Journal of Gakugei, Tokushima University
Vol. X. 1959

ON THE COMPLETE TENSOR PRODUCT OF MODULES

By

Motoyoshi SAKUMA

(Received September 30, 1959)

In our previous paper [2], we introduced the notion of the complete tensor product of modules. Namely, let E and E' be, respectively, finite modules over an $\mathfrak{m}$- and an $\mathfrak{m}'$-adic Zariski rings A and A'. Assume A and A' contain a common subfield K. Put $G_n = E/\mathfrak{m}^n E \otimes_K E'/\mathfrak{m}'^n E'$. Then the system $\{G_n, \phi_n\}$ $(n = 1, 2, \dots)$ constitutes an inverse system of $A \otimes_K A'$-modules, where ϕ_n denotes the canonical homomorphisms $G_n \to G_{n-1}$. Its projective limit $E \hat{\otimes}_K E'$ is referred to as a complete tensor product of E and E' over K. In this note, we shall mainly investigate, following closely the recent work of Satô [4], the relation between the multiplicities $e_E(\mathfrak{q})$, $e_{E'}(\mathfrak{q}')$ and $e_{E\hat{\otimes}E'}((\mathfrak{q}, \mathfrak{q}')(A\hat{\otimes} A'))$ in the case when A and A' are, respectively, local rings, where we denote by $\mathfrak{q}$ and $\mathfrak{q}'$ primary ideals belonging to the maximal ideals of A and A' respectively.[1)]

This relation was studied first, in a restricted case, by Samuel [3] and continued by Nagata [1] and Satô [4] in the case of rings.

1. General remarks on the complete tensor product of modules.

We start with the following proposition which is fundamental in this note.

PROPOSITION 1. *Let A and A' be, respectively, an $\mathfrak{m}$-adic and an $\mathfrak{m}'$-adic Zariski rings which contain a common subfield K and let E, F and G be finite A-modules such that*

$$0 \to F \to E \to G \to 0 \quad \text{(exact)}.$$

Then, for any finite A'-module E', we have the following exact sequence of finite $A\hat{\otimes}_K A'$-modules :

$$0 \to F\hat{\otimes}_K E' \to E\hat{\otimes}_K E' \to G\hat{\otimes}_K E' \to 0.$$

And we have

$$E\hat{\otimes}_K E' \approx (E \otimes_K E') \otimes_{A\otimes A'} (A \hat{\otimes} A').^{2)}$$

For the proof we refer the reader to [2].

1) For the notations and terminology we refer the reader to [2].

2) In the following we shall omit K and $A\otimes A'$ if any confusion does not occur.

COROLLARY. (*With the same notations and assumptions*). *For any submodules F and G* (*resp. F' and G'*) *of E* (*resp. E'*), *we have*

i) $(F+G)\hat{\otimes}(F'+G') = F\hat{\otimes}F'+F\hat{\otimes}G'+G\hat{\otimes}F'+G\hat{\otimes}G'$.

ii) $(E/G)\hat{\otimes}(E'/G') \approx (E\hat{\otimes}E')/(G\hat{\otimes}E'+E\hat{\otimes}G')$.

iii) $(F\cap G)\hat{\otimes}(F'\cap G') = F\hat{\otimes}F'\cap F\hat{\otimes}G'\cap G\hat{\otimes}F'\cap G\hat{\otimes}G'$.

iv) $(F:G)\hat{\otimes}A' = (F\hat{\otimes}A'):(G\hat{\otimes}A')$ *and* $A\hat{\otimes}(F':G') = (A\hat{\otimes}F'):(A\hat{\otimes}G')$.

PROOF. Since the functor $T(F, F') = (F\otimes F')\otimes_{A\otimes A'}(A\hat{\otimes}A')$ is a covariant additive exact functor in both variables, we can prove the corollary in the same way as was given in Lemma 2 in [2].

Remark: Let again F and F' be submodules of E and E' respectively, then by Proposition 1, the canonical mappings $F\hat{\otimes}E' \to E\hat{\otimes}E'$ and $F\hat{\otimes}F' \to F\hat{\otimes}E'$ are injective. Therefore the composed mapping $F\hat{\otimes}F' \to E\hat{\otimes}E'$ is also injective. Hence if we restrict our attention to submodules $M_\lambda (\lambda \in I)$ of $E\otimes E'$ such that M_λ is a finite sum of type $F\otimes F'$, the functor $T(\cdot) = \cdot \otimes_{A\otimes A'}(A\hat{\otimes}A')$ is exact.

PROPOSITION 2. *Let E and E' be finite modules over an* $\mathfrak{m}$*-adic and an* $\mathfrak{m}'$*-adic Zariski rings A and A' respectively, and assume A and A' contain a common subfield K. Then, for any submodules F of E and F' of E' and ideals* $\mathfrak{a}$ *of A and* $\mathfrak{a}'$ *of A'*, we have

i) $F\otimes F'$ *is a closed submodule of* $E\otimes E'$.

ii) $(\mathfrak{a}, \mathfrak{a}')(F\hat{\otimes}F')\cap(F\otimes F') = (\mathfrak{a}, \mathfrak{a}')(F\otimes F')$.

iii) $F\otimes F' = (F\otimes E')\cap(E\otimes F')$ *and* $F\hat{\otimes}F' = (F\hat{\otimes}E')\cap(E\hat{\otimes}F')$.

iv) $(\mathfrak{a}\,\mathfrak{a}')(F\otimes F') = (\mathfrak{a}F)\otimes(\mathfrak{a}'F') = (\mathfrak{a}F\otimes E')\cap(E\otimes\mathfrak{a}'F')$ *and* $(\mathfrak{a}\,\mathfrak{a}')(F\hat{\otimes}F') = (\mathfrak{a}F)\hat{\otimes}(\mathfrak{a}'F') = (\mathfrak{a}F\hat{\otimes}E')\cap(E\hat{\otimes}\mathfrak{a}'F')$.

We assume further that $A/\mathfrak{m}\hat{\otimes}A'/\mathfrak{m}'$ *is Noetherian, then*

v) $(\mathfrak{a}, \mathfrak{a}')(F\hat{\otimes}F') = (\mathfrak{a}F)\hat{\otimes}F'+F\hat{\otimes}(\mathfrak{a}'F')$.

vi) $(F\hat{\otimes}F')/(\mathfrak{a}, \mathfrak{a}')(F\hat{\otimes}F') \approx (F/\mathfrak{a}F)\hat{\otimes}(F'/\mathfrak{a}'F')$.

PROOF. i) Consider the sequence of submodules of $E\otimes E'$: $E\otimes E' \supset F\otimes E' \supset F\otimes F'$. Then, to prove i) it is enough to show $F\otimes E'$ is a closed subspace of $E\otimes E'$. Since we proved in the proof of Proposition 3 in [2] that there exists an integer r such that

$$(\mathfrak{m}, \mathfrak{m}')^{2n}(F\otimes E') \subseteq (\mathfrak{m}^n, \mathfrak{m}'^n)(E\otimes E')\cap(F\otimes E') \subseteq (\mathfrak{m}^{n-r}, \mathfrak{m}'^{n-r})(F\otimes E') \subseteq (\mathfrak{m}, \mathfrak{m}')^{n-r}(F\otimes E')$$

for any $n\geq r$, $F\otimes E'$ is a subspace of $E\otimes E'$. Therefore it remains to show $F\otimes E'$ is closed in $E\otimes E'$. To see this we remark first that

$$\bigcap_n ((F+\mathfrak{m}^nE)\otimes E') = F\otimes E' \quad\text{and}\quad \bigcap_n ((F\otimes E')+(E\otimes\mathfrak{m}'^nE')) = F\otimes E'.$$

In fact, let ξ be an element in $\bigcap_n ((F+\mathfrak{m}^nE)\otimes E')$, then it can be written as

$$\xi = y_1 \otimes y_1' + \dots + y_t \otimes y_t' \text{ with } y_i \in E \text{ and } y_i' \in E',$$

and we may assume $y_1', \dots, y_t'$ are linearly independent over K. Therefore $y_i \in F + \mathfrak{m}^n E$ for any n, hence $\xi \in F \otimes E'$ since $\bigcap_n (F + \mathfrak{m}^n E) = F$ which proves the first equality. As for the second, by passing to the residue module, we may assume $F = 0$ and by the consideration similar to the first part, we get $\bigcap_n (E \otimes \mathfrak{m}'^n E') = 0$.

Now, by virtue of these remarks, we have

$$\text{closure of } F \otimes E' \text{ in } E \otimes E' = \bigcap_n ((F \otimes E') + (\mathfrak{m}, \mathfrak{m}')^n (E \otimes E'))$$
$$= \bigcap_n (F \otimes E' + (\mathfrak{m}^n, \mathfrak{m}'^n)(E \otimes E')) = \bigcap_n ((F \otimes E') + (\mathfrak{m}^n E \otimes E') + (E \otimes \mathfrak{m}'^n E'))$$
$$= \bigcap_n ((F + \mathfrak{m}^n E) \otimes E' + E \otimes \mathfrak{m}'^n E') \subseteq \bigcap_i \bigcap_n ((F + \mathfrak{m}^i E) \otimes E' + E \otimes \mathfrak{m}'^n E')$$
$$= \bigcap_i ((F + \mathfrak{m}^i E) \otimes E') = F \otimes E'.$$

ii) This follows from $(\mathfrak{a}, \mathfrak{a}')(F \hat{\otimes} F') \cap (F \otimes F') =$ closure of $(\mathfrak{a}, \mathfrak{a}')(F \otimes F')$ in $F \otimes F' = (\mathfrak{a}, \mathfrak{a}')(F \otimes F')$ by i).

iii) We take a base $\{x_i\}_{i \in I}$ (resp. $\{x_i'\}_{i \in I'}$) of F (resp. F') over K and extend this base to a base $\{x_i, y_j\}_{i \in I\ j \in J}$ (resp. $\{x'_i, y'_j\}_{i \in I'\ j \in J'}$) of E (resp. E') over K. Then the set $\{y_i \otimes x_{i'}, y_j \otimes y_{j'}, x_i \otimes y_{j'}, y_j \otimes x_{i'}\}_{i \in I, i' \in I', j \in J, j' \in J'}$ foms a base of $E \otimes E'$ over K. By making use of this base we see easily that $(F \otimes E') \cap (E \otimes F') \subseteq F \otimes F'$. Converse inclusion is obvious. The second equality follows from

$$(F \hat{\otimes} F') = (F \otimes F') \otimes_{A \otimes A'} (A \hat{\otimes} A') = ((F \otimes E') \cap (E \otimes F')) \otimes (A \hat{\otimes} A')$$
$$= ((F \otimes E') \otimes (A \hat{\otimes} A')) \cap ((E \otimes F') \otimes (A \hat{\otimes} A')) = (F \hat{\otimes} E') \cap (E \hat{\otimes} F')$$

by the remark stated after Corollary to Proposition 1.

iv) We have $\mathfrak{a}\,\mathfrak{a}'(F \otimes F') = (\mathfrak{a} \otimes A')(A \otimes \mathfrak{a}')(F \otimes F') = (\mathfrak{a} F) \otimes (\mathfrak{a}' F') = (\mathfrak{a} F \otimes E') \cap (E \otimes \mathfrak{a}' F')$ by iii), and $\mathfrak{a}\,\mathfrak{a}'(F \hat{\otimes} F') = \mathfrak{a}\,\mathfrak{a}'((F \otimes F') \otimes (A \hat{\otimes} A')) = ((\mathfrak{a} F) \otimes (\mathfrak{a}' F')) \otimes (A \hat{\otimes} A') = \mathfrak{a} F \hat{\otimes} \mathfrak{a}' F' = (\mathfrak{a} F \hat{\otimes} E') \cap (E \hat{\otimes} \mathfrak{a}' F')$.

v) Clearly it is enough to show that $\mathfrak{a}(F \hat{\otimes} F') = (\mathfrak{a} F) \hat{\otimes} F'$. Since $A \hat{\otimes} A'$ is complete in an $(\mathfrak{m}, \mathfrak{m}')$-adic topology, and since $A/\mathfrak{m} \hat{\otimes} A'/\mathfrak{m}'$ is Noetherian, $A \hat{\otimes} A'$ is also Noetherian [3, Corollary to Proposition 1, p. 21], therefore a Zariski ring. Hence $\mathfrak{a}(F \hat{\otimes} F')$ is closed in $F \hat{\otimes} F'$. Whence $(\mathfrak{a} F) \hat{\otimes} F' =$ closure of $(\mathfrak{a} F) \otimes F'$ in $F \hat{\otimes} F' \subseteq \mathfrak{a}(F \hat{\otimes} F')$. The coverse inclusion is obvious.

vi) Since $(F/\mathfrak{a} F) \hat{\otimes} (F'/\mathfrak{a}' F') \approx F \hat{\otimes} F'/((\mathfrak{a} F) \hat{\otimes} F' + F \hat{\otimes} (\mathfrak{a}' F'))$, by Corollary to Proposition 1, the assetion follows by virtue of v),

2. Multiplicities.

In his paper [4], Satô studied the relations between prime divisors and primary components of ideals $\mathfrak{a}$ and $\mathfrak{a}'$ of Zariski rings A and A' and those of $(\mathfrak{a}, \mathfrak{a}')(A \hat{\otimes} A')$. For our purpose, the following lemma, due to Satô, is necessary.

LEMMA 1. *Let* $(A, \mathfrak{m})$ *and* $(A', \mathfrak{m}')$[3] *be, respectively, local rings of rank*

3) By a local ring $(A, \mathfrak{m})$ we mean a local ring A with the maximal ideal $\mathfrak{m}$.

d and d' which contain a common subfield K. Assume $A/\mathfrak{m} \otimes A'/\mathfrak{m}'$ is an Artin ring. Then $A \hat{\otimes} A'$ becomes a semi-local ring[4] *of rank $d+d'$, $(\mathfrak{q}, \mathfrak{q}')(A \hat{\otimes} A')$ is a defining ideal of $A \hat{\otimes} A'$, any prime divisor of $(\mathfrak{q}, \mathfrak{q}')(A \hat{\otimes} A')$ is isoloted and the lengths of its primary components are the same and are equal to $l(\mathfrak{q})l(\mathfrak{q}')c$ where $l(\mathfrak{q})$ (resp. $l(\mathfrak{q}')$) stands for the length of primary ideal $\mathfrak{q}$ (resp. $\mathfrak{q}'$) and c stands for the common length of primary components of $(\mathfrak{m}, \mathfrak{m}')(A \hat{\otimes} A')$.*

Remark that in the case when A and A' are fields, we have $A \hat{\otimes} A' = A \otimes A'$.

LEMMA 2. *Let E and E' be, respectively, finite dimensional vector spaces over the fields L and L'. Assume that both L and L' are the extensions of a field K and $L \otimes L'$ is an Artin ring. Then*

$$l(E \otimes E') = \dim_L E \dim_{L'} E' \, l(L \otimes L')$$

where $l(E \otimes E')$ (resp. $l(L \otimes L')$) means the length as the finite module $E \otimes E'$ over the Artin ring $L \otimes L'$ (resp. the length of the Artin ring $L \otimes L'$).

PROOF. Put $s = \dim_L E$, $s' = \dim_{L'} E'$ and $l = l(L \otimes L')$. In the case when $s=1$, we can proceed by applying induction to s' as follows : Since our lemma is trivially valid in the case when $s'=1$, we may assume $s' > 1$. Let E' be a subspace of E' such that $\dim_{L'} E_1' = s'-1$. Then $E'/E_1' \approx L'$ and by our induction hypothesis we have $l(L \otimes E_1') = (s'-1)l$. Therefore

$$l(L \otimes E') = l(L \otimes E_1') + l(L \otimes (E'/E_1')) = l(L \otimes E_1') + l(L \otimes L') = sl,$$

which is to be shown. General case follows from this in the same way by applying induction to s.

PROPOSITION 3. *Let $\mathfrak{q}$ and $\mathfrak{q}'$ be, respectively, primary ideals belonging to the maximal ideals of local rings $(A, \mathfrak{m})$ and $(A', \mathfrak{m}')$ which contain a common subfield K. Assume $A/\mathfrak{m} \otimes A'/\mathfrak{m}'$ is an Artin ring. Then, for any finite A- and A'-module E and E', we have*

$$l((E/\mathfrak{q} E) \hat{\otimes} (E'/\mathfrak{q}' E')) = l(E/\mathfrak{q} E) \, l(E'/\mathfrak{q}' E') \, l(A/\mathfrak{m} \otimes A'/\mathfrak{m}').$$

PROOF. First we consider the case when $\mathfrak{q} = \mathfrak{m}$ and proceed by applying induction to the length of $\mathfrak{q}'$. Since our proposition is true, by Lemma 2, in the case when $l(\mathfrak{q}') = 1$, i.e., $\mathfrak{q}' = \mathfrak{m}'$, we may assume $l(\mathfrak{q}') > 1$. Let $\mathfrak{m}' = \mathfrak{q}_1' \supset \mathfrak{q}_2' \supset \dots \supset \mathfrak{q}_t' = \mathfrak{q}'$ be a chain of $\mathfrak{m}'$-primary ideals and asuume that each inclusion is strict and no $\mathfrak{m}'$-primary ideals can be inserted between $\mathfrak{q}_i'$ and $\mathfrak{q}_{i+1}'$ ($i = 1, \dots, t-1$). From the exact sequence

$$0 \to \mathfrak{q}_{t-1}' E'/\mathfrak{q}_t' E' \to E'/\mathfrak{q}_t' E' \to E'/\mathfrak{q}'_{t-1} E' \to 0,$$

4) In this case $A \hat{\otimes} A'$ is Noetherian as we remarked in the proof of v) of Proposition 2, hterefore semi-local [3, §1 e, p. 7].

we have, by Proposition 1, the exact sequence of $A\hat{\otimes}A'$-modules:

$$0\to(E/\mathfrak{m}E)\hat{\otimes}(\mathfrak{q}_{t-1}'E'/\mathfrak{q}_t'E')\to(E/\mathfrak{m}E)\hat{\otimes}(E'/\mathfrak{q}_t'E')$$
$$\to(E/\mathfrak{m}E)\hat{\otimes}(E'/\mathfrak{q}_{t-1}E')\to 0,$$

hence
$$l((E/\mathfrak{m}E)\hat{\otimes}(E'/\mathfrak{q}_t'E'))=l((E/\mathfrak{m}E)\hat{\otimes}(\mathfrak{q}_{t-1}'E'/\mathfrak{q}_t'E'))$$
$$+l((E/\mathfrak{m}E)\hat{\otimes}(E'/\mathfrak{q}_{t-1}'E')).$$

Since $\mathfrak{m}'\mathfrak{q}_{t-1}'\subset\mathfrak{q}_t'$ and $\mathfrak{q}_{t-1}'=(\mathfrak{q}_t',x)$ for any element $x\in\mathfrak{q}_{t-1}'$, $x\notin\mathfrak{q}_t'$, $\mathfrak{q}_{t-1}'/\mathfrak{q}_t'$ is isomorphic to $A'/\mathfrak{m}'$, hence $\mathfrak{q}_{t-1}'E'/\mathfrak{q}_t'E'\approx E'/\mathfrak{m}'E'$. Therefore

$$l((E/\mathfrak{m}E)\hat{\otimes}(\mathfrak{q}_{t-1}'E'/\mathfrak{q}_t'E'))=l((E/\mathfrak{m}E)\hat{\otimes}(E'/\mathfrak{m}'E'))=l(E/\mathfrak{m}E)l(E'/\mathfrak{m}'E')l$$

where $l=l(A/\mathfrak{m}\otimes A'/\mathfrak{m}')$. On the other hand, by our induction hypothesis, we have

$$l((E/\mathfrak{m}E)\hat{\otimes}(E'/\mathfrak{q}_{t-1}'E'))=l(E/\mathfrak{m}E)l(E'/\mathfrak{q}_{t-1}'E')l.$$

Therefore, by combining these relations, we get

$$l((E/\mathfrak{m}E)\hat{\otimes}(E'/\mathfrak{q}'E'))=l(E/\mathfrak{m}E)l(E'/\mathfrak{q}'E')l.$$

Now the general case follows from this relation by applying induction to the length of $\mathfrak{q}$ in the same way as above.

COROLLARY. $l((A/\mathfrak{q})\hat{\otimes}(A'/\mathfrak{q}'))=l(A/\mathfrak{q})\,l(A'/\mathfrak{q}')\,l(A/\mathfrak{m}\otimes A'/\mathfrak{m}')$.

LEMMA 3. *Let E and E' be, respectively, finite modules over an $\mathfrak{m}$-adic and an $\mathfrak{m}'$-adic Zariski rings A and A' and let $\mathfrak{a}$ and $\mathfrak{a}'$ be ideals of A and A' respectively. Assume A and A' contain a common subfield K and $A/\mathfrak{m}\hat{\otimes}A'/\mathfrak{m}'$ is Noetherian. Then we have*

$$(\mathfrak{a}^n,\ \mathfrak{a}^{n-1}\mathfrak{a}',\ \dots,\ \mathfrak{a}^{n-i+1}\mathfrak{a}'^{i-1},\ \mathfrak{a}^{n-i-1}\mathfrak{a}'^{i+1},\ \dots,\ \mathfrak{a}\,\mathfrak{a}'^{n-1},\ \mathfrak{a}'^n)(E\hat{\otimes}E')$$
$$\cap(\mathfrak{a}^{n-i}\mathfrak{a}'^i)(E\hat{\otimes}E')\subseteq(\mathfrak{a}^{n-i+1}\mathfrak{a}'^i,\ \mathfrak{a}^{n-i}\mathfrak{a}'^{i+1})(E\hat{\otimes}E').$$

PROOF. We take a special base of the vector space E over K as follows: First we take a base $\{x_{n\lambda};\ \lambda\in\Lambda_n\}$ of $\mathfrak{a}^nE$ over K, and then extend this base to the base $\{x_{n\lambda},\ x_{n-1\mu};\ \lambda\in\Lambda_n,\ \mu\in\Lambda_{n-1}\}$ of $\mathfrak{a}^{n-1}E$ over K. Continuing this process we obtain a base $\{x_{n\lambda},\ x_{n-1\mu},\ x_{n-2\nu},\ \dots;\ \lambda\in\Lambda_n,\ \mu\in\Lambda_{n-1},\ \nu\in\Lambda_{n-2},\ \dots\}$ of E over K. In the same way, we construct a base $\{x'_{n\lambda'},\ x'_{n-1\mu'},\ x_{n-2\nu'},\ \dots;\ \lambda'\in\Lambda_n',\ \mu'\in\Lambda'_{n-1},\ \nu'\in\Lambda'_{n-2},\ \dots\}$ of E' over K. By making use of these bases, we see easily that

$$(\mathfrak{a}^n,\mathfrak{a}^{n-1}\mathfrak{a}',\dots,\mathfrak{a}^{n-i+1}\mathfrak{a}'^{i-1},\ \mathfrak{a}^{n-i-1}\mathfrak{a}'^{i+1},\ \dots,\ \mathfrak{a}\,\mathfrak{a}'^{n-1},\mathfrak{a}'^n)(E\otimes E')$$
$$\cap(\mathfrak{a}^{n-i}\mathfrak{a}'^i)(E\otimes E')\subseteq(\mathfrak{a}^{n-i+1}\mathfrak{a}'^i,\ \mathfrak{a}^{n-i}\mathfrak{a}'^{i+i})(E\otimes E').$$

Operating $\otimes_{A\otimes A'}(A\hat{\otimes}A')$ to the both side of this relation, we get a required relation by virtue of the remark stated after Corollary to Proposition 1.

LEMMA 4. *Let E be a finite module over a Noetherian ring A and E_1, $\dots$, E_n be submodules of E. And let $\mathfrak{a}$ be an ideal of A such that corank $\mathfrak{a}=0$. Put $F_i=\mathfrak{a}E_i(i=1,\ \dots,\ n)$. Assume $(E_1+\dots+E_{i-1}+E_{i+1}+\dots+E_n)\cap E_i\subseteq F_i\ (i=1,\ \dots,\ n)$.* Then we have

$$l((E_1+\ldots+E_n)/(F_1+\ldots+F_n)) = l(E_1/F_1)+\ldots+l(E_n/F_n).$$

PROOF. Consider a sequence of submodules of $E_1+\ldots+E_n$: $E_1+\ldots+E_n \supset E_1+\ldots+E_{n-1}+F_n \supset E_1+\ldots+E_{n-2}+F_{n-1}+F_n \supset \ldots \supset E_1+F_2+\ldots+F_n \supset F_1+\ldots+F_n$. Since

$$\begin{aligned} &l((E_1+\ldots+E_i+F_{i+1}+\ldots+F_n)/(E_1+\ldots+E_{i-1}+F_i+F_{i+1}+\ldots+F_n)) \\ &=l((E_1+\ldots+E_{i-1}+F_i+\ldots+F_n)+E_i/(E_1+\ldots+E_{i-1}+F_i+\ldots+F_n)) \\ &=l(E_i/E_i\cap(E_1+\ldots+E_{i-1}+F_i+\ldots+F_n)) \\ &=l(E_i/(E_i\cap(E_1+\ldots+E_{i-1}+F_{i+1}+\ldots+F_n))+F_i) = l(E_i/F_i), \end{aligned}$$

by our assumption, therefore we have

$$l((E_1+\ldots+E_n)/(F_1+\ldots+F_n)) = \sum_{i=1}^{n} l((E_1+\ldots+E_i+F_{i+1}+\ldots+F_n)/ (E_1+\ldots+E_{i-1}+F_i+F_{i+1}+\ldots+F_n)) = \sum_{i=1}^{n} l(E_i/F_i).$$

Now we shall prove the theorem which is the main purpose of this note.

THEOREM. *Let $(A, \mathfrak{m})$ and $(A', \mathfrak{m}')$ be local rings which contain a common subfield K and assume $A/\mathfrak{m}\otimes A'/\mathfrak{m}'$ is an Artin ring. And let E and E' be finite modules over A and A' respectively. Then, for any $\mathfrak{m}$-primary ideal $\mathfrak{q}$ and $\mathfrak{m}'$-primary ideal $\mathfrak{q}'$, we have*

$$e_{E\hat{\otimes}E'}((\mathfrak{q}, \mathfrak{q}')(A\hat{\otimes}A')) = e_E(\mathfrak{q})e_{E'}(\mathfrak{q}')\,l(A/\mathfrak{m}\otimes A'/\mathfrak{m}').$$

PROOF. We first show that $l(E\hat{\otimes}E'/(\mathfrak{q}, \mathfrak{q}')^n(E\hat{\otimes}E')) = \sum_{i+j<n} l((\mathfrak{q}^iE/\mathfrak{q}^{i+1}E)\hat{\otimes}(\mathfrak{q}'^jE'/\mathfrak{q}'^{j+1}E'))$, for any integer n. In fact, since $(E\hat{\otimes}E')/(\mathfrak{q}, \mathfrak{q}')(E\hat{\otimes}E') \approx (E/\mathfrak{q}E)\hat{\otimes}(E'/\mathfrak{q}'E')$, by Proposition 2, our assertion is true in the case when $n=1$. We assume our assertion is true in the case when $n=r$, and consider the case when $n=r+1$. Then

$$\begin{aligned} &l(E\hat{\otimes}E'/(\mathfrak{q}, \mathfrak{q}')^{r+1}(E\hat{\otimes}E')) \\ &= l(E\hat{\otimes}E'/(\mathfrak{q}, \mathfrak{q}')^r(E\hat{\otimes}E')) + l((\mathfrak{q}, \mathfrak{q}')^r(E\hat{\otimes}E')/(\mathfrak{q}, \mathfrak{q}')^{r+1}(E\hat{\otimes}E')) \\ &= \sum_{i+j<r} l((\mathfrak{q}^iE/\mathfrak{q}^{i+1}E))\hat{\otimes}(\mathfrak{q}'^jE'/\mathfrak{q}'^{j+1}E')) + l((\mathfrak{q}, \mathfrak{q}')^r(E\hat{\otimes}E')/(\mathfrak{q}, \mathfrak{q}')^{r+1}(E\hat{\otimes}E')) \end{aligned}$$

(by our induction hypothesis)

$$= \sum_{i+j<r} l((\mathfrak{q}^iE/\mathfrak{q}^{i+1}E)\hat{\otimes}(\mathfrak{q}'^jE'/\mathfrak{q}'^{j+1}E')) + \sum_{s+t=r} l((\mathfrak{q}^sE/\mathfrak{q}^{s+1}E)\hat{\otimes}(\mathfrak{q}'^tE'/\mathfrak{q}'^{t+1}E'))$$

(by Lemma 3 and 4)

$$= \sum_{i+j<r+1} l(\mathfrak{q}^iE/\mathfrak{q}^{i+1}E)\hat{\otimes}(\mathfrak{q}'^jE'/\mathfrak{q}'^{j+1}E')).$$

Therefore, by Proposition 3, we have

$$l(E\hat{\otimes}E'/(\mathfrak{q}, \mathfrak{q}')^n(E\hat{\otimes}E')) = \sum_{i+j<n} l(\mathfrak{q}^iE/\mathfrak{q}^{i+1}E)\,l(\mathfrak{q}'^jE'/\mathfrak{q}'^{j+1}E')\,l(A/\mathfrak{m}\otimes A'/\mathfrak{m}')$$

$$= \sum_{i+j<n} f(i)g(j)l,$$

where $f(i) = l(\mathfrak{q}^iE/\mathfrak{q}^{i+1}E)$, $g(j) = l(\mathfrak{q}'^jE'/\mathfrak{q}'^{j+1}E')$ and $l = l(A/\mathfrak{m}\otimes A'/\mathfrak{m}')$. This formula enables us to calculate the multiplicity of the defining ideal $(\mathfrak{q}, \mathfrak{q}')$

$(A \hat{\otimes} A')$ of the semi-local ring $A \hat{\otimes} A'$ in $E \hat{\otimes} E'$ in the same way as was given in [3] replacing $e(\mathfrak{q})$ and $e(\mathfrak{q}')$ by $e_E(\mathfrak{q})$ and $e_{E'}(\mathfrak{q}')$.

References

[1] Nagata, M., *The theory of multiplicity in general local rings*, Proceedings of the International Symposium on Algebraic Number Theory, Tokyo and Nikko, 1955 (1956), 191—226.

[2] Sakuma, M., *On the theory of multiplicities in finite modules over semi-local rings*, J. Sci. Hiroshima Univ., (A) **23** (1959), 1—17.

[3] Samuel, P., *Algèbre locale*, Mém. Sci. Math. **123** (1953).

[4] Satô, H., *Different Noetherian rings in some axiomatic relations*, J. Sci. Hiroshima Univ. (A) **22** (1958), 1—14.