* $\mathfrak{S}$*-property.*

Proof. Let g be the order of G. Then the order of S is

$$n = glm + 1.$$

When $l = m = 1$, S is a group with zero adjoined; then $g > 1$ since we have assumed $n > 2$. But the order g of a proper subsemigroup G is not a divisor of $n = g + 1$.

When at least one of l and m is $\geqq 2$ e. g. $l \geqq 2$, there is a proper subsemigroup T of S

$$T = \{(x\ ;\ i\ j) \mid x \in G,\ \ i=1, \ldots, m\ ;\ \ j=1, \ldots, l-1\} \cup \{0\}$$[2]

whose order is $n' = gm\,(l-1)+1$. On the other hand, we see

$$2n' - n = gm\,(l-2) + 1 > 0$$

whence n' is not a divisor of n. Therefore S has no $\mathfrak{S}$-property if $n > 2$. q. e. d.

Lemma 4. *A finite non-simple semigroup has no $\mathfrak{S}$-property.*

Proof. Let I be a maximal ideal of a finite non-simple semigroup S. Then the difference semigroup $(S : I) = D$ is a simple semigroup with zero. Set $D = \{G,\ 0\ ;\ (p_{ji})\ i = 1, \ldots, m\ ;\ j = 1, \ldots, l\}$ and let g, i, d and n be the orders of G, I, D, and S respectively. Then $d = glm + 1$, $n = i + d - 1 = glm + i$. We may assume $n > 2$, $i > 1$.

First, if $lm = 1$ (i. e. $l = m = 1$), D is a group with zero adjoined, so that S is the union of I and a group G:

$$S = I \cup G, \quad I \cap G = \emptyset, \quad IG \subseteqq I, \quad GI \subseteqq I$$

where $n = i + g$. Then S contains a proper subsemigroup T of order $> n/2$. In fact

$T = I$ if $i > g$

$T = I \cup \{e\}$ where e is a unit of G if $i = g$

$T = G$ if $i < g$.

Second, if $lm > 1$, e. g. $l > 1$, then the proof of Lemma 3 shows that D contains a subsemigroup D' of order $d' = gm\,(l-1) + 1$. Let S' be the inverse image of D' under the homomorphism $S \rightarrow D$. Evidently S' is a proper subsemigroup of S such that $(S' : I) = D'$. Then the order n' of S' is greater than $n/2$, because, from $n' = gm\,(l-1) + i$

we get $$2n' - n = gm\,(l-2) + i > 0.$$

In all cases S has no $\mathfrak{S}$-property.

Lemma 5. *Let $S = Simp.\,(G\ ;\ (p_{ji})\ i = 1, \ldots, m\ ;\ j = 1, \ldots, l)$. If S is an $\mathfrak{S}$-semigroup, then S has one of the following structures:*

(1) *a finite group*

(2) $Simp.\,(G\ ;\ \binom{e}{e})$

(3) $Simp.\,(G\ ;\ (ee))$

(4) $Simp.\,(G\ ;\ \binom{ee}{ea})$, $a \neq 0$.

2) T is not always simple.

Proof. Let g be the order of the ground group G of S, then the order n of S is

$$n = glm.$$

Suppose that S has $\mathfrak{S}$-property and at least one of l and m is $\geqq 3$ e. g. $l \geqq 3$, and consider

$$T = \{(x\,;\ i\ j)\,|\,x \in G,\ i = 1,\ \dots,\ m\,;\ j = 1,\ \dots,\ l-1\}.$$

T is clealy a subsemigroup of S and its order n' is

$$n' = gm\,(l-1)$$

and

$$2n' - n = gm\,(l-2) > 0$$

whence n' is not a factor of n. This contradicts $\mathfrak{S}$-property of S. Therefore we must have the following four cases of S

(i) $l = m = 1,$

(ii) $l = 2, \quad m = 1,$

(iii) $l = 1, \quad m = 2,$

(iv) $l = 2, \quad m = 2.$

If (i), S is a group. According to Rees' theory [1] the defining matrix is equivalent to $\binom{e}{e}$ in the case of (ii), equivalent to $(e\ e)$ in the case of (iii), equivalent to $\binom{e\,e}{e\,a}$ in the case of (iv).

Combining Lemmas 3, 4, and 5 with Lemmas 1, 2 and Corollary 1, we have the following theorem.

Theorem *A finite semigroup S is an $\mathfrak{S}$-semigroup of order $\geqq 2$ if and only if S has one of the following structures :*

(0) *a semigroup of order* 2[3)]

(1) *a group of order* $\geqq 2$

(2) *Simp.* $(G\,;\ \binom{e}{e})$

(3) *Simp.* $(G\,;\ (e\ e))$

(4) *Simp.* $(G\,;\ \binom{e\,e}{e\,a})$

where e is a unit of G, $0 \neq a \in G$, and the order g of G is $\geqq 1$.

A subsemigroup T of S is called proper if it is neither S itself nor a subsemigroup composed of only an idempotent. As a special case of $\mathfrak{S}$-semigroups, we have

Corollary 2. *A finite semigroup which contains no proper subsemigroup is either a semigroup of order at most 2 or a cyclic group of prime order.*

Proof. Let S be a semigroup satisfying this condition. According to Theorem, if the order g of G is $\geqq 2$, the simple semigroups (2), (3), (4) contain

3) In detail, (0) is either

	0	a
0	0	0
a	0	0

or

	a	b
a	a	a
b	a	b.

Besides them, there are 3 types, which belong to (1) ~ 3).

proper subsemigroups e. g. *G*. Hence, apart from the case of order 2, *S* must be a group. The theory of groups teaches us that an $\mathfrak{S}$-group is a cyclic group of order prime.

Added Note Corollary 2 holds even if the condition "finite" is exclnded.

Bibliography

[1] D. Rees On semigroups, Proc. Cambridge Philos. Soc., 36, 387-400.