Journal of Gakugei, Tokushima University
Vol. X. 1959

SOME EXCEPTIONAL EXAMPLES TO STUDENT'S DISTRIBUTION

By

Yoshikatsu WATANABE

(Received March 30, 1959)

The famous Student's test in stochastics is only applicable so far the population distribution is normal. In the present note the author intends to show that, if the universe is not properly normal, e. g. it may be truncatedly normal[1)] or Laplace's truncated distribution, the Student-like ratio of the sample mean to the sample S. D. $t = \bar{x}/s$, distributes quite differently from the ordinary t-distribution.

§ **1.** *Some Preliminary Remarks on the Simplex.* Let some sample taken from an universe with a non-negative variable be $\{x_1, x_2, \ldots, x_n\}$ and its sample mean be

$$(1.1) \qquad \bar{x} = \sum_1^n x_i/n \ \ (= \text{determinate} > 0).$$

The space occupied by these sample-points forms a simplex of the $(n-1)$-th order, S_{n-1}, whose vertices $A_i (i = 1, 2, \ldots, n)$ have co-ordinates such as all $x_j = 0$ except only one $x_i = n\bar{x}$. Really every point P on any side A_iA_j has two positive co-ordinates $x_i = n\bar{x}/(1+\lambda)$, $x_j = \lambda n\bar{x}/(1+\lambda)$ with $\lambda > 0$ and $P \varepsilon S_{n-1}$, while for every point Q on produced parts of $A_i A_j (\lambda < 0)$ at least one of x_i, x_j becomes negative and $Q \bar{\varepsilon} S_{n-1}$. Hence the length of one side is $a = n\bar{x}\sqrt{2}$. In general, let $S_{m-1} (0 < m < n)$ be the simplex formed by all points whose non-negative co-ordinates $x_1, \ldots, x_m$ amount to $\sum_1^m x_i = n\bar{x}$, but $x_{m+1} = \ldots = x_n = 0$. If a vertex $A_j (j > m)$ be adjoined to S_{m-1}, the simplex thus obtained $S_m \subset S_{n-1}$. For, P being any point of S_{m-1}, the co-ordinates of any point on the join PA_j are $X_i = x_i/(1+\lambda) (i = 1, \ldots, m)$ and $X_j = \lambda n\bar{x}/(1+\lambda)$, but the remaining $X_k = 0$ $(k \neq i, j)$, so that $\sum_1^n X_i = n\bar{x}$. Moreover, X_i, X_j are non-negative so far $\lambda > 0$, while if $\lambda < 0$ at least one of X_i, X_j becomes negative. Hence all points of $S_m \varepsilon S_{n-1}$. Similarly $S_l \subset S_m$ if $l < m < n-1$. Consequently

$$(1.2) \qquad S_0 \,(\text{vertex}) \subset S_1 \,(\text{side}) \subset S_2 \,(\text{face}) \subset S_3 (\text{tetrahedron}) \subset S_4 \subset \ldots \subset S_{m-1} \subset \ldots \subset S_{n-1}.$$

Clearly all simplexes $S_1, S_2, \ldots, S_{n-1}$ are compact and convex. In fact, if $P_1(x_{11}, x_{12}, \ldots, x_{1n})$ and $P_2(x_{21}, \ldots, x_{2n})$ be any two boundary points of S_{n-1}, i. e. all of these co-ordinates be non-negative and $\sum x_{1i} = \sum x_{2i} = n\bar{x}$, but $x_{1j} = x_{2k} = 0$ $(j \neq k)$, $x_{2j} > 0$, $x_{1k} > 0$, then every point Q which lies on the join

$\overline{P_1P_2}$ also belongs to S_{n-1}, but every point Q' that is on the produced parts of $\overline{P_1P_2}$ cannot belong to S_{n-1}, because, the co-ordinates of Q' being $x'_i = \frac{x_{1i}+\lambda x_{2i}}{1+\lambda}$ with $\lambda < 0$, either $x'_j = \frac{\lambda x_{2j}}{1+\lambda}$ or $x'_k = \frac{x_{1k}}{1+\lambda}$ becomes necessarily negative.

Further G_{m-1} the centroid of S_{m-1} has as its co-ordinates $n-m$ zeros and m co-ordinates with each $n\bar{x}/m$. Specially for $G_{n-1} \equiv G$, the centroid of the whole S_{n-1}, it is $(\bar{x}, \bar{x}, \dots, \bar{x})$.

Now, on excluding one vertex, say A_1, we obtain a subsimplex S_{n-2} formed by all points whose co-ordinates are $x_1 = 0$ and $\sum_2^n x_i = n\bar{x}$ with non-negative x_2, $\dots$, x_n. To find the minimal distance from A_1 to S_{n-2}, we have to ask the relative minimum of the squared distance

$$y = (n\bar{x})^2 + \bar{x}_2^2 + \dots + x_n^2$$

under condition that $\sum_2^n x_i = n\bar{x}$, or making use of the undetermined multiplier λ, the absolute minimum of

$$z = y - 2\lambda(\sum_2^n x_i - n\bar{x}).$$

Hence, on putting $\frac{\partial z}{\partial x_i} = 2x_i - 2\lambda = 0\ (i=2, \dots, n)$, we obtain $\lambda = x_2 = \dots = x_n = \sum_2^n x_i/(n-1) = n\bar{x}/(n-1)$. Therefore the required point is $G_{n-2}(0, \lambda, \dots, \lambda)$, i. e. the centroid of S_{n-2}, and the minimal distance becomes

$$\sqrt{(n\bar{x})^2+(n-1)\lambda^2} = n\bar{x}\sqrt{n/(n-1)}.$$

Moreover, this line A_1G_{n-2} is really normal to S_{n-2}. For, if $P(0, x_2, \dots, x_n)$ be any point on S_{n-2}, we have

$$\overline{PG}_{n-2}^2 = \sum_2^n (x_i-\lambda)^2 = \sum_2^n x_i^2-(n-1)\lambda^2 \quad \text{and} \quad \overline{PA_1^2} = (n\bar{x})^2 + \sum_2^n x_i^2$$

while $\overline{A_1G_{n-2}^2} = (n\bar{x})^2+(n-1)\lambda^2$, so that $\overline{A_1G_{n-2}^2} + \overline{PG_{n-2}^2} = \overline{PA_1^2}$.

Thus A_1G_{n-2} being perpendicular to every line PG_{n-2} drawn through G_{n-2} in the base simplex S_{n-2}, it may be called the height of S_{n-1} against the base simplex S_{n-2} and its length is

$$h_{n-1} = n\bar{x}\sqrt{n/(n-1)}. \tag{1. 3}$$

Further, if the straight line $\overline{A_1G_{n-2}}$ be divided internally in the ratio $n-1$: 1, the point of division Q has the co-ordinates

$$x_1 = (1.\ n\bar{x}+0)/n = \bar{x}, \qquad x_i = [0+(n-1)\lambda]/n = \bar{x}(i = 2, \dots n),$$

and thus Q coincides with G. Consequently

$$\overline{GG_{n-2}} = h_{n-1}/n = \bar{x}\sqrt{n/(n-1)},$$

which should be the shostest distance between G and S_{n-2}, because GG_{n-2} is perpendicular to S_{n-2}. More generally, if we treat a subsimplex $S_{m-1}(m < n)$, we can prove that the shortest distance from G to S_{m-1} is the central join GG_{m-1}, which is normal to S_{m-1}, and

$$\overline{GG_{m-1}} = \bar{x}\sqrt{n(n-m)/m}, \tag{14}$$

where G and G_{m-1} are the centroids of S_{n-1} and S_{m-1} respectively. Thus the centroids of each in set (1. 2): S_{n-1}, S_{n-2}, ... , S_{m-1}, ... , $S_0(=A)$ are apart away from the whole centroid

$$(1.5)\quad 0,\ \bar{x}\sqrt{n/(n-1)},\ \bar{x}\sqrt{2n/(n-2)},\ \ldots\ldots,\ \bar{x}\sqrt{(n-m)n/m},\ \ldots,\ \bar{x}\sqrt{n(n-2)/2},\ \bar{x}\sqrt{n(n-1)},$$

respectively.

Lastly let us find the volume of the simplex S_{n-1}. If we join A_1 into all points of the base simplex S_{n-2} and divide all of these joins in a same fractional ratio $r:1$, all the resulting points form again a simplex S'_{n-2}, which is parallel to S_{n-2} and accordingly its measure is $r^{n-2}\ S_{n-2}$. The height $h=\overline{A_1G_{n-2}}$ being normal to S'_{n-2}, we get as an elementary volume $r^{n-2}S_{n-2}\,\varDelta\,h$. Hence the required volume shall be

$$S_{n-1} = \lim_{\varDelta h\to 0}\sum r^{n-2}S_{n-2}\,\varDelta h.$$

Making $r=m/N$, $0<m<N\to\infty$, $N\varDelta h=h$, we obtain

$$(1.6)\quad S_{n-1} = S_{n-2}\,h\lim_{\varDelta h\to 0}\frac{1}{N}\sum\left(\frac{m}{N}\right)^{n-2} = S_{n-2}h\int_0^1 r^{n-2}dr = S_{n-2}\,h_{n-1}/(n-1).$$

This formula holds equally good for $n-2$, $n-3$, ... up to 2. In fact, when $n=2$, it reduces to $S_1=S_0h_1$. But, (1. 3) renders, $h_1=n\bar{x}\sqrt{2}$, what can be translated as the height of a linear simplex S_1 with two vertices, because its length $n\bar{x}\sqrt{2}$ may be conveniently deemed as its height with one end point S_0 as base. On the other hand the zero-dimensional S_0 may be measured as $S_0=1^0=1$. Consequently $S_1=n\bar{x}\sqrt{2}=h_1S_0$ is still consistent.

Now, writing down the recurring formula thus obtained (1. 6) successively, we get

$$\begin{aligned} S_{n-1} &= S_{n-2}\,h_{n-1}/(n-1),\\ S_{n-2} &= S_{n-3}\,h_{n-2}/(n-2),\\ &\cdots\cdots\cdots\\ S_2 &= S_1\,h_2/2,\\ S_1 &= S_0\,h_1. \end{aligned}$$

Multiplying all these equations sides by sides, cancelling the same factors and applying (1. 3), we attain

$$S_{n-1} = h_{n-1}\,h_{n-2}\ldots\,h_2h_1/(n-1)! = (n\bar{x})^{n-1}\sqrt{n}/(n-1)! \tag{1.7}$$

Or, if the length of one side $a=\sqrt{2}n\bar{x}$ be substituted, we obtain

$$S_{n-1} = (a/\sqrt{2})^{n-1}\sqrt{n}/(n-1)! \tag{1.8}$$

e. g. $S_2=a^2\sqrt{3}/4$, $S_3=a^3/6\sqrt{2}$, which may be readily verified directly.

§ **2.** *The Distribution of the Sample Mean from a Positively Truncated Universe.* Let $f(x)$ be the frequency function of an universe U with a non-negative variable x, e. g. as a truncated Laplace distribution

(2. 1) $$f(x) = e^{-x/\sigma}/\sigma \qquad (x > 0)$$

whose mean and S. D. are both σ. Now take a random sample $\{x_1, \ldots, x_n\}$ from U, and form the sample mean and variance

(2. 2) $$\bar{x} = \sum_1^n x_i/n > 0,$$

(2. 3) $$s^2 = \sum_1^n (x_i - \bar{x})^2/n.$$

We are to discover the frequency function $f(\bar{x})$, the probability element being

(2. 4) $$dp = f(x_1) f(x_2) \ldots . f(x_n) \, dx_1 \ldots . dx_n.$$

Firstly we assume that the product $f(x_1) \ldots . f(x_n)$ reduces to some function of $\bar{x}$ alone, $g(\bar{x})$ say, what is the case for (2. 1). Ignoring s, therefore, we may only evaluate

$$dP = f(\bar{x}) \, d\bar{x} = g(\bar{x}) dV, \qquad dV = \int dx_1 \ldots dx_n,$$

where the integration is extended over all $\{x_1, \ldots, x_n\}$ satisfying (2. 2) only, and the volume element dV has, as its base S_{n-1}, and height $d(\sqrt{n}\,\bar{x})$. Really (2. 2) being a n-dimensional hyperplane H, it may be written as

(2. 5) $$\sum_1^n x_i/\sqrt{n} = \sqrt{n}\,\bar{x},$$

so its normal from origin has direction cosines $1/\sqrt{n}$, $1/\sqrt{n}$, $\ldots$, $1/\sqrt{n}$ and the normal length $\sqrt{n}\,\bar{x}$. Naturally this normal is also perpendicular to S_{n-1}, because $S_{n-1} \subset H$. Indeed, we see that the co-ordinates of $G(\bar{x}, \ldots, \bar{x})$ satisfy (2. 5) and on joining the co-ordinate origin O into G and any point $P(x_1, \ldots, x_n)$ on (2. 5), it follows that

$$\overline{OG}^2 + \overline{GP}^2 = n\bar{x}^2 + \sum_1^n (x_i - \bar{x})^2 = \sum_1^n x_i^2 = \overline{OP}^2.$$

Furthermore, since for a second hyperplane H': $\sum_1^n x_i / \sqrt{n} = \sqrt{n}(\bar{x} + d\bar{x})$, the same holds, so H and H' are parallel to each other, and the normal from origin is common in direction, only their lengths differ by $\sqrt{n}\,d\bar{x}$. Therefore dV has its base S_{n-1} and height $d(\sqrt{n}\,\bar{x})$. Accordingly we obtain

(2. 6) $$dV = S_{n-1}\, d(\sqrt{n}\,\bar{x}) = (n\bar{x})^{n-1}\, n d\bar{x}/(n-1)\,!$$

in view of (1. 7). Thus, e. g. if $f(x) = e^{-x/\sigma}/\sigma$, we have

(2. 7) $$dP = e^{-n\bar{x}}\left(\frac{n\bar{x}}{\sigma}\right)^{n-1} d\left(\frac{n\bar{x}}{\sigma}\right)/(n-1)\,!$$

and

(2. 8) $$f(\bar{x}) = e^{-n\bar{x}}(n\bar{x}/\sigma)^{n-1} n/\sigma(n-1)\,! \qquad (n = 1, 2, \ldots)$$

which is a gamma distribution and gives the frequency function of the sample mean $\bar{x}$. Consequently

$$E(\bar{x}^k) = \int_0^\infty \bar{x}^k f(\bar{x})\,d\bar{x} = \frac{\Gamma(n+k)}{\Gamma(n)}\left(\frac{\sigma}{n}\right)^k, \quad (k = 0, 1, 2, \ldots).$$

In particular, $E(\bar{x}) = \sigma$, $E(\bar{x}^2) = \dfrac{n+1}{n}\sigma^2$, $D^2(\bar{x}) = \sigma^2/n$, so that $\bar{x}$ is still an unbiased estimate of the population mean. For the normal population, we know that not only the sample mean $\bar{x}$ but also the sample median $\tilde{x}$ is an unbiased estimate of the parent mean m. However, now with the truncated Laplace distribution, this would not hold, e. g. for $n = 3$, we get

$$f(\tilde{x}) = \frac{3!}{\sigma} e^{-\tilde{x}/\sigma} \int_0^{\tilde{x}} e^{-\tilde{x}/\sigma} dx/\sigma \int_{\tilde{x}}^\infty e^{-x/\sigma} dx/\sigma, \qquad \int_0^\infty \tilde{x} f(\tilde{x})\,d\tilde{x} = \frac{5}{6}\sigma\,(\neq \sigma = m).$$

The critical lower and upper limits for a significant test would be found from

$$\int_0^{x_0} f(\bar{x})\,d\bar{x} = \alpha \quad \text{and} \quad \int_{x_1}^{\beta} f(\bar{x})\,d\bar{x} = \beta,$$

where α, β the levels of significance are e. g. 0. 05, 0. 025, 0. 01, 0. 005, &c. For the truncated Laplace distribution (2. 8) these limits may be found from Pearson's tables of the incomplete gamma function by

$$\frac{1}{\Gamma(n)}\int_0^{nx_0/\sigma} e^{-t}\, t^{n-1}\,dt = \alpha \quad \text{and} \quad \frac{1}{\Gamma(n)}\int_{nx_1/\sigma}^\infty e^{-t}\, t^{n-1}\,dt = \beta,$$

assumed the parent mean σ as known.

For a large sample, however, in virtue of the central limit theorem, the standardized variable $\xi = (\bar{x} - \sigma)\sqrt{n}/\sigma$ distributes asymtotically normally. Really, on substituting $\bar{x} = \sigma + \sigma\xi/\sqrt{n}$ in (2. 8) and approximating $\Gamma(n)$ by Stirling, it reduces to

$$f(\bar{x})\,d\bar{x} \cong \frac{1}{\sqrt{2\pi}} \exp\{-\xi^2/2\}\,d\xi \quad (-\sqrt{n} < \xi < \infty),$$

i. e. a truncated normal distribution, so that the usual normal test may be applied, at least on the upper side.

Again, let the population be a truncated normal distribution

$$f(x) = \sqrt{2/\pi\sigma^2}\,\exp\{-x^2/2\sigma^2\} \qquad (x > 0) \tag{2. 9}$$

with $E(x) = \sqrt{\dfrac{2}{\pi}}\,\sigma$ and $D^2(x) = \sigma^2\left(1 - \dfrac{2}{\pi}\right)$. Given a sample $\{x_1, \ldots, x_n\}$ from (2. 9) and formed (2. 2) and (2. 3), the probability element now becomes

$$dP = \left(\frac{1}{\sigma}\sqrt{\frac{2}{\pi}}\right)^n \exp\left\{-\frac{n}{2\sigma^2}(\bar{x}^2 + s^2)\right\} dV = f(\bar{x}, s)\,d\bar{x}\,ds, \tag{2. 10}$$

where dV denotes the measure of the aggregate of $\{x_1, \ldots, x_n\}$ satisfying both (2. 2) and (2. 3), and it contains possibly both of $\bar{x}$ and s against the foregoing, and it shall be discussed in the following sections.

§ **3.** *The Joint Distribution of the Sample Mean and Variance taken from a Positively Truncated Universe.* Let a random sample $\{x_1, \ldots, x_n\}$ be drawn from an universe with a non-negative variable x, and the sample mean and variance be (2. 2) and (2. 3). Now in the probability element $dp = f(x_1) \dots f(x_n)\, dx_1 \dots dx_n$ assuming that the product $f(x_1) \dots f(x_n)$ reduces to some function $g(\bar{x}, s)$ as is the case for (2. 1) or (2. 10) and consequently

(3. 1) $$dP = f(\bar{x}, s)\, d\bar{x}\, ds = g(\bar{x}, s)\, dV,$$

and it needs to find

(3. 2) $$dV = \int dx_1 \dots dx_n,$$

where the integration is to be extended over the aggregate of points $\{x_1, \dots, x_n\}$ which make mean (2. 2) between $\bar{x}$ and $\bar{x}+d\bar{x}$ and variance (2. 3) between s^2 and s^2+ds^2 (or approximately S. D. between s and $s+ds$). As already mentioned, dV has its height $\sqrt{n}\, d\bar{x}$. However, now its base is not the whole S_{n-1}, but only its portion whose points besides (2. 2) satisfy (2. 3). Now (2. 3) is a n-dimensional hypersphere K_n (as surface) of the radius $\sqrt{n}s$ with the center $G(\bar{x}, \bar{x}, \dots, \bar{x})$. Hence a boundary of dV is the intersection of S_{n-1} and K_n, which is a $(n-1)$-dimensional sphere K_{n-1} still with center G and radius $\sqrt{n}s$. The second sphere K'_{n-1} of radius $\sqrt{n}s + d\sqrt{n}s$ being concentric with K_{n-1}, the required base is a $(n-1)$-dimensional spherical shell with thickness $\sqrt{n}ds$. The volume of the $(n-1)$-dimensional sphere of radius r $(=\sqrt{n}s)$ being $v = (\sqrt{\pi}r)^{n-1} \Big/ \Gamma\left(\frac{n+1}{2}\right)$, that of the spherical shell is, as differential of v,

$$\sqrt{\pi}^{n-1}(n-1)r^{n-2}\,dr \Big/ \Gamma\left(\frac{n+1}{2}\right) = \sqrt{n\pi}^{n-1}(n-1)s^{n-2}ds \Big/ \Gamma\left(\frac{n+1}{2}\right).$$

This being multiplied by the height $\sqrt{n}\, d\bar{x}$, we obtain

(3. 3) $$dV = \frac{\sqrt{n}^n \sqrt{\pi}^{n-1}}{\Gamma((n+1)/2)}(n-1)\, s^{n-2}\, ds d\bar{x}.$$

The above is a simple imitation to Fisher's deduction in case when there is no limitation about $\bar{x}$. However, it will equally hold, if s be small compared to $\bar{x}$, i. e. if K_{n-1} lies wholly within S_{n-1}, or if the radius $\sqrt{n}s$ of K_{n-1} be smaller than the central distance $\overline{GG_{n-2}} = \sqrt{n}\bar{x}/\sqrt{n-1}$, that is, if

$$0 < s \leqq \bar{x}/\sqrt{n-1} \quad \text{or} \quad 0 < s/\bar{x} \equiv \tau \leqq 1/\sqrt{n-1}.$$

However, if $\tau > 1/\sqrt{n-1}$, so K_{n-1} protrudes partly outside of S_{n-1}, because then $GG_{n-2} < \sqrt{n}s$ and consequently GG_{n-2} produced to $\sqrt{n}s$, the points at end shall have negative co-ordinates. Therefore we must subtract these protruded parts. Indeed when s increases there occur several circumstances. If the radius of K_{n-1} be of magnitude between GG_{m-1} and GG_{m-2}, then by (1. 4), $\bar{x}\sqrt{n(n-m)/m} < \sqrt{n}s < \bar{x}\sqrt{n(n-m+1)/(m-1)}$ i. e. $\sqrt{(n-m)/m} < \tau < \sqrt{(n-m+1)/(m-1)}$, where $m = n, n-1, \dots, 2, 1$. Thus there are the following n subcases:

(3. 4) $$0 < \tau < \frac{1}{\sqrt{n-1}}, \quad \frac{1}{\sqrt{n-1}} < \tau < \sqrt{\frac{2}{n-2}}, \quad \dots, \quad \sqrt{\frac{n-2}{2}} < \tau < \sqrt{n-1} \quad \text{and}$$
$$\text{lastly } \sqrt{n-1} < \tau < \infty.$$

Of course, the last n-th case $\sqrt{n-1}\,\bar{x} < s < \infty$ means that the whole S_{n-1} lies within K_{n-1} and there is no point of intersection, that is no point of dV, so that we have nothing to consider. Among the remaining $n-1$ cases, the first case I : $0 < \tau < 1/\sqrt{n-1}$ was discusscd above. Next, if II : $1/\sqrt{n-1} < \tau < \sqrt{2/(n-2)}$, the protruded parts consist of n calottes (spherical segments) having no common portion. Hence, to solve this subcase, we have only to refer to the formula for the $(n-2)$-dimensional calotte :

$$C_{n-2} = \int_0^{r_0} \sec\gamma\, dK_{n-2}$$

where γ is the angle between the direction GG_{n-2} (i. e. AG_{n-2}) and GP drawn from G to any point P on the swelling spherical surface of K_{n-1}, while K_{n-2} denotes the volume of sphere with center G_{n-2} and radius $r = G_{n-2}P'$ where P' being the projection of P on S_{n-2}, $0 \leqq r \leqq r_0 = \sqrt{ns^2 - \overline{GG}^2_{n-2}} = \sqrt{n(s^2 - \bar{x}^2/(n-1))}$. Thus

$$C_{n-2} = \int_0^{r_0} \frac{\sqrt{n}s}{\sqrt{ns^2 - r^2}} \frac{\sqrt{\pi}^{n-2}}{\Gamma(n/2)} (n-2) r^{n-3}\, dr = \frac{2\sqrt{n\pi s^2}^{n-2}}{\Gamma(n/2-1)} \int_0^{\theta_0} \sin^{n-3}\theta d\theta, \tag{3. 5}$$

where $\theta_0 = \sin^{-1} r_0/\sqrt{n}s = \cos^{-1}\bar{x}/s\sqrt{n-1}$. Hence, e. g. $C_1 = 2\sqrt{3}\, s \cos \bar{x}/s\sqrt{2}$, $C_2 = 8\pi s^2(1 - \bar{x}/s\sqrt{3})$, which is the celebrated Archimedes' theorem, and $C_3 = 10\sqrt{5}\pi s^3\left(\cos^{-1}\frac{\bar{x}}{2s} - \frac{\bar{x}}{2s}\sqrt{1 - \frac{\bar{x}^2}{4s^2}}\right)$, which shall be verified in (3. 19) later on; also compare the reference cited at the end of this note[2].

Therefore, in the case II : $1/\sqrt{n-1} < \tau < \sqrt{2/(n-2)}$ we obtain dV by subtracting $n^2\, C_{n-2}\, ds d\bar{x}$ from (3. 3) :

$$dV = \frac{\sqrt{n}^n \sqrt{\pi}^{n-1}(n-1)s^{n-2}}{\Gamma((n+1)/2)} \left[1 - \frac{n\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2}-1\right)} \int_0^{\cos^{-1}\bar{x}/s\sqrt{n-1}} \sin^{n-3}\theta d\vartheta\right] ds d\bar{x}. \tag{3. 6}$$

To proceed similarly to the subcase III : $\sqrt{2/(n-2)} < \tau < \sqrt{3/(n-3)}$, &c., the matters become much more complex, now that the calottes have some common portions and further corrections are necessary. However, there is really a general method to find inductively the results for case $n = k+1$ with all its k subcases from those of case $n = k$ with $k-1$ subcases. But, this is preferably to be illustrated well by example. Therefore before to show it, we need to recapitulate each case $n = 2, 3, 4$ separately, as it would at the same time make the facts much more clear.

Case $n = 2$. In this trivial case, the linear simplex S_1 is a linear segment of length $2\sqrt{2}\bar{x}$ with centroid $G(\bar{x}, \bar{x})$, while the linear sphere K_1 consists of two points, either of which are $\sqrt{2}s$ apart from the center G (Fig. 1). Hence the elementar y volume becomes

$$dV = 2\, d(\sqrt{2}\bar{x})\, d(\sqrt{2}s) = 4 ds d\bar{x}, \tag{3. 7}$$

which agrees with what follows by putting $n = 2$ in (3. 3). The series (3. 4) reduces to only one adoptable subcase; $0 < s/\bar{x} < 1$.

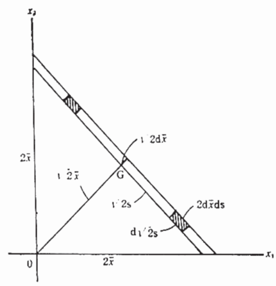

Fig. 1

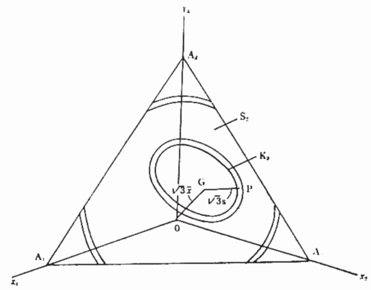

Fig. 2

Case $n=3$. In this case the simplex S_2 is an equilateral triangle of side $3\sqrt{2}\bar{x}$ and centroid $G(\bar{x}, \bar{x}, \bar{x})$, while K_2 is a circle of radius $\sqrt{3}s$ with center G. There are only two non-trivial subcases:

I: $0 < s/\bar{x} = \tau < 1\sqrt{2}$. Observing the concentric circles in Fig. 2 directly, or by (3. 3), we get

$$dV = 2\pi(\sqrt{3}s)\,d(\sqrt{3}s)\,d(\sqrt{3}\bar{x}) = 6\sqrt{3}\pi s\,dsd\bar{x} = 6\sqrt{3}\pi\bar{x}^2\,d\bar{x}\ \tau d\tau. \tag{3. 8}$$

II: $1\sqrt{2} < \tau < \sqrt{2}$. The circular arc of K_2 is cut into three pieces. On calculating the circular arc directly, or using (3. 6), we have

$$\begin{aligned} dV &= 18\sqrt{3}\,s\,(\pi/3-\cos^{-1}\bar{x}/s\sqrt{2})\,dsd\bar{x} \\ &= 18\sqrt{3}\bar{x}^2 d\bar{x}.\ \tau(\pi/3-\cos^{-1}1/\tau\sqrt{2})\,d\tau. \end{aligned} \tag{3. 9}$$

Case $n=4$. The simplex S_3 is a tetrahedron having as face the equilateral triangle with side $4\sqrt{2}\bar{x}$, centroid $G(\bar{x}, \bar{x}, \bar{x}, \bar{x})$, while K_3 is a sphere of radius $2s$, center G.

I: $0 < s/\bar{x} = \tau < 1/\sqrt{3}$. By (3. 3), or directly

$$dV = 4\pi(2s)^2\,2ds.\ 2d\bar{x} = 64\pi s^2\,dsd\bar{x} = 64\pi\bar{x}^3\,d\bar{x}.\ \tau^2\,d\tau. \tag{3. 10}$$

II: $1/\sqrt{3} < \tau < 1$. By (1. 4) $GG_2 = \bar{x}\sqrt{4/3}$, so that the height of the calotte C_2 swelled outside S_3 is $2s - 2\bar{x}/\sqrt{3}$. Its surface is after Archimedes $2\pi(2s)(2s-2\bar{x}/\sqrt{3}) = 8\pi s(s-\bar{x}/\sqrt{3})$ coinciding with (3. 5). There are 4 faces. Hence $128\pi s(s-\bar{x}/\sqrt{3})dsd\bar{x}$ being subtracted from (3. 8), or else by (3. 6), we obtain

$$dV = 64\pi s(2\bar{x}/\sqrt{3}-s)\,dsd\bar{x} = 64\pi\bar{x}^3 d\bar{x}.\ (2\tau/\sqrt{3}-\tau^2)\,d\tau. \tag{3. 11}$$

III: $1 < s/\bar{x} < \sqrt{3}$. Now the radius $2s$ of K_3 being between $GG_1 = 2\bar{x}$ and $GA = 2\sqrt{3}\bar{x}$ only some portions of the spherical surface S_3 contribute to integration. Let the tetrahedron S_3 be $ABCD$ with centroid G, height $AG_2 = 8\bar{x}/\sqrt{3}$ (Fig. 3). Taking conveniently G as origin, GA as ζ-axis and GE, a parallel to G_2G_1, as ξ-axis, complete $\xi\eta\zeta$-rectangular axes. Then the equation of the face ABC shall be expressed by $\xi/GE+\zeta/GA=1$. But $GE = \frac{3}{4}\,G_2G_1 = \sqrt{6}\bar{x}/2$, $GA = 2\sqrt{3}\bar{x}$, so that the equation becomes $\zeta = 2\sqrt{3}\bar{x} - 2\sqrt{2}\xi$, or in cylindrical co-ordinates $\zeta = 2\sqrt{3}\bar{x} - 2\sqrt{2}\rho\cos\theta$. The equation of K_3 is $\rho^2+\zeta_2 = 4s^2$. These two equations combined together, express their intersection curves

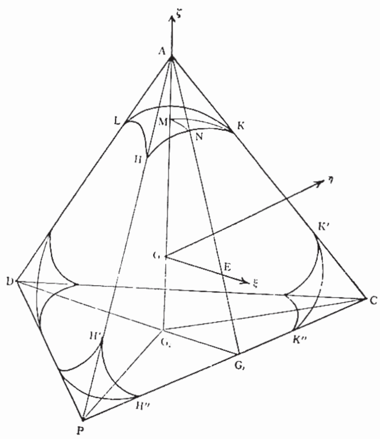

Fig. 3

HK, $H'K'$ &c. We are to evaluate $4 \times$ surface HKL or $24 \times$ surface MNK($=F$ say), where MN and MK are the intersections of the sphere K_3 with the planes AGE and AG_2C, respectively. But

$$F = F(\bar{x}, s) = \int_0^{\pi/3} \int_0^{\rho_1} \sqrt{1+\left(\frac{\partial \zeta}{\partial \rho}\right)^2} \rho d\rho d\theta,$$

where $\zeta = \sqrt{4s^2-\rho^2}$ and ρ_1 is found by solving $\rho^2+(2\sqrt{3}\bar{x}-2\sqrt{2}\rho \cos\theta)^2 = 4s^2$ to be

$$\rho_1 = 2\ [2\sqrt{6}\bar{x} \cos\theta - \sqrt{(1+8\cos^2\theta)s^2-3\bar{x}^2}] \,/\, (1+8\cos^2\theta),$$

the double signs $\pm$ being chosen to be negative, since the larger one corresponds to $H'H''K''K'$ in Fig. 3. Now the inner integral of the above double integral reduces to

$$\int_0^{\rho_1} \sqrt{1+\rho^2/(4s^2-\rho^2)}\, \rho d\rho = 2s \int_0^{\rho_1} \rho d\rho \,/\, \sqrt{4s^2-\rho^2} = 4s^2 - 2s\sqrt{4s^2-\rho_1^2},$$

which integrated with respect to θ, yields

$$F = \frac{4\pi s^2}{3} - 4s \int_0^{\pi/3} \frac{\sqrt{s^2(1+8\cos^2\theta)^2 - \{2\sqrt{6}\bar{x}\cos\theta - \sqrt{(1+8\cos^2\theta)s^2-3\bar{x}^2}\}^2}}{1+8\cos^2\theta}\, d\theta$$

$$= \frac{4\pi s^2}{3} - 4s \int_0^{\pi/3} \frac{\sqrt{3}\bar{x} + \sqrt{8}\cos\theta\sqrt{(1+8\cos^2\theta)s^2-3\bar{x}^2}}{1+8\cos^2\theta}\, d\theta.$$

Performing the integration, we attain finally

$$F = \frac{2\pi s}{3}\left(\frac{2\bar{x}}{\sqrt{3}} - s\right) - \frac{4s\bar{x}}{\sqrt{3}} \tan^{-1}\sqrt{\frac{3(s^2-\bar{x}^2)}{2\bar{x}^2}} + 4s^2 \tan^{-1}\sqrt{\frac{s^2-\bar{x}^2}{2s^2}}\,.$$

Hence

(3.12) $$dV = 24F(\bar{x}, s) 2ds\, 2d\bar{x} = 96F(\bar{x}, \bar{x}\tau)\, \bar{x} d\tau d\bar{x} \quad (s = \bar{x}\tau)$$

$$= 128\bar{x}^3 d\bar{x} \left\{\frac{\pi}{2}\left(\frac{2}{\sqrt{3}}\tau - \tau^2\right) - \sqrt{3}\tau \tan^{-1}\sqrt{\frac{3}{2}(\tau^2-1)} + 3\tau^2 \tan^{-1}\sqrt{\frac{1}{2}\left(1-\frac{1}{\tau^2}\right)}\right\} d\tau.$$

So far we have discussed the problem geometrically. However, even when $n=5$, S_4 and K_4 being four dimensional, the matter becomes less intuitional, and mucn less for cases $n>5$. Hence, we are oblidged to proceed analytically below.

In Case $n=5$, there are 4 subcases; I : $0<\tau<1/2$, II : $1/2<\tau<\sqrt{2/3}$, III $\sqrt{2/3}<\tau<\sqrt{3/2}$, IV : $\sqrt{3/2}<\tau<2$. For subcase I we get immediately by (3. 3)

$$dV = 50\sqrt{5}\pi^2 s^3\, ds d\bar{x}, \tag{3. 13}$$

and for subcase II (3. 6) may be employed.

However, in order to explain the general method before mentiened methodologcially, let us treat this subcase II by the very general method, that can be quite similarly applied to any $n=k+1$, if the case $n=k$ were already solved.

Putting one variable x_1 aside for a while, the remaining four variables x_2, x_3, x_4, x_5 form, as their mean and variance,

$$\bar{x}' = \sum_2^5 x_i/4, \qquad s'^2 = \sum_2^5 (x^i - \bar{x}')^2/4$$

with 3 subcases : I$'$: $0<s'/\bar{x}'=\tau'<1/\sqrt{3}$, II$'$: $1/\sqrt{3}<\tau'<1$, III$'$: $1<\tau'<\sqrt{3}$. It is easy to show that there exist the relations

$$\bar{x}' = \frac{1}{4}(5\bar{x}-x_1), \qquad s'^2 = \frac{5}{16}\left[4s^2-(x_1-\bar{x})^2\right], \tag{3. 14}$$

where $\bar{x}'$, s' being real and non-negative, and

$$0<x_1<5\bar{x}, \qquad \bar{x}-2s<x_1<\bar{x}+2s(=\gamma). \tag{3. 15}$$

Given $\bar{x}$ and s so $s/\bar{x}=\tau$ also, we wish to grasp how the variable x_1 runs its course. For this purpose, we draw the graph of

$$\tau' = \frac{s'}{\bar{x}'} = \frac{\sqrt{5(4s^2-(x_1-\bar{x})^2)}}{5\bar{x}-x_1} \tag{3. 16}$$

for several values of $\tau(=s/\bar{x})$ (Fig. 4). We need not consider those points outside

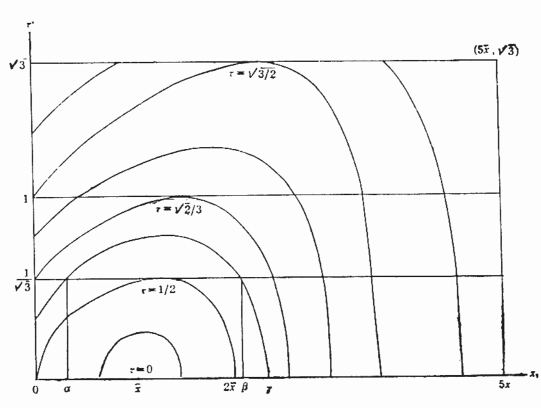

Fig. 4

the rectangle with sides $5\bar{x}$ and $\sqrt{3}$, because of (3. 15) and $0<\tau'<\sqrt{3}$. The curve (3. 16) has, as its τ'- and x_1-intercept, $\tau'_0=\sqrt{(4\tau^2-1)/5}$ and $\gamma=\bar{x}+2s=\bar{x}(1+2\tau)$, while its maximum arises at the point $((1+\tau^2)\bar{x},\ \sqrt{5}\tau/\sqrt{4-\tau^2})$. Hence, we obtain the following table:

τ	0	1/2	$\sqrt{2/3}$	$\sqrt{3/2}$	2
τ'-intercept $=\tau'_0$	imag.	0	$1/\sqrt{3}$	1	$\sqrt{3}$
x_1-intercept $=\gamma$	$\bar{x}$	$2\bar{x}$	$2.63\cdots\bar{x}$	$3.45\cdots\bar{x}$	$5\bar{x}$
mode m_0	$\bar{x}$	$5\bar{x}/4$	$5\bar{x}/3$	$5\bar{x}/2$	$5\bar{x}$
maximum τ_m	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞

In particular, if $\tau=0$, the τ'-curve degenerates to a single point $x_1=\bar{x}$, and it grows up larger and larger as τ increases up to $\tau=2$, in which case, however, the τ'-curve becomes needless.

Now, if e. g. $1/2<\tau<\sqrt{2/3}$, the corresponding τ'-curve lies actually between those corresponding to $\tau=1/2$ and $\tau=\sqrt{2/3}$. It starts from a point $(0,\ \tau'_0)$, such as $0<\tau'_0<1/\sqrt{3}$ and first ascending to a maximum, that lies between two parallels $\tau'=1/\sqrt{3}$, $\tau'=1$ and then descends up to $(\gamma,\ 0)$. Its points of intersection with the parallel $\tau'=1/\sqrt{3}$ is found by solving the equation (3. 16) for $\tau'=1/\sqrt{3}$ to be

$$\alpha,\ \beta=\frac{1}{4}\left[5x\mp\sqrt{15\,(4s^2-\bar{x}^2)}\right]. \tag{3. 17}$$

Hence, the elementary volume in the $\bar{x}'$-s' distribution is given by (3. 10) and (3. 11) as

$$dV'_1=64\pi s'^2 d\,s'd\bar{x}' \quad \text{for} \quad 0<x_1<\alpha \ \text{ as well as } \ \beta<x_1<\gamma,$$

and

$$dV'_2=64\pi s'\left(\frac{2\bar{x}'}{\sqrt{3}}-s'\right)ds'd\bar{x}' \quad \text{for} \quad \alpha<x_1<\beta.$$

Transforming the variables $\bar{x}'$, s' into $\bar{x}$, s by (3. 14) with the Jacobian

$$J=\frac{\partial(\bar{x}',\ s')}{\partial(\bar{x},\ s)}=\frac{25}{16}\frac{s}{s'},$$

and integrating about x_1 in the above described interals, we get

$$dV_1=\int_{x_1} dV'_1\,dx_1=100\pi\,s\,ds\,d\bar{x}\int_{x_1} s'dx_1,$$

where the integrsals are really $\int_0^\alpha+\int_\beta^\gamma$, as well as

$$dV_2=\int_{x_1} dV'_2dx_1=100\pi\ s\,ds\,d\bar{x}\int_\alpha^\beta(2\bar{x}'/\sqrt{3}-s')\,dx_1.$$

Upon substituting (3. 14) in these integrals and integrating, we attain finally

(3. 18) $$dV = dV_1 + dV_2 = 250\sqrt{5}\pi s\left[\frac{\bar{x}}{4}\sqrt{4s^2-\bar{x}^2}+\left(\frac{\pi}{5} - \cos^{-1}\frac{\bar{x}}{2s}\right)s^2\right] ds d\bar{x}$$

for the subcase II: $1/2 < s/\bar{x} < \sqrt{2/3}$, and this coincides with what follows from (3. 6).

Remark. If we subtract (3. 18) from (3. 13) which in II may comprise five 3-dimensional calottes, where some x_i's become certainly negative but still $\sum_1^5 x_i = 5\bar{x}$ and $s^2 = \sum_1^5 (x_i-\bar{x})^2/5$ hold, we obtain the superfluous volume

$$250\sqrt{5}\pi s[\cos^{-1}\bar{x}/2s - \bar{x}\sqrt{4s^2-\bar{x}^2}/4] ds d\bar{x}.$$

Therefore, if this be divided by $5d\sqrt{5}sd\sqrt{5}\bar{x}$, we shall get, as the volume of the 3-dimentional calotte

(3. 19) $$C_3 = 10\sqrt{5}\pi s\ [s^2\cos^{-1}\bar{x}/2s - \bar{x}\sqrt{4s^2-\bar{x}^2}/4],$$

which precisely coincides with (3. 5) for $n-2=3$, and thus the very formula is not a result of mere formal extension, but has an actual conformity.

§ **4.** *Truncated Laplace Distribution.* By the results in the foregoing section it is possible to write down the volume element:

(4. 1) $$dV = dV_{n-1}(\bar{x}, s) = g_{n-1}(\bar{x})d\bar{x}.\ h_{n-1}(\tau)d\tau \quad \text{with } \tau = s/\bar{x},$$

where the coefficients are factorized so as to hold for every n the identity (4. 6) below holds. Therefore, if the universe be e. g. a truncated Laplace distribution $f(x) = e^{-x}(x>0)$, the probability element for the joint distribution of the n-sized sample mean $\bar{x}$ and S. D. s shall be given by

(4. 2) $$dP = f_{n-1}(\bar{x}, s)d\bar{x}ds = e^{-n\bar{x}}g_{n-1}(\bar{x})d\bar{x}.\ h_{n-1}(\tau)d\tau,$$

so that $\bar{x}$ and τ are independent. Really for $n=2, 3, 4,$ we obtain the following:

(4. 3) $$f_1(\bar{x}, s)\, d\bar{x}ds = 4e^{-3\bar{x}}\bar{x}\cdot d\tau$$

(4. 4) $$f_2(x, s)dxds = \frac{27}{2}e^{-3x}x^2dx\cdot\frac{4\pi}{3\sqrt{3}}\tau d\tau \text{ or } \frac{27}{2}e^{-3x}\ \bar{x}^2dx$$

$$\frac{4}{\sqrt{3}}\left(\frac{\pi}{3} - \cos^{-1}\frac{1}{\tau\sqrt{2}}\right)\tau d\tau\ (0<\tau<1/\sqrt{2} \text{ or } 1/\sqrt{2}<\tau<\sqrt{2});$$

(4. 5) $$f_3(\bar{x}, s)d\bar{x}ds = \frac{128}{3}e^{-4\bar{x}}\bar{x}^3\,d\bar{x},\ \frac{3\pi}{2}\tau^2d\tau \text{ or } \frac{128}{3}e^{-4\bar{x}}\bar{x}^3\,d\bar{x},\ \frac{3\pi}{2}\left(\frac{2\tau}{\sqrt{3}} - \tau^2\right)d\tau$$

$$(0<\tau<1/\sqrt{3} \text{ or } 1/\sqrt{3}<\tau<1), \text{ or } \frac{128}{3}e^{-4\bar{x}}\bar{x}^3\,d\bar{x},\ \frac{3\pi}{2}\left(\frac{2\tau}{\sqrt{3}} - \tau^2\right) - 3\sqrt{3}\tau$$

$$\tan^{-1}\sqrt{\frac{3}{2}(\tau^2-1)} + 9\tau^2\tan^{-1}\sqrt{\frac{1}{2}\left(1-\frac{1}{\tau^2}\right)}\quad (1<\tau<\sqrt{3}).$$

And we have

(4. 6) $$\int_0^\infty e^{-n\bar{x}}g_{n-1}(\bar{x})d\bar{x} = 1, \quad \text{where} \quad g_{n-1}(\bar{x}) = \frac{n^n}{\Gamma(n)}\bar{x}^{n-1},$$

as well as $\displaystyle\int_0^{\sqrt{n-1}} h_{n-1}(\tau)d\tau = 1,$ where

$h_1(\tau) = 1 \quad \text{in} \quad 0<\tau<1;$

$h_2(\tau)= \frac{4\pi}{3\sqrt{3}}\tau \quad \text{in} \quad 0<\tau<1/\sqrt{2}, \quad \text{but} \ \frac{4}{\sqrt{3}}\tau\left(\frac{\pi}{3}- \cos^{-1} 1/\tau\sqrt{2}\right) \ \text{in} \ 1/\sqrt{2}<\tau<\sqrt{2};$

$h_3(\tau)= \frac{3\pi}{2}\tau^2 \quad \text{in} \quad 0<\tau<1/\sqrt{3} \quad \text{and} \ \frac{3\pi}{2}\left(\frac{2\tau}{\sqrt{3}} -\tau^2\right) \ \text{in} \ 1/\sqrt{3}<\tau<1, \text{ and lastly}$

$$\frac{3\pi}{2}\left(\frac{2\tau}{\sqrt{3}} -\tau^2\right)-3\sqrt{3}\tau \tan^{-1}\sqrt{\frac{3}{2}(\tau^2-1)} + 9\tau^2 \tan^{-1}\sqrt{\frac{\tau^2-1}{2\tau^2}} \ \text{in} \ 1<\tau<\sqrt{3}, \text{\&c.}$$

Consequently we may utilize the quantity $\tau=s/\bar{x}$ for testing its significance, whether the universe is really $f(x)=e^{-x}(x>0)$ or not. The lower limit τ_0 of significance level $\alpha(=0.01$ or 0.05, &c.) is found by use of (3. 3) from

(4. 7) $$\int_0^{\tau_0} h_{n-1}(\tau)\, d\tau = \frac{\Gamma(n)}{\Gamma((n+1)/2)}\sqrt{\frac{\pi}{n}}^{\,n-1} \frac{\tau_0^{n-1}}{\sqrt{n}} = \alpha$$

to be

$$\tau_0 = \frac{n}{\pi}\left[\frac{\alpha\sqrt{n}\ \Gamma((n+1)/2}{\Gamma(n)}\right]^{1/(n-1)}.$$

If however this value exceeds $1/\sqrt{n-1}$, we must refer to the second subinterval II, &c. Thus, e. g. for $n=4$, we shall get

(4. 8) $\tau_0 = 0.1853$ or 0.4144 according as $\alpha=0.01$ or 0.05.

As to the upper limit τ_1, we have to find it from

(4. 9) $$\int_{\tau_1}^{\sqrt{n-1}} h_{n-1}(\tau)\, d\tau = \alpha,$$

what is a pretty intricate. We shall obtain for case $n=4$ by means of Newton's successive approximation

(4. 10) $\tau_1 = 1.4212,$ or 1.2513 for $\alpha=0.01$ or 0.05.

However, the classical Student's ratio being in fact

$$t=\frac{\bar{x}-m}{s}\sqrt{n-1}, \quad \text{or} \quad t=\frac{\bar{x}}{s}\sqrt{n-1} \quad \text{if} \quad m=0,$$

our $\tau=s/\bar{x}$ is the reciprocal of t multiplied by $\sqrt{n-1}$

(4. 11) $\tau=\sqrt{n-1}/t,$ i. e. $t=\sqrt{n-1}/\tau=\sqrt{n-1}\,\bar{x}/s.$

Hence, the previous $h_{n-1}(\tau)$ if expressed by t becomes $f_{n-1}(t)$ and in details:

(4. 12) $f_1(t) = t^{-2} \qquad (1<t<\infty);$

(4. 13) $f_2(t) = \frac{8\pi}{3\sqrt{3}}t^{-3}$ or $\frac{8}{\sqrt{3}}\left(\frac{\pi}{3} -\cos^{-1}\frac{t}{2}\right)t^{-3} \qquad (2<t<\infty,$ or $1<t<2);$

(4. 14) $f_3(t) = \frac{9\sqrt{3}\pi}{2}t^{-4}$ or $9\sqrt{3}\pi\,(t/3-1/2)t^{-4} \qquad (3<t<\infty,$ or $\sqrt{3}<t<3),$

$$\text{or} \quad 9\sqrt{3}\left[\pi\left(\frac{t}{3}-\frac{1}{2}\right)-t\tan^{-1}\sqrt{\frac{3}{2}\left(\frac{3}{t^2}-1\right)}+3\tan^{-1}\sqrt{\frac{1}{2}\left(1-\frac{t^2}{3}\right)}\right]t^{-4}$$

$(1<t<\sqrt{3});$

and generally

$$\int_0^\infty f_{n-1}(t)\,dt = \int_1^\infty f_{n-1}(t)\,dt = 1.$$

For the sake of comparison, if the figures of (4. 8), (4. 10) be expressed in t by (4. 11), we obtain as the upper limits 9. 347, 4. 180 and as the lower limits, 1.219, 1.384 corresponding to $\alpha = 0.01$, 0.05 respectively, while the Student's Table for $n=4$ delivers ± 4.541 and ± 2.353 by reason of the symmetry.

We have argued on such a truncated Laplacian population as: A whole Laplace distribution $f(X) = \frac{1}{2\sigma}\exp\{-|X-m|/\sigma\}$ is truncated into half at $X=m$, only the part $X>m$ adopted, and the factor 1/2 removed in order to make the resulting expression furnish a frequency function, and finally the variable X transformed into x by $(X-m)/\sigma = x$, so that $f(x) = e^{-x}(x>0)$ holds. Hence, if the original distribution be regarded, of course, the Student-like ratio $\frac{X-m}{s}\sqrt{n-1}$ should be consulted.

§ **5.** *Truncated Normal Distribution as Universe.* Lastly we shall consider the sample distribution in case that the universe is, as in (2.9),

$$f(x) = \sqrt{\frac{2}{\pi}}\exp\left\{-\frac{x^2}{2}\right\} \quad (x>0). \tag{5.1}$$

The probability element which yields the n-sized sample mean $\bar{x}$ and S. D. s so that $\tau = s/\bar{x}$ is

$$dP = \sqrt{\frac{2}{\pi}}^{\,n}\exp\left\{-\frac{n}{2}\bar{x}^2(1+\tau^2)\right\} g_{n-1}(\bar{x})\,h_{n-1}(\tau)\,d\bar{x}d\tau,$$

where $g_{n-1}(\bar{x})$ and $h_{n-1}(\tau)$ are those given in (4. 6). Therefore, this time, $\bar{x}$ and τ are by no means independent, However, on considering τ as fixed, and integrating about $\bar{x}$, we obtain

$$\int_0^\infty \exp\left\{\frac{n\bar{x}^2}{2}(1+\tau^2)\right\} g_{n-1}(\bar{x})\,d\bar{x} = \frac{1}{2}\,\frac{\Gamma(n/2)}{\Gamma(n)}\left(\frac{2n}{1+\tau^2}\right)^{n/2},$$

so that the frequency function of τ shall be given by

$$\psi_{n-1}(\tau) = \frac{1}{2}\,\frac{\Gamma(n/2)}{\Gamma(n)}\sqrt{\frac{4n}{\pi}}^{\,n}\frac{h_{n-1}(\tau)}{(1+\tau^2)^{n/2}}. \tag{5.2}$$

More in details:

$$\psi_1(\tau) = 4/\pi(1+\tau^2) \qquad (0<\tau<1); \tag{5.3}$$

$$\psi_2(\tau) = 4\tau(1+\tau^2)^{-3/2} \text{ or } 4\tau(1+\tau^2)^{-3/2}\left(1-\frac{3}{\pi}\cos^{-1}1/\tau\sqrt{2}\right) \qquad (0<\tau<1/\sqrt{2} \text{ or } 1/\sqrt{2}<\tau<\sqrt{2}); \tag{5.4}$$

$$\begin{aligned}\psi_3(\tau) &= \frac{32\tau^2}{\pi}(1+\tau^2)^{-2} \text{ or } \frac{32}{\pi}\left(\frac{2\tau}{\sqrt{3}}-\tau^2\right)(1+\tau^2)^{-2} \qquad (0<\tau<1/\sqrt{3} \text{ or } 1/\sqrt{3}<\tau<1),\\ &= \frac{32}{\pi^2}(1+\tau^2)^{-2}\left[\pi\left(\frac{2}{\sqrt{3}}-\tau\right)\tau - 2\sqrt{3}\tau\tan^{-1}\sqrt{\frac{3}{2}(\tau^2-1)}\right.\\ &\qquad\left. + 6\tau^2\tan^{-1}\sqrt{\frac{1}{2}(1-1/\tau^2)}\,\right] \qquad (1<\tau<\sqrt{3})^{3)}.\end{aligned} \tag{5.5}$$

Or, if $\tau = \sqrt{n-1}/t$ be adopted, we shall get, as the Student-like distribution,

(5. 6) $f_1(t) = 4/\pi(1+t^2)$ $(1<t<\infty)$;

(5. 7) $f_2(t) = 2\sqrt{2}\,(1+t^2/2)^{-3/2}$ or $2\sqrt{2}\left(1-\frac{3}{\pi}\cos^{-1}\frac{t}{2}\right)\left(1+\frac{t^2}{2}\right)^{-3/2}$

$(2<t<\infty$ or $1<t<2)$;

(5. 8) $f_3(t) = \frac{32}{\pi\sqrt{3}}(1+t^2/3)^{-2}$ or $\frac{32}{\pi\sqrt{3}}\left(\frac{2}{3}t-1\right)(1+t^2/3)^{-2}$

$(3<t<\infty$ or $\sqrt{3}<t<3)$,

$$= \frac{32}{\pi^2\sqrt{3}}(1+t^2/3)^{-2}\left[\pi\left(\frac{2}{3}t-1\right)-2t\tan^{-1}\sqrt{(3-t^2)/2t^2}\right.$$
$$\left.+6\tan^{-1}\sqrt{(3-t^2)/6}\right] \qquad (1<t<\sqrt{3});$$

whereas the classical Student's distributions deliver

$$s_1(t) = \frac{1}{\pi}(1+t^2)^{-1}, \quad s_2(t) = \frac{1}{2\sqrt{2}}(1+t^2)^{-3/2}, \quad s_3(t) = \frac{2}{\sqrt{3}\pi}(1+t^2/3)^{-2},$$

and in general

$$s_{n-1}(t) = \frac{1}{\sqrt{(n-1)\pi}}\frac{\Gamma(n/2)}{\Gamma((n-1)/2)}\left(1+\frac{t^2}{n-1}\right)^{-n/2}, \qquad (-\infty<t<\infty).$$

The significant upper or lower limits with level α of our T. N. D. can be found from

$$\int_{t_1}^{\infty} f_{n-1}(t)\,dt = \alpha \quad \text{or} \quad \int_1^{t_0} f_{n-1}(t)\,dt = \alpha,$$

of which the former is readily computed, while for the latter it requires generally Newton's method of successive approximation. The following table shows a comparison between the significant lower- or upper-limit t_0, t_1 of our T. N. D. and those of Student's ordinary N. D.

		$\alpha=0.05$		$\alpha=0.025$		$\alpha=0.01$		$\alpha=0.005$	
		ours	Student	ours	Student	ours	Student	ours	Student
$n=2$	t_0	1.082	−6.314	1.040	−12.706	1.016	−31.821	1.008	−63.657
	t_1	25.452	+6.314	50.926	+12.706	127.321	+31.821	254.996	+63.657
$n=3$	t_0	1.381	−2.920	1.261	−4.303	1.161	−6.065	1.114	−9.925
	t_1	5.248	+2.920	7.471	+4.303	12.805	+6.065	18.130	+9.925

§ **6.** *Concluding Remark.* Whatever the universe may be if its mean and variance exist, and when the sample size so large that the central limit theorem holds for the distribution of sample mean, the ordinary normal test will convert to use. It is probable that our $f_{n-1}(t)$ shall also follow that theorem just as it is the case for the ordinary Student ratio. But, to prove this strictly, we have to find the general expression for $f_{n-1}(t)$, or at least to show the existence of its mean and variance, what however seems plausible by the general argument done in section 3. Since, however, with large samples the classical normal test will do at any rate, there is little need to know the exact form $f_{n-1}(t)$ for large n. On the contrary, the exact sampling distribution with small size forms certainly a subject of discussion.

Our examples in this note were rather simple. To conceive a little more copmplex case, e. g. let the universe be

(6. 1) $$f(x) = \alpha^{m+1}x^m e^{-\alpha x}/\Gamma(m+1), \qquad (x>0, \quad m=1,2,\dots),$$

or more practically, the χ^2_k -distribution, i. e.

(6. 2) $$f(x) = x^{\frac{k}{2}-1} e^{-\frac{x}{2}} \Big/ 2^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right) \qquad (x>0, \quad k=1,\ 2,\ \dots).$$

With the exact sample of size n we have the probability element for (6. 1)

(6. 3) $$dP = e^{-n\bar{x}} \prod_1^n x_i^m \, dV/\Gamma(m+1) \quad \text{where} \quad dV = \int \prod_1^n dx_i.$$

We may calculate the sample moments

(6. 4) $$\nu_k = \sum_1^n x_i^k/n \qquad k = 1, 2, \dots, n,$$

or, more concretely the central moments

(6. 5) $$\mu_k = \sum_1^n (x_i - \bar{x})^k/n,$$

which, namely, give the sample mean $\bar{x} = \nu_1 (\mu_1 = 0)$, variance $s^2 = \mu_2$, skewness $\alpha_3 = \mu_3/s^3$ and kurtosis $\alpha_4 = \mu_4/s^4$, &c. We should express the probability element (6. 3) so as

(6. 6) $$dP = f(\bar{x},\ s,\ \mu_3,\ \dots,\ \mu_n)\, d\bar{x} ds d\mu_3 \dots d\mu_n,$$

or else, parallel to $s = \sqrt{\mu_2}$ writing $\sqrt[k]{\mu_k}$ as variables for every $k \geq 3$ also.

The expression $\prod_1^n x_i^m$ may be denoted by a combinaticn of $x, s, \mu_3, \dots, \mu_n$, while $\int \prod_1^n dx_i$ should be expressed as $g(\bar{x}, s, \mu_3, \dots)\, d\bar{x} ds\, d\mu_3 \dots d\mu_n$, and in particular accessibly[4)] for cases $n = 2, 3, 4$.

However, the present author will leave the remaining work unfinished to any investigator who is interested in this theme. Of course, if merely the frequency function for sample mean be required, it could be readily obtained by a simple application of the convolution theory to the Γ- or X^2-distribution.

References: 1) Herald Cramér, Mathematical Methods of Statistics, p. 247 (1946).

2) Émile Borel, Introduction géométrique, quelques théories physiques (1914). Using the notations in that text, pp. 63, 64, we obtain, as the area of the spherical calotte,

$$C = \frac{S}{2} - S_1 = 2\pi R^{m-1} \int_0^\theta \sin^{m-2} \varphi_1 \, d\varphi_1 \int_0^\pi \sin^{m-3} \varphi_1 \, d\varphi_1 \dots \int_0^\pi \sin \varphi_{m-2} \, d\varphi_{m-2}$$

$$= \frac{2(\pi R^2)^{\frac{m-1}{2}}}{\Gamma\left(\frac{m-1}{2}\right)} \int_0^\theta \sin^{m-2} \varphi_1 \, d\varphi_1,$$

where $\cos\ \theta = \alpha/R$ and α denotes the central distance of the base, and this result really coincides exactly with our (3.5) if we put $m = n-1$, $R^2 = ns^2$ and $\cos\ \theta = \bar{x}/s\sqrt{n-1}$.

3) The author has verified that the total probabilities always become unity for several distributions which were treated in this note. From that point of view, however, it requires for (5.5)

$$\int_0^\infty \psi_3(\tau)d\tau = \frac{96}{\pi^2}\int_1^{\sqrt{3}} \tan^{-1}\sqrt{\frac{1}{2}\left(1-\frac{1}{\tau^2}\right)}\ \frac{d\tau}{1+\tau^2} = 1,$$

what could really be ascertained by Gauss' method of selected ordinates for numerical integrations. Also he tried to prove it by means of the theory of functions, though yet without finishing completely: Y. Watanabe, Eine Integralformel, the present volume of this Journal.

4) E.g. the case $n = 2$ for X_k^2-distribution may be readily treated by means of (3.7): The fr. f. for the X_k^2-distribution being $f(x) = x^{\frac{k}{2}-1}e^{-\frac{x}{2}}\Big/2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)$ $(x > 0)$, it follows that after (3.7)

$$dP = \frac{4}{2^k\Gamma\left(\frac{k}{2}\right)^2}\ (x_1\ x_2)^{\frac{k}{2}-1}\ e^{-\bar{x}}ds d\bar{x}\ (0 < s < \bar{x},\quad 0 < \bar{x} < \infty),\ \text{where}\ x_1 x_2 = \bar{x}^2 - s^2.$$

Or, writing $\bar{x}\sqrt{n-1}/s = t$, $1 < t = \bar{x}/s < \infty$ for $n = 2$, the fr. f. $f_2(t)$ is found to become

$$f_2(t)dt = \left(1-\frac{1}{t^2}\right)^{\frac{k}{2}-1}\frac{dt}{t^2}\int_0^\infty \frac{1}{\bar{x}}^{k-1} e^{-\bar{x}}d\bar{x}/2^{k-2}\Gamma\left(\frac{k}{2}\right)^2 = 2\Gamma\left(\frac{k+1}{2}\right)\left(1-\frac{1}{t^2}\right)^{\frac{k}{2}-1}\frac{dt}{t^2}\Big/\sqrt{\pi}\ \Gamma\left(\frac{k}{2}\right),$$

because of $\Gamma(k) = \frac{2^{k-1}}{\sqrt{\pi}}\ \Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{k+1}{2}\right)$, a multiplication theorem of the gammafunction, and consequently $\int_1^\infty f_2(t)\,dt = 1$. Hence, the lower- and upper-limit t_0 and t_1 for the significance level 0.05 say, such that

$$\int_1^{t_0} f(t)\,dt = 0.05\quad \text{and}\quad \int_{t_1}^\infty f_2(t)\,dt = 0.05$$

are found to be

k	1	2	3	and so on
t_0	1.003	1.053	1.072	
t_1	12.745	20.000	25.286	

while the corresponding ordinary Student's ratios are $t = \mp 6.314$.

Similarly, with the parent distribution $f(x) = x^m e^{-x}/m\,!$ $(x > 0,\ m = 1, 2, 3, \ldots.)$ and the sample size $n = 2$, it follows that $dP = \frac{4}{m\,!^2}\ (\bar{x}^2 - s^2)^m e^{-2\bar{x}}\,ds d\bar{x}$, and the fr. f. $f_2(t) = \frac{4\,(2m+1)\,!}{m\,!^2}\left(1-\frac{1}{t^2}\right)^m\frac{1}{t^2}$ in $1 < t = \frac{\bar{x}}{s} < \infty$.