Journal of Gakugei, Tokushima University
Vol. X. 1959

NOTES ON GENERAL ANALYSIS (VIII)

By

Isae SHIMODA

(Received September 30, 1959)

In the preceding paper*), we proved the next theorem. *Let the family of functions* $\{f(x)\}$ *from* E_1 *to* E_2 *satisfy following conditions* : (1) *each function* $f(x)$ *is analytic in* $\|x\| < 1$ *in* E_1 *and is a one-to-one mapping to a domain* D_f *in* E_2 *and its inverse function* $f^{-1}(x)$ *is also analytic in* D_f, (2) $\{f(x)\}$ *are bounded, that is,* $\|f(x)\| \leq M$, (3) *the norms of linear parts* $\{g_1(x)\}$ *of* $\{f^{-1}(x)\}$ *are bounded, that is,* $\|g_1\| \leq K$, (4) $f(\theta) = \vartheta$, *then each domain* D_f *includes the sphere whose radius is constant.*

In this note, we discuss the case where E_1 is composed of complex numbers and E_2 is complex Banach spaces.

Lemma 1. *If* $x = f(\alpha)$ *is an* E_2*-valued function defined in the unit circle* $|\alpha| < 1$ *in complex plane and analytic there, then*

$$\|f'(\alpha)\|(1-|\alpha|^2) = \|F'(\beta)\|(1-|\beta|^2),$$

where $F(\beta) = f\left(\dfrac{\beta+\alpha_0}{1+\beta\bar{\alpha}_0}\right)$ *and* $\beta = \dfrac{\alpha-\alpha_0}{1-\alpha\bar{\alpha}_0}$.

Proof. Since $\beta = \dfrac{\alpha-\alpha_0}{1-\alpha\bar{\alpha}_0}$, we have $\dfrac{|d\alpha|}{1-|\alpha|^2} = \dfrac{|d\beta|}{1-|\beta|^2}$.

On the other hand, $F'(\beta) = f'(\alpha)\dfrac{d\alpha}{d\beta}$.

Then we have $\|F'(\beta)\| = \|f'(\alpha)\|\dfrac{|d\alpha|}{|d\beta|} = \|f'(\alpha)\|\dfrac{1-|\alpha|^2}{1-|\beta|^2}$, and we see that

$$\|F'(\beta)\|(1-|\beta|^2) = \|f'(\alpha)\|(1-|\alpha|^2).$$

Lemma 2. *If an* E_2*-valued function* $f(\alpha)$ *defined in* $|\alpha| < 1$ *in complex plane satisfies* $f(0) = \theta$, $\|f'(0)\| = 1$ *and is a one-to-one mapping to a domain* D *in* E_2 *and its inverse function* $f^{-1}(x)$ *is also analytic in* D, *then the norm of the linear part* $g_1(x)$ *of* $f^{-1}(x)$ *is* 1.

Proof. Since $f^{-1}(x)$ is analytic in D and $f(0) = \theta$, we have

$$f^{-1}(x) = \sum_1^\infty g_n(x),$$

where $g_n(x)$ is a complex valued homogeneous polynomial of degree n for $n = 1$, 2, 3, ... Let β be complex variables, then

$$\alpha = f^{-1}(\beta x) = \sum_1^\infty g_n(x)\beta^n.$$

Since $f^{-1}(\beta x)$ is one-to-one mapping and analytic in D, $\sum_1^\infty g_n(x)\beta^n$ converges uniformly for $|\beta| \leq r < 1$ at least and $g_1(x) \neq 0$. This shows that a neighbourhood U of $\beta = 0$ is mapped to a neighbourhood V of $\alpha = 0$. If E_2 is composed of at least two elements x, x', where x and x' are linearly independent, we have on the same way

$$\alpha = \sum_1^\infty g_n(x')\beta^n.$$

By this relation, we see that a neighbourhood U' of $\beta = 0$ is mapped to a neighbourhood V$'$ of $\alpha = 0$. That is, a set $\beta x'$, where $\beta \in U'$, is mapped to V'. Then the intersection $V \cdot V'$ is mapped to different sets Ux and $U'x'$ simultaneously. Ux is a set of βx, where $\beta \in U$, and $U'x'$ is a set of $\beta x'$, where $\beta \in U'$. This contradicts that $f(\alpha)$ is a one-to-one mapping, and we see that E_2 is composed of one element which is linearly independent.

On the other hand, we have

$$y = f(\alpha) = \sum_1^\infty a_n \alpha^n,$$

where $a_n \in E_2$, since $f(\alpha)$ is analytic in $|\alpha| < 1$. Then we have

$$\begin{aligned} \alpha = f^{-1}(x) &= \sum_1^\infty g_n(x) \\ &= \sum_1^\infty g_n\Big(\sum_1^\infty a_n \alpha^n\Big) \\ &= g_1(a_1)\alpha + \dots\dots \end{aligned}$$

Comparing the coefficients of α, we have $g_1(a_1) = 1$. Since $a_1 \in E_2$, E_1 is composed of points βa_1. Then, points on $\|x\| = 1$ are expressed as $a_1 e^{i\theta}$ (where $0 \leq \theta \leq 2\pi$), so we have

$$\|g_1\| = \sup_{0 \leqq \theta \leqq 2\pi} |g_1(a_1 e^{i\theta})| = 1.$$

This completes the proof.

Theorem. *If the family of functions $\{f(\alpha)\}$ from $|\alpha| < 1$ in complex plane to E_2 are analytic in $|\alpha| < 1$ and one-to-one mapping to $\{D_f\}$ in E_2 separately and satisfy $\{\|f'(o)\| = 1\}$ and their inverse functions $\{f^{-1}(x)\}$ are also analytic in $\{D_f\}$, then each domain D_f includes the sphere whose radius is constant.*

Proof. First of all, we assume that $f(\alpha)$ is analytic on $|\alpha| \leq 1$. Since $f'(\alpha)$ is analytic on $|\alpha| \leq 1$, $\|f'(\alpha)\| \cdot (1 - |\alpha|^2)$ is continuons on $|\alpha| \leq 1$. Then, there exists a point α_0 such that

$$\|f'(\alpha_0)\| \cdot (1 - |\alpha_0|^2) = \underset{|\alpha| \leqq 1}{Max.} \|f'(\alpha)\| \cdot (1 - |\alpha|^2).$$

When $|\alpha| = 1$, $\|f'(\alpha)\| (1 - |\alpha|^2) = 0$. Therefore, $|\alpha_0| < 1$.
Put $M = \|f'(\alpha_0)\| (1 - |\alpha_0|^2)$, then $M \geqq 1$, because, $\|f'(\alpha)\| (1 - |\alpha|^2) = \|f'(o)\| = 1$, when $\alpha = 0$. By the transformation $\beta = \dfrac{\alpha - \alpha_0}{1 - \alpha\bar{\alpha}_0}$, let $f(\alpha) = F(\beta)$.
By Lemma 1, we have $\|F'(\beta)\| (1 - |\beta|^2) = \|f'(\alpha)\| (1 - |\alpha|^2)$. Then

$$\|F'(o)\| = \|f'(\alpha_0)\| (1 - |\alpha_0|^2) = M.$$

Put $\varphi(\beta)=\frac{1}{M}(F(\beta)-F(o))$, we have $\varphi(o)=0$ and $\|\varphi'(o)\|=\frac{1}{M}\|F'(o)\|=\frac{M}{M}=1$. $\varphi(\beta)$ is also analytic in $|\beta|\leq 1$ and clearly one-to-one mapping to a domain D' in E_2, since $f(\alpha)$, $\frac{\alpha-\alpha_0}{1-\alpha\bar{\alpha}_0}$ and $\frac{1}{M}(x-x_0)$ are one-to-one mappings. From the relation $\varphi'(\beta)=\frac{1}{M}F'(\beta)$, we see that $\|\varphi'(\beta)\|=\frac{1}{M}\|F'(\beta)\|$ and then

$$\begin{aligned}\|\varphi'(\beta)\| &= \frac{1}{M}\cdot\frac{\|f'(\alpha)\|\cdot(1-|\alpha|^2)}{1-|\beta|^2}\\ &\leq \frac{M}{M(1-|\beta|^2)}\\ &= \frac{1}{1-|\beta|^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\varphi(\beta)\| &= \left\|\int_0^\beta \varphi'(\beta)\,d\beta\right\|\\ &\leq \int_0^\beta \|\varphi'(\beta)\|\cdot|d\beta|\\ &\leq \int_0^\beta \frac{1}{1-|\beta|^2}\,|d\beta|\\ &= \frac{1}{2}\log\frac{1+|\beta|}{1-|\beta|}.\end{aligned}$$

Put $|\beta|\leq R<1$, we have $\|\varphi(\beta)\|\leq\log\frac{1+R}{1-R}$. Put $\log\frac{1+R}{1-R}=K$ and $\varphi_1(\beta)=\frac{1}{R}\varphi(R\beta)$, then $\varphi_1(\beta)$ is analytic in $|\beta|\leq 1$ and one-to-one mapping to D'' and $\|\varphi_1'(o)\|=\frac{1}{R}\|R\varphi'(o)\|=\|\varphi'(o)\|=1$ and $\varphi_1(o)=\frac{1}{R}\varphi(o)=0$.

Therefore, we have $\|\varphi_1(\beta)\|\leqq K$, when $|\beta|\leqq 1$. Appealing to Lemma 2, we see that the norm of the linear part $g_1(x)$ of the inverse function $\varphi_1^{-1}(x)$ is 1, since $\varphi_1^{-1}(x)$ is also analytic, one-to-one mapping, $\varphi_1(o)=0$ and $\|\varphi_1'(o)\|=1$. Thus, by the theorem written at the biginning of this paper, we see that D'' includes the sphere whose radius is constant. Therefore, D includes also the sphere whose radius is constant. It is easy as well as the usual way that the assumption $|\alpha|\leqq 1$ is removed.

References

*). I. Shimoda : Notes on general analysis (VII), Journal of Gakugei, Tokushima University, Vol. IX 1958.