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ZUR LAPLACESCHEN ASYMPTOTISCHEN FORMEL

Von

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(Eingegangen am 30. Sept. 1958)

Einer der Verfässer hat bereits die Laplacesche asymptotische Formel für das Integral von Potenz mit großem Indexe, wenn die Basisfunktion im Innern des Integrationsbereiches maximal wird, zum Falle mehrer Variablen erweitert und vielmehr daraus denjenigen einer Variable bewiesen[1]. Da aber bisweilen das Maximum am Rande auftritt, diesmal wollen wir zunächst solchen besonderen Fällen betrachten und sondern woraus den üblichen Fall herleiten.

§ **1.** Die Einfachheit wegen beschränken wir uns auf einer reellen Variable. Wir auflösen die

Aufgabe 1. *Die reellen Funktionen* $\varphi(x)$ *und* $f(x)$ *seien im Intervalle* $I: a \leqq x < b$, *wobei* a *endlich aber* b *entweder endlich oder unendlich sein möge, definiert und den folgenden Bedingungen unterworfen*:

1° *Die Funktionen* $\varphi(x)[f(x)]^n$ *für* $n = 0, 1, 2, \cdots$ *oder wenigstens für alle* $n \geqq$ *passend großes* $m(>0)$ *seien absolut integrabel in* I.

2° *Die Funktion* $f(x)$ *sei nicht negativ und* $F(x) = \log f(x)$ *erreiche am unteren Randpunkte* a *ihr Maximum und zwar sei die obere Grenz von* $F(x)$ *in jedem abgeschlossenen Intervall, das* a *nicht enthält, kleiner als* $F(a)$ $(=F(a+0)$ *streng ausgesprochen). Ferner sei* $F(x)$ *im Intervalle* $U: a \leqq x < a+\delta$ *stetig differenzierbar bis zu einer passenden Ordnung.*

3° *Auch existiere etwa* $\varphi^{(4)}(x)$ *in* U *und sei es dort stetig.*

Man suche die asymptotische Formel für das Integral

$$\int_a^b \varphi(x)[f(x)]^n dx = \int_a^b \varphi(x) \exp[nF(x)] dx \qquad \text{bei } n \to \infty \,. \tag{1.1}$$

Lösung. Betrachtet man einen rechts offenen innerhalb I liegenden kleinen Intervall $U: a \leqq x < a+\delta$, so erreicht die Funktion $F(x) - F(a)$ wegen 2° im abgeschlossenen Intervall $I - U$ ein Maximum (<0) und folglich dort $\exp\{F(x) - F(a)\}$ auch einen Maximalwert ρ, wo $0 < \rho < 1$ ist. Setzt man

$$\int_a^b \varphi(x) \exp\{n[F(x) - F(a)]\} dx = \int_a^{a+\delta} + \int_{a+\delta}^b = \text{(i)} + \text{(ii)} \,, \tag{1.2}$$

1) Y. Ichijô, Über die Laplacesche asymptotische Formel für das Integral von Potenz mit großem Indexe, dieses Journ., Vol. VI (1955), S. 63. Auch vgl. G. Pólya und Szegö, Aufgaben und Lehrsätze aus der Analysis, I, SS. 78, 244.

so erhält man nach 1°

$$|(\text{ii})| < \rho^n \int_{a+\delta}^{b} |\varphi(x)|\,dx = O(\rho^n) = 0\left(\frac{1}{n^\omega}\right)$$

bei genügend großes n, wieviel groß aber fest ω genommen werden mag[2]. Daher hat man (i) allein zu abschätzen. Dazu schreibe man nach 3° im beschränkten Intervalle $U: a \leqq x < a+\delta$ etwa

$$(1.3) \quad \varphi(x) = \sum_{\mu=0}^{3} \frac{\varphi^{(\mu)}(a)}{\mu!}(x-a)^\mu + R, \quad \text{mit}\,|R| = \frac{1}{4!}|\varphi^{(4)}(a+(x-a)\vartheta)|\,(x-a)^4 < M\delta^4.$$

Ebenso auch wegen 2° hat man in demselben Intervalle eine Taylorsche Entwickelung

$$(1.4) \qquad F(x) - F(a) = \sum_{\nu=0}^{3} \frac{F^{(h+\nu)}(a)}{\underline{|h+\nu}}(x-a)^{h+\nu} + R_1,$$

wobei $F^{(h)}(a) \neq 0$ mit $h \geqq 1$, $R_1 = F^{(h+4)}(a+(x-a)\vartheta_1)(x-a)^{h+4}/\underline{|h+4}$ und $|R_1| < M\delta^{h+4}$ sind. Da aber $F(x)$ am Randpunkt a maximal wird, so muß $F^{(h)}(a)(x-a)^h/h! < 0$ für genug kleines $x-a$ (>0) und demnach

$$(1.5) \qquad F^{(h)}(a) < 0$$

sein. Setzt man der Kürze halber

$$(1.6) \qquad A = \left[\frac{F^{(h)}(a)}{-h!}\right]^{1/h}, \quad An^{1/h} = N, \quad x-a = \frac{u}{N},$$

$$A_\nu = \frac{F^{(h+\nu)}(a)}{\underline{|h+\nu}} \Big/ \frac{F^{(h)}(a)}{-\underline{|h}}, \quad A_0 = -1,$$

und folglich

$$\frac{nF^{(h+\nu)}(a)}{\underline{|h+\nu}}\left(\frac{u}{N}\right)^{h+\nu} = A_\nu \frac{u^{h+\nu}}{N^\nu} \qquad (\nu = 0, 1, 2, \cdots),$$

so ergibt sich

$$(\text{i}) = \int_0^{N\delta}\left[\varphi(a) + \varphi'(a)\frac{u}{N} + \frac{\varphi''(a)}{2}\left(\frac{u}{N}\right)^2 + \frac{\varphi'''(a)}{6}\left(\frac{u}{N}\right)^3 + \frac{\varphi''''(\xi)}{\underline{|4}}\left(\frac{u}{N}\right)^4\right]$$

$$\times \exp\left[-u^h + A_1 u^h \frac{u}{N} + A_2 u^h\left(\frac{u}{N}\right)^2 + A_3 u^h\left(\frac{u}{N}\right)^3 + A_4(\xi_1)u^h\left(\frac{u}{N}\right)^4\right]\frac{du}{N}.$$

Werden nochmals

$$(1.7) \qquad u^h = v, \quad \frac{u}{N} = \frac{1}{A}\left(\frac{v}{n}\right)^{1/h}, \quad \frac{du}{N} = \frac{1}{hA}\left(\frac{v}{n}\right)^{1/h}\frac{dv}{v}$$

geschrieben, so gilt für $n \to \infty$

2) Falls $\varphi(x)f(x)^n$ nicht für $n=0$, aber nur für $n \geqq$ passend großes $m > 0$, absolut integrabel in I ist, so betrachten wir wie folgt: $|(\text{ii})| < \frac{\rho^{n-m}}{f(a)^m}\int_{a+\delta}^{b}|\varphi(x)f(x)^m|\,dx = O(\rho^{n-m}) = O\left(\frac{1}{n^\omega}\right)$ bei genügend großes n.

$$(1.8)\quad \text{(i)} \simeq \int_0^\infty \left[\varphi(a)+\frac{\varphi'(a)}{A}\left(\frac{v}{n}\right)^{1/h}+\frac{\varphi''(a)}{2A^2}\left(\frac{v}{n}\right)^{2/h}+\frac{\varphi'''(a)}{6A^3}\left(\frac{v}{n}\right)^{3/h}+\frac{\varphi''''(\xi)}{24A^4}\left(\frac{v}{n}\right)^{4/h}\right]$$
$$\times e^{-v}\prod_{\nu=1}^{3}\exp\left\{\frac{A_\nu}{A^\nu}v\left(\frac{v}{n}\right)^{\nu/h}\right\}\cdot\exp\left\{\frac{A_4(\xi_1)}{A^4}v\left(\frac{v}{n}\right)^{4/h}\right\}\frac{1}{hA}\left(\frac{v}{n}\right)^{1/h}\frac{dv}{v}.$$

Man entwickle jedes $\exp\left\{\frac{A_\nu}{A^\nu}v\left(\frac{v}{n}\right)^{\nu/h}\right\}$ nach den Potenzen von $\left(\frac{v}{n}\right)^{1/h}$ bis zu den Gliedern vierter Ordnung. Wird die Multiplikation ausgeführt und die Formel der Gammafunktion

$$\int_0^\infty e^{-v}v^{\nu/h+\lambda}\,dv=\Gamma\left(\frac{\nu}{h}+\lambda+1\right)$$

angewandt, so liefert es, als gesuchte asymptotische Formel, die folgende

$$(1.9)\quad \int_a^b\varphi(x)\exp[nF(x)]\,dx\simeq\frac{\exp[nF(a)]}{A}\left(\frac{1}{n}\right)^{1/h}\left\{\Gamma\left(1+\frac{1}{h}\right)\varphi(a)\right.$$
$$+\Gamma\left(1+\frac{2}{h}\right)\left[\frac{\varphi'(a)}{2}+\frac{\varphi(a)A_1}{h}\right]\frac{1}{A}\left(\frac{1}{n}\right)^{1/h}+\Gamma\left(1+\frac{3}{h}\right)\left[\frac{\varphi''(a)}{6}+\frac{\varphi'(a)}{h}A_1\right.$$
$$\left.+\varphi(a)\left(\left(1+\frac{3}{h}\right)\frac{A_1^2}{2h}+\frac{A_2}{h}\right)\right]\frac{1}{A^2}\left(\frac{1}{n}\right)^{2/h}$$
$$+\Gamma\left(1+\frac{4}{h}\right)\left[\frac{\varphi'''(a)}{24}+\frac{\varphi''(a)}{2h}A_1+\varphi'(a)\left(\left(1+\frac{4}{h}\right)\frac{1}{2h}A_1^2+\frac{1}{h}A_2\right)\right.$$
$$\left.\left.+\varphi(a)\left(\left(2+\frac{4}{h}\right)\left(1+\frac{4}{h}\right)\frac{A_1^3}{6h}+\left(1+\frac{4}{h}\right)\frac{A_1A_2}{h}+\frac{A_3}{h}\right)\right]\frac{1}{A^3}\left(\frac{1}{n}\right)^{3/h}+O\left(\frac{1}{n}\right)^{4/h}\right\}.$$

Oder, noch ausführlich geschrieben,

$$(1.10)\quad \int_a^b\varphi(x)\exp[nF(x)]\,dx\simeq\frac{\exp[nF(a)]}{(-nF^{(h)}(a)/\underline{|h})^{1/h}}\left\{\Gamma\left(1+\frac{1}{h}\right)\varphi(a)\right.$$
$$+\frac{\Gamma(1+2/h)}{(-nF^{(h)}(a)/\underline{|h})^{1/h}}\left[\frac{\varphi'(a)}{2}-\frac{\varphi(a)}{h(h+1)}\frac{F^{(h+1)}(a)}{F^{(h)}(a)}\right]$$
$$+\frac{\Gamma(1+3/h)}{(-nF^{(h)}(a)/\underline{|h})^{2/h}}\left[\frac{\varphi''(a)}{6}-\frac{\varphi'(a)}{h(h+1)}\frac{F^{(h+1)}(a)}{F^{(h)}(a)}\right.$$
$$\left.+\frac{\varphi(a)}{h(h+1)}\left(\frac{h+3}{2h(h+1)}\left(\frac{F^{(h+1)}(a)}{F^{(h)}(a)}\right)^2-\frac{F^{(h+2)}(a)}{(h+2)F^{(h)}(a)}\right)\right]$$
$$+\frac{\Gamma(1+4/h)}{(-nF^{(h)}(a)/\underline{|h})^{3/h}}\left[\frac{\varphi'''(a)}{24}-\frac{\varphi''(a)}{2h(h+1)}\frac{F^{(h+1)}(a)}{F^{(h)}(a)}\right.$$
$$+\frac{\varphi'(a)}{h(h+1)}\left(\frac{h+4}{2h(h+1)}\left(\frac{F^{(h+1)}(a)}{F^{(h)}(a)}\right)^2-\frac{F^{(h+2)}(a)}{(h+2)F^{(h)}(a)}\right)$$
$$-\frac{\varphi(a)}{h(h+1)}\left(\frac{(h+2)(h+4)}{3h^2(h+1)^2}\left(\frac{F^{(h+1)}(a)}{F^{(h)}(a)}\right)^3-\frac{h+4}{h(h+1)(h+2)}\frac{F^{(h+1)}(a)F^{(h+2)}(a)}{F^{(h)}(a)^2}\right.$$
$$\left.\left.\left.+\frac{F^{(h+3)}(a)}{(h+2)(h+3)F^{(h)}(a)}\right)\right]+O\left(\frac{1}{n}\right)^{4/h}\right\},$$

wobei jeder Quotient in den äußersten geschweiften Klammern sämtlich null-dimensional in bezug auf Ordnungszahlen des Derivatives, d.h. der Ausdruck zwischen geschweiften Klammern zwar homogen vom Grade Null inbezugauf diese Zahlen ist.

Ins besondere für $h=1$ erhält man

$$(1.11)\quad \int_a^b \varphi(x)\exp[nF(x)]\,dx \simeq \frac{\exp[nF(a)]}{-nF'(a)}\Bigg\{\varphi(a)-\frac{1}{nF'(a)}\Big[\varphi'(a)-\varphi(a)\frac{F''(a)}{F'(a)}\Big]$$
$$+\frac{1}{n^2F'(a)^2}\Big[\varphi''(a)-\frac{3\varphi'(a)F''(a)}{F'(h)}+\varphi(a)\Big(\frac{3F''(a)^2}{F'(a)^2}-\frac{F'''(a)}{F'(a)}\Big)$$
$$-\frac{1}{n^3F'(a)^3}\Big[\varphi'''(a)-6\varphi''(a)\frac{F''(a)}{F'(a)}+\varphi'(a)\Big(\frac{15F''(a)^2}{F'(a)^2}-\frac{4F'''(a)}{F'(a)}\Big)$$
$$-\varphi(a)\Big(\frac{15F''(a)^3}{F'(a)^3}-\frac{10F''(a)F'''(a)}{F'(a)^2}+\frac{F''''(a)}{F'(a)}\Big)\Big]+O\Big(\frac{1}{n^4}\Big)\Bigg\},$$

so wie für $h=2$,

$$(1.12)\quad \int_a^b \varphi(x)\exp\{nF(x)\}\,dx \simeq \frac{1}{2}\sqrt{\frac{-2\pi}{nF''(a)}}\exp\{nF(a)\}\Bigg\{\varphi(a)+\sqrt{\frac{-2}{n\pi F''(a)}}\Big[\varphi'(a)$$
$$-\frac{\varphi(a)}{3}\frac{F'''(a)}{F''(a)}\Big]+\frac{1}{-2nF''(a)}\Big[\varphi''(a)-\varphi'(a)\frac{F'''(a)}{F''(a)}+\frac{\varphi(a)}{4}\Big(\frac{5}{3}\frac{F'''(a)^2}{F''(a)^2}-\frac{F''''(a)}{F''(a)}\Big)\Big]$$
$$+\frac{1}{3}\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{-nF''(a)}^3}\Big[\varphi'''(a)-2\varphi''(a)\frac{F'''(a)}{F''(a)}+\varphi'(a)\Big(2\frac{F'''(a)^2}{F''(a)^2}-\frac{F''''(a)}{F''(a)}\Big)$$
$$-\varphi(a)\Big(\frac{8}{9}\frac{F'''(a)^3}{F''(a)^3}-\frac{F'''(a)F''''(a)}{F''(a)^2}+\frac{F'''''(a)}{5F''(a)}\Big)\Big]+O\Big(\frac{1}{n}\Big)^2\Bigg\}.$$

§ **2.** In Analogie zum vorigen Abschnitt kann man betrachten die

Aufgabe 2. *Die Funktionen $\varphi(x)$ und $f(x)(\geqq 0)$ seien im Intervalle $I: a<x\leqq b$, definiert, wobei b endlich aber a entweder endlich oder unendlich sein möge, und den folgenden Bedingungen unterworfen:*

1° *Die Funktion $\varphi(x)[f(x)]^n$ für $n=0, 1, 2, \cdots$ oder sonst für alle $n\geqq$ passend großes $m(>0)$ seien absolut integrabel in I.*

2° *Die Funktion $f(x)$ sei nicht negativ und $F(x)=\log f(x)$ erreiche ihr Maximum am oberen Randpunkt b und zwar sei die obere Grenz von $F(x)$ in jedem abgeschlossenen Intervalle, das b nicht enthält, kleiner als $F(b)$. Ferner sei $F(x)$ im Intervalle $U: b\geqq x>b-\varepsilon$ stetig derivable bis zu passender Ordnung.*

3° *Auch existiere stwa $\varphi^{(4)}(x)$ in U und sei es dort stetig.*

Man suche die asymptotische Formel für das Integral

$$(2.1)\qquad \int_a^b \varphi(x)[f(x)]^n dx = \int_a^b \varphi(x)\exp[nF(x)]\,dx\,.$$

Lösung. Setzt man ganz analog wie in § 1

$$(2.2)\qquad \int_a^b \varphi(x)\exp\{n[F(x)-F(b)]\}\,dx = \int_a^{b-\delta}+\int_{b-\delta}^b=\text{(i)}+\text{(ii)}\,,$$

so braucht es nun (ii) allein zu abschätzen. Wieder nach Voraussetzungen gelten die folgenden Entwickelungen im beschränkten Intervalle $U: b \geqq x \geqq b-\delta$

(2.3) $$\varphi(x) = \sum_{\mu=0}^{3} \frac{\varphi^{(\mu)}(b)}{\mu!}(x-b)^{\mu} + R \qquad \text{mit } |R| < M\delta^4 \text{ so wie}$$

(2.4) $$F(x) = F(b) + \sum_{\nu=0}^{3} \frac{F^{(k+\nu)}(b)}{\underline{|k+\nu}}(x-b)^{k+\nu} + R_1 \qquad \text{mit } |R_1| < M\delta^{k+4},$$

wo $k \geqq 1$ ist. Da aber $F(x)$ an $x=b$ maximal wird, so soll

$$\frac{F^{(k)}(b)}{k!}(x-b)^k < 0$$

sein, und wegen $x-b<0$

(2.5) $$(-1)^{k-1}F^{(k)}(b) > 0\,.$$

Setzt man ferner

(2.6) $$B = \left(\frac{F^{(k)}(b)}{(-1)^{k-1}\underline{|k}}\right)^{1/k}, \quad Bn^{1/k} = N, \quad x-b = -\frac{u}{N}, \qquad (\text{so daß } u>0)$$

$$B_\nu = \frac{F^{(k+\nu)}(b)}{\underline{|k+\nu}} \Bigg/ \frac{F^{(k)}(b)}{(-1)^{k-1}\underline{|k}}, \quad B_0 = (-1)^{k-1},$$

und folglich

$$\frac{nF^{(k+\nu)}(b)}{\underline{|k+\nu}}\left(\frac{-u}{N}\right)^{k+\nu} = B_\nu \frac{(-u)^{k+\nu}}{N^\nu},$$

so erhält man

$$\text{(ii)} \simeq \int_0^{N\delta}\left[\varphi(b) - \varphi'(b)\frac{u}{N} + \frac{\varphi''(b)}{2}\left(\frac{u}{N}\right)^2 - \frac{\varphi'''(b)}{6}\left(\frac{u}{N}\right)^3 + \frac{\varphi''''(\xi)}{24}\left(\frac{u}{N}\right)^4\right]$$
$$\times \exp\left[-u^k - B_1(-u)^k\frac{u}{N} + B_2(-u)^k\left(\frac{u}{N}\right)^2 - B_3(-u)^k\left(\frac{u}{N}\right)^3 + B_4(\xi_1)(-u)^k\left(\frac{u}{N}\right)^4\right]\frac{du}{N}.$$

Schreibt man schließlich

(2.7) $$u^k = v\,, \quad \frac{u}{N} = \frac{1}{B}\left(\frac{v}{n}\right)^{1/k}, \quad \frac{du}{N} = \frac{1}{kB}\left(\frac{v}{n}\right)^{1/k}\frac{dv}{v},$$

so gilt für $n \to \infty$

(2.8) $$\text{(ii)} \simeq \int_0^\infty\left[\varphi(b) - \frac{\varphi'(b)}{B}\left(\frac{v}{n}\right)^{1/k} + \frac{\varphi''(b)}{2B^2}\left(\frac{v}{n}\right)^{2/k} - \frac{\varphi'''(b)}{6B^3}\left(\frac{v}{n}\right)^{3/k} - \frac{\varphi''''(\xi)}{24B^4}\left(\frac{v}{n}\right)^{4/k}\right]$$
$$\times e^{-v}\prod_{\nu=1}^{3}\exp\left\{(-1)^{k+\nu}\frac{B_\nu}{B^\nu}v\left(\frac{v}{n}\right)^{\nu/k}\right\}\exp\left\{(-1)^k\frac{B_4(\xi_1)}{B^4}v\left(\frac{v}{n}\right)^{4/k}\right\}\frac{1}{kB}\left(\frac{v}{n}\right)^{1/k}\frac{dv}{v}.$$

Wie vom Vergleiche der beiden Ausdrücke (1.8) und (2.8) ersichtbar ist, ergibt sich der letztere aus dem vorigen dadurch, daß man A^ν als Koeffizient im Integrande, und A_ν mit $(-B)^\nu$ und $(-1)^k B_\nu$ bzw. vertauscht. Also besteht es parallel mit (1.9)

$$(2.9)\quad \int_a^b \varphi(x)\exp\{nF(x)\}\,dx \simeq \frac{\exp[nF(b)]}{B}\Big(\frac{1}{n}\Big)^{1/k}\Big\{\Gamma\Big(1+\frac{1}{k}\Big)\varphi(b)$$
$$-\frac{\Gamma(1+2/k)}{B}\Big(\frac{1}{n}\Big)^{1/k}\Big[\frac{\varphi'(a)}{2}+\varphi(b)\frac{(-1)^k}{k}B_1\Big]$$
$$+\frac{\Gamma(1+3/k)}{B^2}\Big(\frac{1}{n}\Big)^{2/k}\Big[\frac{\varphi''(b)}{6}+\varphi'(b)\frac{(-1)^k}{k}B_1+\varphi(b)\Big(\Big(1+\frac{3}{k}\Big)\frac{B_1^2}{2k}+(-1)^k\frac{B_2}{k}\Big)\Big]$$
$$-\frac{\Gamma(1+4/k)}{B^3}\Big(\frac{1}{n}\Big)^{3/k}\Big[\frac{\varphi'''(b)}{24}+\frac{\varphi''(b)}{2k}(-1)^kB_1$$
$$+\varphi'(b)\Big(\Big(1+\frac{4}{k}\Big)\frac{1}{2k}B_1^2+\frac{(-1)^k}{k}B_2\Big)+\varphi(b)\Big(\Big(2+\frac{4}{k}\Big)\Big(1+\frac{4}{k}\Big)\frac{(-1)^4B_1^3}{6k}$$
$$+\Big(1+\frac{4}{k}\Big)\frac{1}{k}B_1B_2+\frac{(-1)^k}{k}B_3\Big)\Big]+O\Big(\Big(\frac{1}{n}\Big)^{4/k}\Big)\Big\}.$$

Es ist besonders für $k=1$

$$(2.10)\quad \int_a^b \varphi(x)\exp[nF(x)]\,dx \simeq \frac{\exp[nF(b)]}{nF'(b)}\Big\{\varphi(b)-\frac{1}{nF'(b)}\Big[\varphi'(b)-\frac{\varphi(b)F''(b)}{F'(b)}\Big]$$
$$+\frac{1}{n^2F'(b)^2}\Big[\varphi''(b)-3\varphi'(b)\frac{F''(b)}{F'(b)}+\varphi(b)\Big(\frac{3F''(b)^2}{F'(b)^2}-\frac{F'''(b)}{F'(b)}\Big)\Big]$$
$$-\frac{1}{n^3F'(b)^3}\Big[\varphi'''(b)-6\varphi''(b)\frac{F''(b)}{F'(b)}+\varphi'(b)\Big(\frac{15F''(b)^2}{F'(b)^2}-\frac{4F'''(b)}{F'(b)}\Big)$$
$$-\varphi(b)\Big(\frac{15F''(b)^3}{F'(b)^3}-\frac{10F''(b)F'''(b)}{F'(b)^2}+\frac{F''''(b)}{F'(b)}\Big)\Big]+O\Big(\frac{1}{n^4}\Big)\Big\},$$

worin der Ausdruck zwischen äußersten geschweiften Klammern zwar ganz dieselbe Gestalt wie (1.11) hat, außer daß a durch b ersetzt und das Zeichen sämtlichen Faktors verändert ist.

Auch für $k=2$ gilt

$$(2.11)\quad \int_a^b \varphi(x)\exp\{nF(x)\}\,dx \simeq \frac{1}{2}\sqrt{\frac{-2\pi}{nF''(b)}}\exp[nF(b)]\Big\{\varphi(b)-\sqrt{\frac{-2}{n\pi F''(b)}}\Big[\varphi'(b)$$
$$-\frac{\varphi(b)}{3}\frac{F'''(b)}{F''(b)}\Big]+\frac{1}{-2nF''(b)}\Big[\varphi''(b)-\varphi'(b)\frac{F'''(b)}{F''(b)}+\frac{\varphi(b)}{4}\Big(\frac{5}{3}\frac{F'''(b)^2}{F''(b)^2}-\frac{F''''(b)}{F''(b)}\Big)\Big]$$
$$-\frac{1}{3}\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{-nF''(b)}^3}\Big[\varphi'''(b)-2\varphi''(b)\frac{F'''(b)}{F''(b)}+\varphi'(b)\Big(\frac{2F'''(b)^2}{F''(b)^2}-\frac{F''''(b)}{F''(a)}\Big)$$
$$-\varphi(b)\Big(\frac{8}{9}\frac{F'''(b)^3}{F''(b)^3}-\frac{F'''(b)F''''(b)}{F''(b)^2}+\frac{1}{5}\frac{F'''''(b)}{F''(b)}\Big)\Big]+O\Big(\frac{1}{n^2}\Big)\Big\},$$

was auch ganz dieselbe Gestalt wie (1.12) hat, außer daß dortige a durch b ersetzt werden und weiter die Zeichen der Glieder mit eckigen Klammern jetzt wechselweise veränderen.

§ **3.** Zusammengelaßen die im vorigen Abschnitte gelösten zwei Aufgaben, haben wir die

Aufgabe 3. *Die reellen Funktionen* $\varphi(x)$ *und* $f(x)$ $(\geqq 0)$ *seien im endlichen oder unendlichen Intervalle* $I: a < x < b$ *definiert und den folgenden Bedingungen unterworfen:*

1° *Die Funktionen* $\varphi(x)[f(x)]^n$ *für* $n=0, 1, 2, \cdots$ *oder sonst für alle* $n \geqq$ *passend großes* $m (>0)$ *seien absolut integrabel in* I.

2° *Die Fnnktion* $f(x)$ *sei nicht negativ und* $F(x) = \log f(x)$ *erreiche am inneren Punkt* $x=c$ *ihr Maximum und zwar sei die obere Grenz von* $F(x)$ *in jedem abgeschlossenen Intervalle, das* c *uicht enthält, kleiner als* $F(c)$. *Ferner sei* $F(x)$ *in jenen nachbarstehenden Unterintervalle* $U: c-\varepsilon < x < c$ *und* $V: c < x < c+\varepsilon'$ *stetig differenzierbar bis zu passender Ordnung, aber möglicherweise* $F^{(\nu)}(c-0) \neq F^{(\nu)}(c+0)$.

3° *Auch existiere* $\varphi^{(4)}(x)$ *schlechthin*[3] *in* $U \cup V = W$ *und dort sei sogar stetig. Man suche die asymptotische Formel für das Integral*

$$\int_a^b \varphi(x)[f(x)]^n dx = \int_a^b \varphi(x) \exp[nF(x)]\,dx \qquad \text{bei } n \to \infty\,. \tag{3.1}$$

Nach (1.9) und (2.9) können wir sofort die Lösung erhalten, wie folgt: Zunächst seien

$$F'(c+0) = \cdots = F^{(h-1)}(c+0) = 0^{4)} \text{ aber } F^{(h)}(c+0) < 0$$
$$F'(c-0) = \cdots = F^{(k-1)}(c-0) = 0 \quad \text{aber } (-1)^{k-1} F^{(k)}(c-0) > 0\,,$$

somit sind die rechtseitigen und linkseitigen Koeffizienten

$$\begin{cases} R = \left(\dfrac{F^{(h)}(c+0)}{-\lfloor h}\right)^{1/h}, \quad R_\nu = \dfrac{F^{(h+\nu)}(c+0)}{\lfloor h+\nu} \Big/ \dfrac{F^{(h)}(c+0)}{-\lfloor h}\,, \\ L = \left(\dfrac{F^{(k)}(c-0)}{(-1)^{k-1}\lfloor k}\right)^{1/k}, \quad L_\nu = \dfrac{F^{(k+\nu)}(c-0)}{\lfloor k+\nu} \Big/ \dfrac{F^{(k)}(c-0)}{(-1)^{k-1}\lfloor k} \quad \text{bzw.} \end{cases} \tag{3.2}$$

Womit ist die Lösung folgendermaßen dargestellt:

$$\begin{aligned} \int_a^b \varphi(x) \exp[nF(x)]\,dx \simeq\ & \frac{\exp[nF(c+0)]}{R}\left(\frac{1}{n}\right)^{1/h}\Big\{\Gamma\Big(1+\frac{1}{h}\Big)\varphi(c) \\ & + \frac{\Gamma(1+2/h)}{R}\Big[\frac{\varphi'(c)}{2} + \frac{R_1}{h}\varphi(c)\Big]\Big(\frac{1}{n}\Big)^{1/h} \\ & + \frac{\Gamma(1+3/h)}{R^2}\Big[\frac{\varphi''(c)}{6} + \frac{\varphi'(c)}{h}R_1 + \frac{\varphi(c)}{h}\Big(\frac{h+3}{2h}R_1^2 + R_2\Big)\Big]\Big(\frac{1}{n}\Big)^{2/h} \\ & + \frac{\Gamma(1+4/h)}{R^3}\Big[\frac{\varphi'''(c)}{24} + \frac{\varphi''(c)}{2h}R_1 + \frac{\varphi'(c)}{h}\Big(\frac{h+4}{2h}R_1^2 + R_2\Big) \\ & + \frac{\varphi(c)}{h}\Big(\frac{(h+2)(h+4)}{3h^2}R_1^3 + \frac{h+4}{h}R_1R_2 + R_3\Big)\Big]\Big(\frac{1}{n}\Big)^{3/h} + O\Big(\frac{1}{n}\Big)^{4/h}\Big\} \\ & + \frac{\exp[nF(c-0)]}{L}\Big(\frac{1}{n}\Big)^{1/k}\Big\{\Gamma\Big(1+\frac{1}{k}\Big)\varphi(c) - \frac{\Gamma(1+2/k)}{L}\Big[\frac{\varphi'(c)}{2} + \frac{(-1)^k}{k}L_1\varphi(c)\Big]\Big(\frac{1}{n}\Big)^{1/k} \end{aligned} \tag{3.3}$$

---

3) Dies ist etwas Einfachheit wegen vorausgesetzt worden.

4) Sonst $h=1$ wenn $F'(c+0) < 0$; oder $k=1$, wenn $F'(c-0) > 0$.

$$+\frac{\Gamma(1+3/k)}{L^2}\left[\frac{\varphi''(c)}{6}+\frac{\varphi'(c)}{k}(-1)^k L_1+\frac{\varphi(c)}{k}\left(\frac{k+3}{2}L_1^2+(-1)^k L_2\right)\right]\left(\frac{1}{n}\right)^{2/k}$$

$$-\frac{\Gamma(1+4/k)}{L^3}\left[\frac{\varphi'''(c)}{24}+\frac{\varphi''(c)}{2k}(-1)^k L_1+\frac{\varphi'(c)}{k}\left(\frac{k+4}{2k}L_1^2+(-1)^k L_2\right)\right.$$

$$\left.+\frac{\varphi(c)}{k}\left(\frac{(-1)^k(k+2)(k+4)}{3k^2}L_1^3+\frac{k+4}{k}L_1L_2+(-1)^k L_3\right)\right]\left(\frac{1}{n}\right)^{3/k}+O\left(\frac{1}{n}\right)^{4/k}\Bigg\}.$$

Im Falle, daß die Funktion $F(x)$ in bezug auf dem Punkte $x=c$ *symmetrisch* sich verfährt, gelten

(3.4) $$h=k=l \quad \text{und sogar} \quad F^{(\nu)}(c-0)=(-1)^\nu F^{(\nu)}(c+0).$$

Überdies ist nach Voraussetzung

(3.5) $$-F^{(h)}(c+0)>0$$

und hieraus von selbst auch

(3.6) $$(-1)^{l-1}F^{(l)}(c-0)=(-1)^{l-1}(-1)^l F^{(l)}(c+0)=-F^{(h)}(c+0)>0.$$

folgt. Also werden

(3.7) $$L=R \quad \text{so wie} \quad L_\nu=(-1)^{l+\nu}R_\nu.$$

Daher gilt für den mit *geradem* l versehenden symmetrischen Fall

(3.8) $$\int_a^b \varphi(x)\exp[nF(x)]\,dx \simeq \frac{\exp[nF(c)]}{R}\left(\frac{1}{n}\right)^{1/l}\Bigg\{2\Gamma\left(1+\frac{1}{l}\right)\varphi(c)$$

$$+\frac{\Gamma(1+2/l)}{R}\frac{2R_1}{l}\varphi(c)\left(\frac{1}{n}\right)^{1/l}$$

$$+\frac{\Gamma(1+3/l)}{R^2}\left[\frac{\varphi''(c)}{3}+\frac{\varphi(c)}{l}\left(\frac{l+3}{l}R_1^2+2R_2\right)\right]\left(\frac{1}{n}\right)^{2/l}$$

$$+\frac{\Gamma(1+4/l)}{R^3}\left[\frac{\varphi''(c)}{l}R_1+\frac{2\varphi(c)}{l}\left(\frac{(l+2)(l+4)}{3l^2}R_1^3\right.\right.$$

$$\left.\left.+\frac{l+4}{l}R_1R_2+R_3\right)\right]\left(\frac{1}{n}\right)^{3/l}+O\left(\frac{1}{n}\right)^{4/l}\Bigg\},$$

während, für den mit *ungeradem* l versehenden symmetrischen Fall,

(3.9) $$\int_a^b \varphi(x)\exp[nF(x)]\,dx \simeq \frac{\exp[nF(c)]}{R}\left(\frac{1}{n}\right)^{1/l}\Bigg\{2\Gamma\left(1+\frac{1}{l}\right)\varphi(c)$$

$$+\frac{\Gamma(1+2/l)}{R}\frac{2R_1}{l}\varphi(c)\left(\frac{1}{n}\right)^{1/l}$$

$$+\frac{\Gamma(1+3/l)}{R^2}\left[\frac{\varphi''(c)}{3}+\frac{\varphi(c)}{l}R_1\left(\frac{l+3}{l}R_1^2+2R_2\right)\right]\left(\frac{1}{n}\right)^{2/l}$$

$$+\frac{\Gamma(1+4/l)}{R^3}\left[\frac{\varphi''(c)}{l}R_1+\frac{2\varphi(c)}{l}\left(\frac{(l+2)(l+4)}{3l^2}R_1^3\right.\right.$$

$$\left.\left.+\frac{l+4}{l}R_1R_2+R_3\right)\right]\left(\frac{1}{n}\right)^{3/l}+O\left(\frac{1}{n}\right)^{4/l}\Bigg\}.$$

Schließlich sei die Funktion $F(x)$ auf eine Umgebung von $x=c$ *regulär*, oder wenigstens stetig differenzierbar zur passenden Ordnung, so daß

$$F^{(\nu)}(c+0)=F^{(\nu)}(c-0)=F^{(\nu)}(c) \tag{3.10}$$

gilt. Da $F'(c)=\cdots=F^{(l-1)}(c)=0$ aber $F^{(l)}(c)\neq 0$ und das Maximum von $F(x)$ tatsächlich an $x=c$ stattfindet, so braucht sich die Ordnungszahl l bekanntlich[5] eine gerade Zahl $(=2m)$ zu sein. Daher erhält man nach (3. 2)

$$R=L=\left(\frac{F^{(2m)}(c)}{-\underline{|2m}}\right)^{1/2m},\quad R_\nu=L_\nu=\frac{F^{(2m+\nu)}(c)}{\underline{|2m+\nu}}\Big/\frac{F^{(2m)}(c)}{-\underline{|2m}}, \tag{3.11}$$

so daß jede ungerade Potenz von $\left(\frac{1}{n}\right)^{1/2m}$ im Ausdruck unter äußersten Klammern von (3. 3) sich aufhebt, und also

$$\int_a^b \varphi(x)\exp[nF(x)]\,dx \simeq \frac{\exp[nF(c)]}{R}\left(\frac{1}{n}\right)^{1/2m}\left\{\frac{1}{m}\Gamma\left(\frac{1}{2m}\right)\varphi(c)\right. + \frac{3}{2m}\Gamma\left(\frac{3}{2m}\right)\frac{1}{R^2}\left[\frac{\varphi''(c)}{3}+\frac{\varphi'(c)}{m}R_1+\frac{\varphi(c)}{m}\left(\frac{2m+3}{4m}R_1^2+R_2\right)\right]\left(\frac{1}{n}\right)^{1/m}\left.+O\left(\frac{1}{n}\right)^{2/m}\right\}. \tag{3.12}$$

Oder, ausführlich gedrückt,

$$\int_a^b \varphi(x)\exp[nF(x)]\,dx \simeq \frac{\exp[nF(c)]}{(nF^{(2m)}(c)/-\underline{|2m})^{1/2m}}\left\{\frac{1}{m}\Gamma\left(\frac{1}{2m}\right)\varphi(c)\right. + \frac{3}{2m}\Gamma\left(\frac{3}{2m}\right)\frac{1}{(nF^{(2m)}(c)/-\underline{|2m})^{1/m}}\left[\frac{\varphi''(c)}{3}-\frac{\varphi'(c)F^{(2m+1)}(c)}{m(2m+1)F^{(2m)}(c)}\right. + \frac{\varphi(c)}{m(2m+1)}\left(\frac{2m+3}{4m(2m+1)}\left(\frac{F^{(2m+1)}(c)}{F^{(2m)}(c)}\right)^2-\frac{F^{(2m+2)}(c)}{2(m+1)F^{(2m)}(c)}\right)\left.\right]\left.+O\left(\frac{1}{n}\right)^{2/m}\right\}. \tag{3.13}$$

Ins besondere für $m=1$

$$\int_a^b \varphi(x)\exp[nF(x)]\,dx \simeq \sqrt{\frac{2\pi}{-nF''(c)}}\exp\{nF(c)\}\left\{\varphi(c)-\frac{1}{2nF''(c)}\left[\varphi''(c)\right.\right. - \frac{\varphi'(c)F'''(c)}{F''(c)}+\varphi(c)\left(\frac{5}{12}\left(\frac{F'''(c)}{F''(c)}\right)^2-\frac{F''''(c)}{4F''(c)}\right)\left.\right]\left.+O\left(\frac{1}{n^2}\right)\right\}, \tag{3.14}$$

was eine schon in früheren Zeiten von einem der Verfässer gefundene Formel ist[6]. Dabei sind die weiteren Korrekturglieder, wenn ferner ausgerechnet:

$$O\left(\frac{1}{n^2}\right) \simeq \frac{1}{8n^2}\left[\frac{\varphi''''}{F''^2}-\frac{1}{F''^3}\left(\frac{10}{3}F'''\varphi'''+\frac{5}{2}F''''\varphi''+F'''''\varphi'+\frac{1}{6}F''''''\varphi\right)\right. + \frac{35}{6F''^4}\left(F'''^2\varphi''+F'''F''''\varphi'+\frac{1}{5}F'''F'''''\varphi+\frac{1}{8}F''''^2\varphi\right) - \frac{35}{2F''^5}\left(\frac{1}{3}F'''^3\varphi'+\frac{1}{4}F'''^2F''''\varphi\right)+\left.\frac{385}{144}\frac{F'''^4}{F''^6}\varphi\right]. \tag{3.14.1}$$

---

5) In der Tat müßen nach unsren Kriterien (1.5) (2.5), beide Ungleichungen

$$F^{(l)}(c+0)=F^{(l)}(c)<0 \text{ und } (-1)^{l-1}F^{(l)}(c-0)=(-1)^{l-1}F^{(l)}(c)>0$$

zugleich bestehen, was meint daß l notwendig gerad ist.

6) Y. Watanabe, Methode der kleinsten Quadrate und Statistik, (1942), S. 556. (im Japanische).

Ebenso tun dieselben Korrekturglieder für den Fall, daß das Maximum am Ende auftritt, d.h. für (1. 12) so wie (2. 11).

§ **4.** Endlich wollen wir noch einige Beispielen hinzufügen. Es sei $\varphi(x)$ nicht nur absolut integrabel[7] in $-\infty<x<\infty$, auch sei regulär um $x=0$.

1. Es sei erstens $f(x)=e^{-x}$ für $0\leqq x<\infty$ definiert. Somit $F(x)=\log f(x)=-x$ und $F'(x)=-1$ aber sonstige $F^{(\nu)}(x)=0$ für alle ν. Trotzdem, weil unsre Voraussetzungen sämtlich erfüllt sind, kann die Lösung der Aufgabe 1 angewandt werden. Zwar nach (1. 11) ist

$$(4.1)\quad \int_0^\infty \varphi(x)\exp\{-nx\}\,dx \cong \frac{1}{n}\left\{\varphi(0)+\frac{1}{n}\varphi'(0)+\frac{1}{n^2}\varphi''(0)+\frac{1}{n^3}\varphi'''(0)+O\left(\frac{1}{n^4}\right)\right\}.$$

Zweitens sei $f(x)=e^{px}$ mit $p>0$ für $x\leqq 0$. Hier sind wiedermals für $F(x)=px$, $F'(x)=p$, aber sonst $F^{(\nu)}(x)=0$ und nach (2. 10) gilt

$$(4.2)\quad \int_{-\infty}^0 \varphi(x)\exp\{npx\}\,dx \cong \frac{1}{np}\left\{\varphi(0)-\frac{1}{np}\varphi'(0)+\frac{1}{n^2p^2}\varphi''(0)-\frac{\varphi'''(0)}{n^3p^3}+O\left(\frac{1}{n^4}\right)\right\},$$

die besonders bei $p=1$

$$\int_{-\infty}^0 \varphi(x)\exp[nx]\,dx \cong \frac{1}{n}\left\{\varphi(0)-\frac{1}{n}\varphi'(0)+\frac{1}{n^2}\varphi''(0)-\frac{1}{n^3}\varphi'''(0)+O\left(\frac{1}{n^4}\right)\right\}$$

wird. Diese mit die vorhergehende zusammengesetzt, liefert

$$(4.2)\quad \int_{-\infty}^\infty \varphi(x)e^{-n|x|}\,dx \cong \frac{2}{n}\left\{\varphi(0)+\frac{1}{n^2}\varphi''(0)+O\left(\frac{1}{n^4}\right)\right\}.$$

Es sei allgemeins die Funktion $f(x)=\exp\{-|x|^{2p+1}\}$, wo p positiv ganz, im Intervalle $-\infty<x<\infty$ symmetrisch, sonach mit ihrem Maximum an $x=0$, definiert. Man setze in (3. 8) $l=2p+1$, $R=1$, $R_\nu=0$ und erhält

$$(4.4)\quad \int_{-\infty}^\infty \varphi(x)\exp\{-n|x|^{2p+1}\}\,dx \cong \left(\frac{1}{n}\right)^{1/(2p+1)}\left\{2\Gamma\left(\frac{2p+2}{2p+1}\right)\varphi(0)\right.$$
$$\left.+\Gamma\left(\frac{2p+4}{2p+1}\right)\frac{\varphi''(0)}{3}\left(\frac{1}{n}\right)^{2/(2p+1)}+\left(\frac{1}{n}\right)^{4/2p+1}\right\}.$$

2. Obgleich die in $-\infty<x<\infty$ definierte Funktion $f(x)=\left(\frac{\sin x}{x}\right)^2$ unendlich viele Maxima besitzt, erreicht sie am Anfangspunkte zwar größten Wert $(=1)$, und unsre Voraussetzungen sämtlich genügt sind. Aus die Entwickelung $F(x)=\log f(x)=-\frac{1}{3}x^2-\frac{1}{90}x^4+\cdots$ folgt, daß $F(0)=0$, $F^{(2\nu+1)}(0)=0$, $F''(0)=-\frac{2}{3}$, $F^{IV}(0)=-\frac{4}{15}$ und damit nach (3. 14) erhält man

$$(4.5)\quad \int_{-\infty}^\infty \varphi(x)\left(\frac{\sin x}{x}\right)^{2n}dx \cong \sqrt{\frac{3\pi}{n}}\left\{\varphi(0)+\frac{3}{4n}\left[\varphi''(0)-\frac{1}{10}\varphi(0)\right]+O\left(\frac{1}{n^2}\right)\right\}.$$

---

7) Mit Ausnahme von drei letzteren Beispielen 7, 8, 9, in welche aber $\varphi(x)f(x)^n$ für alle $n\geqq$ gewißes $m>0$ absolut integrabel und daher unsre Formeln noch gültig sind.

3. Die Funktion $f(x)=e^{-x^2}(1+x^2)$ hat ihren Maximalwert $(=1)$ am Anfangspunkte, weil $f'(x)=-2x^3e^{-x^2}$ ist. Aus lauter Entwickelung $F(x) =\log f(x)=-\frac{x^4}{2}+\frac{x^6}{3}+\cdots$ sieht man sogleich, daß $F^{(2\nu+1)}(0)=0$, $F''(0)=0$, $F''''(0)=-12$, $F^{\mathrm{VI}}(0)=240$. Also $l=2m=4$ und erhält nach (3. 13)

$$\text{(4. 6)}\quad \int_{-\infty}^{\infty}\varphi(x)e^{-nx^2}(1+x^2)^n dx \cong \left(\frac{2}{n}\right)^{1/4}\left\{\frac{1}{2}\Gamma\left(\frac{1}{4}\right)\varphi(0)\right.$$
$$\left.+\sqrt{\frac{2}{n}}\frac{1}{4}\Gamma\left(\frac{3}{4}\right)\left[\varphi''(0)+\varphi(0)\right]+O\left(\frac{1}{n}\right)\right\}.$$

4. Die Funktion $f(x)=\exp[-x^2(x^2-2)]$ wird an den symmetrischen Punkte $x=\pm1$ doppelt maximal, wie aus $f'(x)=4x(1-x^2)\exp[-x^2(x^2-2)]$ gesehen werden kann. Jedoch möge man sich aufs Halbintervalle $0<x<\infty$ denken, wo das Maximum nur einmal am Punkt $x=1$ geschieht. Nun folgt aus $F(x)=\log f(x)=2x^2-x^4$, daß, der Reihe nach, $F'(x)=4x(1-x^2)$, $F''(x) =4(1-3x^2)$, $F'''(x)=-24x$, $F''''(x)=-24$, womit $F(1)=1$, $F'(1)=0$, $F''(1) =-8$, $F'''(1)=-24$, $F''''(1)=-24$. Daher, wieder nach (3. 14) erhält man

$$\text{(4. 7)}\quad \int_{0}^{\infty}\varphi(x)\exp\{-nx^2(x^2-2)\}\,dx \cong \frac{1}{2}\sqrt{\frac{\pi}{n}}\,e^n\left\{\varphi(1)\right.$$
$$\left.+\frac{1}{16n}[\varphi''(1)-3\varphi'(1)+3\varphi(1)]+O\left(\frac{1}{n^2}\right)\right\}.$$

Da nun die Funktion symmetrisch in bezug auf den Anfangspunkt liegt, so lautet das Integral, welches längs negativer Achse erstreckt wird,

$$\int_{-\infty}^{0}\varphi(x)\exp\{-nx^2(x^2-2)\}\,dx \simeq \frac{1}{2}\sqrt{\frac{\pi}{n}}\,e^n\left\{\varphi(-1)\right.$$
$$\left.+\frac{1}{16n}[\varphi''(-1)+3\varphi'(-1)+3\varphi(-1)]+O\left(\frac{1}{n^2}\right)\right\},$$

und daher

$$\text{(4. 8)}\quad \int_{-\infty}^{\infty}\varphi(x)\exp\{-nx^2(x^2-2)\}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{n}}\,e^n\left\{\varphi(1)+\varphi(-1)\right.$$
$$\left.+\frac{1}{16n}\Big[\varphi''(1)+\varphi''(-1)-3(\varphi'(1)-\varphi'(-1))+3(\varphi(1)+\varphi(-1))\Big]+O\left(\frac{1}{n^2}\right)\right\}.$$

5. Für die analogu mit der vorhergehende aber nun unsymmetrisch verlaufende Funktion $f(x)=\exp\left\{x^2+\frac{x^3}{3}-\frac{x^4}{4}\right\}$, $F(x)=x^2+\frac{x^3}{3}-\frac{x^4}{4}$, treten die Maxima an $x=-1$ und $x=2$ auf. Man sieht leicht daß $F(-1)=\frac{5}{12}$, $F'(-1)=0$, $F''(-1)=-3$, $F'''(-1)=8$, $F''''(-1)=-6$, so wie $F(2)=\frac{8}{3}$, $F'(2)=0$, $F''(2)=-6$, $F'''(2)=-10$, $F''''(2)=-6$. Mit Benützung von (3. 14) wiedermals, ergeben sich

$$(4.9)\quad \int_{-\infty}^{0} \varphi(x) \exp\left\{n\left(x^2+\frac{x^3}{3}-\frac{x^4}{4}\right)\right\}dx \simeq \sqrt{\frac{2\pi}{3n}} \exp\left(\frac{5n}{12}\right)\left\{\varphi(-1)\right.$$
$$\left. +\frac{1}{6n}\left[\varphi''(-1)+\frac{8}{3}\varphi'((-1)+\frac{133}{54}\varphi(-1)\right]+O\left(\frac{1}{n^2}\right)\right\},$$

$$(4.10)\quad \int_{0}^{\infty} \varphi(x) \exp\left\{n\left(x^2+\frac{x^3}{3}-\frac{x^4}{4}\right)\right\}dx \simeq \sqrt{\frac{\pi}{3n}} \exp\left(\frac{8n}{3}\right)\left\{\varphi(2)\right.$$
$$\left. -\frac{1}{2n}\left[\varphi''(2)-\frac{5}{3}\varphi'(2)+\frac{49}{54}\varphi(2)\right]+O\left(\frac{1}{n^2}\right)\right\}.$$

Weil $\exp\left\{\frac{5}{12}n\right\} \simeq 0\left(\exp\left\{\frac{8}{3}n\right\}\right)$ für genug größes n befriedigt ist, so ist (4.9) im Vergleich zu (4.10) vernachlässigbar. Wenn tatsächlich man das verbundene Intervall $-\infty < x < \infty$ als ganzes betrachtet, und (3.14) anwendet, so erhält man, als asymtotischer Wert, bloß den rechts stehenden Ausdruck von (4.10).

6. Die in $-\infty < x < \infty$ definierte Funktion $f(x)=\exp\{-x^{2/3}\}$ deutlich erreicht ihr Maximum an $x=0$. Da aber $F(x)=\log f(x)=-x^{2/3}$, und $F'(x) =-\frac{2}{3}x^{-1/3}\to\infty$ für $x\to 0$ zustrebt, so stellt sich heraus, daß unsre Lösung nutzlos wird. Führt jedoch man durch die Gleichung $x=\xi^3$ eine neue Variable ξ ein und setzt $\varphi(x)dx=3\xi^2\varphi(\xi^3)d\xi=\varphi_1(\xi)d\xi$, so kann man schreiben

$$\int_{-\infty}^{\infty} \varphi(x) \exp\{-nx^{2/3}\}\,dx = \int_{-\infty}^{\infty} \varphi_1(\xi) \exp\{-n\xi^2\}\,d\xi\,.$$

Vorausgesetzt, daß $\varphi(x)=0\ (|x|^{\omega})$ bei wieviel großem ω für $|x|\to\infty$ gilt, so bleibt es auch dasselbe für neue Funktion $\varphi_1(\xi)$, und unsre Methode anwendbar. Wir erhalten das einfache Resultat

$$(4.11)\quad \int_{-\infty}^{\infty} \varphi(x) \exp\{nx^{-2/3}\}\,dx = \int_{-\infty}^{\infty} \varphi_1(\xi) \exp\{-n\xi^2\}\,d\xi \simeq \sqrt{\frac{\pi}{n}}\left\{\frac{\varphi(0)}{2n}+O\left(\frac{1}{n^2}\right)\right\}.$$

Wir wollen noch drei weitere praktisch angewandte Beispiele notieren.

7. Man soll die Stirlingsche asymptotische Formel der Gammafunktion $\Gamma(n+1)$ wiederfinden. Aus

$$\Gamma(n+1)=\int_0^{\infty} t^n e^{-t}\,dt = n^{n+1}\int_0^{\infty} x^n e^{-nx}\,dx \qquad (t=nx)$$
$$= n^{n+1}\int_0^{\infty} \exp\left[-n(x-\log x)\right]dx$$

folgt, daß $\varphi(x)=1$ und $F(x)=-x+\log x$ und die letztere Funktion für $x=1$ maximal mit

$$F(1)=-1\,,\quad F'(1)=0\,,\quad F''(1)=-1,\quad F'''(1)=2\,,\quad F''''(1)=-6$$

wird. Daher erhält man gemüß (3.14)

(4. 12) $$\Gamma(n+1) \simeq n^{n+1}\sqrt{\frac{2\pi}{n}}\,e^{-n}\left\{1+\frac{1}{12n}+O\left(\frac{1}{n^2}\right)\right\}$$
$$= \sqrt{2\pi n}\,n^n \exp\left\{-n+\frac{1}{12n}+O\left(\frac{1}{n^2}\right)\right\}.$$

Das letzte Resultat wird dadurch versichert, daß von (3. 14. 1) aus $O(1/n^2) \simeq 1/288n^2$ gezeigt werden kann.

8. Es seien die n Paare von Abweichungen $x_i-m_1=u_i$, $y_i-m_2=v_i$, $(u_1, v_1)\cdots, (u_n, v_n)$ in einer aus normaler Bevölkerung $N(x, y, m_1, m_2, \sigma_1, \sigma_2, \rho)$ genommenen Stichprobe. Bekanntlich ist der empirische Korrelationskoeffizient durch

$$r=\sum_{i=1}^{n} u_i v_i / \sqrt{\sum u_i^2 \sum v_i^2}$$

mit der Wahrscheinlichkeitsdichte

(4. 13) $$f_n(r)=\frac{n-2}{\pi}(1-\rho)^{(n-1)/2}(1-r^2)^{(n-4)/2}\int_0^1 \frac{t^{n-2}}{(1-\rho r t)^{n-1}}\frac{dt}{\sqrt{1-t^2}}$$

gegeben[8]. Dieses Integral wird für $t=\operatorname{sech} s$ lediglich auf

$$\int_0^\infty \frac{ds}{(\cosh s-\rho r)^{n-1}}$$

reduziert, das die nämliche Gestalt (1. 1) hat, wofür

$$\varphi(s)=\cosh s-\rho r\,, \quad F(s)=\log f(s)=\log 1/(\cosh s-\rho r)\,.$$

Zwar erreicht $F(s)$ am anfänglichen Punkte $s=0$ ihr Maximum $F(0) = \log 1/(1-\rho r)$; und

$$F^{(2\nu+1)}(0)=0\,,\quad F''(0)=\frac{-1}{1-\rho r}\,,\quad F''''(0)=\frac{2+\rho r}{(1+\rho r)^2}\,,\quad F''''''(0)=-\frac{16+13\rho r+\rho^2 r^2}{(1-\rho r)^3}\,,$$

während $\varphi(0)=1-\rho r$, $\varphi^{(2\nu+1)}(0)=0$, $\varphi^{(2\nu)}(0)=1$ bei $\nu>0$. Damit haben wir nach (1. 12)

$$\int_0^\infty \simeq \sqrt{\frac{\pi}{2n}}\frac{1}{(1-\rho r)^{n-3/2}}\left\{1+\frac{6+\rho r}{8n}+O\left(\frac{1}{n^2}\right)\right\},$$

und deshalb geht (4. 13) in

(4. 14) $$f_n(r)\simeq\frac{n-2}{\sqrt{2n\pi}}\frac{(1-\rho^2)^{(n-1)/2}(1-r^2)^{(n-4)/2}}{(1-\rho r)^{n-3/2}}\left\{1+\frac{1}{8n}(6+\rho r)+O\left(\frac{1}{n^2}\right)\right\}$$

über. Dabei betrågen die Glieder $O(1/n^2)$ vermöge (3. 14. 1) um

(4. 14. 1) $$O\left(\frac{1}{n^2}\right)\simeq\frac{100+36\rho r+9\rho^2 r^2}{128n^2}.$$

Aber sind diese Korrekturglieder für jetzt vernachlässigt, jedoch im nachstehenden Beispiele 9 spielen sehr wichtige Rolle.

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8) Herald Cramér, Mathematical Methods of Statistics (1946), p. 398.

Jetzt schätzen wir den Mittelwert von r^k für genügend große n ab:

$$(4.15)\quad M(r^k)=\frac{n-2}{\sqrt{2n\pi}(1-\rho^2)}\int_{-1}^{1}\left(\frac{\sqrt{1-\rho^2}\sqrt{1-r^2}}{1-\rho r}\right)^n\frac{\sqrt{1-\rho r}^3\, r^k}{(1-r^2)^2}\left\{1+\frac{6+\rho r}{8n}\right\}dr\,,$$

das auch als (3.14) mit

$$F(r)=\log f(r)=\frac{1}{2}\log(1-\rho^2)+\frac{1}{2}\log(1-r^2)-\log(1-\rho r)$$

und

$$\varphi(r)=r^k\frac{(1-\rho r)^{3/2}}{(1-r^2)^2}\left\{1+\frac{1}{8n}(6+\rho r)\right\}$$

angesehen werden darf. Also liefert die Gleichung $F'(r)=0$ den Maximalpunkt $r=\rho$ und

$$F(\rho)=0\,,\quad F''(\rho)=\frac{-1}{(1-\rho^2)^2},\quad F'''(\rho)=\frac{-6\rho}{(1-\rho^2)^3},\quad F^{IV}(\rho)=\frac{-6(1+6\rho^2)}{(1-\rho^2)^4}\,,$$

mit

$$\varphi(\rho)=\frac{\rho^k}{\sqrt{1-\rho^2}}\left\{1+\frac{6+\rho^2}{8n}\right\},\quad \frac{\varphi'}{\varphi}(\rho)=\frac{k}{\rho}+\frac{5\rho}{2(1-\rho^2)}\,,$$

$$\frac{\varphi''}{\varphi}(\rho)=\frac{k(k-1)}{\rho^2}+\frac{5k}{1-\rho^2}+\frac{16+35\rho^2}{4(1-\rho^2)^2}\,.$$

Durch Einsetzung dieser Werte in (3.14) findet man

$$(4.16)\qquad M(r^k)\simeq\rho^k\left\{1+\frac{k(1-\rho^2)(k-1-k\rho^2)}{2n\rho^2}\right\},\quad k=0,1,2,\cdots.$$

Somit

$$(4.17)\quad\begin{cases} M(r^0)=1\,,\quad M(r)=\rho\left(1-\dfrac{1-\rho^2}{2n}\right),\quad M(r^2)=\rho^2+\dfrac{(1-\rho^2)(1-2\rho^2)}{n}\,,\\ D^2(r)=M(r^2)-M(r)^2=\dfrac{(1-\rho^2)^2}{n}\,,\qquad \sigma_r=\dfrac{1-\rho^2}{\sqrt{n}}\,.\end{cases}$$

Mit hilfe von diesen Ergebnisse kann man den zentralen Grenzwertsatz in bezug auf r leicht beweisen: Führt man nämlich unter Berücksichtigung von (4.17)

$$(4.18)\quad \xi=\left[r-\rho\left(1-\frac{1-\rho^2}{2n}\right)\right]\Big/\frac{1-\rho^2}{\sqrt{n}}\,,\quad \text{d.h. } r=\rho+(1-\rho^2)\left(\frac{\xi}{\sqrt{n}}-\frac{\rho}{2n}\right)$$

als neue Variable ein, so sieht man, daß der Ausdruck

$$(4.19)\qquad f_n(r)\frac{dr}{d\xi}\simeq\frac{1}{\sqrt{2\pi}}\left[\frac{(1-\rho^2)(1-\rho r)^{3/2}}{(1-r^2)^2}\right]\left[\frac{(1-\rho^2)(1-r^2)}{(1-\rho r)^2}\right]^{n/2}$$

gegen $\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{\xi^2}{2}\right\}$ strebt. Denn, ihr zweiter Faktor strebt gegen $1+O\left(\frac{1}{\sqrt{n}}\right)$, während der Logarithmus des dritten Faktors

$$\frac{n}{2}\log\frac{1-2\rho\left(\frac{\xi}{\sqrt{n}}-\frac{\rho}{2n}\right)-(1-\rho^2)\frac{\xi^2}{n}+O\left(\frac{1}{n\sqrt{n}}\right)}{1-2\rho\left(\frac{\xi}{\sqrt{n}}-\frac{\rho}{2n}\right)+\frac{\rho^2\xi^2}{n}+O\left(\frac{1}{n\sqrt{n}}\right)}\cong-\frac{\xi^2}{2}+O\left(\frac{1}{\sqrt{n}}\right)$$

wird.

9. Schließlich versuchen wir aus dem vorhergehenden Beispiele die berühmte Fishersche Formel zu herleiten[9]. Führt man nach Fisher $r=\tanh z$, $\rho=\text{tahn}\ \zeta$ ein, so drückt sich (4.14) folgendergestalt aus:

(4.20) $$f_n(r)\,dr=(n-2)\sqrt{\frac{\operatorname{sech}\zeta}{2n\pi}}\frac{\sqrt{\cosh z\cosh^3(z-\zeta)}}{\cosh^n(z-\zeta)}\left[1+\frac{6+\rho\tanh z}{8n}+O\left(\frac{1}{n^2}\right)\right]dz\,,$$

wobei nach (4.14.1)

(4.20.1) $$O\left(\frac{1}{n^2}\right)\cong\frac{1}{128n^2}(100+36\rho\tanh z+9\rho^2\tanh^2 z)$$

ist. Wir haben den Mittelwert

(4.21) $$M(z^m)=(n-2)\sqrt{\frac{\operatorname{sech}\zeta}{2n\pi}}\int_{-\infty}^{\infty}\frac{z^m\sqrt{\cosh z\cosh^3(z-\zeta)}}{\cosh^n(z-\zeta)}\times\left[1+\frac{6+\rho\tanh z}{8n}+O\left(\frac{1}{n^2}\right)\right]dz$$

zu ermitteln. Das Integral gehört noch zum Typus (3.14) mit

$$F(z)=-\log\cosh(z-\zeta)\,,$$

was an $z=\zeta$ den Maximalwert 0 erreicht. Weiter sind $F^{(2\nu+1)}(\zeta)=0$, $F''(\zeta)=-1$, $F''''(\zeta)=2$, $F''''''(\zeta)=-16$. Anderseits ist

$$\varphi(z)=z^m\sqrt{\cosh z\cosh^3(z-\zeta)}\left[1+\frac{6+\rho\tanh z}{8n}+\frac{100+36\rho\tanh z+9\rho^2\tanh^2 z}{128n^2}\right],$$

so daß

$$\varphi(\zeta)=\zeta^m\sqrt{\cosh\zeta}\left[1+\frac{6+\rho^2}{8n}+\frac{100+36\rho^2+9\rho^4}{128n^2}\right].$$

Durch wiedermalige Differenziationen nach z und Einsetzung $z=\zeta$ erhält man nach einander

$$\frac{\varphi'}{\varphi}(\zeta)=\frac{m}{\zeta}+\frac{\rho}{2}+\frac{\rho(1-\rho^2)}{8n}\,,$$

$$\frac{\varphi''}{\varphi}(\zeta)=\frac{m(m-1)}{\zeta^2}+\frac{m\rho}{\zeta}+\left(2-\frac{\rho^2}{4}\right)+\frac{1}{4n}\rho(1-\rho^2)\left[\frac{m}{\zeta}-\frac{\rho}{2}\right],$$

$$\frac{\varphi'''}{\varphi}(\zeta)=\frac{m(m-1)(m-2)}{\zeta^3}+\frac{3m(m-1)\rho}{2\zeta^2}+\frac{6m}{\zeta}\left(1-\frac{\rho^2}{8}\right)+2\rho\left(1+\frac{3}{16}\rho^2\right),$$

$$\frac{\varphi''''}{\varphi}(\zeta)=\frac{m(m-1)(m-2)(m-3)}{\zeta^4}+\frac{2m(m-1)(m-2)}{\zeta^3}+\frac{12m(m-1)}{\zeta^2}\left(1-\frac{\rho^2}{8}\right)+\frac{8m\rho}{\zeta}\left(1+\frac{3}{16}\rho^2\right)+8-\rho^2-\frac{15}{16}\rho^4\,.$$

9) Cramér, loc. cit. pp. 399–400.

Damit gewinnt man vermöge (3. 14) und (3. 14. 1)

$$(4.22)\quad M(z^m) \cong \frac{n-2}{n}\sqrt{1-\rho^2}\varphi(\zeta)\left[1+\frac{1}{2n}\left(\frac{1}{2}+\frac{\varphi''}{\varphi}\right)+\frac{1}{8n^2}\left(\frac{1}{4}+\frac{5\varphi''}{\varphi}+\frac{\varphi''''}{\varphi}\right)\right]$$

$$\cong \zeta^m\left\{1+\frac{m}{2n\zeta}\left(\frac{m-1}{\zeta}+\rho\right)+\frac{m}{8n^2\zeta}\left[\frac{(m-1)(m-2)(m-3)}{\zeta^3}+\frac{2(m-1)(m-2)}{\zeta^2}\right.\right.$$

$$\left.\left.+\frac{m-1}{\zeta}(12-\rho^2)+(9+\rho^2)\rho\right]\right\}.$$

Insbesondere

$$(4.23)\quad \begin{cases} M(z^0)=1\,, \quad M(z)=\zeta+\dfrac{\rho}{2n}+\dfrac{\rho}{8n^2}(9+\rho^2)\,, \\ M(z^2)=\zeta^2+\dfrac{1}{n}(1+\rho\zeta)+\dfrac{1}{4n^2}[12-\rho^2+(9+\rho^2)\rho\zeta]\,. \end{cases}$$

Und daher

$$(4.24)\quad D^2(z)=M(z^2)-M(z)^2 \cong \frac{1}{n}+\frac{3}{n^2}\left(1-\frac{\rho^2}{6}\right).$$

Läßt man näherungsweise $M(z)=\zeta+\rho/2n$, $D^2(z)=1/n$, so sind die Korrekturgrößen ungefähr ρ/n^2 bzw. $3/n^2$. Das illustriert warum Fisher schicklich

$$(4.25)\quad M(z)=\zeta+\frac{\rho}{2(n-1)}\,, \quad D^2(z)=\frac{1}{n-3}$$

als bessere Schätzung aufstellte[10].

Auch kann wieder der zentrale Grenzwertsatz für die Variable z leicht gefolgert werden. Gesetzt nämlich nach (4. 25)

$$\left[z-\zeta-\frac{\rho}{2(n-1)}\right]\Big/\frac{1}{\sqrt{n-3}}=\xi\,, \quad \text{d.h.} \quad z=\zeta+\frac{\xi}{\sqrt{n}}+O\left(\frac{1}{n}\right),$$

so formt (4. 20) folgendermaßen um:

$$(4.26)\quad f_n(r)\,dr$$

$$=\frac{1}{\sqrt{2\pi}}\sqrt{\frac{\cosh(\zeta+\xi/\sqrt{n})}{\cosh\zeta}}\exp\left[-\left(n-\frac{3}{2}\right)\log\cosh\frac{\xi}{\sqrt{n}}\right]\left(1+O\left(\frac{1}{n}\right)\right)d\xi\,.$$

Da aber

$$\frac{\cosh(\zeta+\xi/\sqrt{n})}{\cosh\zeta}=\cosh\frac{\xi}{\sqrt{n}}+\tanh\zeta\,\sinh\frac{\xi}{\sqrt{n}}\cong 1+O\left(\frac{1}{\sqrt{n}}\right),$$

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10) Wenn man anstatt des wahren Wertes $\frac{\rho}{2n}+\frac{\rho}{8n^2}(9+\rho^2)+\cdots$ Fishersche Schätzung $\frac{\rho}{2(n-1)}$ $\left(=\frac{\rho}{2n}+\frac{\rho}{2n^2}+\cdots\right)$ einnimmt, so scheint die Hauptkorrektur $\rho/2n^2$ noch etwas kleiner als die wirkliche Korrekturgröße $\frac{\rho}{n^2}\left(\frac{9+\rho^2}{8}\right)\doteqdot\frac{\rho}{n^2}$. Jedoch ist praktisch n zwar nicht so groß, so daß der annehmliche Wert $\frac{\rho}{2(n-2)}\left(=\frac{\rho}{2n}+\frac{\rho}{n^2}+\cdots\right)$ vielleicht allzuviel sein möchte.

und

$$\text{der Exponent} \cong -n\left[\frac{\xi^2}{2n}+O\left(\frac{1}{n^2}\right)\right]\left[1+O\left(\frac{1}{n}\right)\right] \cong -\frac{\xi^2}{2}+O\left(\frac{1}{n}\right)$$

sind, sofort die Behauptung sich ergibt. Damit der Exponent $\cong -\xi^2/2$ gilt, braucht es für jetziges (4. 26) nur $O\left(\frac{1}{n}\right) \cong 0(1)$, während bei (4. 19) noch $O\left(\frac{1}{\sqrt{n}}\right) \cong 0(1)$. Also schneller ist die normale Annäherung im Falle des z als diejenige mit r.

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NOTES ON GENERAL ANALYSIS (VII)

By

Isae SHIMODA

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In this note, a theorem of complex valued functions is extended to the case of complex Banach spaces. Let E_1 and E_2 be complex Banach spaces.

Theorem. *Let the family of functions* $\{f(x)\}$ *from* E_1 *to* E_2 *satisfy following conditions*: (1) *each function* $f(x)$ *is analytic in* $\|x\|<1$ *in* E_1 *and is a one-to-one mapping to a domain* D_f *in* E_2 *and its inverse function* $f^{-1}(x)$ *is also analytic in* D_f, (2) $\{f(x)\}$ *are bounded, that is,* $\|f(x)\|\leqq M$, (3) *the norms of linear parts* $\{g_1(x)\}$ *of* $\{f^{-1}(x)\}$ *are bounded, that is,* $\|g_1\|\leqq K$, (4) $f(\theta)=\theta$, *then each domain* D_f *includes the sphere whose radius is constant.*

Proof. Since $f(\theta)=\theta$, we have

$$f^{-1}(x)=\sum_{n=1}^{\infty} g_n(x) \text{ and } f(x)=\sum_{n=1}^{\infty} f_n(x),$$

where $f_n(x)$ and $g_n(x)$ are homogeneous polynomials of degree n, for $n=1, 2, 3, \cdots$. Then,

$$\begin{aligned} x &= f^{-1}(f(x)) \\ &= \sum_{n=1}^{\infty} g_n(f(x)) \\ &= \sum_{n=1}^{\infty} g_1(f_n(x)) + \sum_{n=2}^{\infty} g_n(f(x)) \\ &= g_1(f_1(x)) + \sum_{n=2}^{\infty} g_1(f_n(x)) + \sum_{n=2}^{\infty} g_n(f(x)). \end{aligned}$$

For an arbitrarily fixed x and an arbitrary complex number α, we have

$$\alpha x = \alpha g_1(f_1(x)) + \sum_{n=2}^{\infty} \alpha^n g_1(f_n(x)) + \sum_{n=2}^{\infty} \alpha^n g_n(f_1(x) + \sum_{m=2}^{\infty} \alpha^{m-1} f_m(x)).$$

Dividing each terms by α,

$$x = g_1(f_1(x)) + \alpha\{\sum_{n=2}^{\infty} \alpha^{n-2} g_1(f_n(x)) + \sum_{n=2}^{\infty} \alpha^{n-2} g_n(f_1(x) + \sum_{m=2}^{\infty} \alpha^{m-1} f_m(x))\}.$$

Put $\alpha=0$ and we have $x=g_1(f_1(x))$. Since x is arbitrary, $g_1(x)$ is an inverse function of $f_1(x)$. That is, $f_1(x)$ has the continuous inverse function $g_1(x)$. Since $\|x\|=\|g_1(f_1(x))\|\leqq\|g_1\|\cdot\|f_1(x)\|$, we have

$$\|f_1(x)\| \geqq \frac{1}{\|g_1\|}\|x\| \geqq \frac{1}{K}\|x\|,$$

from the assumption (3).

On the other hand, $f_n(x)=\frac{1}{2\pi i}\int_C \frac{f(\alpha x)}{\alpha^{n+1}}d\alpha,\ n=1, 2, 3, \cdots$, where C is a circle with radius r satisfying $r\|x\|\leqq\delta<1$ and $r>1$.
From the assumption (2), $\|f(\alpha x)\|\leqq M$, when $\|\alpha x\|\leqq\delta$.

Thus we see that $$\|f_n(x)\|\leqq\frac{M}{r^n} \quad \text{for } n=1, 2, \cdots .$$

Then $$\|\sum_{n=2}^{\infty}f_n(x)\|\leqq\sum_{n=2}^{\infty}\frac{M}{r^n}=\frac{M}{r(r-1)}.$$

Taking a positive number δ_1 such that

$$0<\delta_1<\delta<1,\ r\delta_1=\delta \text{ and } KM\delta_1^2<\delta^3(\delta-\delta_1),$$

we have $$\frac{M}{r(r-1)}=\frac{M\delta_1}{\delta(\delta-\delta_1)}<\frac{\delta^2}{K}.$$

For an arbitrary y in $\|y\|\leqq\delta$, there exist x and α such that $y=\alpha x$, $|\alpha|\leqq r$, $\|x\|=\delta_1$ and $\|\alpha x\|\leqq\delta$.

$$\|\sum_{n=2}^{\infty}f_n(\alpha x)\|=|\alpha|^2\cdot\|\sum_{n=2}^{\infty}f_n(x)\alpha^{n-2}\|,$$

then $$\frac{\|\sum_{n=2}^{\infty}f_n(\alpha x)\|}{|\alpha|^2}=\|\sum_{n=2}^{\infty}f_n(x)\alpha^{n-2}\|.$$

Since $\sum_{n=2}^{\infty}f_n(x)\alpha^{n-2}$ is an analytic function of α, its norm takes the maximum on the boundary $|\alpha|=r$ and so we have

$$\frac{\|\sum_{n=2}^{\infty}f_n(\alpha x)\|}{|\alpha|^2}\leqq\frac{1}{r^2}\cdot\frac{M_1}{r(r-1)}.$$

Putting $\alpha x=y$, $\|\sum_{n=2}^{\infty}f_n(y)\|\leqq\frac{M}{r^2\cdot r(r-1)}|\alpha|^2\leqq\frac{M}{r(r-1)}\cdot\frac{\|y\|^2}{r^2\|x\|^2}=\frac{M}{r(r-1)\delta^2}\|y\|^2.$

Thus we see that
$$\begin{aligned}\|f(x)\|&\geqq\|f_1(x)\|-\|\sum_{n=2}^{\infty}f_n(x)\|\\ &\geqq\frac{1}{K}\|x\|-\frac{M}{r(r-1)\delta^2}\|x\|^2,\end{aligned}$$

for $\|x\|\leqq\delta_1$. Letting $\|x\|=\delta_1$, we have

$$\begin{aligned}\|f(x)\|&\geqq\frac{\delta_1}{K}-\frac{M}{r(r-1)\delta^2}\cdot\delta_1^2\\ &=\delta_1\left(\frac{1}{K}-\frac{M\delta_1}{r(r-1)\delta^2}\right),\end{aligned}$$

which is a positive number, since $0<\delta_1<1$ and $\frac{1}{K}>\frac{M}{r(r-1)\delta^2}>\frac{M\delta_1}{r(r-1)\delta^2}$. Put $\delta_1\left(\frac{1}{K}-\frac{M\delta_1}{r(r-1)\delta^2}\right)=\rho$, then we see that

$$D_f\supset U(\rho),$$

which is the sphere with radius ρ. This completes the proof.

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CHARACTERIZATION OF CERTAIN ADDITIVE SEMIGROUPS BY DISTRIBUTIVE MULTIPLICATIONS

By

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§1. Introduction.

Let S_+ be a semigroup with addition $+$ defined in a set S. We introduce another operation "multiplication" into the same set S, which is symbolized as $S_\times$, not necessarily associative, such that

$$\begin{aligned} &(1) \qquad x(a+b) = xa+xb \\ &(2) \qquad (x+y)a = xa+ya \end{aligned} \quad \text{for every } x, y, a, b \in S.$$

Then we say that S_+ has a multiplication (multiplicative system) $S_\times$. In the previous paper [1] we proved the two theorems:

Theorem (A). A right singular or left singular[1] semigroup S_+ has all arbitrary multiplications.

Theorem (B). Let S_+ be a semigroup defined as $x+y=0$ for all $x, y \in S$. S_+ has a multiplication $S_\times$ if and only if $S_\times$ has 0 as the two-sided zero.

In the present note, we shall prove that Theorem (A) characterizes a right or left singular semigroup S_+, but Theorem (B) does not characterize the semigroup S_+ defined as $x+y=0$, and we shall have the following Theorems 1, 2 under weaker conditions. Hereafter, by "a semigroup S has a multiplication (multiplicative system) $S_\times$" we mean "(1) holds i.e. the multiplication $S_\times$ is distributive to the addition S_+ with respect to the only one-side." Of course we assume that S is non-trivial, i.e. it contains two elements at least.

Theorem 1. *If a semigroup S_+ has all arbitrary multiplications, then S_+ is a right or left singular semigroup.*

Theorem 2. *If a semigroup S_+ has all multiplicative systems $S_\times$ with a right zero 0 and has nothing but such multiplications, then S_+ is either a semigroup with $x+y=0$ for all $x, y \in S$ or a group of order 2.*

§2. Proof of Theorem 1.

1. *$x_0 \neq y_0$ implies either $x_0+y_0=x_0$ or $x_0+y_0=y_0$.*

Proof. We shall prove that $x_0 \neq y_0$ and $x_0+y_0 \neq x_0$ imply $x_0+y_0=y_0$. Let

---

1) S is called a right singular or left singular if S is defined as $x+y=y$ or $x+y=x$ for any x, y respectively.

$z_0 = x_0 + y_0 \neq x_0$, and corresponding to x_0, y_0 and z_0, consider a multiplication which satisfies

$$x_0^2 = x_0, \quad x_0 y_0 = x_0 z_0 = y_0 .$$

Then we get $x_0 + y_0 = x_0^2 + x_0 y_0 = x_0(x_0 + y_0) = x_0 z_0 = y_0$.

2. *If $x_0 \neq y_0$ and $x_0 + y_0 = y_0$ for some x_0, y_0, then $x + y = y$ for all x, y.*

Proof. Take any x and y. Consider a multiplication satisfying

$$x_0^2 = x \text{ and } x_0 y_0 = y ,$$

then we get

$$x + y = x_0^2 + x_0 y_0 = x_0(x_0 + y_0) = x_0 y_0 = y .$$

Similarly we obtain

3. *If $x_0 \neq y_0$ and $x_0 + y_0 = x_0$ for some x_0, y_0, then $x + y = x$ for all x, y.*

Gathering together 1, 2, and 3, it has been proved that non-trivial S_+ is right or left singular.

§ 3. Proof of Theorem 2.

1. $0 + 0 = 0$.

Proof. For a special multiplication $S_\times$ with two-sided zero 0,

$$0 + 0 = 0^2 + 0^2 = 0(0 + 0) = 0 .$$

2. *Either $0 + x = x$ for all x, or $0 + x = 0$ for all x.*

Proof. We shall prove that $0 + x = 0$ for all x if $0 + x_0 \neq x_0$ for some x_0. From $0 + x_0 \neq x_0$, we see easily $x_0 \neq 0$ by **1**. Let x be any element of S, let $u_0 = 0 + x_0$, and consider a multiplication satisfying

$$x_0^2 = x, \quad x_0 u_0 = 0, \quad x0 = 0 \text{ for all } x .$$

Then we have $0 + x = x_0 0 + x_0^2 = x_0(0 + x_0) = x_0 u_0 = 0$.
Hence $0 + x_0 \neq x_0$ implies $0 + x = 0$ for all x.

Similarly we can prove

3. *Either $x + 0 = x$ for all x, or $x + 0 = 0$ for all x.*

4. *Either $0 + x = x + 0 = x$ for all x or $0 + x = x + 0 = 0$ for all x.*

Proof. Suppose both $0 + x = x$ for all x and $x + 0 = 0$ for all x. Then

$$x + y = x + (0 + y) = (x + 0) + y = 0 + y = y$$

for every x, y, which concludes that S_+ is a right singular semigroup. This contradicts the assumption of this theorem because of Theorem 1. Similarly it is false that both $0 + x = 0$ for all x and $x + 0 = x$ for all x.

From **5** to **7** we assume that S contains three elements at least.

5. *$x_0 \neq 0$, $y_0 \neq 0$, $x_0 \neq y_0$ imply $x_0 + y_0 \neq x_0$ and $x_0 + y_0 \neq y_0$.*

Proof. Suppose $x_0 + y_0 = x_0$. In case $0 + x = x + 0 = x$ for all x, considering a multiplication which satisfies for x_0, y_0

(3) $$x_0^2 = 0, \quad x_0 y_0 = x_0, \quad z0 = 0 \text{ for all } z.$$

we get $$0 + x_0 = x_0^2 + x_0 y_0 = x_0(x_0 + y_0) = x_0^2 = 0,$$

which contradicts the above assumption.
In case $0+x=x+0=0$ for all x, a multiplication satisfying

(4) $$x_0^2 = x_0, \quad x_0 y_0 = 0, \quad z0 = 0 \text{ for all } z$$

leads to $$x_0 + 0 = x_0^2 + x_0 y_0 = x_0(x_0 + y_0) = x_0^2 = x_0,$$

contradicting the assumption. Hence it has been proved that $x_0 + y_0 \neq x_0$. Similarly we can prove $x_0 + y_0 \neq y_0$. Under the supposition of $x_0 + y_0 = y_0$ we may use (4) in case $0+x=x+0=x$ for all x, (3) in case $0+x=x+0=0$ for all x.

6. $x_0 \neq 0$, $y_0 \neq 0$, $x_0 \neq y_0$ *imply* $x_0 + y_0 = 0$.

Proof. Suppose $u_0 = x_0 + y_0 \neq 0$. We can consider a multiplication which fulfils

$$x_0^2 = x_0 y_0 = 0, \quad x_0 u_0 \neq 0, \quad z0 = 0 \text{ for all } z.$$

Possibility of such a multiplication follows from 5, i.e. $x_0 + y_0 \neq x_0$ and $x_0 + y_0 \neq y_0$.

Then $$0 \neq x_0 u_0 = x_0(x_0 + y_0) = x_0^2 + x_0 y_0 = 0 + 0 = 0,$$

arriving at contradiction. Therefore $x_0 + y_0 = 0$.

7. *If S_+ contains three elements at least, then S_+ is given as $x+y=0$ for all x, y.*

Proof. It is sufficient to prove only $0+x=x+0=0$ for all x. By the assumption, there are x_0, y_0 such that $x_0 \neq 0$, $y_0 \neq 0$, $x_0 \neq y_0$. For x, x_0, y_0, consider a multiplication fulfilling

$$x x_0 = 0, \quad x y_0 = x, \quad z0 = 0 \text{ for all } z.$$

Then $$0 + x = x x_0 + x y_0 = x(x_0 + y_0) = x0 = 0$$
$$x + 0 = x y_0 + x x_0 = x(y_0 + x_0) = x0 = 0 \quad \text{by 6.}$$

As consequence of 7, we have

8. *If $0+x=x+0=x$ for all x, then non-trivial S_+ is of order 2.*

Accordingly the type of S_+ is either a group of order 2

	0	a
0	0	a
a	a	0

or a semilattice

(5)

	0	a
0	0	a
a	a	a

9. *The addition S_+ of order 2 which satisfies the assumption of this theorem is nothing but a group of order 2.*

Proof. At first, the semilattice S_+ (5) of order 2 has a multiplication $S_\times$ defined as

	0	a
0	a	a
a	a	a

Because $x(y+z)=a$, $xy+xz=a+a=a$ whenever x, y, z are 0 or a. Hence the required S_+ is a group. Conversely if S_+ is a group of order 2, all the multiplications which S_+ has are proved to be

	0	a
0	0	0
a	0	0

	0	a
0	0	0
a	0	a

by Theorem 1 of the previous paper [1].

References

[1] T. Tamura etc.: Distributive multiplications to semigroup operations, Jour. of Gakugei, Tokushima Univ., Vol VIII, 1957, 91–101.

Correction to the Previous Papers.

"On a Special Semilattice with a Minimal Condition"
This Journal, Vol. VII, pp. 9-17.

Page 11, line 15 from the bottom, Lemma 4.
For "*a dispersed semilattice*" read "*an above bounded dispersed semilattice*".

We express many thanks to Professor J. Hartmanis for his kind advice.

"Distributive Multiplications to Semigroup Operations"
This Journal, Vol. VIII, pp. 91-101.

Page 94, line 9. For "*is a semigroup with*" read "*has*".

Page 94, line 11. Put "*arbitrarily*" betwen "*are*" and "*chosen*".

Page 94, line 11. Put "; *and moreover* $S_{\times}$ *is a semigroup if and only if* u_{ijk} $(i, j, k=1, \cdots, r)$ *are chosen*" between "*chosen*" and "*such*".

Page 94, line 14. For "*Consequently*" read "*Then*".

Page 94, line 14. Eliminate "*commutative*".

"All Semigroups of Order at Most 5"
This Journal Vol. VI, pp 19-39.

Page	Title	Error	Correction
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ON SOME PROPERTIES OF PLANE CURVES IN RIEMANN SPACES

By

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§ 1. Consider in an m-dimensional Riemann space V_m a curve denoted by

$$y^\lambda = y^\lambda(s) \qquad (\lambda, \mu, \nu = 1, 2, \cdots, m),$$

where we denote by s arc length of this curve.
Then we see by Frenet's formula

$$\frac{dy^\lambda}{ds} = \underset{(1)}{\xi^\lambda}, \quad \frac{\delta \underset{(1)}{\xi^\lambda}}{ds} = \kappa_1 \underset{(2)}{\xi^\lambda}, \quad \frac{\delta \underset{(2)}{\xi^\lambda}}{ds} = -\kappa_1 \underset{(1)}{\xi^\lambda} + \kappa_2 \underset{(3)}{\xi^\lambda}, \cdots, \frac{\delta \underset{(m)}{\xi^\lambda}}{ds} = -\kappa_{m-1} \underset{(m-1)}{\xi^\lambda}.$$

Now we define a plane curve as a curve satisfying

$$\frac{\delta \underset{(2)}{\xi^\lambda}}{ds} = -\kappa_1 \underset{(1)}{\xi^\lambda}. \tag{1, 1}$$

Then from the relation

$$-\kappa_1 \underset{(1)}{\xi^\lambda} = \frac{\delta}{ds}\left(\frac{1}{\kappa_1} \frac{\delta \underset{(1)}{\xi^\lambda}}{ds}\right),$$

the equation (1, 1) has a form

$$\frac{\delta^3 y^\lambda}{ds^3} + \kappa_1 \frac{d\left(\frac{1}{\kappa_1}\right)}{ds} \frac{\delta^2 y^\lambda}{ds^2} + \kappa_1^2 \frac{\delta y^\lambda}{ds} = 0. \tag{1, 2}$$

Putting

$$\begin{cases} q = \kappa_1^2 = g_{\lambda\mu} \dfrac{\delta^2 y^\lambda}{ds^2} \dfrac{\delta^2 y^\mu}{ds^2}, \\ p = \dfrac{d}{ds}\left(\log \dfrac{1}{\sqrt{q}}\right) = \kappa_1 \dfrac{d\left(\frac{1}{\kappa_1}\right)}{ds}, \end{cases} \tag{1, 3}$$

we can rewrite (1, 2) in

$$\frac{\delta^3 y^\lambda}{ds^3} + p \frac{\delta^2 y^\lambda}{ds^2} + q \frac{dy^\lambda}{ds} = 0. \tag{1, 4}$$

§ 2. Consider in V_m an n-dimensional subspace V_n whose current point is given by a system of coordinate (x^i)[1]. If the curve $y^\lambda = y^\lambda(s)$ is contained in V_n, then we must have

1) Hereafter we shall denote by $\alpha, \beta, \gamma, \delta, \mu, \nu$, the suffices which take the value $1, 2, \cdots m$; by a, b, c, i, j, k, those which take the value $1, 2, \cdots, n$ and P, Q, R, those which take the value $\dot{n}+\dot{1}, \dot{n}+\dot{2}, , \cdots, \dot{m}$.

$$(2,1)\qquad \begin{cases} \dfrac{dy^\lambda}{ds} = B_i^{\cdot\lambda}\dfrac{dx^i}{ds}, \\ \dfrac{\delta^2 y^\lambda}{ds^2} = H^\lambda_{ij}\dfrac{dx^i}{ds}\dfrac{dx^j}{ds} + B_i^{\cdot\lambda}\dfrac{\delta^2 x^i}{ds^2} \\ \dfrac{\delta^3 y^\lambda}{ds^3} = H^\lambda_{ij;k}\dfrac{dx^i}{ds}\dfrac{dx^j}{ds}\dfrac{dx^k}{ds} + 3H^\lambda_{ij}\dfrac{\delta^2 x^i}{ds^2}\dfrac{dx^j}{ds} + B_i^{\cdot\lambda}\dfrac{\delta^3 x^i}{ds^3}, \end{cases}$$

where we put

$$B_i^{\cdot\lambda} = \frac{\partial y^\lambda}{\partial x^i},\quad H^\lambda_{ij} = B^{\cdot\lambda}_{i;j} = \sum_P H^P_{ij}\xi^\lambda_P,$$

and $(m-n)$ normal vectors of V_n in V_m ξ^α_P must be satisfied

$$\xi^\alpha_{P;k} = -g^{ij}B_i^{\cdot\alpha}H^P_{jk} + \sum_Q L_{PQ|k}\xi^\alpha_Q .$$

Hence we easily obtain the relations

$$H^\alpha_{ij;k} = \sum_P H^P_{ij;k}\xi^\alpha_P - \sum_P (H^P_{ij}H^P_{bk})g^{ab}B_a^{\cdot\alpha} + \sum_P\sum_Q L_{PQ|k}\xi^\alpha_Q H^P_{ij},$$

and

$$(2,2)\qquad \frac{\delta^3 y^\lambda}{ds^3} = \left[\frac{\delta^3 x^a}{ds^3} - \sum_P H^P_{ij}H^P_{bk}g^{ba}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}\right]B_a^{\cdot\lambda}$$
$$+\sum_P\left[H^P_{ijk}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds} + 3H^P_{ij}\frac{\delta^2 x^i}{ds^2}\frac{dx^j}{ds} + \sum_Q H^Q_{ij}L_{PQ|k}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}\right]\xi^\lambda_P .$$

The quantities H^P_{ijk} are components of a tensor given by

$$H^P_{ijk} = H^P_{(ij;k)}$$

Putting (2, 2) into (1, 4) we obtain the equation of a plane curve contained in V_n immersed in V_m as

$$(2,3)\qquad \begin{cases} \dfrac{\delta^3 x^h}{ds^3} - \sum_P H^P_{ij}\dfrac{dx^i}{ds}\dfrac{dx^j}{ds}H^P_{bk}\dfrac{dx^k}{ds}g^{hb} + p\dfrac{\delta^2 x^h}{ds^2} + q\dfrac{dx^h}{ds} = 0, \\ pH^P_{ij}\dfrac{dx^i}{ds}\dfrac{dx^j}{ds} + H^P_{ijk}\dfrac{dx^i}{ds}\dfrac{dx^j}{ds}\dfrac{dx^k}{ds} + 3H^P_{ij}\dfrac{\delta^2 x^i}{ds^2}\dfrac{dx^j}{ds} + \sum_Q L_{PQ|k}H^Q_{ij}\dfrac{dx^i}{ds}\dfrac{dx^j}{ds}\dfrac{dx^k}{ds} = 0 . \end{cases}$$

The curves, however, whose equations are given by

$$(2,4)\qquad H^P_{ijk}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds} + 3H^P_{ij}\frac{\delta^2 x^i}{ds^2}\frac{dx^j}{ds} + \sum_Q L_{PQ|k}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

are called[2)] Darboux lines of the third kind contained in V_n immersed in V_m (in brief Darboux lines in V_n in V_m) and plane curves such as at each point $p=0$, that is $\kappa_1=$const., $\kappa_2=0$, are called geodesic circles or Riemann circles.[3)]

2) M. Prvanovitch; Ligne de Darboux dans l'espace riemannien. (Bull. Sci. Math. (2), 78, 1954, p.p. 9–14).

Y. Ichijô; On Darboux lines contained in a Riemannian space. (Journ. Gakugei, Tokushima Univ. Japan Vol. VIII, 1957, p.p. 27–32).

3) K. Yano; Concircular geometry I (Proc. Imp. Acad. Japan 16, 1940, p.p. 195–200).

Hence we conclude, by (2, 3) *that if plane curves in V_n in V_m are Darboux lines in V_n in V_m, then they are asymptotic curves or geodesic circles.*

§ **3.** As V_n is an n-dimensional Riemann space, along the curve in V_n in V_m we shall consider $\bar{p}$ and $\bar{q}$ in this Riemann space V_n.
By (2, 1) and $q=g_{\lambda\mu}\dfrac{\delta^2 y^\lambda}{ds^2}\dfrac{\delta^2 y^\mu}{ds^2}$
we see

$$q=\bar{q}+\sum_P\left(H^P_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right)^2 \tag{3, 1}$$

$$p=\bar{p}-\frac{\bar{p}\cdot\sum_P\left(H^P_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right)^2+\sum_P\left\{\left(H^P_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right)\cdot\frac{d}{ds}\left(H^P_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right)\right\}}{\bar{q}+\sum_P\left(H^P_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right)^2}, \tag{3, 2}$$

where we put

$$\bar{q}=g_{ij}\frac{\delta^2x^i}{ds^2}\frac{\delta^2x^j}{ds^2},\quad\text{and}\quad \bar{p}=\frac{d}{ds}\log\frac{1}{\sqrt{\bar{q}}}.$$

Putting (3, 1) and (3, 2) we obtain the equations of plane curves

$$\tag{3, 3}\left\{\begin{aligned}
&\frac{\delta^3x^h}{ds^3}+\bar{p}\frac{\delta^2x^h}{ds^2}+\bar{q}\frac{dx^h}{ds}-\sum_P A^PH^P_{ij}g^{ih}\frac{dx^k}{ds}\\
&\qquad-\frac{\bar{p}\sum_P(A^P)^2+\sum_P A^P\frac{dA^P}{ds}}{\bar{q}+\sum_P(A^P)^2}\cdot\frac{\delta^2x^h}{ds^2}+\sum_P(A^P)^2\frac{dx^h}{ds}=0,\\
&H^P_{ijk}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}+3H^P_{ij}\frac{\delta^2x^i}{ds^2}\frac{dx^j}{ds}+\sum_Q L_{PQ|k}A^Q\frac{dx^k}{ds}\\
&\qquad+\left(\bar{p}-\frac{\bar{p}(\sum_P(A^P)^2)+\sum_P A^P\frac{dA^P}{ds}}{\bar{q}+\sum_P(A^P)^2}\right)A^P=0,
\end{aligned}\right.$$

where we put $A^P=H^P_{ij}\dfrac{dx^i}{ds}\dfrac{dx^j}{ds}$.

Hence we have that *if V_n is totally geodesic subspace in V_m, then the plane curve in V_m is a plane curve in V_n and, at the same time, is a Darboux lime in V_n in V_m.*

We have also by (3, 3) that *when a plane curve in V_m is contained in V_m, if it is a plane curve in V_n, then the following equation must be satisfied*

$$\sum_P A^PH^P_{ij}g^{ih}\frac{dx^k}{ds}+\frac{\bar{p}\sum_P(A^P)^2+\sum_P A^P(A^P)'}{\bar{q}+\sum_P(A^P)^2}\frac{\delta^2x^h}{ds^2}-\sum_P(A^P)^2\frac{dx^h}{ds}=0. \tag{3, 4}$$

§ **4.** In this paragraph we consider the case $m=n+1$.
The equations of plane curves (3, 3) are written as

$$(4,1)\quad \begin{cases} \dfrac{\delta^3 x^h}{ds^3}+\bar{p}\dfrac{\delta^2 x^h}{ds^2}+\bar{q}\dfrac{dx^h}{ds}-AH_{ik}g^{hi}\dfrac{dx^k}{ds}-\dfrac{\bar{p}A^2+A\dfrac{dA}{ds}}{\bar{q}+A^2}\dfrac{\delta^2 x^h}{ds^2}+A^2\dfrac{dx^h}{ds}=0\,, \\ H_{ijk}\dfrac{dx^i}{ds}\dfrac{dx^j}{ds}\dfrac{dx^k}{ds}+3H_{ij}\dfrac{\delta^2 x^i}{ds^2}\dfrac{dx^j}{ds}+\left(\bar{p}-\dfrac{\bar{p}A^2+A\dfrac{dA}{ds}}{\bar{q}+A^2}\right)A=0\,. \end{cases}$$

Hence we have: *the necessary and sufficient condition that a plane curve in* V_{n+1} *be a plane curve in* V_n *is either this curve be an asymptotic curve or satisfy the relation*

$$(4,2)\qquad H_{ik}g^{ih}\frac{dx^k}{ds}+\frac{\bar{p}A+\dfrac{dA}{ds}}{\bar{q}+A^2}\frac{\delta^2 x^h}{ds^2}-A\frac{dx^h}{ds}=0\,.$$

In the case where V_n is perfectly totally umbilic hypersurface in V_{n+1}, that is $H_{ij}=\rho g_{ij}$ ($\rho=$const.), the second member of the equations (4, 1) has the following form

$$\bar{p}=\frac{\bar{p}\rho^2}{q+\rho^2}\,,$$

and moreover $H_{ijk}=0$.

Hence we have $\bar{p}=0$, that is, $\bar{q}=$const..

By K. Yano's theorem[4)] the geodesic circle in V_n is, at the same time, geodesic circle in V_{n+1}.

Hence we have *if a plane curve of* V_{n+1} *is contained in a perfectly totally umbilic hypersurface, then it is a geodesic circle of* V_{n+1}, *and at the same time, of this hypersurface.*

By (4, 1) we easily see a sufficient condition that every plane curve in V_{n+1} is a plane curve in V_n when it is contained in V_n, is that V_n is a totally geodesic hypersuface in V_{n+1}.

This conclusion, however, is not necessary. For example, when we consider a perfectly totally umbilic hypersurface in V_{n+1}, the equations (4, 2) have the following form

$$\frac{\bar{p}A}{\bar{q}+\rho^2}=0\,.$$

Therefore we must have $q=$const..

However we must have seen that if a plane curve in V_{n+1} is contained in a perfectly umbilic hypersurface, it must satisfy $q=$const..

Hence we have: *a sufficient condition that every plane curve in* V_{n+1} *be plane curve in* V_n *when it is contained in* V_n, *is either* V_n *be totally geodesic or perfectly totally umbilic hypersurface in* V_{n+1}.

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4) K. Yano; Concircular Geometry III. (Proc. of Imp. Acad. Japan. 16 1940 p.p. 447.)

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SOME CONTRIBUTIONS TO ORDER STATISTICS (Continued)

By

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Contents

Introduction. We had endeavoured in Part I and II of the previous paper[1] to obtain systematically the first and second moments of order statistics up to the sample-size $n=7$. In the present note we shall continue it to find the third and fourth moments, but now reducing to sizes $n \leqq 5$, because of a pretty great number of various combinations.

We shall use the before made abbreviations, and besides, as a shorter notation of integral power $\Phi^i/i!$ simply $\underline{\Phi^i}$ by omitting the factorial in denominator, because this facilitates the differentiation as $\frac{d^p}{d\Phi^p}\underline{\Phi^i} = \underline{\Phi^{i-p}}$; of course, it means zero, if the index i be a negative integer, since then $1/\Gamma(i+1)=0$. Thus, e.g.

$$E(t^p_{i|n} t^q_{k|n}) = n! \int \underline{(1-\Phi)^{n-k}} \varphi t^q dt \int^t \underline{(\Phi-\Phi_1)^{k-i-1}} \underline{\Phi^{i-1}} \varphi_1 t_1^p dt_1 ,$$

$$E(t_{i|n} t_{j|n} t_{k|n}) = -n! \int \underline{(1-\Phi)^{n-k}} \varphi' dt \int^t \underline{(\Phi-\Phi_1)^{k-j-1}} \varphi_1' dt_1 \int^{t_1} \underline{(\Phi_1-\Phi_2)^{j-i-1}} \underline{\Phi_2^{i-1}} \varphi_2' dt_2 .$$

1) Y. Watanabe and others, Some Contributions to Order Statistics, this Journ. Vol. VIII (1957), pp. 41–90.

Our main task is firstly to make these expressions of moments free from t and t-derivatives by repeating integrations by parts (§§ 14–15) ; and next to write them in sums of auxiliary integrals

$$J_{\lambda}^{\alpha}=\int\Phi^{\lambda}\varphi^{\alpha}dt\,,\quad J_{\mu\nu}^{\alpha\beta}=\int\Phi^{\mu}\varphi^{\alpha}dt\int^{t}\Phi_{1}^{\nu}\varphi_{1}^{\beta}dt_{1}\,,\quad J_{\lambda\mu\nu}^{\alpha\beta\gamma}=\int\Phi^{\lambda}\varphi^{\alpha}dt\int^{t}\Phi_{1}^{\mu}\varphi_{1}^{\beta}dt_{1}\int^{t_{1}}\Phi_{2}^{\nu}\varphi_{2}^{\gamma}dt_{2}\,,$$

which were already partly treated in Part II, however, now in § 17 shall be further supplemented. Consequently the required moments could be explicitly expressed and numerically evaluated (§ 18), for which some check formulas should be constructed (§ 16). Lastly the features of frequency functions $f(t_{i|n})$ and their asymptotic behaviours toward N.D. are discussed (§§19–20).

PART III

§ **13. Some Preliminary Formulas.** We begin to set up some preliminary formulas which shall be useful in the subsequent paragraphs. The evident identity

$$\int F(\Phi)\varphi^{\alpha}\varphi' dt=\frac{1}{\alpha+1}\int F'(\Phi)\varphi^{\alpha+2}dt\quad (F(\Phi)\text{ being a polynomial of }\Phi)$$

is most often availed. Also, on differentiating the fundamental equality $\varphi'=-\varphi t$ about t, we have $\varphi''=-\varphi' t-\varphi$, and whence

$$\varphi\varphi''+\varphi^{2}=-\varphi\varphi' t=\varphi'^{2}\,. \tag{13. 0}$$

follows. By aid of this relation an integrand having φ'^{2} as factor could be derived from that containing $\varphi\varphi''$. In fact, we have

$$A\equiv\int F(\Phi)\varphi^{\alpha}\varphi'' dt=\frac{1}{(\alpha+1)(\alpha+2)}\int F''(\Phi)\varphi^{\alpha+3}dt-\frac{\alpha}{\alpha+1}\int F(\Phi)\varphi^{\alpha+1}dt\,. \tag{13. 1}$$

For, considering φ'' as $(\varphi')'$ and repeating integrations by parts, we obtain

$$\begin{aligned}A&=-\int F'(\Phi)\varphi^{\alpha+1}\varphi' dt-\alpha\int F(\Phi)\varphi^{\alpha-1}\varphi'^{2}dt\\&=\frac{1}{\alpha+2}\int F''(\Phi)\varphi^{\alpha+3}dt-\alpha\int F(\Phi)\varphi^{\alpha}\varphi'' dt-\alpha\int F(\Phi)\varphi^{\alpha+1}dt\,,\end{aligned}$$

because of (13. 0). On transposing the integral last before ($=-\alpha A$) into the left side and dividing out by $\alpha+1$, we get (13. 1). Moreover, by addition of $\int F(\Phi)\varphi^{\alpha+1}dt$ to both sides of (13. 1), we get

$$\int F(\Phi)\varphi^{\alpha-1}\varphi'^{2}dt=\frac{1}{(\alpha+1)(\alpha+2)}\int F''(\Phi)\varphi^{\alpha+3}dt+\frac{1}{\alpha+1}\int F(\Phi)\varphi^{\alpha+1}dt\,. \tag{13. 2}$$

In particular, letting $\alpha=1, 2, 3, \cdots$, we obtain

$$\int F(\Phi)\varphi'^{2}dt=\frac{1}{6}\int F''\varphi^{4}dt+\frac{1}{2}\int F\varphi^{2}dt\,, \tag{13. 2. 1}$$

(13. 2. 2) $$\int F(\Phi)\varphi\varphi'^2 dt=\frac{1}{12}\int F''\varphi^5 dt+\frac{1}{3}\int F\varphi^3 dt\,,$$

(13. 2. 3) $$\int F(\Phi)\varphi^2\varphi'^2 dt=\frac{1}{20}\int F''\varphi^6 dt+\frac{1}{4}\int F\varphi^4 dt\,.$$

Example 1. Show that

(13. 3) $$\int F(\Phi)\varphi'^2 t\,dt=\frac{2}{3}\int F'\varphi^3 dt+\frac{1}{24}\int F'''\varphi^5 dt\,.$$

Observing that $\varphi'^2 t=-\varphi t^2\varphi'$ and integrating by parts, we get

$$-\int F\varphi t^2\varphi' dt=\frac{2}{2}\int F\varphi^2 t\,dt+\frac{1}{2}\int F'\varphi^3 t^2 dt=-\int F\varphi\varphi' dt+\frac{1}{2}\int F'\varphi\varphi'^2 dt$$

$$=\frac{1}{2}\int F'\varphi^3 dt+\frac{1}{24}\int F'''\varphi^5 dt+\frac{1}{6}\int F'\varphi^3 dt \quad \text{(by (13. 2. 2))}$$

which, however, nothing but the right side of (13. 3), Q.E.D.

Example 2. Make $\int F(\Phi)\varphi'^3 dt$ free from t and t-derivatives.

Considering φ'^3 as $(\varphi^2+\varphi\varphi'')\varphi'=\varphi^2\varphi'+\frac{1}{2}\varphi(\varphi'^2)'$, we get

$$\int F\varphi'^3 dt=\int F\varphi^2\varphi' dt+\frac{1}{2}\int F\varphi(\varphi'^2)' dt=-\frac{1}{3}\int F'\varphi^4 dt-\frac{1}{2}\int F'\varphi^2\varphi'^2 dt-\frac{1}{2}\int F\varphi'^3 dt\,.$$

Transposing the last integral into left and applying (13. 2. 3) on the integral before last, and finally multiplying the whole by $\frac{2}{3}$, we find that

(13. 4) $$\int F(\Phi)\varphi'^3 dt=-\frac{11}{36}\int F'(\Phi)\varphi^4 dt-\frac{1}{60}\int F'''(\Phi)\varphi^6 dt\,,\quad \text{Q.E.I.}$$

Next, for a double integral containing besides $F(\Phi)$ also a polynomial of Φ, Φ_1, say $G(\Phi, \Phi_1)$, holds

(13. 5) $$B\equiv\int F(\Phi)\varphi^\alpha\varphi'' dt\int^t G(\Phi,\Phi_1)\varphi_1^\beta dt_1=\frac{1}{(\alpha+1)(\alpha+2)}\Big[\int F''\varphi^{\alpha+3}dt\int^t G\varphi_1^\beta dt_1$$

$$+2\int F'\varphi^{\alpha+3}dt\int^t \mathfrak{D}G\cdot\varphi_1^\beta dt_1+\int F\varphi^{\alpha+3}dt\int \mathfrak{D}^2G\cdot\varphi_1^\beta dt_1+\int\{F'g+F\mathfrak{g}'\}\varphi^{\alpha+\beta+2}dt\Big]$$

$$+\frac{1}{(\alpha+1)(\alpha+\beta+1)}\int D(Fg)\varphi^{\alpha+\beta+2}dt-\frac{\alpha}{\alpha+1}\int F\varphi^{\alpha+1}dt\int^t G\varphi_1^\beta dt_1\,,$$

where $D=\frac{d}{d\Phi}$, $\mathfrak{D}=\frac{\partial}{\partial\Phi}$, $\mathfrak{D}^2=\frac{\partial^2}{\partial\Phi^2}$ and[2] $g=G(\Phi,\Phi)$, $g'=DG(\Phi,\Phi)$ and $\mathfrak{g}'=\mathfrak{D}G(\Phi,\Phi_1)\,|_{t_1=t}$.

To prove this, consider φ'' as $(\varphi')'$ and integrate by parts. We have

2) It should be noticed that $\mathfrak{g}'\neq g'$ i.e. $\mathfrak{D}G(\Phi,\Phi_1)\,\big|_{t_1=t}\neq DG(\Phi,\Phi)$, so that $F'g+F\mathfrak{g}'\neq D(Fg)$.

$$B = -\int F'\varphi^{\alpha+1}\varphi' dt \int^t G\varphi_1^\beta dt_1 - \alpha \int F\varphi^{\alpha-1}\varphi'^2 dt \int^t G\varphi_1^\beta dt_1$$
$$-\int F\varphi^{\alpha+1}\varphi' dt \int^t \mathfrak{D}G\varphi_1^\beta dt_1 - \int Fg\varphi^{\alpha+\beta}\varphi' dt,$$

whose second integral in view of (13. 0) equals $-\alpha B - \alpha \int F\varphi^{\alpha+1} dt \int^t G\varphi_1^\beta dt_1$. On transposing this $-\alpha B$ into the left side and repeating integrations by parts and finally dividing out by $\alpha+1$, we obtain (13. 5). Moreover, again in view of (13. 0), by adding $\int F\varphi^{\alpha-1}\varphi'^2 dt \int^t G\varphi_1^\beta dt_1$ to both sides, we see that

(13. 6) $\int F(\Phi)\varphi^{\alpha-1}\varphi'^2 dt \int^t G(\Phi, \Phi_1)\varphi_1^\beta dt_1 =$ "the expression on the right side of (13. 5) but having the last integral with the coefficient $\frac{1}{\alpha+1}$ $\left(\text{instead of } \frac{-\alpha}{\alpha+1}\right)$".

In particular, for $\alpha = 1, 2, \cdots$, we get

(13. 6. 1) $$\int F(\Phi)\varphi'^2 dt \int^t G(\Phi, \Phi_1)\varphi_1^\beta dt_1 = \frac{1}{6}\Big[\int F''\varphi^4 dt \int^t G\varphi_1^\beta dt_1 + 2\int F'\varphi^4 dt \int^t \mathfrak{D}G\varphi_1^\beta dt_1$$
$$+ \int F\varphi^4 dt \int^t \mathfrak{D}^2 G\varphi_1^\beta dt_1 + \int (F'g + Fg')\varphi^{\beta+3} dt\Big] + \frac{1}{2(\beta+2)}\int D(Fg)\varphi^{\beta+3} dt + \frac{1}{2}\int F\varphi^2 dt \int^t G\varphi_1^\beta dt_1,$$

(13. 6. 2) $$\int F(\Phi)\varphi\varphi'^2 dt \int^t G(\Phi, \Phi_1)\varphi_1^\beta dt_1 = \frac{1}{12}\Big[\int F''\varphi^5 dt \int^t G\varphi_1^\beta dt_1 + 2\int F'\varphi^5 dt \int^t \mathfrak{D}G\varphi_1^\beta dt_1$$
$$+ \int F\varphi^5 dt \int^t \mathfrak{D}^2 G\varphi_1^\beta dt_1 + \int (F'g + Fg')\varphi^{\beta+4} dt\Big] + \frac{1}{3(\beta+3)}\int D(Fg)\varphi^{\beta+4} dt + \frac{1}{3}\int F\varphi^3 dt \int^t G\varphi_1^\beta dt_1,$$

and so on.

Thirdly, let the double integral be of the form $\int F(\Phi)\varphi^\alpha dt \int^t G(\Phi, \Phi_1)\varphi_1^{\beta-1}\varphi_1'^2 dt_1$. To make the inner integral free from t and t-derivatives, we may again treat as above: An integration by parts yields

$$C \equiv \int^t G(\Phi, \Phi_1)\varphi_1^\beta \varphi_1'' dt_1 = G(\Phi, \Phi)\varphi^\beta \varphi' - \int \mathfrak{D}_1 G\cdot\varphi_1^{\beta+1}\varphi_1' dt_1 - \beta\int^t G\varphi_1^{\beta-1}\varphi'^2 dt_1,$$

where $\mathfrak{D}_1 = \dfrac{\partial}{\partial\Phi_1}$.

On substituting $\varphi_1'^2 = \varphi_1\varphi_1'' + \varphi_1^2$ in the last integral and transposing thus obtained $-\beta C$ into the left side, and repeating integrations by parts, we obtain

(13. 7) $$\int^t G(\Phi, \Phi_1)\varphi_1^\beta \varphi_1'' dt_1 = \frac{1}{\beta+1} G(\Phi, \Phi)\varphi^\beta\varphi' - \frac{\beta}{\beta+1}\int^t G\varphi_1^{\beta+1} dt_1$$
$$-\frac{1}{(\beta+1)(\beta+2)}\Big[\mathfrak{D}_1 G(\Phi, \Phi_1)\Big|_{t_1=t}\varphi^{\beta+2} - \int^t \mathfrak{D}_1^2 G\cdot\varphi_1^{\beta+3} dt_1\Big].$$

Whence, again by aid of (13. 0), we find that

(13. 8) $\int^t G(\Phi, \Phi_1)\varphi_1^\beta \varphi_1'^2 dt_1 =$ "the same expression on the right side of (13. 7), but now having the second integral with the coefficient $\dfrac{1}{\beta+1}$ instead of $\dfrac{-\beta}{\beta+1}$".

In particular, for $\beta=1, 2, 3, \cdots$,

(13. 8. 1) $$\int^t G(\Phi, \Phi_1)\varphi_1'^2 dt_1 = \frac{1}{2}G(\Phi, \Phi)\varphi\varphi' + \frac{1}{2}\int^t G(\Phi, \Phi_1)\varphi_1^2 dt_1 - \frac{1}{6}\mathfrak{D}_1 G\Big|_{t_1=t}\varphi^3 + \frac{1}{6}\int^t \mathfrak{D}_1^2 G\cdot\varphi_1^4 dt_1,$$

(13. 8. 2) $$\int^t G(\Phi, \Phi_1)\varphi_1\varphi_1'^2 dt_1 = \frac{1}{3}G(\Phi, \Phi)\varphi^2\varphi' + \frac{1}{3}\int G(\Phi, \Phi_1)\varphi_1^3 dt_1 - \frac{1}{12}\mathfrak{D}_1 G\Big|_{t_1=t}\varphi^4 + \frac{1}{12}\int^t \mathfrak{D}_1^2 G\cdot\varphi_1^5 dt_1,$$

(13. 8. 3) $$\int^t G(\Phi, \Phi_1)\varphi_1^2\varphi_1'^2 dt_1 = \frac{1}{4}G(\Phi, \Phi)\varphi^3\varphi' + \frac{1}{4}\int G(\Phi, \Phi_1)\varphi_1^4 dt_1 - \frac{1}{20}\mathfrak{D}_1 G\Big|_{t_1=t}\varphi^5 + \frac{1}{20}\int^t \mathfrak{D}_1^2 G\cdot\varphi_1^6 dt_1,$$ and so on.

Remark. If the double integral be of the form

$$\int F(\Phi)\varphi^\alpha\varphi'^2 dt \int^t G(\Phi, \Phi_1)\varphi_1^\beta\varphi_1'^2 dt_1,$$

we may first employ (13. 8) and reduce it to the sum of forms (13. 2), (13. 6) and then make the whole integrals dashfree.

Example 3. Write the double integral

$$\int F(\Phi)\varphi^\alpha dt \int^t G(\Phi, \Phi_1)t_1^3\varphi_1 dt_1$$

in a form whose integrand is free from t_1 and t_1-derivatives.

From the equality $\varphi_1'^2 = t_1^2\varphi_1^2 = \varphi_1^2 + \varphi_1\varphi_1''$, we get $t_1^2\varphi_1 = \varphi_1 + \varphi_1''$. This yields on differentiation $t_1^2\varphi_1' + 2t_1\varphi_1 = \varphi_1' + \varphi_1'''$, i.e. $t_1^3\varphi_1 = -3\varphi_1' - \varphi_1'''$. Hence

$$\int^t G(\Phi, \Phi_1)\,t_1^3\varphi_1 dt_1 = -3\int^t G(\Phi, \Phi_1)\varphi_1' dt_1 - \int^t G(\Phi, \Phi_1)(\varphi_1'')' dt_1$$
$$= -3G(\Phi, \Phi)\varphi + 3\int^t \mathfrak{D}_1 G\cdot\varphi_1^2 dt_1 - \underline{G(\Phi, \Phi)\varphi''} + \int^t \mathfrak{D}_1 G\cdot\varphi_1\varphi_1'' dt_1.$$

The last integral being of the form (13. 7) with $\beta=1$, equals

$$\underline{\frac{1}{2}\mathfrak{D}_1 G\Big|_{t_1=t}\varphi\varphi'} - \frac{1}{2}\int^t \mathfrak{D}_1 G\cdot\varphi_1^2 dt_1 - \frac{1}{6}\mathfrak{D}_1^2 G\Big|_{t_1=t}\varphi^3 + \frac{1}{6}\int^t \mathfrak{D}_1^3 G\cdot\varphi_1^4 dt_1.$$

Thus the given double integral attains the required form, if those corresponding to the underlined two:

$$-\int F(\Phi)G(\Phi, \Phi)\varphi^\alpha\varphi'' dt \quad\text{and}\quad \frac{1}{2}\int F(\Phi)\mathfrak{D}_1 G\Big|_{t_1=t}\varphi^{\alpha+1}\varphi' dt$$

do so. For the former it is so by (13. 1), while for the latter it is clear. And thus we get

$$(13.9)\quad \int F(\Phi)\varphi^{\alpha}dt\int^{t}G(\Phi,\Phi_1)t_1^3\varphi_1dt_1=-3\int F(\Phi)G(\Phi,\Phi)\varphi^{\alpha+1}dt$$

$$-\frac{1}{(\alpha+1)(\alpha+2)}\int D^2[F(\Phi)G(\Phi,\Phi)]\varphi^{\alpha+3}dt+\frac{\alpha}{\alpha+1}\int F(\Phi)G(\Phi,\Phi)\varphi^{\alpha+1}dt$$

$$-\frac{1}{2(\alpha+2)}\int D[F(\Phi)]\cdot[\mathfrak{D}_1G]_{t_1=t}\varphi^{\alpha+3}dt-\frac{1}{6}\int F(\Phi)[\mathfrak{D}_1^2G]_{t_1=t}\varphi^{\alpha+3}dt$$

$$+\frac{5}{2}\int F(\Phi)\varphi^{\alpha}dt\int^{t}\mathfrak{D}_1G\cdot\varphi_1^2dt_1+\frac{1}{2}\int F(\Phi)\varphi^{\alpha}dt\int^{t}\mathfrak{D}_1^3G\cdot\varphi_1^4dt_1,\quad \text{Q.E.I.}$$

(D) It requires frequently to be found those values of $G=G(\Phi,\Phi_1)$, $\mathfrak{D}G$, $\mathfrak{D}_1G$ and $\mathfrak{D}_1^2G$, when $t_1=t$. In the following the actual expression of $G(\Phi,\Phi_1)$ is often of the form

$$G(\Phi,\Phi_1)=\frac{(\Phi-\Phi_1)^j\Phi_1^{i-1}}{j!\,(i-1)!}\equiv\underline{(\Phi-\Phi_1)^j\Phi_1^{i-1}}\,.$$

Hence, if $j\geqq 1$, $G(\Phi,\Phi)=0$. Moreover, if $j>k\geqq 1$, all $\mathfrak{D}^l\mathfrak{D}_1^{k-l}G(\Phi,\Phi_1)$ for $l=0, 1,\cdots,k$ do vanish when $t_1=t$. However, if $j\leqq k$, it is not necessarily so. In the most frequent case $j=1$, we have by Leibnitz's formula

$$(13.10)\qquad \mathfrak{D}_1^k[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]_{t_1=t}=-k\underline{\Phi^{i-k}}\qquad\text{for}\quad k=0,1,2,\cdots.$$

E.g. $\mathfrak{D}_1[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]_{t_1=t}=-\underline{\Phi^{i-1}}$, $\mathfrak{D}_1^2[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]=-2\underline{\Phi^{i-2}}$, $\cdots$,

$\mathfrak{D}_1^i[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]=-i$,

and if $k>i$, of course, $\mathfrak{D}_1^k[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]=-k\underline{\Phi^{i-k}}=0$, because, then the denominator $\Gamma(i-k+1)=\infty$.

(E) Any formal expression obtained on the way should be reduced to a concrete one if possible. For example, it is really

$$(13.11)\qquad \int^{t}\underline{(\Phi-\Phi_1)^{n-i-1}}\,\underline{\Phi_1^{i-1}}\varphi_1dt_1=\underline{\Phi^{k-1}}\,.$$

For, upon putting $\Phi_1=\Phi v$, $d\Phi_1=\varphi_1dt_1=\Phi dv$, the integral reduces to

$$\frac{\Phi^{k-1}}{\lfloor k-i-1\,\lfloor i-1}\int_0^1(1-v)^{k-i-1}v^{i-1}dv=\frac{\Phi^{k-1}}{\lfloor k-1}=\underline{\Phi^{k-1}}\,.$$

In particular, if $k=i+1$, $\int^{t}\underline{\Phi_1^{i-1}}\varphi_1dt_1=\underline{\Phi^{i}}$; if $t=\infty$, $\Phi=1$ and $\int\underline{(1-\Phi_1)^{k-i-1}}\times\underline{\Phi^{i-1}}\varphi_1dt_1=1$. It is evident that $\int F(\Phi)\varphi dt=\int_0^1F(\Phi)d\Phi$. On denoting $\underline{\Phi^{i-1}}\underline{(1-\Phi)^{n-i}}\equiv F$ and $D=\dfrac{d}{d\Phi}$, we had in (1.3) (2.4) (2.6) (2.7) (2.8) of Part I:

$$E(t_{i|n}^0)=1=\lfloor n\int F\varphi dt,\quad E(t_{i|n})=\lfloor n\int DF\cdot\varphi^2dt,\quad E(t_{i|n}^2)=1+\frac{1}{2}\lfloor n\int D^2F\cdot\varphi^3dt,$$

$$E(t_{i|n}^3)=\frac{5}{2}E(t_{i|n})+\frac{\lfloor n}{\lfloor 3}\int D^3F\cdot\varphi^4dt,\quad E(t_{i|n}^4)=\frac{13}{3}E(t_{i|n}^2)-\frac{4}{3}+\frac{\lfloor n}{\lfloor 4}\int D^4F\cdot\varphi^5dt,$$

so that $\int D^k F\cdot\varphi^{k+1}dt$ $(k=0, 1, 2, \cdots)$ may be expressed in terms of $E(t_{i|n}^{k-j})$.

Example 4. Show that

$$1-n!\int\underline{(1-\Phi)^\alpha\Phi^{\beta+\gamma-1}}\varphi^\delta dt = n!\int\underline{(1-\Phi)^{n-k}}\varphi\, dt\int^t\underline{(\Phi-\Phi_1)^{k-i-1}}\Phi_1^{i-1}\varphi_1 dt_1$$
$$-n!\int\underline{(1-\Phi)^\alpha}\varphi^\delta dt\int^t\underline{(\Phi-\Phi_1)^{\beta-1}}\Phi_1^{\gamma-1}\varphi_1 dt_1\,.$$

This is evident: For, the first double integral on the right side is nothing but

$$\iint f(t_{i|n},\, t_{k|n})dt_{i|n}dt_{k|n}=1\,,$$

while, the second reduces to the single integral on the left side because of (13. 11).

§ 14. Expectations of Products $t_{i|n}^p t_{k|n}^q$ $(p+q=3$ or 4 and $1\leqq i<k\leqq n)$.

Practically it need not obtain the formal expression of moments, especially for small samples such as $n=2, 3$: We may solely proceed to calculate directly every value separately. However, for the sake of theoretical completeness and eventual use even in case of large sample, we have prepared the detailed formulas, which are free from t and t-derivatives. By symmetrical property obtained in (5. 1): $E(t_{n-i+1}^p t_{n-k+1}^q)=(-)^{p+q}E(t_{i|n}^p t_{k|n}^q)$, we have only to consider those cases $p\geqq q$.

$$1^\circ\quad E(t_{i|n}^2 t_{k|n})=\underline{|n}\int\underline{(1-\Phi)^{n-k}}\varphi' dt\int^t\underline{(\Phi-\Phi_1)^{k-i-1}}\Phi_1^{i-1}\varphi_1' t_1 dt_1\,.$$

Firstly for $k=i+1$

$$E(t_{i|n}^2 t_{i+1|n})=\underline{|n}\int\underline{(1-\Phi)^{n-i-1}}\varphi' dt\int^t\Phi_1^{i-1}t_1\varphi_1' dt_1$$

By integrations by parts the inner integral reduces to

$$I(t)\equiv\int^t\Phi_1^{i-1}t_1\varphi_1' dt_1=-\Phi^i-\Phi^{i-1}\varphi'+\frac{1}{2}\Phi^{i-2}\varphi^2-\frac{1}{2}\int^t\Phi_1^{i-3}\varphi_1^3 dt_1\,. \tag{14.1.0}$$

This being substituted in $E(t_{i|n}^2 t_{i+1|n})=\underline{|n}\int\underline{(1-\Phi)^{n-i-1}}I(t)\varphi' dt$ and repeatedly integrated by parts, we find[3)]

$$\begin{aligned}E(t_{i|n}^2 t_{i+1|n}) &= -\underline{|n}\int\underline{(1-\Phi)^{n-i-2}\Phi^i}\varphi^2 dt+\frac{\underline{|n}}{2}\int\underline{(1-\Phi)^{n-i-1}\Phi^{i-1}}\varphi^2 dt\\ &-\frac{\underline{|n}}{6}\int\underline{(1-\Phi)^{n-i-3}\Phi_1^{i-1}}\varphi^4 dt+\frac{\underline{|n}}{2}\int\underline{(1-\Phi)^{n-i-2}\Phi^{i-2}}\varphi^4 dt\\ &+\frac{\underline{|n}}{6}\int\underline{(1-\Phi)^{n-i-1}\Phi^{i-3}}\varphi^4 dt-\frac{\underline{|n}}{2}\int\underline{(1-\Phi)^{n-i-2}}\varphi^2 dt\int^t\Phi_1^{i-3}\varphi_1^3 dt_1\,,\end{aligned} \tag{14.1.1}$$

where the first integral becomes $E(t_{i+1|n})$.

---

3) As check, we may examine the dimension of each integral to be n, e.g. that of the first integral is really $n-i-2+i+2=n$, &c.

Secondly, for $k \geqq i+2$, the inner integral $I(t)$ becomes now

$$I(t)=\int^{t}(\Phi-\Phi_1)\Phi_1^{i-1}\varphi_1{}'t_1dt_1=-\Phi^{i+1}-\frac{1}{2}\Phi^{i-1}\varphi^2-\frac{1}{2}\int^{t}\mathfrak{D}_1^2[(\Phi-\Phi_1)\Phi_1^{i-1}]\varphi_1^3dt_1\,,$$

and we obtain

$$(14.1.2)\quad E(t_{i|n}^2t_{i+2|n})=\underline{|n}\int D[(1-\Phi)^{n-i-2}\Phi^{i+1}]\varphi^2dt-\underline{|n}\int(1-\Phi)^{n-i-2}\Phi^{i-2}\varphi^4dt$$
$$+\frac{\underline{|n}}{6}\int D[(1-\Phi)^{n-i-2}\Phi^{i-1}]\varphi^4dt+\frac{\underline{|n}}{2}\int(1-\Phi)^{n-i-2}\varphi^2dt\int^{t}\Phi_1^{i-3}\varphi_1^3dt_1$$
$$-\frac{\underline{|n}}{2}(1-\Phi)^{n-i-3}\varphi^2dt\int^{t}\mathfrak{D}_1^2[(\Phi-\Phi_1)\Phi_1^{i-1}]\varphi_1^3dt_1\,.$$

where the first integral is $E(t_{i+2|n})$.

Thirdly for $k \geqq i+3$, the inner integral becomes

$$I(t)=-\Phi^{k-1}-\frac{1}{2}\int^{t}\mathfrak{D}_1^2[(\Phi-\Phi_1)^{k-i-1}\Phi^{i-1}]\varphi_1^3dt_1$$

and we get

$$(14.1.3)\quad E(t_{i|n}^2t_{k|n})=\underline{|n}\int D[(1-\Phi)^{n-k}\Phi^{k-1}]\varphi^2dt-\frac{\underline{|n}}{2}\int(1-\Phi)^{n-k-1}\varphi^2dt$$
$$\times\int^{t}\mathfrak{D}_1^2[(\Phi-\Phi_1)^{k-i-1}\Phi_1^{i-1}]\varphi_1^3dt_1+\frac{\underline{|n}}{2}\int(1-\Phi)^{n-k}\varphi^2dt\int^{t}\mathfrak{D}_1^2[(\Phi-\Phi_1)^{k-i-2}\Phi_1^{i-1}]\varphi_1^3dt_1$$
$$+\delta_{i+3}^{\kappa}\frac{\underline{|n}}{2}\int(1-\Phi)^{n-k}\Phi^{i-1}\varphi^4dt\,,$$

where the first integral is $E(t_{k|n})$ and δ_{i+3}^{k} denotes Kronecker's delta.

$$2^\circ\quad E(t_{i|n}^2t_{k|n}^2)=\underline{|n}\int(1-\Phi)^{n-k}\varphi't\,dt\int^{t}(\Phi-\Phi_1)^{k-i-1}\Phi_1^{i-1}\varphi_1{}'t_1dt_1\,.$$

Firstly for $k=i+1$,

$$E(t_{i|n}^2t_{i+1|n}^2)=\underline{|n}\int(1-\Phi)^{n-i-1}\varphi't\,dt\int^{t}\Phi_1^{i-1}\varphi_1{}'t_1dt_1\,,$$

so that the inner integral $I(t)$ is the same as in (14.1.0), and $I'(t)=\Phi^{i-1}\varphi't$. Hence, if we write $\underline{|n}(1-\Phi)^{n-i-1}=F(\Phi)$,

$$E(t_{i|n}^2t_{i+1|n}^2)=\int F(\Phi)I(t)\varphi't\,dt=-\int F(\Phi)I(t)\varphi\,dt-\int F'(\Phi)I(t)\varphi^2tdt-\int F(\Phi)I'(t)\varphi\,t\,dt$$
$$=(\mathrm{i})+(\mathrm{ii})+(\mathrm{iii})\,.$$

On, substituting the values of $I(t)$ and $I'(t)$, we obtain

$$(\mathrm{i})=1-\frac{1}{2}\int D(F\Phi^{i-1})\varphi^3dt-\frac{1}{2}\int F\Phi^{i-2}\varphi^3dt+\frac{1}{2}\int F\varphi\,dt\int^{t}\Phi_1^{i-3}\varphi_1^3dt_1$$

$$(\mathrm{ii})=\frac{1}{2}\int D(F'\Phi^{i})\varphi^3dt-\int F'\Phi^{i-1}\varphi\varphi'^2dt-\frac{1}{8}\int D(F'\Phi^{i-2})\varphi^5dt$$
$$+\frac{1}{4}\int F''\varphi^3dt\int^{t}\Phi_1^{i-3}\varphi_1^3dt_1+\frac{1}{4}\int F'\Phi^{i-3}\varphi^5dt \qquad \cdots\cdots\cdots(a)$$

where for the second integral (13. 2. 2) may be employed[4]. And by (13. 3)

$$(\text{iii}) = \int F\underline{\Phi^{i-1}}\varphi'^2 dt = \frac{2}{3}\int D(F\underline{\Phi^{i-1}})\varphi^3 dt + \frac{1}{24}\int D^3(F\underline{\Phi^{i-1}})\varphi^5 dt .$$

Hence, we obtain

$$(14.2.1)\quad E(t^2_{i|n}t^2_{i+1|n}) = 1 + \lfloor n \int\Big\{\frac{1}{2}\underline{(1-\Phi)^{n-i-3}\Phi^i} - \frac{1}{3}\underline{(1-\Phi)^{n-i-2}\Phi^{i-1}} - \frac{1}{3}\underline{(1-\Phi)^{n-i-1}\Phi^{i-2}}\Big\}\varphi^3 dt + \frac{\lfloor n}{24}\int\{\underline{(1-\Phi)^{n-i-4}\Phi^{i-1}} - 4\underline{(1-\Phi)^{n-i-3}\Phi^{i-2}} - 4\underline{(1-\Phi)^{n-i-2}\Phi^{i-3}} + \underline{(1-\Phi)^{n-i-1}\Phi^{i-4}}\}\varphi^5 dt + \frac{\lfloor n}{2}\int\underline{(1-\Phi)^{n-i-1}}\varphi\, dt\int^t \underline{\Phi_1^{i-3}}\varphi_1^3 dt_1 + \frac{\lfloor n}{4}\int\underline{(1-\Phi)^{n-i-3}}\varphi^3 dt\int^t \underline{\Phi_1^{i-3}}\varphi_1^3 dt_1 .$$

Secondly for $k=i+2$, on treating similarly as in (2. 1), but now with the inner integral of (1. 2): $I(t) = -\underline{\Phi^{i+1}} - \frac{1}{2}\int\underline{\Phi^{i-1}}\varphi^2 - \frac{1}{2}\int^t \mathfrak{D}_1^2[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]\varphi_1^3 dt_1$, so that $I'(t) = -\underline{\Phi^i}\varphi - \underline{\Phi^{i-1}}\varphi\varphi' + \frac{1}{2}\Phi^{i-2}\varphi^3 - \frac{1}{2}\int^t \underline{\Phi_1^{i-3}}\varphi_1^3 dt_1$, we find

$$(14.2.2)\quad E(t^2_{i|n}t^2_{k|n}) = 1 + \lfloor n \int\Big\{\frac{1}{2}\underline{(1-\Phi)^{n-i-4}\Phi^{i+1}} - [\underline{(1-\Phi)^{n-i-3}\Phi^i}] + \frac{2}{3}[\underline{(1-\Phi)^{n-i-2}\Phi^{i-1}}]\Big\}\varphi^3 dt + \lfloor n \int\Big\{\frac{1}{24}\underline{(1-\Phi)^{n-i-4}\Phi^{i-1}} + \frac{2}{3}\underline{(1-\Phi)^{n-i-3}\Phi^{i-2}} + \frac{1}{24}\underline{(1-\Phi)^{n-i-2}\Phi^{i-3}}\Big\}\varphi^5 dt - \frac{\lfloor n}{2}\int\underline{(1-\Phi)^{n-i-3}}\varphi^3 dt\int^t \underline{\Phi_1^{i-3}}\varphi_1^3 dt_1 + \frac{\lfloor n}{2}\int\underline{(1-\Phi)^{n-i-2}}\varphi\, dt\int^t \mathfrak{D}_1^2[(\Phi-\Phi_1)\Phi_1^{i-1}]\varphi_1^3 dt_1 + \frac{\lfloor n}{4}\int\underline{(1-\Phi)^{n-i-4}}\varphi^3 dt\int^t \mathfrak{D}_1^2[(\Phi-\Phi_1)\Phi_1^{i-1}]\varphi_1^3 dt_1 .$$

Thirdly for $k \geqq i+3$, the inner integral is the same as in (1. 3): $I(t) = \int^t (\Phi-\Phi_1)^{n-i-1}\Phi_1^{i-1}\varphi_1' dt_1 = -\underline{\Phi^{k-1}} - \frac{1}{2}\int^t \mathfrak{D}_1^2[\underline{(\Phi-\Phi_1)^{k-i-1}}\underline{\Phi_1^{i-1}}]\varphi_1^3 dt_1$, so that $I'(t) = -\Phi^{k-2}\varphi - \frac{\varphi}{2}\int\mathfrak{D}_1^2[\underline{(\Phi-\Phi_1)^{k-i-2}}\underline{\Phi_1^{i-1}}]\varphi_1^3 dt_1$.

---

4) Otherwise:

$$(\text{ii}) = \int F'I\varphi\varphi' dt = -\frac{1}{2}\int F''I\varphi^3 dt - \frac{1}{2}\int F'I'\varphi^2 dt = \frac{1}{2}\int F''\underline{\Phi^i}\varphi^3 dt - \frac{1}{8}\int D[F''\underline{\Phi^{i-1}}]\varphi^5 dt - \frac{1}{4}\int F''\underline{\Phi^{i-2}}\varphi^5 dt + \frac{1}{4}\int F''\varphi^3 dt\int^t \underline{\Phi_1^{i-3}}\varphi_1^3 dt_1 + \frac{1}{2}\int F'\underline{\Phi^{i-1}}\varphi\varphi'^2 dt, \qquad \cdots (b)$$

which appears to differ from the before obtained result (a). However, this difference is only superficial, because, if (a) be equated to (b), we shall thereupon obtain $\frac{3}{2}\int F'\underline{\Phi^{i-1}}\varphi\varphi'^2 dt = \frac{1}{8}\int\Big\{F'''\underline{\Phi^{i-1}} + 2F''\underline{\Phi^{i-2}} + F'\underline{\Phi^{i-3}}\Big\}\varphi^5 dt + \frac{1}{2}\int F'\underline{\Phi^{i-1}}\varphi^3 dt$ which agrees materially with (13. 2. 2). Such alike occurs frequently.

And we get

(14. 2. 3) $E(t_{i|n}^2 t_{i|n}^2) \quad (k \geqq i+3)$

$$
\begin{aligned}
&= 1 + \frac{\underline{|n}}{2}\int (1-\Phi)^{n-k-2}\Phi^{k-1}\varphi^3 dt - \underline{|n}\int (1-\Phi)^{n-k-1}\Phi^{k-2}\varphi^3 dt + \frac{\underline{|n}}{2}\int (1-\Phi)^{n-k}\Phi^{k-3}\varphi^3 dt \\
&\quad + \frac{\underline{|n}}{2}\int (1-\Phi)^{n-k}\varphi dt \int^t \mathfrak{D}_1^2[(\Phi-\Phi_1)^{k-i-1}\Phi_1^{i-1}]\varphi_1^3 dt_1 - \frac{\underline{|n}}{2}\int (1-\Phi)^{n-k-1}\varphi^3 dt \\
&\quad \times \int \mathfrak{D}_1^2[(\Phi-\Phi_1)^{k-i-2}\Phi_1^{i-1}]\varphi_1^3 dt_1 + \frac{\underline{|n}}{4}\int (1-\Phi)^{n-k-2}\varphi^3 dt \int^t \mathfrak{D}_1^2[(\Phi-\Phi_1)^{k-i-1}\Phi_1^{i-1}]\varphi_1^3 dt_1 \\
&\quad - \frac{\underline{|n}}{4}\int (1-\Phi)^{n-k}\varphi^3 dt \int^t \mathfrak{D}_1^2[(\Phi-\Phi_1)^{k-i-3}\Phi_1^{i-1}]\varphi_1^3 dt_1 - \delta_{i+3}^k \frac{3\underline{|n}}{8}\int \{(1-\Phi)^{n-k}\Phi^{i-2} \\
&\quad + (1-\Phi)^{n-k-1}\Phi^{i-1}\}\varphi^5 dt + \delta_{k+4}^k \frac{\underline{|n}}{4}\int (1-\Phi)^{n-k}\Phi^{i-1}\varphi^5 dt .
\end{aligned}
$$

3° $E(t_{i|n}^3 t_{k|n}) = \underline{|n}\int (1-\Phi)^{n-k}\varphi' dt \int^t (\Phi-\Phi_1)^{k-i-1}\Phi_1^{i-1}\varphi_1' t_1^2 dt_1 .$

Firstly for $k=i+1$

$$
E(t_{i|n}^3 t_{i+1|n}) = \underline{|n}\int (1-\Phi)^{n-i-1}\varphi' dt \int^t \Phi_1^{i-1}\varphi_1' t_1^2 dt_1 = \underline{|n}\int (1-\Phi)^{n-i-1}\varphi' I(t) dt .
$$

Here the inner integral being

$$
I(t) = \int^t \Phi_1^{i-1}\varphi_1' t_1^2 dt_1 = -\Phi^{i-1}\varphi' t - \int^t \Phi_1^{i-2}\varphi_1'^2 dt_1 - 2\int \Phi_1^{i-1}\varphi_1' dt_1 ,
$$

$$
\begin{aligned}
E(t_{i|n}^3 t_{i+1|n}) &= -\underline{|n}\int (1-\Phi)^{n-i-1}\Phi^{i-1}\varphi'^2 t\, dt - \underline{|n}\int (1-\Phi)^{n-i-1}\varphi' dt \int^t \Phi_1^{i-1}\varphi_1'^2 dt_1 \\
&\quad + 2\underline{|n}\int (1-\Phi)^{n-i-1}\varphi' dt \int^t \Phi_1^{i-1}\varphi_1' dt_1 .
\end{aligned}
$$

Hence, making use of (13. 3), (13. 8. 1) (13. 2. 3) &c., we find[5]

(14. 3. 1) $E(t_{i|n}^3 t_{i+1|n}) = \underline{|n}\int \left\{\frac{2}{3}(1-\Phi)^{n-i-1}\Phi^{i-2} + \frac{5}{3}(1-\Phi)^{n-i-2}\Phi^{i-1}\right\}\varphi^3 dt$

$$
\begin{aligned}
&+ \frac{\underline{|n}}{24}\int [(1-\Phi)^{n-i-4}\Phi^{i-1} - 4(1-\Phi)^{n-i-3}\Phi^{i-2} + 6(1-\Phi)^{n-i-2}\Phi^{i-3} + (1-\Phi)^{n-i-1}\Phi_1^{i-4}]\varphi^5 dt \\
&- \frac{5}{2}\underline{|n}\int (1-\Phi)^{n-i-2}\varphi^2 dt \int^t \Phi_1^{i-2}\varphi_1^2 dt_1 - \frac{\underline{|n}}{6}\int (1-\Phi)^{n-i-2}\varphi^2 dt \int^t \Phi_1^{i-4}\varphi_1^4 dt_1 .
\end{aligned}
$$

Secondly for $k=i+2$, $E(t_{i|n}^3 t_{i+2|n}) = \underline{|n}\int (1-\Phi)^{n-i-2}\varphi' dt \int^t (\Phi-\Phi_1)\Phi_1^{i-1}\varphi_1' t_1^2 dt_1$, and we find the inner integral to be

---

5) Otherwise, we may utilize e. g. (13. 9) directly and obtain a result that is apparently different from (14. 3. 1) in the integrand containing φ^5 as follows:

$$
\begin{aligned}
&\underline{|n}\int \Big[\frac{1}{24}D^3\{(1-\Phi)^{n-i-1}\Phi^{i-2}\} + \frac{1}{12}D^2\{(1-\Phi)^{n-i-2}\Phi^{i-1}\} + \frac{1}{8}D\{(1-\Phi)^{n-i-2}\Phi^{i-2}\} \\
&\quad + \frac{1}{6}(1-\Phi)^{n-i-2}\Phi^{i-3}\Big]\varphi^5 dt.
\end{aligned}
$$

However, they are really equal, since it is readily seen that the expressions under both square brackets are the same polynomial: $\frac{1}{8}(1-\Phi)^{n-i-2}\Phi^{i-3}$.

$$I(t)=-\int^t \mathfrak{D}_1 G\cdot\varphi_1'^2 dt_1-2\int^t \mathfrak{D}_1 G\cdot\varphi_1^2 dt_1 \quad (\text{if} \quad G=G(\Phi,\Phi_1)=(\Phi-\Phi_1)\Phi_1^{i-1})$$

$$=\frac{1}{2}\Phi^{i-1}\varphi\varphi'-\frac{1}{3}\Phi^{i-2}\varphi^3-\frac{5}{2}\int^t \mathfrak{D}_1 G\cdot\varphi_1^2 dt_1-\frac{1}{6}\int \mathfrak{D}_1^3 G\cdot\varphi_1^4 dt_1$$

on applying (13. 8. 1). Hence, we get

(14. 3. 2) $E(t_{i|n}^3 t_{i+2|n})=|\underline{n}\int(1-\Phi)^{n-i-2}I(t)\varphi' dt$

$$=-\frac{7}{3}|\underline{n}\int(1-\Phi)^{n-i-2}\Phi^{i-1}\varphi^3 dt+\frac{|\underline{n}}{24}\int[(1-\Phi)^{n-i-4}\Phi^{i-1}-4(1-\Phi)^{n-i-3}\Phi^{i-2}$$

$$-9(1-\Phi)^{n-i-2}\Phi^{i-3}]\varphi^5 dt+\frac{5}{2}|\underline{n}\int(1-\Phi)^{n-i-2}\varphi^2 dt\int^t \Phi_1^{i-2}\varphi_1^2 dt_1$$

$$-\frac{5}{2}|\underline{n}\int(1-\Phi)^{n-i-3}\varphi^2 dt\int^t \mathfrak{D}_1[(\Phi-\Phi_1)\Phi_1^{i-1}]\varphi_1^2 dt_1+\frac{1}{6}|\underline{n}\int(1-\Phi)^{n-i-2}\varphi^2 dt\int^t \Phi_1^{i-4}\varphi_1^4 dt_1$$

$$-\frac{1}{6}|\underline{n}\int(1-\Phi)^{n-i-3}\varphi^2 dt\int^t \mathfrak{D}_1^3[(\Phi-\Phi_1)\Phi_1^{i-1}]\varphi_1^4 dt_1\,.$$

Thirdly for $k\geqq i+3$, we have the integral $I(t)=\int^t(\Phi-\Phi_1)^{k-i-1}\Phi_1^{i-1}\varphi_1' t_1^2 dt_1=$
$-\frac{5}{2}\int^t \mathfrak{D}_1[(\Phi-\Phi_1)^{k-i-1}\Phi_1^{i-1}]\varphi_1^2 dt_1-\frac{1}{6}\int^t \mathfrak{D}_1^3[(\Phi-\Phi_1)^{k-i-1}\Phi_1^{i-1}]\varphi_1^4 dt_1+\delta_{i+3}^k\frac{1}{6}\Phi^{i-1}\varphi^3.$

Hence, we get

(14. 3. 3) $E(t_{1|n}^3 t_{k|n})=|\underline{n}\int(1-\Phi)^{n-k}\varphi' I(t)dt \quad (k\geqq i+3)$

$$=\frac{5}{2}|\underline{n}\int(1-\Phi)^{n-k}\varphi^2 dt\int^t \mathfrak{D}_1[(\Phi-\Phi_1)^{k-i-2}\Phi_1^{i-1}]\varphi_1^2 dt_1-\frac{5}{2}|\underline{n}\int(1-\Phi)^{n-k-1}\varphi^2 dt$$

$$\times\int^t \mathfrak{D}_1[(\Phi-\Phi_1)^{k-i-1}\Phi_1^{i-1}]\varphi_1^2 dt_1+\frac{1}{6}|\underline{n}\int(1-\Phi)^{n-k}\varphi^2 dt\int^t \mathfrak{D}_1^3[(\Phi-\Phi_1)^{k-i-2}\Phi_1^{i-1}]\varphi_1^4 dt_1$$

$$-\frac{1}{6}|\underline{n}\int(1-\Phi)^{n-k-1}\varphi^2 dt\int^t \mathfrak{D}_1^3[(\Phi-\Phi_1)^{k-i-1}\Phi_1^{i-1}]\varphi_1^4 dt_1$$

$$+\delta_{i+3}^k\frac{|\underline{n}}{24}\int\{11(1-\Phi)^{n-k}\Phi^{i-2}+(1-\Phi)^{n-k-1}\Phi^{i-1}\}\varphi^5 dt-\delta_{i+4}^k\frac{|\underline{n}}{6}\int(1-\Phi)^{n-k}\Phi^{i-1}\varphi^5 dt\,.$$

§ 15. Expectation of Products $t_{i|n}^p t_{j|n}^q t_{k|n}^r$ for $p+q+r=3$ or 4 and $1\leqq i<j<k\leqq n$.

1° $E(t_{i|n}t_{j|n}t_{k|n})$

$$=-|\underline{n}\int(1-\Phi)^{n-k}\varphi' dt\int^t(\Phi-\Phi_1)^{k-j-1}\varphi_1' dt_1\int^{t_1}(\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}\varphi_2' dt_2\,.$$

Firstly, let $j=i+1$, $k=i+2$:

$$E(t_{i|n}t_{i+1|n}t_{i+2|n})=-|\underline{n}\int(1-\Phi)^{n-i-2}\varphi' dt\int^t \varphi_1' dt_1\int^{t_1}\Phi_2^{i-1}\varphi_2' dt_2\,.$$

Hence, the inner integrals are

$$I_1(t_1)=\int^{t_1}\Phi_2^{i-1}\varphi_2' dt_2=\Phi_1^{i-1}\varphi_1-\int^{t_1}\Phi_2^{i-2}\varphi_2^2 dt_2\,,$$

and

$$I(t)=\int^t I_1(t_1)\varphi_1'dt_1=\frac{1}{2}\underline{\Phi^{i-1}}\varphi^2-\varphi\int^t\underline{\Phi_1^{i-2}}\varphi_1^2dt_1+\frac{1}{2}\int^t\underline{\Phi_1^{i-2}}\varphi_1^3dt_1\,.$$

Therefore

$$(15.1.1)\quad E(t_{i|n}t_{i+1|n}t_{i+2|n})=-|n\int\underline{(1-\Phi)^{n-i-2}}\varphi' I(t)dt$$
$$=\frac{|n}{6}\int D[\underline{(1-\Phi)^{n-i-2}}\underline{\Phi^{i-1}}]\varphi^4dt-\frac{|n}{2}\int\underline{(1-\Phi)^{n-i-3}}\varphi^2dt\int^t\underline{\Phi_1^{i-2}}\varphi_1^3dt_1$$
$$+\frac{|n}{2}\int\underline{(1-\Phi)^{n-i-3}}\varphi^3dt\int^t\underline{\Phi_1^{i-2}}\varphi_1^2dt\,.$$

Secondly, let still $j=i+1$ but $k=i+3$; then the innermost integral $I_1(t_1)$ is the same as before. However, now

$$I(t)=\int^t(\Phi-\Phi_1)\varphi_1'I_1(t_1)dt_1=\int^t\underline{\Phi_1^{i-1}}\varphi_1^3dt_1-\int^t\varphi_1^2dt_1\int^{t_1}\underline{\Phi_2^{i-2}}\varphi_2^2dt_2$$
$$+\frac{1}{2}\int^t\mathfrak{D}_1[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]\varphi_1^3dt_1$$

Therefore

$$(15.1.2)\quad E(t_{i|n}t_{i+1|n}t_{i+3|n})=\frac{|n}{2}\int\underline{(1-\Phi)^{n-i-3}}\underline{\Phi^{i-1}}\varphi^4dt-|n\int\underline{(1-\Phi)^{n-i-4}}\varphi^2dt$$
$$\times\int^t\underline{\Phi_1^{i-1}}\varphi_1^3dt_1+\frac{|n}{2}\int\underline{(1-\Phi)^{n-i-3}}\varphi^2dt\int^t\underline{\Phi_1^{i-2}}\varphi_1^3dt-|n\int\underline{(1-\Phi)^{n-i-3}}\varphi^3dt\int^t\underline{\Phi_1^{i-2}}\varphi_1^2dt_1$$
$$-\frac{|n}{2}\int\underline{(1-\Phi)^{n-i-4}}\varphi^2dt\int^t\mathfrak{D}_1[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]\varphi_1^3dt_1$$
$$+|n\int\underline{(1-\Phi)^{n-i-4}}\varphi^2dt\int^t\varphi_1^2dt_1\int^{t_1}\underline{\Phi_2^{i-2}}\varphi_2^2dt_2\,.$$

Thirdly, let $j=i+2$ but $k\geqq i+4$. The innermost integral is still the same as before, but now

$$I(t)=\int^t\underline{(\Phi-\Phi_1)^{k-i-2}}I_1(t_1)\varphi_1'dt_1\quad(I_1(t_1)=\underline{\Phi_1^{i-1}}\varphi_1-\int^{t_1}\underline{\Phi_2^{i-2}}\varphi_2^2dt_2)$$
$$=-\frac{1}{2}\int^t\mathfrak{D}_1[\underline{(\Phi-\Phi_1)^{k-i-2}}\underline{\Phi_1^{i-1}}]\varphi_1^3dt_1-\int^t\underline{(\Phi-\Phi_1)^{k-i-3}}\varphi_1^2dt_1\int^t\underline{\Phi_2^{i-2}}\varphi_2^2dt_2$$
$$+\int^t\underline{(\Phi-\Phi_1)^{k-i-2}}\underline{\Phi_1^{i-2}}\varphi_1^3dt_1$$

and $\quad I'(t)=\varphi\int^t\underline{(\Phi-\Phi_1)^{k-i-3}}I_1(t_1)\varphi_1'dt_1$;

so that $I'(t)$ is obtained by replacing[6] k by $k-1$ in the expression of $I(t)$, and multiplying by φ. Hence

6) If the power index obtained by this replacement becomes negative, of course, that term vanishes and therefore need not be written down. This remark shall be used so frequently below.

$$E(t_{i|n}t_{i+1|n}t_{k|n}) = -\underline{|n} \int \underline{(1-\Phi)^{n-k}}\varphi' I(t)dt = -\underline{|n} \int \underline{(1-\Phi)^{n-k-1}}\varphi^2 I(t)dt$$

$$+\underline{|n} \int \underline{(1-\Phi)^{n-k}}\varphi I'(t)dt = -\underline{|n} \int \underline{(1-\Phi)^{n-k-1}}\varphi^2 I(t)dt + \underline{|n} \int \underline{(1-\Phi)^{n-(k-1)-1}}\varphi I'(t)dt$$

$$= (\mathrm{i}) + (\mathrm{ii}) .$$

Therefore, (ii) is obtainable by simply writing $k-1$ for k in the resulting expression of (i), then changing the sign, and thus the labour is reduced by half.

(15. 1. 3) $E(t_{i|n}t_{i+1|n}t_{k|n}) \quad (k \geqq i+4)$

$$= -\frac{|n}{2} \int \underline{(1-\Phi)^{n-k-1}}\varphi^2 dt \int^t \{\underline{(\Phi-\Phi_1)^{k-i-2}}\underline{\Phi_1^{i-2}} + \underline{(\Phi-\Phi_1)^{k-i-3}}\underline{\Phi_1^{i-1}}\}\varphi_1^3 dt_1$$

$$+\frac{|n}{2} \int \underline{(1-\Phi)^{n-k}}\varphi^2 dt \int^t \{\underline{(\Phi-\Phi_1)^{k-i-3}}\underline{\Phi_1^{i-2}} + \underline{(\Phi-\Phi_1)^{k-i-4}}\underline{\Phi_1^{i-1}}\}\varphi_1^3 dt_1$$

$$+|n \int \underline{(1-\Phi)^{n-k-1}}\varphi^2 dt \int^t \underline{(\Phi-\Phi_1)^{k-i-3}}\varphi_1^2 dt_1 \int^{t_1} \underline{\Phi_2^{i-2}}\varphi_2^2 dt_2$$

$$-|n \int \underline{(1-\Phi)^{n-k}}\varphi^2 dt \int^t \underline{(\Phi-\Phi_1)^{k-i-4}}\varphi_1^2 dt_1 \int^{t_1} \underline{\Phi_2^{i-2}}\varphi_2^2 dt_2 .$$

Fourthly we proceed to the case that $j=i+2$. At first letting $k=i+3$,

$$E(t_{i|n}t_{i+2|n}t_{i+3|n}) = -\underline{|n} \int \underline{(1-\Phi)^{n-i-3}}\varphi' dt \int^t \varphi_1' dt_1 \int^{t_1} (\Phi_1-\Phi_2)\underline{\Phi_2^{i-1}}\varphi_2' dt_2 .$$

Here $\qquad I_1(t_1) = \int^{t_1} (\Phi_1-\Phi_2)\underline{\Phi_2^{i-1}}\varphi_2' dt_2 = -\int^{t_1} \mathfrak{D}_2[(\Phi_1-\Phi_2)\underline{\Phi_2^{i-1}}]\varphi_2^2 dt_2$

and $\qquad I_1'(t_1) = \varphi_1 \int^{t_1} \underline{\Phi_2^{i-1}}\varphi_2' dt_2 = \underline{\Phi_1^{i-1}}\varphi_1^2 - \varphi_1 \int^{t_1} \underline{\Phi_2^{i-2}}\varphi_2^2 dt_2 .$

So that

$$I(t) = \int^t \varphi_1' I_1(t_1)dt_1 = -\varphi \int^t \mathfrak{D}_1[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]\varphi_2^2 dt_2 - \int^t \underline{\Phi_1^{i-1}}\varphi_1^3 dt_1 + \int^t \varphi_1^2 dt_1 \int^{t_1} \underline{\Phi_2^{i-2}}\varphi_2^2 dt_2 ,$$

and $\qquad I'(t) = \varphi' I_1(t) = -\varphi' \int^t \mathfrak{D}_1[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]\varphi_1^2 dt_1 .$

These being substituted in

$$E(t_{i|n}t_{i+2|n}t_{i+3|n}) = -\underline{|n} \int \underline{(1-\Phi)^{n-i-3}}\varphi' I(t)dt = -|n \int \underline{(1-\Phi)^{n-i-4}}\varphi^2 I(t)dt$$

$$+\underline{|n} \int \underline{(1-\Phi)^{n-i-3}}\varphi I'(t)dt ,$$

we get

(15. 1. 4) $\quad E(t_{i|n}t_{i+2|n}t_{i+3|n}) = -\frac{|n}{2} \int \underline{(1-\Phi)^{n-i-3}}\underline{\Phi^{i-1}}\varphi^4 dt$

$$+|n \int \underline{(1-\Phi)^{n-i-4}}\varphi^2 dt \int^t \underline{\Phi_1^{i-1}}\varphi_1^3 dt_1 + \frac{|n}{2} \int \underline{(1-\Phi)^{n-i-3}}\varphi^3 dt \int^t \underline{\Phi_1^{i-2}}\varphi_1^2 dt_1$$

$$+\frac{|n}{2} \int \underline{(1-\Phi)^{n-i-4}}\varphi^3 dt \int^t \mathfrak{D}_1[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]\varphi_1^2 dt_1$$

$$-|n \int \underline{(1-\Phi)^{n-i-4}}\varphi^2 dt \int^t \varphi_1^2 dt_1 \int^{t_1} \underline{\Phi_2^{i-2}}\varphi_2^2 dt_2 .$$

Fifthly, letting $j=i+2,\ k\geqq i+4,$

$$E(t_{i|n}t_{i+2|n}t_{k|n})=-\underline{|n}\int\underline{(1-\Phi)^{n-k}}\varphi'dt\int^{t}\underline{(\Phi-\Phi_1)^{k-i-3}}\varphi_1'dt_1\int^{t_1}\underline{(\Phi_1-\Phi_2)}\Phi_2^{i-1}\varphi_2'dt_2\,.$$

The innermost integral is the same as before, i.e. $I_1(t_1)=-\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)\Phi_2^{i-1}]\varphi_2^2dt_2$, but

$$I(t)=\int^{t}\underline{(\Phi-\Phi_1)^{k-i-3}}I_1(t_1)\varphi_1'dt_1\quad(k-i-3\geqq 1)$$

$$=-\int^{t}\underline{(\Phi-\Phi_1)^{k-i-4}}\varphi_1^2dt_1\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)\Phi_2^{i-1}]\varphi_2^2dt_2-\int^{t}\underline{(\Phi-\Phi_1)^{k-i-3}}\underline{\Phi_1^{i-1}}\varphi_1^3dt_1$$

$$+\int^{t}\underline{(\Phi-\Phi_1)^{k-i-3}}\varphi_1^2dt_1\int^{t_1}\underline{\Phi_2^{i-2}}\varphi_2^2dt_2$$

with $I'(t)=\varphi\int^{t}\underline{(\Phi-\Phi_1)^{k-i-4}}I_1(t_1)\varphi_1'dt_1$. Hence, to obtain $I'(t)$, we have to replace k in the integrand of $I(t)$ by $k-1$, multiply by φ, and write down the eventual additional term $\delta_{i+4}^{k}I_1(t)\varphi^2$. Thus

$$I'(t)=-\varphi\int^{t}\underline{(\Phi-\Phi_1)^{k-i-5}}\varphi_1^2dt_1\int^{t}\mathfrak{D}_2[(\Phi_1-\Phi_2)\underline{\Phi_2^{i-1}}]\varphi_2^2dt_2-\varphi\int^{t}\underline{(\Phi-\Phi_1)^{k-i-4}}\underline{\Phi_1^{i-1}}\varphi_1^3dt_1$$

$$+\varphi\int^{t}\underline{(\Phi-\Phi_1)^{k-i-4}}\varphi_1^2dt_1\int^{t_1}\underline{\Phi_2^{i-2}}\varphi_2^2dt_2-\delta_{i+4}^{k}\int^{t}\mathfrak{D}_1[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]\varphi_1^2dt_1\,.$$

Therefore

$$E(t_{i|n}t_{i+2|n}t_{k|n})=-\underline{|n}\int\underline{(1-\Phi)^{n-k}}\varphi'I(t)dt=-\underline{|n}\int\underline{(1-\Phi)^{n-k-1}}\varphi^2I(t)dt$$

$$+\underline{|n}\int\underline{(1-\Phi)^{n-k}}\varphi I'(t)dt=\text{(i)}+\text{(ii)}\,,$$

where (ii) is obtained simply by replacing k in (i) by $k-1$ and changing the sign, yet with the additional Kronecker's term. Thus we find

(15. 1. 5) $\quad E(t_{i|n}t_{i+2|n}t_{k|n})\ (k\geqq i+4)$

$$=\underline{|n}\int\underline{(1-\Phi)^{n-k-1}}\varphi^2dt\int^{t}\underline{(\Phi-\Phi_1)^{k-i-3}}\underline{\Phi_1^{i-1}}\varphi_1^3dt_1-\underline{|n}\int\underline{(1-\Phi)^{n-k}}\varphi^2dt$$

$$\times\int^{t}\underline{(\Phi-\Phi_1)^{k-i-4}}\underline{\Phi_1^{i-1}}\varphi_1^3dt_1-\underline{|n}\int\underline{(1-\Phi)^{n-k-1}}\varphi^2dt\left[\int^{t}\underline{(\Phi-\Phi_1)^{k-i-3}}\varphi_1^2dt_1\right.$$

$$\left.\times\int^{t_1}\underline{\Phi_2^{i-2}}\varphi_2^2dt_2-\int^{t}\underline{(\Phi-\Phi_1)^{k-i-4}}\varphi_1^2dt_1\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)\underline{\Phi_2^{i-1}}]\varphi_2^2dt_2\right]$$

$$+\underline{|n}\int\underline{(1-\Phi)^{n-k}}\varphi^2dt\left[\int^{t}\underline{(\Phi-\Phi_1)^{k-i-4}}\varphi_1^2dt_1\int^{t_1}\underline{\Phi_2^{i-2}}\varphi_2^2dt_2-\int^{t}\underline{(\Phi-\Phi_1)^{k-i-5}}\varphi_1^2dt_1\right.$$

$$\left.\times\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)\underline{\Phi_2^{i-1}}]\varphi_2^2dt_2\right]-\delta_{i+4}^{k}\underline{|n}\int\underline{(1-\Phi)^{n-k}}\varphi^3dt\int^{t}\mathfrak{D}_1[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]\varphi_1^2dt_1.$$

Sixthly, if $j\geqq i+3,\ k=j+1$, we have

$$E(t_{i|n}t_{j|n}t_{j+1|n})=-\underline{|n}\int\underline{(1-\Phi)^{n-j-1}}\varphi'dt\int^{t}\varphi_1'dt_1\int^{t_1}\underline{(\Phi_1-\Phi_2)^{j-i-1}}\underline{\Phi_2^{i-1}}\varphi_2'dt_2\,.$$

The inner integral are

$$I_1(t_1)=\int^{t_1}(\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}\varphi_2'dt_2=-\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}]\varphi_2^2dt_2 \quad \text{with}$$

$$I_1'(t_1)=-\varphi_1\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)^{j-i-2}\Phi_2^{i-1}]\varphi_2^2dt_2\,,$$

and
$$I(t)=\int^{t}\varphi_1'I_1(t_1)dt_1=-\varphi\int^{t}\mathfrak{D}[(\Phi-\Phi_1)^{j-i-1}\Phi_1^{i-1}]\varphi_1^2dt_2+\int^{t}\varphi_1^2dt_1 \times\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}]\varphi_2^2dt_2$$

with
$$I'(t)=\varphi'I_1(t)=-\varphi'\int^{t}\mathfrak{D}_1[(\Phi-\Phi_1)^{j-i-1}\Phi_1^{i-1}]\varphi^2dt_1\,.$$

Therefore

(15.1.6) $E(t_{i|n}t_{j|n}t_{j+1|n})\ (j\geqq i+3)$

$$=\frac{\underline{|n}}{2}\int(1-\Phi)^{n-j-2}\varphi^3dt\int^{t}\mathfrak{D}_1[(\Phi-\Phi_1)^{j-i-1}\Phi_1^{i-1}]\varphi_1^2dt_1+\frac{\underline{|n}}{2}\int(1-\Phi)^{n-j-1}\varphi^3dt$$
$$\times\int^{t}\mathfrak{D}_1[(\Phi-\Phi_1)^{j-i-2}\Phi_1^{i-1}]\varphi_1^2dt_1-\underline{|n}\int(1-\Phi)^{n-j-2}\varphi^2dt\int^{t}\varphi_2^2dt_2$$
$$\times\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)^{j-i-2}\Phi_2^{i-1}]\varphi_2^2dt_2\,.$$

Seventhly, if $j\geqq i+3$ and $k\geqq j+2$, we have

$$E(t_{i|n}t_{j|n}t_{k|n})=-\underline{|n}\int(1-\Phi)^{n-k}\varphi'dt\int^{t}(\Phi-\Phi_1)^{k-j-1}\varphi_1'dt_1\int^{t_1}(\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}\varphi_2'dt_2\,.$$

Here the innermost integal is

$$I_1(t_1)=-\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}]\varphi_2^2dt_2 \text{ with } I_1'(t_1)=-\varphi_1\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)^{j-i-2}\Phi_2^{i-1}]\varphi_2^2dt_2\,.$$

Therefore

$$I(t)=\int^{t}(\Phi_1-\Phi_2)^{k-j-1}\varphi_1'I_1(t_1)dt_1=-\int^{t}(\Phi-\Phi_1)^{k-j-2}\varphi_1^2dt_1\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}]\varphi_2^2dt_2$$
$$+\int^{t}(\Phi-\Phi_1)^{k-j-1}\varphi_1^2dt_1\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)^{j-i-2}\Phi_2^{i-1}]\varphi_2^2dt_2,$$

and

$$I'(t)=\varphi\int^{t}(\Phi-\Phi_1)^{k-j-2}\varphi_1'I_1(t_1)dt_1$$
$$=\varphi\int^{t}(\Phi-\Phi_1)^{k-j-3}\varphi_1^2I_1(t_1)dt_1-\varphi\int^{t}(\Phi-\Phi_1)^{k-j-2}\varphi_1I_1'(t_1)dt_1+\delta^k_{j+2}\varphi^2I_1(t)$$
$$=-\varphi\int(\Phi-\Phi_1)^{k-j-3}\varphi_1^2dt_1\int^{t}\mathfrak{D}_2[(\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}]\varphi_2^2dt_2+\varphi\int(\Phi-\Phi_1)^{k-j-2}\varphi_1^2dt_1$$
$$\times\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)^{j-i-2}\Phi_2^{i-1}]\varphi_2^2dt_2-\delta^k_{j+2}\varphi^2\int^{t}\mathfrak{D}_1[(\Phi-\Phi_1)^{j-i-1}\Phi_1^{i-1}]\varphi_1^2dt_1\,.$$

And we have to find

$$E(t_{i|n}t_{j|n}t_{k|n}) = -\underline{|n}\int\underline{(1-\Phi)}^{n-k}\varphi' I(t)dt = -\underline{|n}\int\underline{(1-\Phi)}^{n-k-1}\varphi^2 I(t)dt$$
$$+\underline{|n}\int\underline{(1-\Phi)}^{n-k}\varphi I'(t)dt\,.$$

Hence, we obtain

(15.1.7) $E(t_{i|n}t_{j|n}t_{k|n})$ $(j\geqq i+3,\ k\geqq j+2)$

$$= \underline{|n}\int\underline{(1-\Phi)}^{n-k-1}\varphi^2 dt\int^t\underline{(\Phi-\Phi_1)}^{k-j-2}\varphi_1^2 dt_1\int^{t_1}\mathfrak{D}_2[\underline{(\Phi_1-\Phi_2)}^{j-i-1}\underline{\Phi_2}^{i-1}]\varphi_2^2 dt_2$$
$$-\underline{|n}\int\underline{(1-\Phi)}^{n-k}\varphi^2 dt\int^t\underline{(\Phi-\Phi_1)}^{k-j-3}\varphi_1^2 dt_1\int^{t_1}\mathfrak{D}_2[\underline{(\Phi_1-\Phi_2)}^{j-i-1}\underline{\Phi_2}^{i-1}]\varphi_2^2 dt_2$$
$$-\underline{|n}\int\underline{(1-\Phi)}^{n-k-1}\varphi^2 dt\int^t\underline{(\Phi-\Phi_1)}^{k-j-1}\varphi_1^2 dt_1\int^{t_1}\mathfrak{D}_2[\underline{(\Phi_1-\Phi_2)}^{j-i-2}\underline{\Phi_2}^{i-1}]\varphi_2^2 dt_2$$
$$+\underline{|n}\int\underline{(1-\phi)}^{n-k}\varphi^2 dt\int^t\underline{(\Phi-\Phi_1)}^{k-j-2}\varphi_1^2 dt_1\int^{t_1}\mathfrak{D}_2[\underline{(\Phi_1-\Phi_2)}^{j-i-2}\underline{\Phi_2}^{i-1}]\varphi_2^2 dt_2$$
$$-\delta_{j+2}^k\underline{|n}\int\underline{(1-\phi)}^{n-k}\varphi^3 dt\int\mathfrak{D}_1[\underline{(\Phi-\Phi_1)}^{j-i-1}\underline{\Phi_1}^{i-1}]\varphi_1^2 dt_1\,.$$

2° $E(t_{i|n}^2 t_{j|n}t_{k|n})$

$$= -\underline{|n}\int\underline{(1-\Phi)}^{n-k}\varphi' dt\int^t\underline{(\Phi-\Phi_1)}^{k-j-1}\varphi_1' dt_1\int^{t_1}\underline{(\Phi-\Phi_2)}^{j-i-1}\underline{\Phi_2}^{i-1}\varphi_2' t_2 dt_2\,.$$

Firstly for $j=i+1$, $k=i+2$,

$$E(t_{i|n}^2 t_{i+1|n}t_{i+2|n}) = -\underline{|n}\int\underline{(1-\Phi)}^{n-i-2}\varphi' dt\int^t\varphi_1' dt_1\int^{t_1}\underline{\Phi_2}^{i-1}\varphi_2' t_2 dt_2\,.$$

The innermost integral is, as in (14.1.1),

$$I_1(t_1) = \int^{t_1}\underline{\Phi_2}^{i-1}\varphi_2' t_2 dt_2 = -\underline{\Phi_1}^{i}-\underline{\Phi_2}^{i-1}\varphi_1' + \frac{1}{2}\underline{\Phi_1}^{i-2}\varphi_1^2 - \frac{1}{2}\int^{t_1}\underline{\Phi_2}^{i-3}\varphi_2^3 dt_2$$

and

$$I_1'(t_1) = \underline{\Phi_1}^{i-1}\varphi_1' t_1\,.$$

Hence, making use of (13.8.1), the next inner integral is found to be

$$I(t) = \int^t I_1(t_1)\varphi_1' dt_1$$
$$= -\underline{\Phi}^i\varphi - \frac{1}{2}\underline{\Phi}^{i-1}\varphi\varphi' + \frac{1}{3}\underline{\Phi}^{i-2}\varphi^3 + \frac{1}{2}\int^t\underline{\Phi_1}^{i-1}\varphi_1^2 dt_1 - \frac{\varphi}{2}\int^t\underline{\Phi_1}^{i-3}\varphi_1^3 dt_1 + \frac{1}{6}\int^t\underline{\Phi_1}^{i-3}\varphi_1^4 dt_1\,.$$

Substituting this in

$$E(t_{i|n}^2 t_{i+1|n}t_{i+2|n}) = -\underline{|n}\int\underline{(1-\Phi)}^{n-i-2}I(t)\varphi' dt$$

and integrating by parts (on the way formula (13.8.1), (13.2.2) &c. may be employed), we find

$$(15.2.1)\quad E(t_{i|n}^2 t_{i+1|n} t_{i+2|n}) = \lfloor n \int\left\{\frac{1}{6}\underline{(1-\Phi)^{n-i-2}}\underline{\Phi^{i-1}}+\frac{1}{2}\underline{(1-\Phi)^{n-i-3}}\underline{\Phi^{i}}\right\}\varphi^3 dt$$

$$+\frac{\lfloor n}{24}\int\left\{\underline{(1-\Phi)^{n-i-4}}\underline{\Phi^{i-1}}-4\underline{(1-\Phi)^{n-i-3}}\underline{\Phi^{i-2}}+\underline{(1-\Phi)^{n-i-2}}\underline{\Phi^{i-3}}\right\}\varphi^5 dt$$

$$-\frac{\lfloor n}{2}\int\underline{(1-\Phi)^{n-i-3}}\varphi^2 dt\int^t \underline{\Phi_1^{i-1}}\varphi_1^2 dt_2-\frac{\lfloor n}{6}\int\underline{(1-\Phi)^{n-i-3}}\varphi^2 dt\int^t \underline{\Phi_1^{i-3}}\varphi_1^4 dt_1$$

$$+\frac{\lfloor n}{4}\int\underline{(1-\Phi)^{n-i-3}}\varphi^3 dt\int^t \underline{\Phi_1^{i-3}}\varphi_1^3 dt_1 .$$

Secondly, for $j=i+1,\ k=i+3$,

$$E(t_{i|n}^2 t_{i+1|n} t_{i+3|n}) = -\lfloor n \int\underline{(1-\Phi)^{n-i-3}}\varphi' dt\int^t \underline{(\Phi-\Phi_1)}\varphi_1' dt_1\int^{t_1}\underline{\Phi_2^{i-1}}\varphi_2' t_2 dt_2 .$$

The innermost integral is the same as before, but now

$$I(t)=\int^t(\Phi-\Phi_1)\varphi_1' I_1(t_1)dt_1=\int^t\varphi_1^2 I_1(t_1)dt_1-\int^t(\Phi-\Phi_1)\varphi I_1'(t_1)dt_1$$

$$=-\int^t\underline{\Phi_1^{i-1}}\varphi_1^2 dt_1-\frac{1}{6}\underline{\Phi^{i-1}}\varphi^3+\frac{5}{6}\int^t\underline{\Phi_1^{i-2}}\varphi_1^4 dt_1-\frac{1}{2}\int^t\varphi_1^2 dt_1\int^{t_1}\underline{\Phi_2^{i-3}}\varphi_2^3 dt_2$$

$$+\frac{1}{2}\int^t(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}\varphi_1^2 dt_1+\frac{1}{6}\int\mathfrak{D}_1^2[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]\varphi_1^4 dt_1 ,$$

in which (13.8.1) has been used. Hence, we have

$$(15.2.2)\quad E(t_{i|n}^2 t_{i+1|n} t_{i+3|n}) = -\lfloor n \int\underline{(1-\Phi)^{n-i-3}}\underline{\Phi^{i-1}}\varphi^3 dt+\frac{\lfloor n}{24}\int\underline{(1-\Phi)^{n-i-4}}\underline{\Phi^{i-1}}\varphi^5 dt$$

$$+\frac{11}{24}\lfloor n\int\underline{(1-\Phi)^{n-i-3}}\underline{\Phi^{i-2}}\varphi^5 dt+\frac{\lfloor n}{2}\int\underline{(1-\Phi)^{n-i-3}}\varphi^2 dt\int^t\underline{\Phi_1^{i-1}}\varphi_1^2 dt_1$$

$$+\lfloor n\int\underline{(1-\Phi)^{n-i-4}}\varphi^2 dt\int^t\underline{\Phi_1^{i}}\varphi_1^2 dt_1-\frac{\lfloor n}{2}\int\underline{(1-\Phi)^{n-i-4}}\varphi^2 dt\int^t\underline{(\Phi-\Phi_1)}\underline{\Phi_1^{i-1}}\varphi_1^2 dt_1$$

$$+\frac{\lfloor n}{6}\int\underline{(1-\Phi)^{n-i-3}}\varphi^2 dt\int^t\underline{\Phi_1^{i-3}}\varphi_1^4 dt_1-\frac{5}{6}\lfloor n\int\underline{(1-\Phi)^{n-i-4}}\varphi^2 dt\int^t\underline{\Phi_1^{i-2}}\varphi_1^4 dt_1$$

$$-\frac{\lfloor n}{6}\int\underline{(1-\Phi)^{n-i-4}}\varphi^2 dt\int^t\mathfrak{D}_1^2[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]\varphi_1^4 dt_1-\frac{\lfloor n}{2}\int\underline{(1-\Phi)^{n-i-3}}\varphi^3 dt$$

$$\times\int^t\underline{\Phi_1^{i-3}}\varphi_1^3 dt_1+\frac{\lfloor n}{2}\int\underline{(1-\Phi)^{n-i-4}}\varphi^2 dt\int^t\varphi_1^2 dt_1\int^{t_1}\underline{\Phi_2^{i-3}}\varphi_2^3 dt_2 .$$

Thirdly for $j=i+1,\ k\geqq i+4$,

$$E(t_{k|n}^2 t_{i+1|n} t_{k|n}) = -\lfloor n\int\underline{(1-\Phi)^{n-k}}\varphi' dt\int^t(\Phi-\Phi_1)^{k-i-2}\varphi_1' dt_1\int^{t_1}\underline{\Phi_2^{i-1}}\varphi_2' t_2 dt_2 .$$

Here $I_1(t_1)$ has still the same form as before. However

$$I(t)=\int^t\underline{(\Phi-\Phi_1)^{k-i-2}}\varphi_1' I_1(t_1)dt_1=\int^t\underline{(\Phi-\Phi_1)^{k-i-3}}\varphi_1^2 I_1(t_1)dt_1-\int^t\underline{(\Phi-\Phi_1)^{k-i-2}}\varphi_1 I_1'(t_1)dt_1$$

$$=-\int^t\underline{(\Phi-\Phi_1)^{k-i-3}}\underline{\Phi_1^{i}}\varphi_1^2 dt_1+\frac{1}{3}\int^t\mathfrak{D}[\underline{(\Phi-\Phi_1)^{k-i-3}}\underline{\Phi_1^{i-1}}]\varphi_1^4 dt_1$$

$$+\frac{1}{2}\int^t \underline{(\Phi-\Phi_1)}^{k-i-3}\underline{\Phi_1}^{i-2}\varphi_1^4 dt_1-\frac{1}{2}\int^t \underline{(\Phi-\Phi_1)}^{k-i-3}\varphi^2 dt\int^t \underline{\Phi_2}^{i-3}\varphi_2^2 dt_2$$

$$+\frac{1}{2}\int^t \underline{(\Phi-\Phi_1)}^{k-i-2}\underline{\Phi_1}^{i-1}\varphi_1^2 dt_1+\frac{1}{6}\int^t \mathfrak{D}[\underline{(\Phi-\Phi_1)}^{k-i-2}\underline{\Phi_1}^{i-1}]\varphi_1^4 dt_1 .$$

And $I'(t)=\varphi\int^t \underline{(\Phi-\Phi_1)}^{k-i-3}\varphi_1' I_1(t_1)dt_1$ has the integral which shall be obtained by replacing k by $k-1$ in $I(t)$. Thus, if $I(t)=H(t, k)$, we get $I'(t)=\varphi H(t, k-1)$. Hence

$$E(t_{i|n}^2 t_{i+1|n} t_{k|n})$$

$$=-\underline{|n}\int \underline{(1-\Phi)}^{n-k}\varphi' I(t)dt=-\underline{|n}\int \underline{(1-\Phi)}^{n-k-1}\varphi^2 I(t)dt+\underline{|n}\int \underline{(1-\Phi)}^{n-k}\varphi I'(t)dt$$

$$=-\underline{|n}\int \underline{(1-\Phi)}^{n-k-1}\varphi^2 H(t, k)dt+\underline{|n}\int \underline{(1-\Phi)}^{n-(k-1)-1}\varphi^2 H(t, k-1)dt=\text{(i)}+\text{(ii)} .$$

Therefore if (i) be obtained, we get immediately (ii). However, those terms that ceases to vanish when k is replaced by $k-1$, should be examined with particular caution. Thus we obtain

(15. 2. 3) $E(t_{i|n}^2 t_{i+1|n} t_{k|n})$ $(k\geqq i+4)$

$$=\underline{|n}\int \underline{(1-\Phi)}^{n-k-1}\varphi^2 dt\Big[\int^t \underline{(\Phi-\Phi_1)}^{k-i-3}\underline{\Phi_1}^{i}\varphi_1^2 dt_1-\frac{1}{2}\int^t \underline{(\Phi-\Phi_1)}^{k-i-2}\underline{\Phi_1}^{i-1}\varphi_1^2 dt_1$$

$$-\frac{1}{2}\int^t \underline{(\Phi-\Phi_1)}^{k-i-3}\underline{\Phi_1}^{i-2}\varphi_1^4 dt_1-\frac{1}{3}\int^t \mathfrak{D}_1[\underline{(\Phi-\Phi_1)}^{k-i-3}\underline{\Phi_1}^{i-1}]\varphi_1^4 dt_1$$

$$-\frac{1}{6}\int^t \mathfrak{D}_1^2[\underline{(\Phi-\Phi_1)}^{k-i-2}\underline{\Phi_1}^{i-1}]\varphi_1^4 dt_1+\frac{1}{2}\int^t \underline{(\Phi-\Phi_1)}^{k-i-3}\varphi_1^2 dt_1\int^t \underline{\Phi_2}^{i-3}\varphi_2^3 dt_2\Big]$$

— "the expression that would be obtained by replacing k by $k-1$ in the above"

$$-\delta_{i+4}^k\frac{\underline{|n}}{6}\int \underline{(1-\Phi)}^{n-k}\underline{\Phi}^{i-1}\varphi^5 dt,$$

because, although on applying (13. 8. 1) to the inner integral

$$\int^t G(\Phi, \Phi_1)\varphi_1'^2 dt_1\equiv\int \underline{(\Phi-\Phi_1)}^{k-i-2}\underline{\Phi_1}^{i-1}\varphi_1'^2 dt_1$$

in (i) for $k\geqq i+4$, really $G(\Phi, \Phi_1)$ as well as $\mathfrak{D}_1 G$ do vanish when $t_1=t$, it is not so in (ii), since the corresponding $G(\Phi, \Phi_1)$ in (ii) being $\underline{(\Phi-\Phi_1)}^{k-i-3}\underline{\Phi_1}^{i-1}$, itself still vanishes when $t_1=t$, but now $\mathfrak{D}_1 G|_{t_1=t}$ for $k=i+4$ becomes $-\underline{\Phi}^{i-1}$ instead of vanishing, and thus there yields $\delta_{i+4}^k\frac{1}{6}\underline{\Phi}^i\varphi^3$ from the inner integral, so that after integration we must add $\delta_{i+4}^k\frac{\underline{|n}}{6}\int \underline{(1-\Phi)}^{n-k}\underline{\Phi}^i\varphi^5 dt$ with the sign changed.

Further, let now $j=i+2$ and $k>j$; then we have

$$E(t_{i|n}^2 t_{i+2|n} t_{k|n})=-\underline{|n}\int \underline{(1-\Phi)}^{n-k}\varphi' dt\int^t \underline{(\Phi-\Phi_1)}^{k-i-3}\varphi_1' dt_1\int^{t_1}(\Phi_1-\Phi_2)\underline{\Phi_2}^{i-1}\varphi_2' t_2 dt_2 .$$

Here the innermost integral being

$$I_1(t_1)=\int^{t_1}(\Phi_1-\Phi_2)\underline{\Phi_2^{i-1}}\varphi_2{}'t_2dt_2=\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)\underline{\Phi_2^{i-1}}]\varphi_2\varphi_2{}'dt_2-\int^{t_1}(\Phi_1-\Phi_2)\underline{\Phi_2^{i-1}}\varphi_2dt_2$$

$$=-\underline{\Phi_1^{i+1}}-\frac{1}{2}\underline{\Phi_1^{i-1}}\varphi_1^2-\frac{1}{2}\int^{t_1}\mathfrak{D}_2^2[(\Phi_1-\Phi_2)\underline{\Phi_2^{i-1}}]\varphi_2^3dt_2$$

with

$$I_1{}'(t_1)=-\underline{\Phi_1^{i}}\varphi_1-\underline{\Phi_1^{i-1}}\varphi_1\varphi_1{}'+\frac{1}{2}\underline{\Phi_1^{i-2}}\varphi_1^3-\frac{1}{2}\varphi_1\int^{t_1}\underline{\Phi_2^{i-3}}\varphi_2^3dt_2\,,$$

the next inner integral becomes

$$I(t)=\int^{t}\underline{(\Phi-\Phi_1)^{k-i-3}}\varphi_1{}'I_1(t_1)dt_1=\delta_{i+3}^{k}\varphi I_1(t)+\int^{t}\underline{(\Phi-\Phi_1)^{k-i-4}}\varphi_1^2I_1(t_1)dt_1$$

$$-\int^{t}\underline{(\Phi-\Phi_1)^{k-i-3}}\varphi_1I_1{}'(t_1)dt_1$$

$$=\delta_{i+3}^{k}\Big[-\underline{\Phi^{i+1}}\varphi-\frac{1}{2}\underline{\Phi^{i-1}}\varphi^3-\frac{1}{2}\varphi\int^{t}\mathfrak{D}_1[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]\varphi_1^3dt_1\Big]-\int^{t}\underline{(\Phi-\Phi_1)^{k-i-4}}\underline{\Phi_1^{i+1}}\varphi_1^2dt_1$$

$$-\frac{1}{2}\int^{t}\underline{(\Phi-\Phi_1)^{k-i-4}}\underline{\Phi_1^{i-1}}\varphi_1^4dt_1-\frac{1}{2}\int^{t}\underline{(\Phi-\Phi_1)^{k-i-4}}\varphi_1^2dt_1\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)\underline{\Phi_2^{i-1}}]\varphi_2^3dt_2$$

$$+\int^{t}\underline{(\Phi-\Phi_1)^{k-i-3}}\underline{\Phi_1^{i}}\varphi_1^2dt_1-\frac{1}{2}\int^{t}\underline{(\Phi-\Phi_1)^{k-i-3}}\underline{\Phi_1^{i-2}}\varphi_1^4dt_1+\frac{1}{2}\int^{t}\underline{(\Phi-\Phi_1)^{k-i-3}}\varphi_1^2dt_1$$

$$\times\int^{t_1}\underline{\Phi_2^{i-3}}\varphi_2^3dt_2-\frac{1}{3}\int^{t}\mathfrak{D}_1[\underline{(\Phi-\Phi_1)^{k-i-3}}\underline{\Phi_1^{i-1}}]\varphi_1^4dt_1+\frac{1}{3}\delta_{i+3}^{k}\underline{\Phi^{i-1}}\varphi^3$$

with

$$I'(t)=\delta_{i+3}^{k}\varphi'I_1(t)+\varphi\int^{t}\underline{(\Phi-\Phi_1)^{k-i-4}}\varphi_1{}'I_1(t_1)dt_1\,.$$

These expressions being substituted in

$$E(t_{i|n}^2t_{i+2|n}t_{k|n})=-\underline{|n}\int\underline{(1-\Phi)^{n-k}}\varphi'I(t)dt$$

$$=-\underline{|n}\int\underline{(1-\Phi)^{n-k-1}}\varphi^2I(t)dt+\underline{|n}\int\underline{(1-\Phi)^{n-k}}\varphi I'(t)dt=\text{(i)}+\text{(ii)}\,,$$

(i) attains a form that is already free from t and t-derivatives, while we need to continue the integrations for (ii).

Now *fourthly*, let $k=j+1=i+3$. Then, since those with negative exponent $k-i-4=-1$ vanish because of their denominator's ∞, both of $I(t)$ and $I'(t)$ become somewhat simpler: Especially $I'(t)=\varphi'I_1(t)$, and (ii)$=\underline{|n}\int\underline{(1-\Phi)^{n-i-3}}I_1(t)\varphi\varphi'dt$ can be completed by once integration by parts. Hence we obtain

$$\text{(15. 2. 4)}\quad E(t_{i|n}^2t_{i+2|n}t_{i+3|n})=\frac{\underline{|n}}{2}\int\underline{(1-\Phi)^{n-i-3}}\underline{\Phi^{i}}\varphi^3dt+\frac{\underline{|n}}{2}\int\underline{(1-\Phi)^{n-i-4}}\underline{\Phi^{i+1}}\varphi^3dt$$

$$-\frac{3\underline{|n}}{8}\int\underline{(1-\Phi)^{n-i-3}}\underline{\Phi^{i-2}}\varphi^5dt+\frac{1}{24}\underline{|n}\int\underline{(1-\Phi)^{n-i-4}}\underline{\Phi^{i-1}}\varphi^5dt$$

$$-\underline{|n}\int(1-\Phi)^{n-i-4}\varphi^2dt\int^t\Phi_1^i\varphi_1^2dt_1+\frac{5}{6}\underline{|n}\int(1-\Phi)^{n-i-4}\varphi^2dt\int^t\Phi_1^{i-2}\varphi_1^4dt_1$$

$$+\frac{\underline{|n}}{4}\int(1-\Phi)^{n-i-3}\varphi^3dt\int^t\Phi_1^{i-3}\varphi_1^3dt_1+\frac{\underline{|n}}{4}\int(1-\Phi)^{n-i-4}\varphi^3dt$$

$$\times\int^t\mathfrak{D}_1^2[(\Phi-\Phi_1)\,\Phi_1^{i-1}]\varphi_1^3dt_1-\frac{\underline{|n}}{4}(1-\Phi)^{n-i-4}\varphi^2dt\int^t\varphi_1^2dt_1\int^{t_1}\Phi_2^{i-3}\varphi dt_2\,.$$

Fifthly, let $j=i+2$ and $k\geqq i+4$. Then the foregoing expressions for $I(t)$ and $I'(t)$ both drop those terms with δ_{i+3}^k, and especially

$$I'(t)=\varphi\int^t(\Phi-\Phi_1)^{k-i-4}\varphi_1'I_1(t_1)dt_1$$

is nothing but which is obtainable from

$$I(t)=\int^t(\Phi-\Phi_1)^{k-i-3}\varphi_1'I_1(t_1)dt_1$$

by replacing k by $k-1$, and multiplying by φ, with caution that those vanishing terms in $I(t)$ might eventually become non-vanishing in $I'(t)$. Hence, we obtain

$$\text{(12. 2. 5. 1)}\quad E(t_{i|n}^2t_{i+2|n}t_{k|n})=-\underline{|n}\int(1-\Phi)^{n-k}\varphi' I(t)dt$$

$$=-\underline{|n}\int(1-\Phi)^{n-k-1}\varphi^2I(t)dt+\underline{|n}\int(1-\Phi)^{n-k}\varphi I'(t)dt=\text{(i)}+\text{(ii)}\,,$$

where

$$\text{(i)}=\underline{|n}\int(1-\Phi)^{n-k-1}\varphi^2dt\int^t(\Phi-\Phi_1)^{k-i-4}\Big\{\Phi_1^{i+1}+\frac{1}{6}\Phi_1^{i-1}\varphi_1^2$$

$$+\frac{1}{2}\int^{t_1}\mathfrak{D}_2^2[(\Phi_1-\Phi_2)\Phi_1^{i-1}]\varphi_2^3dt_2\Big\}\varphi_1^2dt_1-\underline{|n}\int(1-\Phi)^{n-k-1}\varphi^2dt\int^t(\Phi-\Phi_1)^{k-i-3}\Big\{\Phi_1^i$$

$$-\frac{5}{6}\Phi_1^{i-2}\varphi_1^2+\frac{1}{2}\int^{t_1}\Phi_2^{i-3}\varphi_2^3dt_2\Big\}\varphi_1^2dt_1\,,$$

(ii) = "the integrals with integrands to be obtained by replacing k by $k-1$ in those of (i) and multiplying by $-\varphi$"

$$-\delta_{i+4}^k\underline{|n}\int(1-\Phi)^{n-k}\Big\{\Phi^{i+1}+\frac{1}{6}\Phi^{i-1}\varphi^2+\frac{1}{2}\int^t\mathfrak{D}_1^2[(\Phi-\Phi_1)\Phi_1^{i-1}]\varphi_1^3dt_1\Big\}\varphi^3dt\,.$$

In particular, for $k=i+4$,

$$\text{(15. 2. 5)}\quad E(t_{i|n}^2t_{i+2|n}t_{i+4|n})=-\underline{|n}\int(1-\Phi)^{n-i-4}\Phi^{i+1}\varphi^3dt-\frac{\underline{|n}}{6}\int(1-\Phi)^{n-i-4}\Phi^{i-1}\varphi^5dt$$

$$+\underline{|n}\int(1-\Phi)^{n-i-4}\varphi^2dt\int^t\Phi_1^i\varphi_1^2dt_1+\underline{|n}\int(1-\Phi)^{n-i-5}\varphi^2dt\int^t\Phi_1^{i+1}\varphi_1^2dt_1$$

$$-\underline{|n}\int(1-\Phi)^{n-i-5}\varphi^2dt\int^t(\Phi-\Phi_1)\,\Phi_1^i\varphi_1^2dt_1-\frac{5}{6}\underline{|n}\int(1-\Phi)^{n-i-4}\varphi^2dt$$

$$\times\int^t\Phi_1^{i-2}\varphi_1^4dt_1+\frac{\underline{|n}}{6}\int(1-\Phi)^{n-i-5}\varphi^2dt\int^t\Phi_1^{i-1}\varphi_1^4dt_1+\frac{5}{6}\underline{|n}\int(1-\Phi)^{n-i-5}\varphi^2dt$$

$$\times\int^{t}(\Phi-\Phi_1)\underline{\Phi_1^{i-2}}\varphi_1^4dt_1-\frac{|\underline{n}}{2}\int\underline{(1-\Phi)^{n-i-4}}\varphi^3dt\int^{t}\mathfrak{D}_1^2[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]\varphi_1^3dt_1$$
$$+\frac{|\underline{n}}{2}\int\underline{(1-\Phi)^{n-i-4}}\varphi^2dt_1\int^{t}\varphi_1^2dt_1\int^{t_1}\underline{\Phi_2^{i-3}}\varphi_2^3dt_2-\frac{|\underline{n}}{2}\int\underline{(1-\Phi)^{n-i-5}}\varphi^2dt$$
$$\times\int^{t}(\Phi-\Phi_1)\varphi_1^2dt_1\int^{t_1}\underline{\Phi_2^{i-3}}\varphi_2^3dt_2+\frac{|\underline{n}}{2}\int\underline{(1-\Phi)^{n-i-5}}\varphi^2dt\int^{t}\varphi_1^2dt_1$$
$$\times\int^{t_1}\mathfrak{D}_2^2[(\Phi_1-\Phi_2)\,\underline{\Phi_2^{i-1}}]\varphi_2^3dt_2\,.$$

Sixthly, for $j=i+3,\ k=i+4$

$$E(t_{i|n}^2t_{i+3|n}t_{i+4|n})=-|\underline{n}\int\underline{(1-\Phi)^{n-i-4}}\varphi'dt\int^{t}\varphi_1'dt_1\int^{t_1}\underline{(\Phi_1-\Phi_2)^2}\underline{\Phi_2^{i-1}}\varphi_2't_2dt_2\,.$$

Here the innermost integral becomes

$$I_1(t_1)=\int^{t_1}(\Phi_1-\Phi_2)^2\underline{\Phi_2^{i-1}}\varphi_2't_2dt_2=-\underline{\Phi_1^{i+2}}-\frac{1}{2}\int^{t_1}\mathfrak{D}_2^2[(\Phi_1-\Phi_2)^2\underline{\Phi_2^{i-1}}]\varphi_2^3dt_2$$

with

$$I'(t_1)=-\underline{\Phi_1^{i+1}}\varphi_1-\frac{1}{2}\underline{\Phi_1^{i-1}}\varphi_1^3-\frac{1}{2}\varphi_1\int^{t_1}\mathfrak{D}_2^2[(\Phi_1-\Phi_2)\underline{\Phi_2^{i-1}}]\varphi_2^3dt_2\,.$$

So that

$$I(t)=\int^{t}\varphi_1'I_1(t_1)dt_1=\varphi I_1(t)-\int^{t}I_1'(t_1)\varphi_1dt_1$$
$$=-\underline{\Phi^{i+2}}\varphi+\int^{t}\underline{\Phi_1^{i+1}}\varphi_1^2dt_1+\frac{1}{2}\int^{t}\underline{\Phi_1^{i-1}}\varphi_1^4dt_1-\frac{1}{2}\varphi\int^{t}\mathfrak{D}_1^2[\underline{(\Phi-\Phi_1)^2}\underline{\Phi_1^{i-1}}]\varphi_1^3dt_1$$
$$+\frac{1}{2}\int^{t}\varphi_1^2dt_1\int^{t_1}\mathfrak{D}_2^2[\underline{(\Phi_1-\Phi_2)^2}\underline{\Phi_2^{i-1}}]\varphi_2^3dt_2$$

and

$$I'(t)=\varphi'I_1(t)=-\underline{\Phi^{i-2}}\varphi'-\frac{1}{2}\varphi'\int^{t}\mathfrak{D}_1^2[\underline{(\Phi-\Phi_1)^2}\underline{\Phi_1^{i-1}}]\varphi_1^3dt_1\,.$$

Hence, we obtain

$$(15.2.6)\quad E(t_{i|n}^2t_{i+3|n}t_{i+4|n})=\frac{|\underline{n}}{2}\int\underline{(1-\Phi)^{n-i-5}}\underline{\Phi^{i+2}}\varphi^3dt+\frac{|\underline{n}}{2}\int\underline{(1-\Phi)^{n-i-4}}\underline{\Phi^{i+1}}\varphi^3dt$$
$$+\frac{|\underline{n}}{4}\int\underline{(1-\Phi)^{n-i-4}}\underline{\Phi^{i-1}}\varphi^5dt-|\underline{n}\int\underline{(1-\Phi)^{n-i-5}}\varphi^2dt\int^{t}\underline{\Phi_1^{i+1}}\varphi_1^2dt_1$$
$$-\frac{|\underline{n}}{2}\int\underline{(1-\Phi)^{n-i-5}}\varphi^2dt\int^{t}\underline{\Phi_1^{i-1}}\varphi_1^4dt_1+\frac{|\underline{n}}{4}\int\underline{(1-\Phi)^{n-i-5}}\varphi^3dt$$
$$\times\int^{t}\mathfrak{D}_1^2[\underline{(\Phi-\Phi_1)^2}\underline{\Phi_1^{i-1}}]\varphi_1^3dt_1+\frac{|\underline{n}}{4}\int\underline{(1-\Phi)^{n-i-4}}\varphi^3dt\int^{t}\mathfrak{D}_1^2[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]\varphi_1^3dt_1$$
$$-\frac{|\underline{n}}{2}\int\underline{(1-\Phi)^{n-i-5}}\varphi^2dt\int^{t}\varphi_1^2dt_1\int^{t_1}\mathfrak{D}_2^2[\Phi_1-\Phi_2)\underline{\Phi_2^{i-1}}]\varphi_2^3dt_2\,.$$

Seventhly, for $j=i+3,\ k\geqq i+5$,

$$E(t_{i|n}^2t_{i+3|n}t_{k|n})=-|\underline{n}\int\underline{(1-\Phi)^{n-k}}\varphi'dt\int^{t}\underline{(\Phi-\Phi_1)^{k-i-3}}\varphi_1'dt_1\int^{t_1}(\Phi_1-\Phi_3)^2\underline{\Phi_2^{i-1}}\varphi_2't_2dt_2\,.$$

Still $I_1(t_1)$, $I_1'(t_1)$ are the same as in the foregoing, and

$$I(t)=\int^t \underline{(\Phi-\Phi_1)}^{k-i-4}\varphi_1' I_1(t_1) \quad (k\geqq i+5)$$
$$=\int^t \underline{(\Phi-\Phi_1)}^{k-i-5}\varphi_1^2 I_1(t_1)dt_1-\int^t \underline{(\Phi-\Phi_1)}^{k-i-4}\varphi_1 I_1'(t_1)dt_1$$

with $$I'(t)=\varphi\int^t \underline{(\Phi-\Phi_1)}^{k-i-5}\varphi_1' I_1(t_1)dt_1 .$$

Hence $I'(t)$ is obtained by replacing k by $k-1$ in $I(t)\equiv H(t, k)$ and multiplying by φ, eventually with the additional term when $k=i+5$. Thus we obtain

$$(15.2.7)\quad E(t^{i}_{2|n}t_{i+3|n}t_{k|n})=-\underline{|n}\int\underline{(1-\Phi)}^{n-k}\varphi' I(t)dt \quad (k\geqq i+5)$$
$$=-\underline{|n}\int\underline{(1-\Phi)}^{n-k-1}\varphi^2 H(t, k)dt+\underline{|n}\int\underline{(1-\Phi)}^{n-k}\varphi^2 H(t, k-1)dt$$
$$+\delta^k_{i+5}\underline{|n}\int\underline{(1-\Phi)}^{n-k}\varphi^2\varphi' I_1(t)dt=(\mathrm{i})+(\mathrm{ii})+(\mathrm{iii}) .$$

where

$$(\mathrm{i})=\underline{|n}\int\underline{(1-\Phi)}^{n-k-1}\varphi^2 dt\int^t\underline{(\Phi-\Phi_1)}^{k-i-5}\left[\underline{\Phi}_1^{i+2}+\frac{1}{2}\int^{t_1}\mathfrak{D}_2^2[\underline{(\Phi_1-\Phi_2)}^2\underline{\Phi}_2^{i-1}]\varphi_2^3 dt_2\right]\varphi_1^2 dt_1$$
$$-\underline{|n}\int\underline{(1-\Phi)}^{n-k-1}\varphi^2 dt\int^t\underline{(\Phi-\Phi_1)}^{k-i-4}\left[\underline{\Phi}_1^{i+1}+\frac{1}{2}\underline{\Phi}_1^{i-1}\varphi_1^2\right.$$
$$\left.+\frac{1}{2}\int^{t_1}\mathfrak{D}_2^2[(\Phi_1-\Phi_2)\underline{\Phi}_2^{i-1}]\varphi_2^3 dt_2\right]\varphi_1^2 dt_1 ,$$

(ii) = – "the integrals to be obtained by replacing k by $k-1$ in those of (i),"

$$(\mathrm{iii})=\delta^k_{i+5}\left\{\frac{|n}{3}\int D[\underline{(1-\Phi)}^{n-k}\underline{\Phi}^{i+2}]\varphi^4 dt-\frac{|n}{6}\int\underline{(1-\Phi)}^{n-k-1}\varphi^4 dt\int^t\mathfrak{D}_1^2[\underline{(\Phi-\Phi_1)}^2\underline{\Phi}_1^{i-1}]\varphi_1^3 dt_1\right.$$
$$\left.+\frac{|n}{6}\int\underline{(1-\Phi)}^{n-k}\underline{\Phi}^{i-1}\varphi^6 dt+\frac{|n}{6}\int\underline{(1-\Phi)}^{n-k}\varphi^4 dt\int^t\mathfrak{D}_1^2[(\Phi-\Phi_1)\underline{\Phi}_1^{i-1}]\varphi_1^3 dt\right\} .$$

Eighthly, for $j\geqq i+4$, $k\geqq j+1$,

$$E(t^2_{i|n}t_{j|n}t_{k|n})=-\underline{|n}\int\underline{(1-\Phi)}^{n-k}\varphi' dt\int^t\underline{(\Phi-\Phi_1)}^{k-j-1}\varphi_1' dt_1\int^{t_1}\underline{(\Phi_1-\Phi_2)}^{j-i-1}\underline{\Phi}_2^{i-1}\varphi_2' t_2 dt_2 .$$

Now

$$I_1(t_1)=\int^{t_1}\underline{(\Phi_1-\Phi_2)}^{j-i-1}\underline{\Phi}_2^{i-1}\varphi_2' t_2 dt_2 \quad (j-i-1\geqq 3)$$
$$=-\underline{\Phi}_1^{j-1}-\frac{1}{2}\int^{t_1}\mathfrak{D}_2^2[\underline{(\Phi_1-\Phi_2)}^{j-i-1}\underline{\Phi}_2^{i-1}]\varphi_2^3 dt_2$$

and

$$I_1'(t_1)=-\underline{\Phi}_1^{j-2}\varphi_1-\frac{1}{2}\varphi_1\int^{t_1}\mathfrak{D}_2^2[\underline{(\Phi-\Phi_2)}^{j-i-2}\underline{\Phi}_2^{i-1}]\varphi_2^3 dt_2 .$$

Hence

$$I(t)=\int^{t}\underline{(\Phi-\Phi_1)^{k-j-1}}\varphi_1'I_1(t_1)dt_1$$

$$=\delta^{k}_{j+1}I_1(t)\varphi+\int^{t}\underline{(\Phi-\Phi_1)^{k-j-2}}\varphi_1^2I_1(t_1)dt_1-\int^{t}\underline{(\Phi-\Phi_1)^{k-j-1}}\varphi_1I_1'(t_1)dt_1\,,$$

and

$$I'(t)=\varphi\int^{t}\underline{(\Phi-\Phi_1)^{k-j-2}}\varphi_1'I_1(t_1)dt_1-\delta^{j}_{j+1}I_1(t)\varphi'.$$

Thus $I'(t)$ is obtained by replacing k by $k-1$ in $I(t)$ and multiplying by φ with the further additional terms. Hence

$$(15.2.8)\quad E(t^2_{i|n}t_{j|n}t_{k|n})=-\underline{|n}\int\underline{(1-\Phi)^{n-k}}\varphi'I(t)dt\quad(j\geqq i+4,\ k\geqq j+1)$$

$$=-\underline{|n}\int\underline{(1-\Phi)^{n-k-1}}\varphi^2I(t)dt+\underline{|n}\int\underline{(1-\Phi)^{n-k}}\varphi I'(t)dt=\text{(i)}+\text{(ii)}\,,$$

where

$$\text{(i)}=\underline{|n}\int\underline{(1-\Phi)^{n-k-1}}\varphi^2dt\int^{t}\underline{(\Phi-\Phi_1)^{k-j-2}}\Big\{\underline{\Phi_1^{j-1}}+\frac{1}{2}\int^{t_1}\mathfrak{D}_2^2[\underline{(\Phi_1-\Phi_2)^{j-i-1}}\underline{\Phi_2^{i-1}}]\varphi_2^3dt_2\Big\}\varphi_1^2dt_1$$

$$-\underline{|n}\int\underline{(1-\Phi)^{n-k-1}}\varphi^2dt\int^{t}\underline{(\Phi-\Phi_1)^{k-j-1}}\Big\{\underline{\Phi_1^{j-2}}+\frac{1}{2}\int^{t_1}\mathfrak{D}_2^2[\underline{(\Phi_1-\Phi_2)^{j-i-2}}\underline{\Phi_2^{i-1}}]\varphi_2^3dt_2\Big\}\varphi_1^2dt_1$$

$$+\delta^{k}_{j+1}\underline{|n}\int\underline{(1-\Phi)^{n-k-1}}\Big\{\underline{\Phi^{j-1}}+\frac{1}{2}\int^{t}\mathfrak{D}_1^2[\underline{(\Phi-\Phi_1)^{j-i-1}}\underline{\Phi_1^{i-1}}]\varphi_1^3dt_1\Big\}\varphi^3dt\,,$$

and accordingly

$$\text{(ii)}=-\underline{|n}\int\underline{(1-\Phi)^{n-k}}\varphi^2dt\int^{t}\underline{(\Phi-\Phi_1)^{k-j-3}}\Big\{\underline{\Phi_1^{j-1}}+\frac{1}{2}\int^{t_1}\mathfrak{D}_2^2[\underline{(\Phi_1-\Phi_2)^{j-i-1}}\underline{\Phi_2^{i-1}}]\varphi_2^3dt_2\Big\}\varphi_1^2dt_1$$

$$+\underline{|n}\int\underline{(1-\Phi)^{n-k}}\varphi^2dt\int^{t}\underline{(\Phi-\Phi_1)^{k-j-2}}\Big\{\underline{\Phi_1^{j-2}}+\frac{1}{2}\int^{t_1}\mathfrak{D}_2^2[\underline{(\Phi_1-\Phi_2)^{j-i-2}}\underline{\Phi_2^{i-1}}]\varphi_2^3dt_2\Big\}\varphi_1^2dt_1$$

$$-\delta^{k-1}_{j+1}\underline{|n}\int\underline{(1-\Phi)^{n-k}}\Big\{\underline{\Phi^{j-1}}+\frac{1}{2}\int^{t}\mathfrak{D}_1^2[\underline{(\Phi-\Phi_1)^{j-i-1}}\underline{\Phi_1^{i-1}}]\varphi_1^3dt_1\Big\}\varphi^3dt$$

$$-\delta^{k}_{j+1}\underline{|n}\int\underline{(1-\Phi)^{n-k}}I_1(t)\varphi\varphi'dt\,.$$

The last term having $-\delta^{k}_{j+1}$ as factor becomes, after integration, in details

$$=\delta^{k}_{j+1}\frac{\underline{|n}}{2}\Big[\int\underline{(1-\Phi)^{n-k-1}}\varphi^3I_1(t)dt-\int\underline{(1-\Phi)^{n-k}}\varphi^2I_1'(t)dt\Big]$$

$$=\delta^{k}_{j+1}\frac{\underline{|n}}{2}\Big[\int\underline{(1-\Phi)^{n-k-1}}\underline{\Phi^{j-1}}\varphi^3dt+\frac{1}{2}\int\underline{(1-\Phi)^{n-k-1}}\varphi^3dt\int^{t}\mathfrak{D}_1^2[\underline{(\Phi-\Phi_1)^{j-i-1}}\underline{\Phi_1^{i-1}}]\varphi_1^3dt_1$$

$$-\int\underline{(1-\Phi)^{n-k}}\underline{\Phi^{j-2}}\varphi^3dt-\frac{1}{2}\int\underline{(1-\Phi)^{n-k}}\varphi^3dt\int^{t}\mathfrak{D}_1^2[\underline{(\Phi-\Phi_1)^{j-i-2}}\underline{\Phi_1^{i-1}}]\varphi_1^3dt_1\Big].$$

3° $E(t_{i|n}t^2_{j|n}t_{k|n})$

$$=-\underline{|n}\int\underline{(1-\Phi)^{n-k}}\varphi'dt\int^{t}\underline{(\Phi-\Phi_1)^{k-j-1}}\varphi_1't_1dt_1\int^{t_1}\underline{(\Phi_1-\Phi_2)^{j-i-1}}\underline{\Phi_2^{i-1}}\varphi_2'dt_2$$

Firstly

$$E(t_{i|n}t^2_{i+1|n}t_{i+2|n}) = -\underline{|n}\int\underline{(1-\Phi)^{n-i-2}}\varphi' dt\int^t\varphi_1' t_1 dt_1\int^{t_1}\underline{\Phi_2^{i-1}}\varphi_2' dt_2 .$$

Here the innermost integral is the same as in (15. 1. 1): i.e.

$$I_1(t_1) = \underline{\Phi_1^{i-1}}\varphi_1 - \int^t\underline{\Phi_2^{i-2}}\varphi dt_2 , \quad I_1'(t_1) = \underline{\Phi_1^{i-1}}\varphi_1' .$$

But

$$I(t) = \int^t I_1(t)\varphi_1' t_1 dt_1 = -I_1(t)\varphi' - \int^t I_1(t)\varphi_1 dt_1 + \int^t I_1'(t_1)\varphi_1' dt_1 .$$

On substituting $I_1(t)$, $I_1'(t_1)$ and integrating by parts, on the way applying e.g. (13. 8. 1), we get

$$I(t) = -\frac{1}{2}\underline{\Phi^{i-1}}\varphi\varphi' + \varphi'\int^t\underline{\Phi_1^{i-2}}\varphi_1^2 dt_1 - \frac{1}{6}\underline{\Phi^{i-2}}\varphi^3 + \frac{1}{6}\int^t\underline{\Phi_1^{i-3}}\varphi_1^4 dt_1 - \frac{1}{2}\int^t\underline{\Phi_1^{i-1}}\varphi_1^2 dt_1$$
$$+\int^t\varphi_1 dt_1\int^{t_1}\underline{\Phi_2^{i-2}}\varphi_2^2 dt_2 ,$$

whose all terms, except the first two, are already free from t-derivative, and

$$I_1'(t) = I_1(t)\varphi' t = -\underline{\Phi^{i-1}}\varphi'^2 - \varphi' t\int^t\underline{\Phi_1^{i-2}}\varphi_1^2 dt_1 .$$

Hence, on applying (13. 3. 2) and (13. 6. 1), we obtain

(15. 3. 1) $$E(t_{i|n}t^2_{i+1|n}t_{i+2|n}) = -\underline{|n}\int\underline{(1-\Phi)^{n-i-3}}\varphi^2 I(t)dt + \underline{|n}\int\underline{(1-\Phi)^{n-i-2}}\varphi I'(t)dt$$
$$= -\frac{\underline{|n}}{3}\int\underline{(1-\Phi)^{n-i-2}}\underline{\Phi^{i-1}}\varphi^3 dt + \frac{\underline{|n}}{24}\int\underline{(1-\Phi)^{n-i-4}}\underline{\Phi^{i-1}}\varphi^5 dt + \frac{\underline{|n}}{4}\int\underline{(1-\Phi)^{n-i-3}}\underline{\Phi^{i-2}}\varphi^5 dt$$
$$+\frac{\underline{|n}}{24}\int\underline{(1-\Phi)^{n-i-2}}\underline{\Phi^{i-3}}\varphi^5 dt + \frac{\underline{|n}}{2}\int\underline{(1-\Phi)^{n-i-2}}\varphi^2 dt\int^t\underline{\Phi_1^{i-2}}\varphi_1^2 dt_1$$
$$+\frac{\underline{|n}}{2}\int\underline{(1-\Phi)^{n-i-3}}\varphi^2 dt\int^t\underline{\Phi_1^{i-1}}\varphi_1^2 dt_1 - \frac{\underline{|n}}{6}\int\underline{(1-\Phi)^{n-i-3}}\varphi^2 dt\int^t\underline{\Phi_1^{i-3}}\varphi_1^4 dt_1$$
$$-\frac{\underline{|n}}{6}\int\underline{(1-\Phi)^{n-i-4}}\varphi^4 dt\int^t\underline{\Phi_1^{i-2}}\varphi_1^2 dt_1 - \underline{|n}\int\underline{(1-\Phi)^{n-i-3}}\varphi^2 dt\int^t\varphi_1 dt_1\int^{t_1}\underline{\Phi_2^{i-2}}\varphi_2^2 dt_2 .$$

Secondly, for $E(t_{i|n}t^2_{i+1|n}t_{i+3|n}) = -\underline{|n}\int\underline{(1-\Phi)^{n-i-3}}\varphi dt\int^t\underline{(\Phi-\Phi_1)}\varphi_1' t_1 dt_1\int^{t_1}\underline{\Phi_2^{i-1}}\varphi_2' dt_2$, the innermost integral being again $I_1(t_1) = \underline{\Phi_1^{i-1}}\varphi_1 - \int^{t_1}\underline{\Phi_2^{i-2}}\varphi_2^2 dt_2$, $I_1'(t_1) = \underline{\Phi_1^{i-1}}\varphi_1'$, we have

$$I(t) = \int^t(\Phi-\Phi_1)I_1(t_1)\varphi_1' t_1 dt_1 = -\int^t I_1(t_1)\varphi_1\varphi_1' dt_1 - \int^t(\Phi-\Phi_1)I_1(t_1)\varphi_1 dt_1$$
$$+\int^t(\Phi-\Phi_1)\Phi_1^{i-1}\varphi'^2 dt_1 ,$$

whose last integral by (13. 8. 1) splits up into

$$\frac{1}{2}\int^t(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}\varphi_1^2 dt_1 + \frac{1}{6}\underline{\Phi^{i-1}}\varphi^3 + \frac{1}{6}\int^t\mathfrak{D}_1^2[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]\varphi_1^4 dt_1 .$$

Also $I'(t)=\varphi\int^{t} I_1(t_1)\varphi_1{}'t_1 dt_1$ and accordingly

$$E(t_{i|n}t^2_{i+1|n}t_{i+2|n})=$$

$$-\underline{|n}\int \underline{(1-\Phi)^{n-i-3}}I(t)\varphi' dt=-\underline{|n}\int\underline{(1-\Phi)^{n-i-4}}\varphi^2 I(t)dt+\underline{|n}\int\underline{(1-\Phi)^{n-i-3}}\varphi I'(t)dt.$$

On substituting $I(t)$ and $I'(t)$ obtained above and integrating, we get

$$(15.3.2)\quad E(t_{i|n}t^2_{i+1|n}t_{i+3|n})=\frac{\underline{|n}}{24}\int\underline{(1-\Phi)^{n-i-4}}\underline{\Phi^{i-1}}\varphi^5 dt-\frac{3\underline{|n}}{8}\underline{(1-\Phi)^{n-i-3}}\underline{\Phi^{i-2}}\varphi^5 dt$$

$$-\frac{\underline{|n}}{2}\int\underline{(1-\Phi)^{n-i-3}}\varphi^2 dt\int^t\underline{\Phi_1^{i-1}}\varphi_1^2 dt_1+\frac{\underline{|n}}{2}\int\underline{(1-\Phi)^{n-i-4}}\varphi^2 dt\int^t(\Phi-\Phi_1)\underline{\Phi^{i-1}}\varphi_1^2 dt_1$$

$$+\frac{\underline{|n}}{6}\int\underline{(1-\Phi)^{n-i-3}}\varphi^2 dt\int^t\underline{\Phi_1^{i-3}}\varphi_1^4 dt_1+\frac{\underline{|n}}{6}\int\underline{(1-\Phi)^{n-i-4}}\varphi^2 dt\int^t\underline{\Phi_1^{i-2}}\varphi_1^4 dt_1$$

$$-\frac{\underline{|n}}{6}\int\underline{(1-\Phi)^{n-i-4}}\varphi^2 dt\int^t\mathfrak{D}_1^2[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]\varphi_1^4 dt_1-\frac{\underline{|n}}{6}\int\underline{(1-\Phi)^{n-i-4}}\varphi^4 dt\int\underline{\Phi_1^{i-2}}\varphi_1^2 dt_1$$

$$+\underline{|n}\int\underline{(1-\Phi)^{n-i-3}}\varphi^2 dt\int^t\varphi_1 dt_1\int^{t_1}\underline{\Phi_2^{i-2}}\varphi_2^2 dt_2-\underline{|n}\int\underline{(1-\Phi)^{n-i-4}}\varphi^2 dt\int^t(\Phi-\Phi_1)\varphi_1 dt_1$$

$$\times\int^{t_1}\underline{\Phi_2^{i-2}}\varphi_2^2 dt_2 .$$

Thirdly, to find $E(t_{i|n}t^2_{i+1|n}t_{k|n})$ for $k\geqq i+4$, $I_1(t_1)$ and $I_1'(t_1)$ have still the same values as before. But now

$$I(t)=\int^t\underline{(\Phi-\Phi_1)^{k-i-2}}I_1(t)\varphi_1{}'t_1 dt_1$$

$$=-\int^t\underline{(\Phi-\Phi_1)^{k-i-3}}I_1(t)\varphi_1\varphi_1{}'dt_1-\int^t\underline{(\Phi_1-\Phi_1)^{k-i-2}}I_1(t)\varphi_1 dt_1$$

$$+\int^t\underline{(\Phi-\Phi_1)^{k-i-2}}\underline{\Phi_1^{i-1}}\varphi_1{}'^2 dt_1$$

with $\qquad I'(t)=\varphi\int^t\underline{(\Phi-\Phi_1)^{k-i-3}}I_1(t)\varphi_1{}'t_1 dt_1$

and

$$E(t_{i|n}t^2_{i+1|n}t_{k|n})=-\underline{|n}\int\underline{(1-\Phi)^{n-k-1}}\varphi^2 I(t)dt+\underline{|n}\int\underline{(1-\Phi)^{n-k}}\varphi I'(t)dt=\text{(i)}+\text{(ii)}.$$

So that $I'(t)$ and accordingly (ii) could be obtained by replacing k in $I(t)$ or (i) by $k-1$ and simply changing the sign (the factor φ^2 being already present in both of (i) and (ii)). However, the eventually arizing Kronecker's term should be considered, and we obtain

$$(15.3.3)\quad E(t_{i|n}t^2_{i+1|n}t_{k|n})\ (k\geqq i+4)$$

$$=\frac{\underline{|n}}{2}\int\underline{(1-\Phi)^{n-k-1}}\varphi^2 dt\int^t\underline{(\Phi-\Phi_1)^{k-i-2}}\underline{\Phi_1^{i-1}}\varphi dt_1+\frac{\underline{|n}}{6}\int\underline{(1-\Phi)^{n-k-1}}\varphi^2 dt$$

$$\times\int^t\{\underline{(\Phi-\Phi_1)^{k-i-4}}\underline{\Phi_1^{i-1}}+3\underline{(\Phi-\Phi_1)^{k-i-3}}\underline{\Phi_1^{i-2}}-\underline{(\Phi-\Phi_1)^{k-i-2}}\underline{\Phi_1^{i-3}}\}\varphi_1^4 dt_1$$

$$-\underline{|n}\int\underline{(1-\Phi)^{n-k-1}}\varphi^2 dt\Big\{\int^t\underline{(\Phi-\Phi_1)^{k-i-2}}\varphi_1 dt_1\int^{t_1}\underline{\Phi_2^{i-2}}\varphi_2^2 dt_2$$

$$+\frac{1}{2}\int^t \underline{(\Phi-\Phi_1)^{k-i-4}}\varphi_1^3\,dt_1\int^t \underline{\Phi_2^{i-2}}\varphi_2^2\,dt_2\Big\}$$

— "the expression to be obtained by replacing k in the above by $k-1$"

$$-\delta^k_{i+4}\frac{\underline{|n}}{6}\int\underline{(1-\Phi)^{n-k}}\,\underline{\Phi^{i-1}}\varphi^5\,dt\,.$$

Fourthly, for $j=i+2$, and at first $k=i+3$,

$$E(t_{i|n}t^2_{i+2|n}t_{i+3|n})=-\underline{|n}\int\underline{(1-\Phi)^{n-i-3}}\varphi'\,dt\int^t\varphi_1't_1\,dt_1\int^{t_1}(\Phi_1-\Phi_2)\underline{\Phi_2^{i-1}}\varphi_2'kt_2\,.$$

Now the innermost integral is that given in 1° also fourthly:

$$I_1(t_1)=-\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)\underline{\Phi_2^{i-1}}]\varphi_2^2\,dt_2\quad\text{with}\quad I_1'(t_1)=\underline{\Phi_1^{i-1}}\varphi_1^2-\varphi_1\int^{t_1}\underline{\Phi_2^{i-2}}\varphi_2^2\,dt_2\,.$$

But now

$$I(t)=\int^t I_1(t_1)\varphi_1't_1\,dt_1=\frac{1}{3}\underline{\Phi^{i-1}}\varphi^3-\frac{\varphi^2}{2}\int^t\underline{\Phi_1^{i-2}}\varphi_1^2\,dt_1+\frac{1}{6}\int^t\underline{\Phi_1^{i-2}}\varphi_1^4\,dt_1$$

and

$$I'(t)=-\varphi't\int^t\mathfrak{D}_1[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]\varphi_1^2\,dt_1\,.$$

Hence

$$(15.3.4)\quad E(t_{i|n}t^2_{i+2|n}t_{i+3|n})=-\frac{3}{8}\underline{|n}\int\underline{(1-\Phi)^{n-i-4}}\,\underline{\Phi^{i-1}}\varphi^5\,dt+\frac{1}{24}\underline{|n}\int\underline{(1-\Phi)^{n-i-3}}\,\underline{\Phi^{i-2}}\varphi^5\,dt$$

$$+\frac{\underline{|n}}{2}\int\underline{(1-\Phi)^{n-i-3}}\varphi^2\,dt\int^t\mathfrak{D}_1[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]\varphi_1^2\,dt_1-\frac{\underline{|n}}{6}\int\underline{(1-\Phi)^{n-i-4}}\varphi^2\,dt\int^t\underline{\Phi_1^{i-2}}\varphi_1^4\,dt_1$$

$$+\frac{\underline{|n}}{2}\int\underline{(1-\Phi)^{n-i-4}}\varphi^4\,dt\int^t\underline{\Phi_1^{i-2}}\varphi_1^2\,dt_1-\frac{\underline{|n}}{6}\int\underline{(1-\Phi)^{n-i-5}}\varphi^4\,dt\int^t\mathfrak{D}_1[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]\varphi_1^2\,dt_1$$

$$-\underline{|n}\int\underline{(1-\Phi)^{n-i-4}}\varphi^2\,dt\int^t\varphi_1\,dt_1\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)\underline{\Phi_2^{i-1}}]\varphi_2^2\,dt_2\,.$$

Fifthly, for $E(t_{i|n}t^2_{i+2|n}t_{i+4|n})$ still with the same $I_1(t)$ as in the above

$$I(t)=\int^t(\Phi-\Phi_1)I_1(t_1)\varphi_1't_1\,dt_1$$

$$=\int^t I_1(t_1)\varphi_1^2t_1\,dt_1-\int^t(\Phi-\Phi_1)I_1(t_1)\varphi_1\,dt_1+\int^t I_1'(t_1)(\Phi-\Phi_1)\varphi_1'\,dt_1$$

$$=\frac{1}{2}\int^t\underline{\Phi^{i-1}}\varphi_1^4\,dt_1+\frac{1}{2}\int^t(\Phi-\Phi_1)\underline{\Phi_1^{i-2}}\varphi_1^4\,dt_1+\frac{1}{2}\varphi^2\int^t\mathfrak{D}_1[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]\varphi_1^2\,dt_1$$

$$-\frac{1}{3}\int^t\mathfrak{D}_1[(\Phi-\Phi_1)\underline{\Phi_1^{i-1}}]\varphi_1^4\,dt_1-\int^t\varphi_1^3\,dt_1\int^{t_1}\underline{\Phi_2^{i-2}}\varphi_2^2\,dt_2+\int^t(\Phi-\Phi_1)\varphi_1\,dt_1$$

$$\times\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)\underline{\Phi_2^{i-1}}]\varphi_2^2\,dt_2\,.$$

And we have

$$(15.3.5)\quad E(t_{i|n}t^2_{i+2|n}t_{i+4|n})=\frac{2|\underline{n}}{3}\int(1-\Phi)^{n-i-4}\Phi^{i-1}\varphi^5dt$$

$$-\frac{1}{2}|\underline{n}\int(1-\Phi)^{n-i-5}\varphi^2dt\int^t\Phi^{i-1}\varphi^4dt-\frac{1}{2}|\underline{n}\int(1-\Phi)^{n-i-5}\varphi^2dt\int^t(\Phi-\Phi_1)\Phi_1^{i-2}\varphi_1^4dt_1$$

$$+\frac{1}{3}|\underline{n}\int(1-\Phi)^{n-i-5}\varphi^2dt\int^t\mathfrak{D}_1[(\Phi-\Phi_1)\Phi_1^{i-1}]\varphi_1^4dt_1$$

$$+\frac{|\underline{n}}{6}\int(1-\Phi)^{n-i-4}\varphi^2dt\int^t\Phi^{i-2}\varphi_1^4dt_1-\frac{5|\underline{n}}{6}\int(1-\Phi)^{n-i-4}\varphi^4dt\int^t\Phi_1^{i-2}\varphi_1^2dt_1$$

$$-\frac{|\underline{n}}{6}\int(1-\Phi)^{n-i-5}\varphi^4dt\int^t\mathfrak{D}_1[(\Phi-\Phi_1)\Phi_1^{i-1}]\varphi_1^2dt_1+|\underline{n}\int(1-\Phi)^{n-i-5}\varphi^2dt\int^t\varphi_1^3dt_1$$

$$\times\int^{t_1}\Phi_2^{i-2}\varphi_2^2dt_2-|\underline{n}\int(1-\Phi)^{n-i-5}\varphi^2dt\int^t(\Phi-\Phi_1)\varphi_1dt_1\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)\Phi_2^{i-1}]\varphi_2^2dt_2$$

$$+|\underline{n}\int(1-\Phi)^{n-i-4}\varphi^2\int^t\varphi_1dt_1\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)\Phi_2^{i-1}]\varphi_2^2dt_2\,.$$

Sixthly, for $j=i+2$, $k\geqq i+5$,

$$E(t_{i|n}t^2_{i+2|n}t_{k|n})=-|\underline{n}\int(1-\Phi)^{n-k}\varphi'dt\int^t(\Phi-\Phi_1)^{n-i-3}\varphi_1't_1dt_1\int^{t_1}(\Phi_1-\Phi_2)\Phi_1^{i-1}\varphi_2'dt_2\,.$$

The inner integral

$$I(t)=\int^t(\Phi-\Phi_1)^{k-i-3}I_1(t_1)\varphi_1't_1dt_1\quad(I_1(t_1)\text{ being still the same as before})$$

$$=\int^t(\Phi-\Phi_1)^{k-i-4}\varphi_1\varphi_1'dt_1\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)\Phi_2^{i-1}]\varphi_2^2dt_2+\int^t(\Phi-\Phi_1)^{k-i-3}\Phi_1^{i-1}\varphi_1^2\varphi_1'dt_1$$

$$-\int^t(\Phi-\Phi_1)^{k-i-3}\varphi_1dt_1\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)\Phi_2^{i-1}]\varphi_2^2dt_2-\int^t(\Phi-\Phi_1)^{k-i-3}\varphi_1\varphi_1'dt_1$$

$$\times\int^{t_1}\Phi_2^{i-2}\varphi_2^2dt_2$$

with

$$I'(t)=\varphi\int^t(\Phi-\Phi_1)^{k-i-4}I_1(t_1)\varphi_1't_1dt_1\,.$$

Therefore $I'(t)$ is obtained by replacing k by $k-1$ in $I(t)$ but with some additional term, which is eventually non-vanishing as it does vanish for k in $I(t)$. Thus

$$(15.3.6)\quad E(t_{i|n}t^2_{i+2|n}t_{k|n})=-|\underline{n}\int(1-\Phi)^{n-k}\varphi'I(t)dt$$

$$=-|\underline{n}\int(1-\Phi)^{n-k-1}\varphi^2I(t)dt+|\underline{n}\int(1-\Phi)^{n-k}\varphi I'(t)dt=(\mathrm{i})+(\mathrm{ii})\,,$$

where

$$(\mathrm{i})=|\underline{n}\int(1-\Phi)^{n-k-1}\varphi^2dt\Big[\int^t\Big\{-\frac{1}{6}(\Phi-\Phi_1)^{k-i-3}\Phi_1^{i-2}-\frac{5}{6}(\Phi-\Phi_1)^{k-i-4}\Phi_1^{i-1}\Big\}\varphi_1^4dt_1$$

$$+\int^t(\Phi-\Phi_1)^{k-i-4}\varphi_1^3dt\int^{t_1}\Phi_2^{i-2}\varphi_2^2dt_2+\int^t(\Phi-\Phi_1)^{k-i-3}\varphi_1dt_1\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)\Phi_2^{i-1}]\varphi_2^2dt_2$$

$$-\frac{1}{2}\int^t(\Phi-\Phi_1)^{k-i-5}\varphi_1^3dt_1\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)\Phi_2^{i-1}]\varphi_2^2dt_2\Big],$$

and (ii) is obtained by replacing k by $k-1$ in (i), changing the sign and further adding

$$\delta_{i+5}^{k}\frac{\underline{|n}}{2}\underline{(1-\Phi)}^{n-k}\varphi^4 dt\int^t \mathfrak{D}_1[\underline{(\Phi-\Phi_1)}\Phi_1^{i-1}]\varphi_1^2 dt_1 .$$

Seventhly for $j \geqq i+3$, $k=j+1$

$$E(t_{i|n}t_{j|n}^2 t_{i+1|n}) = -\underline{|n}\int \underline{(1-\Phi)}^{n-j-1}\varphi' dt\int^t \varphi_1' t_1 dt_1 \int^{t_1} \underline{(\Phi_1-\Phi_2)}^{j-i-1}\underline{\Phi_2^{i-1}}\varphi_2' dt_2 .$$

Here the inner integral are

$$I_1(t_1) = \int^{t_1} \underline{(\Phi_1-\Phi_2)}^{j-i-1}\underline{\Phi_2^{i-1}}\varphi_2' dt_2 = -\int^{t_1} \mathfrak{D}_2[\underline{(\Phi_1-\Phi_2)}^{j-i-1}\underline{\Phi_2^{i-1}}]\varphi_2^2 dt_2 ,$$

with

$$I'(t_1) = \varphi_1 \int^{t_1} \underline{(\Phi_1-\Phi_2)}^{j-i-2}\underline{\Phi_2^{i-1}}\varphi_2' dt_2 = -\varphi_1 \int^{t_1} \mathfrak{D}_2[\underline{(\Phi-\Phi_2)}^{j-i-2}\underline{\Phi^{i-1}}]\varphi_2^2 dt_2$$

and

$$I(t) = \int^t I_1(t_1)\varphi_1' t_1 dt_1 = -\varphi' \int^t \mathfrak{D}_1 G(\Phi_1\ \Phi_1)\varphi_1^2 dt_1 + \int^t \varphi_1 dt_1 \int^{t_1} \mathfrak{D}_2 G(\Phi_1, \Phi_2)\varphi_2^2 dt_2$$
$$-\int^t \varphi_1\varphi_1' dt_1 \int^{t_1} \mathfrak{D}_1\mathfrak{D}_2 G(\Phi_1, \Phi_2)\varphi_2^2 dt_2 ,$$

where $G'(\Phi_1, \Phi_2) = \underline{(\Phi_1-\Phi_2)}^{j-i-1}\underline{\Phi_2^{i-1}}$, and the last double integral splits into

$$-\frac{1}{2}\int^t \mathfrak{D}_1[\mathfrak{D}G(\Phi, \Phi_1)]\varphi_1^2 dt_1 + \frac{1}{2}\int^t \varphi_1^3 dt_1 \int^{t_1} \mathfrak{D}_2\mathfrak{D}_1^2 G(\Phi_1, \Phi_2)\varphi_2^2 dt_2 - \frac{1}{2}\delta_{i+3}^{j}\int^t \underline{\Phi_1^{i-1}}\varphi_1^4 dt_1 .$$

Therefore

(15. 3. 7) $E({}_{i|n}t_{j|n}^2 t_{j+1|n})$ $(j \geqq i+3)$

$$= \frac{\underline{|n}}{2}\int \underline{(1-\Phi)}^{n-j-1}\varphi^2 dt \int^t \mathfrak{D}_1[\underline{(\Phi-\Phi_1)}^{i-j-1}\underline{\Phi_1^{i-1}}]\varphi_1^2 dt_1 + \frac{\underline{|n}}{2}\int \underline{(1-\Phi)}^{n-j-2}\varphi^4 dt$$
$$\times \int^t \mathfrak{D}_1[\underline{(\Phi-\Phi_1)}^{j-i-2}\underline{\Phi_1^{i-1}}]\varphi_1^2 dt_1 - \frac{\underline{|n}}{6}\int \underline{(1-\Phi)}^{n-j-3}\varphi^4 dt \int^t \mathfrak{D}_1[\underline{(\Phi-\Phi_1)}^{j-i-1}\underline{\Phi_1^{i-1}}]\varphi_1^2 dt_1$$
$$+\frac{\underline{|n}}{6}\int \underline{(1-\Phi)}^{n-j-1}\varphi^4 dt \int^t \mathfrak{D}_1[\underline{(\Phi-\Phi_1)}^{j-i-3}\underline{\Phi_1^{i-1}}]\varphi_1^2 dt_1 - \underline{|n}\int \underline{(1-\Phi)}^{n-j-2}\varphi^2 dt \int^t \varphi_1 dt_1$$
$$\times \int^{t_1} \mathfrak{D}_2[\underline{(\Phi_1-\Phi_2)}^{j-i-1}\Phi_2^{i-1}]\varphi_2^2 dt_2 - \frac{\underline{|n}}{2}\int \underline{(1-\Phi)}^{n-j-2}\varphi^2 dt \int^t \varphi_1^3 dt_1 \int^{t_1} \mathfrak{D}_2[\underline{(\Phi_1-\Phi_2)}^{j-i-3}\underline{\Phi_2^{i-1}}]\varphi_2^2 dt_2$$
$$+\delta_{i+3}^{j}\frac{\underline{|n}}{2}\Big[\int \underline{(1-\Phi)}^{n-j-2}\varphi^2 dt \int^t \underline{\Phi_1^{i-1}}\varphi_1^4 dt_1 - \int \underline{(1-\Phi)}^{n-j-1}\Phi^{i-1}\varphi^5 dt\Big] .$$

Eigthlyh, for $j \geqq i+3$, $k \geqq j+2$,

$$E(t_{i|n}t_{j|n}t_{k|n}) = -\underline{|n}\int \underline{(1-\Phi)}^{n-k}\varphi' dt \int^t \underline{(\Phi-\Phi_1)}^{k-j-1}\varphi_1' t_1 dt_1 \int^{t_1} \underline{(\Phi_1-\Phi_2)}^{j-i-1}\underline{\Phi_2^{i-1}}\varphi_2' dt_2 .$$

Here the innermost integral being

$$I_1(t_1) = -\int^{t_1} \mathfrak{D}_2[\underline{(\Phi_1-\Phi_2)}^{j-i-1}\underline{\Phi_2^{i-1}}]\varphi_2^2 dt_2$$

with

$$I_1'(t_1) = -\varphi_1 \int^{t_1} \mathfrak{D}_2[(\Phi_1-\Phi_2)^{j-i-2}\Phi_2^{i-1}]\varphi_2^2 dt_2$$

and

$$I''(t_1)=\delta^j_{i+3}\Phi_1^{i-1}\varphi_1^3-\varphi_1'\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)^{j-i-2}\Phi_2^{i-1}]\varphi_2^2dt_2+\varphi_1\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)^{j-i-3}\Phi_2^{i-1}]\varphi_2^2dt_2,$$

the inner integral becomes

$$
\begin{aligned}
I(t) &= \int^t (\Phi-\Phi_1)^{k-j-1} I_1(t_1)\varphi_1' t_1 dt_1 \\
&= -\int^t (\Phi-\Phi_1)^{k-j-2} I_1(t_1)\varphi_1\varphi_1' dt_1 - \int^t (\Phi-\Phi_1)^{k-j-1} I_1(t_1)\varphi_1 dt_1 + \int^t (\Phi-\Phi_1)^{k-j-1} I_1'(t_1)\varphi_1' dt_1 \\
&= \delta^k_{j+2}\frac{\varphi^2}{2}\int^t \mathfrak{D}_1[(\Phi-\Phi_1)^{j-i-1}\Phi_1^{i-1}]\varphi_1^2 dt_1 + \frac{1}{2}\int^t (\Phi-\Phi_1)^{k-j-3}\varphi_1^3 dt_1 \\
&\quad \times \int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}]\varphi_2^2 dt_2 - \frac{3}{2}\int^t (\Phi-\Phi_1)^{k-j-2}\varphi_1^3 dt_1 \int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)^{j-i-2}\Phi_2^{i-1}]\varphi_2^2 dt_2 \\
&\quad + \int^t (\Phi-\Phi_1)^{k-j-1}\varphi_1 dt_1 \int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}]\varphi_2^2 dt_2 - \int^t (\Phi-\Phi_1)^{k-j-1}\varphi_1^2 dt_1 \\
&\quad \times \int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}]\varphi_2^2 dt_2 - \delta^j_{i+3}\int^t (\Phi-\Phi_1)^{k-j-1}\Phi_1^{i-1}\varphi_1^4 dt_1 \\
&\quad + \int^t (\Phi-\Phi_1)^{k-j-1}\varphi_1\varphi_1' dt_1 \int^t [(\Phi_1-\Phi_2)^{j-i-2}\Phi_2^{i-1}]\varphi_2^2 dt_2,
\end{aligned}
$$

where the latest double integral again splits up into

$$
\begin{aligned}
&\frac{1}{2}\int^t (\Phi-\Phi_1)^{k-j-2}\varphi_1^3 dt_1 \int^t \mathfrak{D}_2[(\Phi_1-\Phi_2)^{j-i-2}\Phi_2^{i-1}]\varphi_2^2 dt_2 - \frac{1}{2}\int^t (\Phi-\Phi_1)^{k-j-1}\varphi_1^3 dt_1 \\
&\quad \times \int^t \mathfrak{D}_2[(\Phi_1-\Phi_2)^{j-i-3}\Phi_2^{i-1}]\varphi_2^2 dt_2 + \frac{1}{2}\delta^j_{i+3}\int^t (\Phi-\Phi_1)^{k-j-1}\Phi_1^{i-1}\varphi_1^4 dt_1
\end{aligned}
$$

(this becomes $-\frac{1}{2}\delta^j_{i+3}$ if combined with the before obtained $-\delta^j_{i+3}$).

Hence

$$
\begin{aligned}
(15.3.8)\qquad E(t_{i|n}t^2_{j|n}t_{k|n}) &= -\lfloor\underline{n} \int (1-\Phi)^{n-k} I(t)\varphi' dt \\
&= -\lfloor\underline{n} \int (1-\Phi)^{n-k-1}\varphi^2 I(t) dt + \lfloor\underline{n} \int (1-\Phi)^{n-k}\varphi I'(t) dt = \text{(i)} + \text{(ii)}.
\end{aligned}
$$

The sum (i) is already free from dash, while (ii) is obtainable by replacing k by $k-1$ in (i) and changing the sign, so that $-\delta^k_{j+2}$ standing at head in (i) should be replaced by $+\delta^{k-1}_{j+2}$ namely

$$\delta^k_{j+3}\frac{\lfloor\underline{n}}{2}\int (1-\Phi)^{n-k-1}\varphi^4 dt \int^t \mathfrak{D}_1[(\Phi-\Phi_1)^{j-i-1}\Phi_1^{i-1}]\varphi_1^2 dt_1,$$

and δ^j_{i+3} standing at tail in (i) by

$$-\delta^j_{k+3}\frac{\lfloor\underline{n}}{2}\int (1-\Phi)^{n-k}\varphi^2 dt \int^t (\Phi-\Phi_1)^{k-j-2}\Phi_1^{i-1}\varphi_1^4 dt_1.$$

So far we have exhausted all the cases in regard to the third and fourth moments up to with 3 arguments. However, we omit general formulas for the fourth moment with 4 arguments, i.e. those about

$$E(t_{i|n}t_{j|n}t_{k|n}t_{l|n}) = \underline{|n} \int \underline{(1-\Phi)^{n-l}}\varphi' dt \int^{t} \underline{(\Phi-\Phi_1)^{l-k-1}}\varphi_1' dt_1 \int^{t_1} \underline{(\Phi_1-\Phi_2)^{k-j-1}}\varphi_2' dt_2$$
$$\times \int^{t_2} \underline{(\Phi_2-\Phi_3)^{j-i-1}}\underline{\Phi_3^{i-1}}\varphi_3' dt_3$$

to make it dashfree, since it is rather simple to calculate them directly for each case that i, j, k, l are numerically given, as long as n is rather small (cf. the end of §18).

§ **16. Check Formulas.** By use of the formulas in the preceding and subsequent paragraphs (§§ 14, 15 and 17), we can obtain the explicit forms or numerical values of moments. However, to ascertain their correctness, we should verify it by means of some check formulas. Some of them were already described in § 5 Part I, among which (5. 10) i.e. $\sum_{k=1}^{n} E(t_{i|n}t_{k|n})=1$ was especially eminent. Therefore we would extend it somehow. Now by means of identities (3. 2) in Part I, we can finish all work to find e.g. $E(t_{i|n}^{p})$ by half: namely, we have only to compute them for $i=1, 2, \cdots, \left[\frac{n+1}{2}\right]$. Also we get identity (3. 1): $\sum_{k=1}^{n} E(t_{i|n}^{p})=0$ or $1\cdot 3 \cdots (n-1)p$ according as p is odd or even, and hence this for even p suffices as check. However, if p be odd, with our determination $E(t_{n-i+1|n}^{p})=-E(t_{i|n}^{p})$, naturally the whole sum becomes zero, which, however, of course, gives no check. Therefore, to verify e.g. $E(t_{i|n}^{3})$, we require other check formulas, which would be found below.

1° Firstly, treating the case of 2 arguments, we shall prove the following identity:

(16. 1) $\qquad \sum_{k=1}^{n} E(t_{i|n}^{p}t_{k|n}) = pE(t_{i|n}^{p-1}) \qquad$ for any fixed $i\,(=1, 2, \cdots, n)$.

Proof. Putting

$$\sum_{k=1}^{n} = \sum_{k=1}^{i-1} + E(t_{i|n}^{p+1}) + \sum_{k=i+1}^{n} = (\mathrm{i}) + (\mathrm{ii}) + (\mathrm{iii}),$$

we have

$$(\mathrm{i}) = -\sum_{k=1}^{i-1} \underline{|n} \int \underline{(1-\Phi)^{n-i}}\varphi t^p dt \int^{t} \underline{(\Phi-\Phi_1)^{i-k-1}}\Phi_1\varphi_1' dt_1$$
$$= -\underline{|n} \int \underline{(1-\Phi)^{n-i}}\varphi t^p dt \int^{t} \sum_{k=1}^{i-1} \underline{(\Phi-\Phi_1)^{i-k-1}}\underline{\Phi_1^{k-1}}\varphi_1' dt_1$$
$$= -\underline{|n} \int \underline{(1-\Phi)^{n-i}}\underline{\Phi^{i-2}}\varphi^2 t^p dt = -\underline{|n} \int \underline{(1-\Phi)^{n-i}} D(\underline{\Phi^{i-1}})\varphi^2 t^p dt;$$

$$(\mathrm{ii}) = -\underline{|n} \int \underline{(1-\Phi)^{n-i}}\underline{\Phi^{i-1}}\varphi' t^p dt$$
$$= \underline{|n} \int D[\underline{(1-\Phi)^{n-i}}\underline{\Phi^{i-1}}]\varphi^2 t^p dt + \underline{|n}\, p \int \underline{(1-\Phi)^{n-i}}\underline{\Phi^{i-1}}\varphi t^{p-1} dt,$$

whose last integral is evidently $pE(t_{i|n}^{p-1})$; and lastly

$$\text{(iii)} = -\sum_{k=i+1}^{n} \underline{|n} \int (1-\Phi)^{n-k}\varphi' dt \int^{t} (\Phi-\Phi_1)^{k-i-1}\Phi_1^{i-1}\varphi_1 t_1^p dt_1$$

$$= -\underline{|n} \int \Phi_1^{i-1}\varphi_1 t_1^p dt_1 \int_{t_1} \sum_{k=i+1}^{n} (1-\Phi)^{n-k}(\Phi-\Phi_1)^{k-i-1}\varphi' dt$$

$$= \underline{|n} \int (1-\Phi)^{n-i-1}\Phi^{i-1}\varphi^2 t^p dt = -\underline{|n} \int D(1-\Phi)^{n-i}\cdot\Phi^{i-1}\varphi^2 t^p dt\,.$$

By adding (i) and (iii) together it amounts to $-\underline{|n} \int D[(1-\Phi)^{n-i}\Phi^{i-1}]\varphi^2 t^p dt$ and therefore the total sum (i)+(ii)+(iii) gives $pE(t_{i|n}^{p-1})$, as asserted, Q.E.D.

Against the foregoing, the evaluation of $\sum_{k=1}^{n} E(t_{i|n}t_{k|n}^q)$ becomes somewhat complicate, so that we treat only those actually needed, i.e. the cases $q=2$ and 3.

$$\sum_{k=1}^{n} E(t_{i|n}t_{k|n}^2) = (n+1)E(t_{i|n})\,. \tag{16.2}$$

Proof. As before, letting $\sum_{k=1}^{n} = \sum_{k=1}^{i-1} + E(t_{i|n}^3) + \sum_{k=i+1}^{n} =$ (i)+(ii)+(iii), we have

$$\text{(i)} = \underline{|n} \int (1-\Phi)^{n-i}\varphi' dt \int^{t} \sum_{k=1}^{i-1} (\Phi-\Phi_1)^{i-k-1}\Phi_1^{k-1}\varphi_1' t_1 dt_1$$

$$= \underline{|n} \int (1-\Phi)^{n-i}\Phi^{i-2}\varphi' dt \int^{t} \varphi_1' t_1 dt_1 \;(=\varphi t-\Phi)$$

$$= -\underline{|n} \int (1-\Phi)^{n-i}\Phi^{i-2}\varphi'^2 dt + (i-1)\underline{|n} \int (1-\Phi)^{n-i}\Phi^{i-1}\varphi t\, dt \;(=(i-1)E(t_{i|n}))\,;$$

$$\text{(ii)} = -|n \int (1-\Phi)^{n-i}\Phi^{i-1}\varphi' t^2 dt$$

$$= |n \int D(1-\Phi)^{n-i}\Phi^{i-1}\varphi^2 t^2 dt + 2\,|n \int (1-\Phi)^{n-i}\Phi^{i-1}\varphi t dt$$

$$= |n \int D(1-\Phi)^{n-i}\Phi^{i-1}\varphi'^2 dt + 2E(t_{i|n})\,;$$

$$\text{(iii)} = \sum_{k=i+1}^{n} |n \int (1-\Phi)^{n-k}\varphi' t\, dt \int^{t} (\Phi-\Phi_1)^{k-i-1}\Phi_1^{i-1}\varphi_1' dt_1$$

$$= |n \int \Phi_1^{i-1}\varphi_1' dt_1 \int_{t_1} \sum_{k=i+1}^{n} (1-\Phi)^{n-k}\cdot(\Phi-\Phi_1)^{k-i-1}\varphi' t\, dt$$

$$= \underline{|n} \int \Phi_1^{i-1}(1-\Phi_1)^{n-i-1}\varphi_1' dt_1 \int_{t_1} \varphi' t\, dt \;\;(=\varphi_1'-(1-\Phi_1))$$

$$= -\underline{|n} \int \Phi^{i-1}D(1-\Phi)^{n-i}\varphi'^2 dt + (n-i)E(t_{i|n})\,.$$

Therefore

$$\text{(i)+(ii)+(iii)} = (n+1)\,E(t_{i|n}),\ \text{ Q.E.D.}$$

Also

$$\sum_{k=1}^{n} E(t_{i|n}t_{k|n}^3) = 2+E(t_{i|n}^2)\,. \tag{16.3}$$

Proof. We put again $\sum_{k=1}^{n}=\sum_{k=1}^{i-1}+E(t_{i|n}^4)+\sum_{k=i+1}^{n}$ (i)+(ii)+(iii), where

$$(\mathrm{i})=|\underline{n}\int(1-\Phi)^{n-i}\varphi' dt\int^{t}\sum_{k=1}^{i-1}(\Phi-\Phi_1)^{i-k-1}\Phi_1^{k-1}\varphi_1' t_1^2 dt_1$$

$$=|\underline{n}\int(1-\Phi)^{n-i}\Phi^{i-2}\varphi' dt\int^{t}\varphi_1' t_1^2 dt_1\,;$$

but, since the inner integral becomes $\varphi t^2-2\int^{t}\varphi t\,dt=-\varphi' t+2\varphi$,

$$(\mathrm{i})=|\underline{n}\int(1-\Phi)^{n-i}D(\Phi^{i-1})\varphi\varphi' t^2 dt+2|\underline{n}\int(1-\Phi)^{n-i}D[\Phi^{i-1}]\varphi\varphi' dt\,;$$

and similarly

$$(\mathrm{iii})=|\underline{n}\int\Phi_1^{i-1}\varphi_1' dt_1\int_{t_1}\sum_{k=i+1}^{n}(1-\Phi)^{n-k}(\Phi-\Phi_1)^{k-i-1}\varphi' t^2 dt$$

$$=|\underline{n}\int\Phi_1^{i-1}(1-\Phi_1)^{n-i-1}[-\varphi_1 t^2-2\varphi_1]\varphi_1' dt_1$$

$$=|\underline{n}\int\Phi^{i-1}D[(1-\Phi)^{n-i}]\varphi\varphi' t^2 dt+2|\underline{n}\int\Phi^{i-1}D[(1-\Phi)^{n-i}]\varphi\varphi' dt\,.$$

Lastly

$$(\mathrm{ii})=|\underline{n}\int(1-\Phi)^{n-i}\Phi^{i-1}(-\varphi' t^3)dt$$

$$=-|\underline{n}\int D[(1-\Phi)^{n-i}\Phi^{i-1}]\varphi\varphi' t^2 dt+3|\underline{n}\int(1-\Phi)^{n-i}\Phi^{i-1}\varphi t^2 dt\quad(=3E(t_{1|n}^2))\,.$$

But
$$E(t_{i|n}^2)=-|\underline{n}\int(1-\Phi)^{n-i}\Phi^{i-1}\varphi' t\,dt$$

$$=|\underline{n}\int(1-\Phi)^{n-i}\Phi^{i-1}\varphi dt+|\underline{n}\int D[(1-\Phi)^{n-i}\Phi^{i-1}]\varphi^2 t\,dt$$

$$=1-|\underline{n}\int D[(1-\Phi)^{n-i}\Phi^{i-1}]\varphi\varphi' dt\,,$$

that is

$$|\underline{n}\int D[(1-\Phi)^{n-i}\Phi^{i-1}]\varphi\varphi' dt=1-E(t_{i|n}^2)\,.$$

Hence, the total sum (i)+(ii)+(iii) yields the right hand side of (16.3), Q.E.D.

Further

$$(16.4)\qquad \sum_{k=1}^{n}E(t_{i|n}^2 t_{k|n}^2)=(n+2)E(t_{i|n}^2)\,,\qquad \sum_{i=1}^{n}\sum_{k=1}^{n}E(t_{i|n}^2 t_{k|n}^2)=n(n+2)$$

Proof. Letting $\sum_{k=1}^{n}=\sum_{k=1}^{i-1}+E(t_{i|n}^4)+\sum_{k=i+1}^{n}=$ (i)+(ii)+(iii),

$$(\mathrm{i})=|\underline{n}\int(1-\Phi)^{n-i}\Phi^{i-2}(\varphi t-\Phi)\varphi' t\,dt$$

$$=-|\underline{n}\int(1-\Phi)^{n-i}D[\Phi^{i-1}]\varphi'^2 t\,dt+(i-1)E(t_{i|n}^2)\,;$$

$$\text{(iii)} = \underline{|n}\int \underline{(1-\Phi)^{n-i-1}}\underline{\Phi^{i-2}}[-\varphi t-(1-\Phi)]\varphi' t\,dt$$

$$= -\underline{|n}\int \underline{\Phi^{i-1}}D[\underline{(1-\Phi)^{n-i}}]\varphi'^2 t\,dt + (n-i)\cdot E(t^2_{i|n})\,.$$

But

$$\text{(ii)} = E(t^4_{i|n}) = \underline{|n}\int \underline{(1-\Phi)^{n-i}}\underline{\Phi^{i-1}}(-\varphi' t^3)dt$$

$$= \underline{|n}\int D[\underline{(1-\Phi)^{n-i}}\underline{\Phi^{i-1}}]\varphi'^2 t\,dt + 3E(t^2_{i|n})\,.$$

Hence their sum (i)+(ii)+(iii) gives $(n+2)E(t^2_{i|n})$, as asserted.

Remark. We may here recapitulate another kind of check formulus, which was already described at (5. 3), Part I:

$$\text{(16. 5)} \qquad \sum_{k=2}^{n}\sum_{i=1}^{k-1} E(t^p_{i|n}t^q_{k|n}) = \frac{n(n-1)}{2\pi}2^{\frac{p+q}{2}}\Gamma\left(\frac{p+q}{2}+1\right)\int_{\pi/4}^{5\pi/4}\cos^p\theta\sin^q\theta\,d\theta\,,$$

so that, in particular,

$$\text{(16. 5. 1)} \qquad \sum_{k=1}^{n}\sum_{i=1}^{k-1} E(t^2_{1|n}t_{k|n}) = \frac{n(n-1)}{4\sqrt{\pi}} \quad\text{and}\quad \sum_{k=2}^{n}\sum_{i=1}^{k-1} E(t_{i|n}t^2_{n|n}) = -\frac{n(n-1)}{4\sqrt{\pi}}\,.$$

$$\text{(16. 5. 2)} \qquad \sum_{k=2}^{n}\sum_{i=1}^{k-1} E(t^3_{i|n}t_{k|n}) = \sum_{k=2}^{n}\sum_{i=1}^{k-1} E(t_{i|n}t^3_{k|n}) = 0\,.$$

$$\text{(16. 5. 3)} \qquad \sum_{k=2}^{n}\sum_{i=1}^{k-1} E(t^2_{i|n}t^2_{k|n}) = \frac{1}{2}n(n-1)\,.$$

2° Next, we consider the case of 3 arguments: we shall prove the identity

$$\text{(16. 6)} \qquad \sum_{k=1}^{n} E(t_{i|n}t_{j|n}t_{k|n}) = E(t_{i|n}) + E(t_{j|n})\,.$$

Proof. When $i=j$, (16. 6) reduces to (16. 1) for $p=2$. Hence, without the loss of generality we may assume that $i<j$. We put

$$\sum_{k=1}^{n} E(t_{i|n}t_{j|n}t_{k|n}) = \sum_{k=1}^{i-1} + E(t^2_{i|n}t_{j|n}) + \sum_{k=i+1}^{j-1} + E(t_{i|n}t^2_{j|n}) + \sum_{k=j+1}^{n} = \text{(i)}+\text{(ii)}+\text{(iii)}+\text{(iv)}+\text{(v)},$$

where (iii) becomes 0, if $i+1=j$. Upon integrating by parts, the successive terms by the assumption $j-i-1>0$ yield

$$\text{(i)} = -\underline{|n}\int \underline{(1-\Phi)^{n-j}}\varphi' dt\int^t \underline{(\Phi-\Phi_1)^{j-i-1}}\varphi_1' dt_1\int^t \sum_{k=1}^{i-1}\underline{(\Phi_1-\Phi_2)^{i-k-1}}\underline{\Phi_2^{k-1}}\varphi_2' dt_2$$

$$= -\underline{|n}\int \underline{(1-\Phi)^{n-j}}\varphi' dt\int^t \underline{(\Phi-\Phi_1)^{j-i-1}}\underline{\Phi_1^{i-2}}\varphi_1\varphi_1' dt_1\,;$$

$$\text{(iii)} = -\sum_{k=i+1}^{j-1}\underline{|n}\int \underline{(1-\Phi)^{n-j}}\varphi' dt\int^t \underline{(\Phi-\Phi_1)^{j-k-1}}\varphi_1' dt_1\int^{t_1}\underline{(\Phi_1-\Phi_2)^{k-i-1}}\underline{\Phi_1^{i-1}}\varphi_2' dt_2$$

$$= -\underline{|n}\int \underline{(1-\Phi)^{n-j}}\varphi' dt\int^t \underline{\Phi_2^{i-1}}\varphi_2' dt_2\int_{t_2}^{t_1}\sum_{k=i+1}^{j-1}\underline{(\Phi_1-\Phi_2)^{j-k-1}}\underline{(\Phi-\Phi_1)^{k-i-1}}\varphi_1' dt_1$$

$$= -\underline{|n}\int \underline{(1-\Phi)^{n-j}}\varphi\varphi' dt\int^t \underline{(\Phi-\Phi_1)^{j-i-2}}\underline{\Phi_1^{i-1}}\varphi_1' dt_1$$

$$+\underline{|n}\int\underline{(1-\Phi)^{n-j}}\varphi'dt\int^{t}\underline{(\Phi-\Phi_1)^{j-i-2}}\underline{\Phi_1^{i-1}}\varphi_1\varphi_1'dt_1 \quad (=0,\quad \text{if } i+1=j)\,;$$

$$(\text{v}) = -\sum_{k=i+1}^{n}\underline{|n}\int\underline{(1-\Phi)^{n-k}}\varphi'dt\int^{t}\underline{(\Phi-\Phi_1)^{k-j-1}}\varphi_1'dt_1\int^{t_1}\underline{(\Phi_1-\Phi_2)^{j-i-1}}\underline{\Phi_2^{i-1}}\varphi_2'dt_2$$

$$= -\underline{|n}\int\varphi_1'dt_1\int_{t_1}\sum_{k=j+1}^{n}\underline{(1-\Phi)^{n-k}}\underline{(\Phi-\Phi_1)^{k-j-1}}\varphi'dt\int^{t_1}\underline{(\Phi_1-\Phi_2)^{j-i-1}}\underline{\Phi_2^{i-1}}\varphi_2'dt_2$$

$$= \underline{|n}\,\underline{(1-\Phi)^{n-j-1}}\varphi\varphi'dt\int^{t}\underline{(\Phi-\Phi_1)^{j-i-1}}\underline{\Phi_1^{i-1}}\varphi_1'dt_1\,.$$

On the other hand, we have

$$(\text{ii}) = \underline{|n}\int\underline{(1-\Phi)^{n-j}}\varphi'dt\int^{t}\underline{(\Phi-\Phi_1)^{j-i-1}}\varphi_1't_1dt_1$$

$$= \underline{|n}\int\underline{(1-\Phi)^{n-j}}\varphi'dt\int^{t}D_1[\underline{(\Phi-\Phi_1)^{j-i-1}}\underline{\Phi_1^{i-1}}]\varphi_1\varphi_1'dt_1$$

$$\underline{-\underline{|n}\int\underline{(1-\Phi)^{n-j}}\varphi'dt\int^{t}\underline{(\Phi-\Phi_1)^{j-i-1}}\underline{\Phi_1^{i-1}}\varphi_1dt_1}\,;$$

$$(\text{iv}) = \underline{|n}\int\underline{(1-\Phi)^{n-j}}\varphi't\,dt\int^{t}\underline{(\Phi-\Phi_1)^{j-i-1}}\underline{\Phi_1^{i-1}}\varphi_1'dt_1$$

$$= -\underline{|n}\int\underline{(1-\Phi)^{n-j-1}}\varphi\varphi'dt\int^{t}\underline{(\Phi-\Phi_1)^{j-i-1}}\underline{\Phi_1^{i-1}}\varphi_1'dt_1$$

$$\underline{-\underline{|n}\int\underline{(1-\Phi)^{n-j}}\varphi\,dt\int^{t}\underline{(\Phi-\Phi_1)^{j-i-1}}\underline{\Phi_1^{i-1}}\varphi_1'dt_1}$$

$$+\underline{|n}\int\underline{(1-\Phi)^{n-j}}\varphi\varphi'dt\int^{t}\underline{(\Phi-\Phi_1)^{j-i-2}}\underline{\Phi_1^{i-1}}\varphi_1'dt_1\,.$$

Therefore, on summing up all of them, they almost cancel two by two, except those underlined two integrals. But, the first underlined integral becomes, on writing $\Phi_1=\Phi v$,

$$-\underline{|n}\int\underline{(1-\Phi)^{n-j}}\varphi'dt\int_0^1\underline{\Phi^{i-1}}\underline{(1-v)^{j-i-1}}\underline{v^{i-1}}dv = \underline{|n}\int\underline{(1-\Phi)^{n-j}}\underline{\Phi^{j-1}}\varphi t\,dt = E(t_{j|n})\,,$$

while the second, on interchanging the order of integrations and then writing $\Phi-\Phi_1=(1-\Phi_1)v$,

$$-\underline{|n}\int\underline{\Phi_1^{i-1}}\varphi_1'dt_1\int\underline{(1-\Phi)^{n-j}}\underline{(\Phi-\Phi_1)^{j-i-1}}\varphi\,dt = -\underline{|n}\int\underline{\Phi^{i-1}}\underline{(1-\Phi)^{n-i}}\varphi'dt = E(t_{i|n})\,.$$

Thus, (16.6) is established.

Adding (16.6) for $j=1,2,\cdots,n$ and applying (3.1), Part I, $\sum_j E(t_{j|n})=0$, we get

$$(16.6.1)\qquad \sum_{j=1}^{n}\sum_{k=1}^{n}E(t_{i|n}t_{j|n}t_{k|n}) = nE(t_{i|n})\,.$$

Further, we have

$$(16.7)\qquad \sum_{k=1}^{n}E(t_{i|n}^2t_{j|n}t_{k|n}) = E(t_{i|n}^2)+2E(t_{i|n}t_{j|n})\,.$$

Proof. First assuming that $i+1<j$, we put, as before,

$$\sum_{k=1}^{n} = \sum_{k=1}^{i-1} + E(t^3_{i|n} t_{j|n}) + \sum_{k=i+1}^{j-1} + E(t^2_{i|n} t^2_{i|n}) + \sum_{k=j+1}^{n} = \text{(i)}+\text{(ii)}+\text{(iii)}+\text{(iv)}+\text{(v)}\,,$$

where

$$\text{(i)} = -\underline{|n} \int (1-\Phi)^{n-j}\varphi' dt \int^t (\Phi-\Phi_1)^{j-i-1}\varphi_1' t_1 dt_1 \int^{t_1} \sum_{k=1}^{i-1} (\Phi_1-\Phi_2)^{i-k-1}\Phi_2^{k-1}\varphi_2' dt_2$$

$$= -\underline{|n} \int (1-\Phi)^{n-j}\varphi' dt \int^t (\Phi-\Phi_1)^{j-i-1}\Phi_1^{i-2}\varphi_1\varphi_1' t_1 dt_1\,;$$

$$\text{(iii)} = -\underline{|n} \sum_{k=i+1}^{j-1} \int (1-\Phi)^{n-j}\varphi' dt \int^t (\Phi-\Phi_1)^{j-k-1}\varphi_1' dt \int^{t_1} (\Phi_1-\Phi_2)^{k-i-1}\Phi_2^{i-1}\varphi_2' t_2 dt_2$$

$$= -\underline{|n} \int (1-\Phi)^{n-j}\varphi' dt \int^t \Phi_2^{i-1}\varphi_2' t_2 dt_2 \int_{t_2}^{t} \sum_{k=i+1}^{j-1} (\Phi-\Phi_1)^{k-i-1}\varphi_1' dt_1$$

$$= -\underline{|n} \int (1-\Phi)^{n-j}\varphi\varphi' dt \int^t (\Phi-\Phi_1)^{j-i-2}\Phi_1^{i-1}\varphi_1' t_1 dt_1$$

$$+\underline{|n} \int (1-\Phi)^{n-j}\varphi' dt \int^t (\Phi-\Phi_1)^{j-i-2}\Phi_1^{i-1}\varphi_1' t_1 dt_1\,;$$

$$\text{(v)} = -\underline{|n} \sum_{k=j+1}^{n} \int (1-\Phi)^{n-k}\varphi' dt \int^t (\Phi-\Phi_1)^{k-j-1}\varphi_1' dt_1 \int^{t_1} (\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}\varphi_2' t_2 dt_2$$

$$= -\underline{|n} \int \varphi_1' dt_1 \int_{t_1} \sum_{k=j+1}^{n} (1-\Phi)^{n-k}(\Phi-\Phi_1)^{k-j-1}\varphi' dt \int^{t_1} (\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}\varphi_2' t_2 dt_2$$

$$= \underline{|n} \int (1-\Phi)^{n-j-1}\varphi\varphi' dt \int^t (\Phi-\Phi_1)^{j-i-1}\Phi_1^{i-1}\varphi_1' t_1 dt_1\,;$$

$$\text{(ii)} = \underline{|n} \int (1-\Phi)^{n-j}\varphi' dt \int^t (\Phi-\Phi_1)^{j-i-1}\Phi_1^{i-1}\varphi_1' t_1^2 dt_1$$

$$= \underline{|n} \int (1-\Phi)^{n-j}\varphi' dt \int^t D_1[(\Phi-\Phi_1)^{j-i-1}\Phi_1^{i-1}]\varphi_1\varphi_1' t_1 dt_1$$

$$\underline{+2\underline{|n} \int (1-\Phi)^{n-j}\varphi' dt \int^t (\Phi-\Phi_1)^{j-i-1}\Phi_1^{i-1}\varphi_1' dt_1\,;}$$

$$\text{(iv)} = \underline{|n} \int (1-\Phi)^{n-j}\varphi' t\, dt \int^t (\Phi-\Phi_1)^{j-i-1}\Phi_1^{i-1}\varphi_1' t_1 dt_1$$

$$= -\underline{|n} \int (1-\Phi)^{n-j-1}\varphi\varphi' dt \int^t (\Phi-\Phi_1)^{j-i-1}\Phi_1^{i-1}\varphi_1' t_1 dt_1$$

$$\underline{-\underline{|n} \int (1-\Phi)^{n-j}\varphi\, dt \int^t (\Phi-\Phi_1)^{j-i-1}\Phi_1^{i-1}\varphi_1' t_1 dt_1}$$

$$+\underline{|n} \int (1-\Phi)^{n-j}\varphi\varphi' dt \int^t (\Phi-\Phi_1)^{j-i-2}\Phi_1^{i-1}\varphi_1' t_1 dt_1\,.$$

On summing up all of them, they almost cancel, except those two underlined integrals, one of which is clearly $2E(t_{j|n} t_{i|n})$, while the other is

$$\underline{|n} \int n(1-\Phi)^{n-j}\varphi\, dt \int^t (\Phi-\Phi_1)^{j-i-1}\Phi_1^{i-1}\varphi_1 t_1^2 dt_1$$

$$= \underline{|n} \int \Phi_1^{i-1}\varphi_1 t_1^2 dt_1 \int_{t_1} (1-\Phi)^{n-j}(\Phi-\Phi_1)^{j-i-1}\varphi\, dt\,;$$

or, on putting $1-\Phi=(1-\Phi_1)v$, it reduces to $\underline{|n}\int\underline{\Phi^{i-1}}\underline{(1-\Phi)^{n-i}}\varphi t^2 dt=E(t^2_{i|n})$, Q.E.D.

The case $j=i+1$ as well as $j<i$ could be quite similarly proved[7]. Thus we see that two summations

$$\sum_{k=1}^{n} E(t^2_{i|n}t_{j|n}t_{k|n})=E(t^2_{i|n})+2E(t_{i|n}t_{j|n}),\quad (i<j)$$

$$\sum_{k=1}^{n} E(t_{i|n}t^2_{j|n}t_{k|n})=E(t^2_{j|n})+2E(t_{i|n}t_{j|n})\quad (i<j)$$

either has the same form. Consequently, we obtain

$$(16.7.1)\quad \sum_{j=1}^{n}\sum_{k=1}^{n} E(t^2_{i|n}t_{j|n}t_{k|n})=nE(t^2_{i|n})+2,\quad \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} E(t^2_{i|n}t_{j|n}t_{k|n})=n(n+2).$$

$$(16.7.2)\quad \sum_{j=1}^{n}\sum_{k=1}^{n} E(t_{i|n}t^2_{j|n}t_{k|n})=n+2.$$

For,

$$\sum_{j=1}^{n}\sum_{k=1}^{n} E(t^2_{i|n}t_{j|n}t_{k|n})=\sum_{j=1}^{i-1}\sum_{k=1}^{n} E(t_j t_i^2 t_k)+\sum_{k=1}^{n} E(t^3_{i|n}t_{k|n})+\sum_{j=i+1}^{n}\sum_{k=1}^{n} E(t_i^2 t_j t_k)$$
$$=\sum_{j=1}^{i-1}[E(t^2_{i|n})+2E(t_i t_j)]+3E(t^2_{1|n})+\sum_{j=i+1}^{n}[E(t^2_{i|n})+2E(t_i t_j)]$$
$$=nE(t^2_{i|n})+2\sum_{j=1}^{n} E(t_{i|n}t_{j|n})=nE(t^2_{i|n})+2$$

in view of (5. 10), Part I. Also, by reason of (3. 1) for $p=2$, and (5. 10)

$$\sum_{i=1}^{n}\sum_{k=1}^{n} E(t_{i|n}t^2_{j|n}t_{k|n})=\sum_{j=i}^{i-1} E(t_{i|n}t^2_{j|n}t_{k|n})+\sum_{k=1}^{n} E(t^3_{i|n}t_{k|n})+\sum_{j=i+1}^{n}\sum_{k=1}^{n} E(t_{i|n}t^2_{j|n}t_{k|n})$$
$$=\sum_{j=1}^{i-1}[E(t^2_{j|n})+2E(t_i t_j)]+3E(t^2_{i|n})+\sum_{j=i+1}^{n}[E(t^2_{j|n})+2E(t_i t_j)]$$
$$=\sum_{j=1}^{n} E(t^2_{j|n})+2\sum_{j=1}^{n} E(t_{i|n}t_{j|n})=n+2.$$

Remark. We may again recapitulate the formulas (5. 6) in Part I:

$$(16.8)\quad \begin{cases} \sum\sum\sum E(t_i^2 t_j t_k)=\sum\sum\sum E(t_i t_j t_k^2)=\dfrac{n(n-1)(n-2)}{12\pi\sqrt{3}}, \\ \sum\sum\sum E(t_i t_j^2 t_k)=-\dfrac{n(n-1)(n-2)}{6\pi\sqrt{3}}, \end{cases}$$

where the summation $\sum\sum\sum$ denotes $\sum_{k=3}^{n}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}$. These may be conveniently employed, because, although the number of summands becomes somewhat larger, here all the summands are, so to speak, pure i.e. only $E(t^2_{i|n}t_{j|n}t_{k|n})$ or $E(t_{i|n}t^2_{j|n}t_{k|n})$ come forth without mixing those degenerated kinds e.g. $E(t_i^2 t_j^2)$.

3° Furthermore, we consider the case of 4 arguments. We have

$$(16.9)\quad \sum_{l=1}^{n} E(t_{i|n}t_{j|n}t_{k|n}t_{l|n})=E(t_{i|n}t_{j|n})+E(t_{i|n}t_{k|n})+E(t_{j|n}t_{k|n}).$$

7) The case $j=i$ reduces to (16.1) with $p=3$, so that (16.7) still holds formally.

Proof. By reason of symmetry we may consider only the case $i<j<k$. Let

$$\sum_{l=1}^{n}=\sum_{l=1}^{i-1}+E(t_i^2 t_j t_k)+\sum_{l=i+1}^{j-1}+E(t_i t_j^2 t_k)+\sum_{l=j+1}^{k-1}+E(t_i t_k t_k^2)+\sum_{l=k+1}^{n}$$
$$=\text{(i)}+\text{(ii)}+\text{(iii)}+\text{(iv)}+\text{(v)}+\text{(vi)}+\text{(vii)}\,.$$

we obtain by summation each by each (details being omitted)

$$\text{(i)}=|\underline{n}\int(1-\Phi)^{n-k}\varphi'dt\int^{t}(\Phi-\Phi_1)^{k-j-1}\varphi_1'dt_1\int^{t_1}(\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-2}\varphi_2\varphi_2'dt_2 \quad\cdots\cdots(\alpha)$$

$$\text{(iii)}=|\underline{n}\int(1-\Phi)^{n-k}\varphi'dt\int^{t}(\Phi-\Phi_1)^{k-j-1}\varphi_1\varphi_1'dt_1\int^{t_1}(\Phi_1-\Phi_2)^{j-i-2}\Phi_2^{i-1}\varphi_2'dt_2 \quad\cdots\cdots(\beta)$$
$$-|\underline{n}\int(1-\Phi)^{n-k}\varphi'dt\int^{t}(\Phi-\Phi_1)^{k-j-1}\varphi_1'dt_1\int^{t_1}(\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}\varphi_2'dt_2 \quad\cdots\cdots(\alpha)$$

$$\text{(v)}=-|\underline{n}\int(1-\Phi)^{n-k}\varphi\varphi'dt\int^{t}(\Phi-\Phi_1)^{k-j-2}\varphi_1'dt_1\int^{t_1}(\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}\varphi_2'dt_2 \quad\cdots(\gamma)$$
$$-|\underline{n}\int(1-\Phi)^{n-k}\varphi'dt\int^{t}(\Phi-\Phi_1)^{k-j-2}\varphi_1\varphi_1'dt_1\int^{t_1}(\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}\varphi_2'dt_2 \quad\cdots(\delta)$$

$$\text{(vii)}=-|\underline{n}\int(1-\Phi)^{n-k-1}\varphi\varphi'dt\int^{t}(\Phi-\Phi_1)^{k-j-1}\varphi_1'dt_1\int^{t_1}(\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}\varphi_2'dt_2 \cdots(\varepsilon)$$

On the otherhand,

$$\text{(ii)}=-|\underline{n}\int(1-\Phi)^{n-k}\varphi'dt\int^{t}(\Phi-\Phi_1)^{k-j-1}\varphi_1'dt_1\int^{t_1}\mathfrak{D}_2[(\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}]\varphi_2\varphi_2'dt_2 \quad\cdots\cdots(\alpha)$$
$$+|\underline{n}\int(1-\Phi)^{n-k}\varphi'dt\int^{t}(\Phi-\Phi_1)^{k-j-1}\varphi_1'dt_1\int^{t_1}(\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}\varphi_2 dt_2 \quad\cdots\cdots(\xi)$$

$$\text{(iv)}=|\underline{n}\int(1-\Phi)^{n-k}\varphi'dt\int^{t}(\Phi-\Phi_1)^{k-j-2}\varphi_1\varphi_1'dt_1\int^{t_1}(\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}\varphi_2'dt_2 \quad\cdots\cdots(\delta)$$
$$+|\underline{n}\int(1-\Phi)^{n-k}\varphi'dt\int^{t}(\Phi-\Phi_1)^{k-j-1}\varphi_1 dt_1\int^{t_1}(\Phi_1-\Phi_1)^{j-i-1}\Phi_2^{i-1}\varphi_2'dt_2 \quad\cdots\cdots(\eta)$$
$$-|\underline{n}\int(1-\Phi)^{n-k}\varphi'dt\int^{t}(\Phi-\Phi_1)^{k-j-1}\varphi_1\varphi_1'dt_1\int^{t_1}(\Phi_1-\Phi_2)^{j-i-2}\Phi_2^{i-1}\varphi_2'dt_2 \cdots\cdots(\beta)$$

$$\text{(vi)}=|\underline{n}\int(1-\Phi)^{n-k-1}\varphi\varphi'dt\int^{t}(\Phi-\Phi_1)^{k-j-1}\varphi_1'dt\int^{t}(\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}\varphi_2'dt_2 \quad\cdots\cdots(\varepsilon)$$
$$+|\underline{n}\int(1-\Phi)^{n-k}\varphi dt\int^{t}(\Phi-\Phi_1)^{k-j-1}\varphi_1'dt_1\int^{t_1}(\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}\varphi_2'dt_2 \quad\cdots\cdots(\zeta)$$
$$-|\underline{n}\int(1-\Phi)^{n-k}\varphi\varphi'dt\int^{t}(\Phi-\Phi_1)^{k-j-2}\varphi_1'dt_1\int^{t_1}(\Phi_1-\Phi_2)^{j-i-1}\Phi_2^{i-1}\varphi_2'dt_2\,. \quad\cdots(\gamma)$$

Since those samely named (α) (β) (γ) (δ) (ε) cancel each other[8] there remain only underlined 2 integrals (ξ) (η) (ζ). But

8) If $j=i+1$, or $k=j+1$, (β) or (γ) and (δ) reduce to naught, respectively, which, however, does not hinder our result.

$$(\xi) = |\underline{n} \int \underline{(1-\Phi)^{n-k}}\varphi' dt \int^{t} \underline{(\Phi-\Phi_1)^{k-j-1}}\varphi_1' dt_1 \int_0^1 \underline{\Phi_1^{j-1}}\underline{(1-v)^{j-i-1}}\underline{v^{i-1}} dv \quad (\text{if } \Phi_2 = \Phi_1 v)$$

$$= |\underline{n} \int \underline{(1-\Phi)^{n-k}}\varphi t dt \int^{t} \underline{(\Phi-\Phi_1)^{k-j-1}}\underline{\Phi_1^{j-1}}\varphi_1 t_1 dt_1 = E(t_{j|n} t_{k|n}) .$$

Also, on interchanging the order of integrations, we have

$$(\eta) = |\underline{n} \int \underline{(1-\Phi)^{n-k}}\varphi' dt \int^{t} \underline{\Phi_2^{i-1}}\varphi_2' dt_2 \int_{t_2}^{t} \underline{(\Phi-\Phi_1)^{k-j-1}}\underline{(\Phi_1-\Phi_2)^{j-i-1}}\varphi_1 dt_1$$

and

$$(\zeta) = |\underline{n} \int \varphi_1' dt_1 \int_{t_1} \underline{(1-\Phi)^{n-k}}\underline{(\Phi-\Phi_1)^{k-j-1}}\varphi dt \int^{t_1} \underline{(\Phi_1-\Phi_2)^{j-i-1}}\underline{\Phi_2^{i-1}}\varphi_2' dt_2 .$$

Hence, if we put $\Phi-\Phi_1=(\Phi-\Phi_2)v$ in the former, and $1-\Phi=(1-\Phi_1)v$ in the latter, they reduce to $E(t_{i|n}t_{k|n})$ and $E(t_{j|n}t_{k|n})$, respectively, Q.E.D.

From the foregoing and (5.10), Part I, it follows that

$$\sum_{k=1}^{n}\sum_{l=1}^{n} E(t_{i|n}t_{j|n}t_{k|n}t_{l|n}) = nE(t_{i|n}t_{j|n})+2 , \tag{16.9.1}$$

and

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} E(t_{i|n}t_{j|n}t_{k|n}t_{l|n}) = 3n . \tag{16.9.2}$$

Remark. If we concern about the fourth moments with genuine 4 arguments, as seen at (5.9), Part I, we know that

$$\sum_{l=4}^{n}\sum_{k=3}^{l-1}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1} E(t_{i|n}t_{j|n}t_{k|n}t_{l|n}) = 0 . \tag{16.10}$$

4° Lastly we ought to extend the symmetric property (5.1) : $E(t^p_{n-i+1|n}t^q_{n-k+1|n}) =(-1)^{p+q}E(t^p_{i|n}t^q_{k|n})$ in order to save the labour to calculate $E(t^p_{i|n}t^q_{j|n}t^r_{k|n})$ for all combinations of i, j, k. In fact, we obtain a similar formula[9]:

$$E(t^p_{n-i+1|n}t^q_{n-j+1|n}t^r_{n-k+1|n}) = (-1)^{p+q+r}E(t^p_{i|n}t^q_{j|n}t^r_{k|n}) . \tag{16.11}$$

Proof. On assuming $i<j<k$ and considering Φ, Φ_1, Φ_2 as independent variables, and t, t_1, t_2 as functions of Φ, Φ_1, Φ_2 respectively, all being monotonely increasing, we have

$$E(t^p_{i|n}t^q_{j|n}t^r_{k|n}) = \delta_{ijk|n}\int_0^1 t^r\underline{(1-\Phi)^{n-k}}d\Phi\int_0^{\Phi} t_1^q\underline{(\Phi-\Phi_1)^{k-j-1}}d\Phi_1\int_0^{\Phi_1} t_2^p\underline{(\Phi_1-\Phi_2)^{j-i-1}}\underline{\Phi_2^{i-1}}d\Phi_2 \qquad \cdots\cdots(\text{i})$$

where $\delta_{ijk|n}=n!/(n-k)!\,(k-j-1)!\,(j-i-1)!\,(i-1)!$. But, for $i'=n-i+1$, $j'=n-j+1$, $k'=n-1$, it follows that $k'<j'<i'$ and $n-i'=i-1$, $j'-k'=k-j$, $i'-j'=j-i$, $k'-1=n-k$ and therefore $\delta_{k'j'i'|n}=\delta_{ijk|n}$. And

9) Of course, if $n=3$ and $p=q=r$, we have only one $E(t^p_{1|3}t^p_{2|3}t^p_{3|3})$, because here the problem concerns with combination but not permutation, so that $E(t_3{}^pt_2{}^pt_1{}^p)$ &c., is nothing else $E(t_1{}^pt_2{}^pt_3{}^p)$, and (16.11) is of no use.

$$E(t^p_{i'|n}t^q_{j'|n}t^r_{k'|n})=\delta_{k'j'i'|n}\int_0^1 t^p\underline{(1-\Phi)^{n-i'}}d\Phi\int_0^{\Phi}t_1^q\underline{(\Phi-\Phi_1)^{i'-j'-1}}d\Phi_1$$
$$\times\int_0^{\Phi_1}t_2^r\underline{(\Phi_1-\Phi_2)^{j'-k'-1}}\Phi_2^{k'-1}d\Phi_2$$
$$=\delta_{ijk|n}\int_0^1 t^p\underline{(1-\Phi)^{i-1}}d\Phi\int_0^{\Phi}t_1^q\underline{(\Phi-\Phi_1)^{j-i-1}}d\Phi_1\int_0^{\Phi_1}t_2^r\underline{(\Phi_1-\Phi_2)^{k-i-1}}\Phi_2^{n-k}d\Phi_2. \quad \cdots\cdots(\text{ii})$$

Hence we have to prove that the above two integrals (i) and (ii) are equal. Now, on interchanging the order of integrations in (ii), it yields

$$\int_0^1 t_1^q d\Phi_1\int_{\Phi_1}^1 t^p\underline{(1-\Phi)^{i-1}}\underline{(\Phi-\Phi_1)^{j-i-1}}d\Phi\int_1^{\Phi_1}t_2^r\underline{(\Phi_1-\Phi_2)^{k-j-1}}\underline{\Phi_2^{n-k}}d\Phi_2$$
$$=\int_0^1 t_1^q d\Phi_1\int_0^{\Phi_1}t_2^r\underline{(\Phi_1-\Phi_2)^{k-j-1}}\underline{\Phi_2^{n-k}}d\Phi_2\int_{\Phi_1}^1 t^p\underline{(1-\Phi)^{i-1}}\underline{(\Phi-\Phi_1)^{j-i-1}}d\Phi$$
$$=\int_0^1 t_2^r\underline{\Phi_2^{n-k}}d\Phi_2\int_{\Phi_2}^1 t_1^q\underline{(\Phi_1-\Phi_2)^{k-j-1}}d\Phi_1\int_{\Phi_1}^1 t^p\underline{(\Phi-\Phi_1)^{j-i-1}}\underline{(1-\Phi)^{i-1}}d\Phi.$$

Instead of the old independent variables Φ, Φ_1, Φ_2 with $\Phi_2<\Phi_1<\Phi$, we take, as new variables $\Psi_2=1-\Phi$, $\Psi_1=1-\Phi_1$, $\Psi=1-\Phi_2$ with the order $\Psi_2<\Psi_1<\Psi$. Every function $t=t(\Phi)$, $t_1=t_1(\Phi_1)$, $t_2=t_2(\Phi_2)$ being each monotonely increasing the functions $T_2=-t$, $T_1=-t_1$, $T_2=-t_2$ are also all monotonely decreasing, in regard to Φ, Φ_1, Φ_2 respectively; or they are all monotonely increasing in regard to new independent variables Ψ_2, Ψ_1, Ψ and vice versa. Hence, the last triple integral becomes

$$(-1)^{p+q+r}\int_0^1 T^r\underline{(1-\Psi)^{n-k}}d\Psi\int_0^{\Psi}T_1^q\underline{(\Psi-\Psi_1)^{k-j-1}}d\Psi_1\int_0^{\Psi_1}T_2^p\underline{(\Psi_1-\Psi_2)^{j-i-1}}\underline{\Psi_2^{i-1}}d\Psi_2,$$

which integral is equal to (i). Therefore, we have, as asserted,

$$E(t^p_{i'|n}t^q_{j'|n}t^r_{k'|n})=(-1)^{p+q+r}E(t^p_{i|n}t^q_{j|n}t^r_{k|n}), \qquad \text{Q.E.D.}$$

§ **17. Computations of J^{α}_{λ}, $J^{\alpha\beta}_{\mu\nu}$ and $J^{\alpha\beta\gamma}_{\lambda\mu\nu}$.** We had already computed J^{α}_{λ} and $J^{\alpha\beta}_{\mu\nu}$ generally in Part II, so that now we have only to tabulate those necessary for calculations of the third and fourth moments.

1° It is required to express

$$E(t^3_{i|n})=\frac{5}{2}E(t_{i|n})+\frac{|n}{6}\int D^3[\Phi^{i-1}\underline{(1-\Phi)^{n-i}}]\varphi^4 dt, \tag{2.7}$$

$$E(t^4_{i|n})=\frac{13}{3}E(t^2_{i|n})-\frac{4}{3}+\frac{|n}{24}\int D^4[\Phi^{i-1}\underline{(1-\Phi)^{n-i}}]\varphi^5 dt \tag{2.8}$$

by $J^{\alpha}_{\lambda}=\int\Phi^{\lambda}\varphi^{\alpha}dt$. We had already obtained (7.4) (7.5) (7.10) and (7.11) in Part II: i.e.

$$J^{\alpha}_0=c_{\alpha}=\frac{1}{\sqrt{\alpha}\sqrt{2\pi}^{\alpha-1}},\ J^{\alpha}_1=\frac{1}{2}c_{\alpha},\ J^{\alpha}_2=\frac{c_{\alpha}}{2}[1-2S(\alpha)],\ J^{\alpha}_3=\frac{c_{\alpha}}{2}[1-3S(\alpha)],\ \&\text{c}.$$

These values are given in the following

Table VI Values of J_λ^α and Allied Constants

α \ λ	$\lambda=0$	$\lambda=1$	$\lambda=2$	$\lambda=3$	Allied constants
$\alpha=1$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$S(2)=0.19591\ 328$
$\alpha=2$	$\frac{1}{2\sqrt{\pi}}=0.28209\ 47918$	$\frac{1}{4\sqrt{\pi}}$	$\frac{1}{4\sqrt{\pi}}[1-2S(2)]$	$\frac{1}{4\sqrt{\pi}}[1-3S(2)]$	$S(3)=0.20978\ 469$
$\alpha=3$	$\frac{1}{2\pi\sqrt{3}}=0.09188\ 81492$	$\frac{1}{4\pi\sqrt{3}}$	$\frac{1}{4\pi\sqrt{3}}[1-2S(3)]$	$\frac{1}{4\pi\sqrt{3}}[1-3S(3)]$	$S(4)=0.21795\ 288$
$\alpha=4$	$\frac{1}{4\pi\sqrt{2\pi}}=0.03174\ 68180$	$\frac{1}{8\pi\sqrt{2\pi}}$	$\frac{1}{8\pi\sqrt{2\pi}}[1-2S(4)]$	$\frac{1}{8\pi\sqrt{2\pi}}[1-3S(4)]$	$S(5)=0.22334\ 979$
$\alpha=5$	$\frac{1}{4\pi^2\sqrt{5}}=0.01132\ 80527$	$\frac{1}{8\pi^2\sqrt{5}}$	$\frac{1}{8\pi^2\sqrt{5}}[1-2S(5)]$	$\frac{1}{8\pi^2\sqrt{5}}[1-3S(5)]$	

2° To computate $E(t_{i|n}^p t_{k|n}^q)$ for $p+q=3$ or 4 and $n\leqq 5$, it needs to know the values of the following

(17.1) $J_{00}^{\alpha\beta}=\frac{1}{2}c_\alpha c_\beta$ with $c_\alpha=\frac{1}{\sqrt{\alpha}\sqrt{2\pi}^{\alpha-1}}$ by (8.6) in Part II;

(17.2) $J_{10}^{\alpha\beta}=J_{00}^{\alpha\beta}[1-S(p-2)-S(q)]$ by (8.9), also cf. (17.7) below;

(17.3) $J_{01}^{\alpha\beta}=J_{00}^{\alpha\beta}-J_{10}^{\beta\alpha}$ by (8.10);

(17.4) $J_{11}^{\alpha\beta}=\frac{1}{4}J_{00}^{\alpha\beta}=\frac{1}{8}c_\alpha c_\beta=J_{11}^{\beta\alpha}$ by (8.12);

(17.5) $J_{20}^{\alpha\beta}=J_{10}^{\alpha\beta}-J_{00}^{\alpha\beta}S(\alpha)$ by (8.14) and (8.9);

(17.6) $J_{02}^{\alpha\beta}=J_{01}^{\alpha\beta}-J_{00}^{\alpha\beta}S(\beta)$ by (8.16) and (8.10).

In fact we have by (8.9)

$$J_{10}^{\alpha\beta}=\frac{1}{2}c_\alpha c_\beta\left[\frac{1}{2}+\frac{1}{\pi}\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}-\frac{1}{\pi}\sin^{-1}\sqrt{\frac{\alpha}{(\alpha+\beta)(\beta+1)}}\right]$$
$$=J_{00}^{\alpha\beta}\left[\frac{1}{2}+S\left(\frac{2(\alpha+\beta+1)}{\alpha\beta-(\alpha+\beta+1)}\right)-S\left(\frac{2\alpha}{(\alpha+\beta+1)\beta-\alpha}\right)\right].$$

However, the second term in the last brackets becomes of a negative argument $-p$, because $\alpha,\ \beta\geqq 1$, $\alpha+\beta+1>\alpha\beta$, if one of α, β be 1 or if $\alpha=\beta\leqq 2$, so that we must contrive to get rid of this inconvenience: Really, since

$$S(-p)+S(p-2)=\frac{1}{2\pi}[\sec^{-1}(1-p)+\sec^{-1}(p-1)]=\frac{1}{2},$$

i.e.

$$\frac{1}{2}+S(-p)=1-S(p-2),$$

we may write the above $J_{00}^{\alpha\beta}$ as[10)]

10) If $\alpha,\ \beta=2,3$, then $\alpha+\beta+1-\alpha\beta=0$ and $p=\frac{2(\alpha+\beta+1)}{\alpha+\beta+1-\alpha\beta}$ becomes ∞, so that $S(\infty)=\frac{1}{4}$ (cf. p. 61, Part. I). We get therefore $J_{1,0}^{2,3}=J_{00}^{23}\left[\frac{3}{4}-S\left(\frac{1}{4}\right)\right]$ and $J_{10}^{32}=J_{00}^{32}\left[\frac{3}{4}-S\left(\frac{2}{3}\right)\right]$. But $S\left(\frac{1}{4}\right)=\frac{1}{\pi}\sin^{-1}\frac{1}{\sqrt{10}}$ and $S\left(\frac{2}{3}\right)=\frac{1}{\pi}\sin^{-1}\frac{1}{\sqrt{5}}=\frac{1}{4}-S\left(\frac{1}{4}\right)$. Hence $J_{10}^{32}=J_{00}^{32}\left[\frac{1}{2}+S\left(\frac{1}{4}\right)\right]$ (Table VII).

(17.7) $J_{10}^{\alpha\beta}=J_{00}^{\alpha\beta}[1-S(p-2)-S(q)]$, where $p=\dfrac{2(\alpha+\beta+1)}{\alpha+\beta+1-\alpha\beta}$, $q=\dfrac{2\alpha}{(\alpha+\beta+1)\beta-\alpha}$.

Furthermore, by reasons of complementary angles, we have $S\left(\frac{2}{7}\right)=\frac{1}{2}-2S(2)$, $S\left(\frac{1}{7}\right)=\frac{1}{2}-2S(3)$, and $S(8)=\left(\frac{1}{2}\right)-2S\left(\frac{1}{2}\right)$. Using these relations, we get the following

Table VII Values of $J_{\mu\nu}^{\alpha\beta}$

α	β	$J_{00}^{\alpha\beta}$	$J_{10}^{\alpha\beta}$	$J_{01}^{\alpha\beta}$	$J_{11}^{\alpha\beta}$	$J_{20}^{\alpha\beta}$	$J_{02}^{\alpha\beta}$
1	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{12}$
1	2	$\frac{1}{4\sqrt{\pi}}$	$\frac{1}{4\sqrt{\pi}}\left[\frac{1}{2}+S(2)\right]$	$\frac{1}{2\sqrt{\pi}}S(2)$	$\frac{1}{16\sqrt{\pi}}$	$\frac{1}{4\sqrt{\pi}}\left[\frac{1}{3}+S(2)\right]$	$\frac{1}{4\sqrt{\pi}}S(2)$
2	1		$\frac{1}{4\sqrt{\pi}}[1-2S(2)]$	$\frac{1}{4\sqrt{\pi}}\left[\frac{1}{2}-S(2)\right]$		$\frac{1}{4\sqrt{\pi}}[1-3S(2)]$	$\frac{1}{4\sqrt{\pi}}\left[\frac{1}{3}-S(2)\right]$
2	2	$\frac{1}{8\pi}$	$\frac{1}{8\pi}\left[\frac{1}{2}+S\left(\frac{1}{2}\right)\right]$	$\frac{1}{8\pi}\left[\frac{1}{2}-S\left(\frac{1}{2}\right)\right]$	$\frac{1}{32\pi}$	$\frac{1}{8\pi}\left[\frac{1}{2}+S\left(\frac{1}{2}\right)-S(2)\right]$	$\frac{1}{8\pi}\left[\frac{1}{2}-S\left(\frac{1}{2}\right)-S(2)\right]$
1	3	$\frac{1}{4\pi\sqrt{3}}$	$\frac{1}{4\pi\sqrt{3}}\left[\frac{1}{2}+S(3)\right]$	$\frac{1}{2\pi\sqrt{3}}S(3)$	$\frac{1}{16\pi\sqrt{3}}$	$\frac{1}{4\pi\sqrt{3}}\left[\frac{1}{3}+S(3)\right]$	$\frac{1}{4\pi\sqrt{3}}S(3)$
3	1		$\frac{1}{4\pi\sqrt{3}}[1-2S(3)]$	$\frac{1}{4\pi\sqrt{3}}\left[\frac{1}{2}-S(3)\right]$		$\frac{1}{4\pi\sqrt{3}}[1-3S(3)]$	$\frac{1}{4\pi\sqrt{3}}\left[\frac{1}{3}-S(3)\right]$
2	3	$\frac{1}{8\pi\sqrt{3\pi}}$	$\frac{1}{8\pi\sqrt{3\pi}}\left[\frac{3}{4}-S\left(\frac{1}{4}\right)\right]$	$\frac{1}{8\pi\sqrt{3\pi}}\left[\frac{1}{2}-S\left(\frac{1}{4}\right)\right]$	$\frac{1}{32\pi\sqrt{3\pi}}$	and so on.	
3	2		$\frac{1}{8\pi\sqrt{3\pi}}\left[\frac{1}{2}+S\left(\frac{1}{4}\right)\right]$	$\frac{1}{8\pi\sqrt{3\pi}}\left[\frac{1}{4}+S\left(\frac{1}{4}\right)\right]$			

3° Finally we touch the triple integral

(17.8) $$J_{\lambda\mu\nu}^{\alpha\beta\gamma}=\int\Phi^{\lambda}\varphi^{\alpha}dt\int^{t}\Phi_1^{\mu}\varphi_1^{\beta}dt_1\int^{t_1}\Phi_2^{\nu}\varphi_2^{\gamma}dt_2\,.$$

Or, on putting $U=\int_0^t\varphi dt$ &c.,

$$J_{\lambda\mu\nu}^{\alpha\beta\gamma}=\int\left(U+\frac{1}{2}\right)^{\lambda}\varphi^{\alpha}dt\int^{t}\left(U_1+\frac{1}{2}\right)^{\mu}\varphi_1^{\beta}dt_1\int^{t_1}\left(U_2+\frac{1}{2}\right)^{\nu}\varphi_2^{\gamma}dt_2$$
$$=\sum_{l=0}^{\lambda}\sum_{m=0}^{\mu}\sum_{n=0}^{\nu}\binom{\lambda}{l}\binom{\mu}{m}\binom{\nu}{n}\left(\frac{1}{2}\right)^{\lambda+\mu+\nu-l-m-n}K_{lmn}^{\alpha\beta\gamma},$$

where

(17.9) $$K_{lmn}^{\alpha\beta\gamma}=\int U^{l}\varphi^{\alpha}dt\int^{t}U_1^{m}\varphi_1^{\beta}dt_1\int^{t_1}U_2^{n}\varphi_2^{\gamma}dt_2\,.$$

Thus, to compute $J_{\lambda\mu\nu}^{\alpha\beta\gamma}$ it is sufficient to know $K_{lmn}^{\alpha\beta\gamma}$ for $l\leqq\lambda$, $m\leqq\mu$, $n\leqq\nu$.

However, the said integrals, especially with positive λ, μ, ν, seldom take place for smaller valves of n, so that we shall consider only the special case

(17.10) $$K_{000}^{\alpha\beta\gamma}=J_{000}^{\alpha\beta\gamma}=\int\varphi^{\alpha}dt\int^{t}\varphi_1^{\beta}dt_1\int^{t_1}\varphi_2^{\gamma}dt_2\,.$$

We express it as the sum of three triple integrals I, II, III corresponding to (i) $t>0$, $t_1<0$, (ii) $t>0$, $t_1>0$ and (iii) $t<0$, respectively. Using polar co-ordinates,

$$
\begin{aligned}
\mathrm{I} &= \int_0^\infty \varphi^\alpha dt \int_{-\infty}^0 \varphi_1^\beta dt_1 \int^{t_1} \varphi_2^\gamma dt_2 \\
&= \frac{1}{\sqrt{2\pi}^{\alpha+\beta+\gamma}} \int_0^\infty \exp\left\{-\frac{\alpha}{2}t^2\right\} dt \int_{-3\pi/4}^{-\pi/2} d\theta \int_0^\infty \exp\left\{-\frac{\rho^2}{2}(\beta\cos^2\theta+\gamma\sin^2\theta)\right\}\rho d\rho \\
&= \frac{1}{\sqrt{2\pi}^{\alpha+\beta+\gamma}} \frac{\sqrt{2\pi}}{2\sqrt{\alpha}} \int_{-3\pi/4}^{-\pi/2} \frac{d\theta}{\beta\cos^2\theta+\gamma\sin^2\theta} = \frac{c_\alpha c_\beta c_\gamma}{4\pi} \int_1^\infty \frac{d\tau}{\beta+\gamma\tau^2} = \frac{c_\alpha c_\beta c_\gamma}{4\pi}\left[\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{\gamma}{\beta}}\right];
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{II} &= \int_0^\infty \varphi^\alpha dt \int_0^t \varphi_1^\beta dt_1 \int^{t_1} \varphi_2^\gamma dt_2 \\
&= \frac{1}{\sqrt{2\pi}^{\alpha+\beta+\gamma}} \int_0^\infty \exp\left\{-\frac{\alpha}{2}t^2\right\} dt \int_{-\pi/2}^{\pi/4} d\theta \int_0^{t\sec\theta} \exp\left\{-\frac{\rho^2}{2}(\beta\cos^2\theta+\gamma\sin^2\theta)\right\}\rho d\rho \\
&= \frac{c_\alpha c_\beta c_\gamma\sqrt{\alpha\beta\gamma}}{\sqrt{2\pi}^3} \int_0^\infty \exp\left\{-\frac{\alpha}{2}t^2\right\} dt \int_{-\pi/2}^{\pi/4} \frac{d\theta}{\beta\cos^2\theta+\gamma\sin^2\theta}\left[1-\exp\left\{-\frac{t^2}{2}(\beta+\gamma\tan^2\theta)\right\}\right] \\
&= \frac{c_\alpha c_\beta c_\gamma\sqrt{\alpha\beta\gamma}}{4\pi}\left[\frac{1}{\sqrt{\alpha}}\int_{-\infty}^1 \frac{d\tau}{\beta+\gamma\tau^2} - \int_{-\infty}^1 \frac{d\tau}{(\beta+\gamma\tau^2)\sqrt{\alpha+\beta+\gamma\tau^2}}\right] \qquad (\tau=\tan\theta) \\
&= \frac{c_\alpha c_\beta c_\gamma}{4\pi}\left[\tan^{-1}\sqrt{\frac{\gamma}{\beta}} + \frac{\pi}{2} - \sin^{-1}\frac{\sqrt{\alpha\gamma}\tau}{\sqrt{(\beta+\gamma\tau^2)(\beta+\gamma\tau^2)}}\Bigg|_{-\infty}^1\right] \\
&= \frac{c_\alpha c_\beta c_\gamma}{4\pi}\left[\tan^{-1}\sqrt{\frac{\gamma}{\beta}} + \frac{\pi}{2} - \sin^{-1}\sqrt{\frac{\alpha\gamma}{(\alpha+\beta)(\beta+\gamma)}} - \sin^{-1}\sqrt{\frac{\alpha}{\alpha+\beta}}\right];
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{III} &= \int_{-\infty}^0 \varphi^\alpha dt \int^t \varphi_1^\beta dt_1 \int_0^{t_1} \varphi_2^\gamma dt_2 \\
&= \frac{1}{\sqrt{2\pi}^{\alpha+\beta+\gamma}} \int_{-\infty}^0 \exp\left\{-\frac{\alpha}{2}t^2\right\} dt \int_{-3\pi/4}^{-\pi/2} d\theta \int_{t\sec\theta}^\infty \exp\left\{-\frac{\rho^2}{2}(\beta\cos^2\theta+\gamma\sin^2\theta\right\}\rho d\rho \\
&= \frac{c_\alpha c_\beta c_\gamma\sqrt{\alpha\beta\gamma}}{\sqrt{2\pi}^3} \int_{-\infty}^0 \exp\left\{-\frac{\alpha}{2}t^2\right\} dt \int_{-3\pi/4}^{-\pi/2} \frac{d\theta}{\beta\cos^2\theta+\gamma\sin^2\theta}\cdot\exp\left\{-\frac{t^2}{2}(\beta+\gamma\tan^2\theta)\right\} \\
&= \frac{c_\alpha c_\beta c_\gamma\sqrt{\alpha\beta\gamma}}{4\pi} \int_1^\infty \frac{d\tau}{(\beta+\gamma\tau^2)\sqrt{\alpha+\beta+\gamma\tau^2}} \qquad (\tau=\tan\theta) \\
&= \frac{c_\alpha c_\beta c_\gamma}{4\pi}\sin^{-1}\sqrt{\frac{\alpha\gamma\tau^2}{(\alpha+\beta)(\beta+\gamma\tau^2)}}\Bigg|_1^\infty = \frac{c_\alpha c_\beta c_\gamma}{4\pi}\left[\sin^{-1}\sqrt{\frac{\alpha}{\alpha+\beta}} - \sin^{-1}\sqrt{\frac{\alpha\gamma}{(\alpha+\beta)(\beta+\gamma)}}\right].
\end{aligned}
$$

Therefore

$$
\mathrm{I}+\mathrm{II}+\mathrm{III} = \frac{c_\alpha c_\beta c_\gamma}{4\pi}\left[\pi - 2\sin^{-1}\sqrt{\frac{\alpha\gamma}{(\alpha+\beta)(\beta+\gamma)}}\right].
$$

Thus we obtain

(17. 11) $$J_{000}^{\alpha\beta\gamma} = \frac{1}{2}c_\alpha c_\beta c_\gamma\left[\frac{1}{2} - S\left(\frac{2\alpha\gamma}{(\alpha+\beta+\gamma)\beta-\alpha\gamma}\right)\right].$$

In particular

(17. 11. 1) $$J_{000}^{\alpha\alpha\alpha} = \frac{1}{2}c_\alpha^3\left[\frac{1}{2} - S(1)\right] = \frac{1}{6}c_\alpha^3.$$

E.g. if $\alpha=\beta=\gamma=1$, $J_{000}^{111}=\frac{1}{6}$ which agrees with the fundamental relation $3!\int\varphi dt\int^{t}\varphi_1 dt_1\int^{t_1}\varphi_2 dt_2=1.$[11]

§18. **Calculations of the Third and Fourth Moments.** By making use of the formulas for $E(t_{i|n}^p)$, $E(t_{i|n}^p t_{k|n}^q)$ and $E(t_{i|n}^p t_{j|n}^q t_{k|n}^r)$ and the foregoing values of J_λ^α, $J_{\mu\nu}^{\alpha\beta}$ and $J_{\lambda\mu\nu}^{\alpha\beta\gamma}$, we have obtained the third and fourth moments for $n\leqq 5$, as in the following Tables VIII–XIV:

Table VIII Values of $E(t_{i|n}^3)$ and $E(t_{i|n}^4)$

n	i	$n-i+1$	$E(t_{i\|n}^3)=-E(t_{n-i+1\|n}^3)$	$E(t_{i\|n}^4)=E(t_{n-i+1\|n}^4)$
2	2	1	$\frac{5}{2\sqrt{\pi}}=1.4104\ 774$	3
3	3	1	$\frac{15}{4\sqrt{\pi}}=2.1157\ 109$	$3+\frac{13}{2\sqrt{3}\pi}=4.1945\ 459$
	2		0	$3-\frac{13}{\sqrt{3}\pi}=0.6109\ 081$
4	4	1	$\frac{15}{2\sqrt{\pi}}[1-2S(2)]+\frac{1}{\pi\sqrt{2\pi}}=2.7004\ 257$	$3+\frac{13}{\sqrt{3}\pi}=5.3890\ 919$
	3	2	$\frac{15}{2\sqrt{\pi}}[-1+6S(2)]-\frac{3}{\pi\sqrt{2\pi}}=0.3615\ 667$	$3-\frac{13}{\sqrt{3}\pi}=0.6109\ 081$
5	5	1	$\frac{25}{2\sqrt{\pi}}[1-3S(2)]+\frac{5}{2\pi\sqrt{2\pi}}=3.2248\ 794$	$3+\frac{65}{2\sqrt{3}\pi}[1-2S(3)]+\frac{\sqrt{5}}{4\pi^2}=6.5233\ 955$
	4	2	$\frac{25}{2\sqrt{\pi}}[-1+6S(2)]-\frac{5}{\pi\sqrt{2\pi}}=0.6026\ 111$	$3+\frac{65}{\sqrt{3}\pi}[-1+4S(3)]-\frac{\sqrt{5}}{\pi^2}=0.8518\ 775$
	3		0	$3+\frac{65}{\sqrt{3}\pi}[1-6S(3)]+\frac{3\sqrt{5}}{2\pi^2}=0.2494\ 541$

Checks: E.g. $\sum_{i=1}^{5}E(t_{i|5}^4)=15(=15.00000\ 01$ as sum of numerical values).

---

11) Although for the presently restricted case that $n\leqq 5$ it suffices with all the above formulas, if e.g. $n=7$, $i=2$ in (15.1.2), we need to calculate $7!\int(1-\Phi)\varphi^2 dt\int_{t_0}^{t}\varphi_1^2 dt_1\int^{t_1}\varphi_2^2 dt_2$, whose integration is somewhat intricate. Really to find $K_{100}^{222}=\int U\varphi^2 dt\int^{t}\varphi_1^2 dt_1\int^{t_1}\varphi_2^2 dt_2$, treating similarly as in (17.10), we get easily $I=\frac{1}{128\pi^2\sqrt{\pi}}\tan^{-1}\frac{1}{\sqrt{2}}$. However $II=\frac{1}{8\pi^3\sqrt{2\pi}}\int_0^\infty e^{-t^2}dt\int_0^t e^{-\tau^2/2}d\tau\int_{-\pi/2}^{\pi/4}d\theta\int_0^{t\sec\theta}e^{-\rho^2}\rho d\rho$ $=\frac{1}{16\pi^3\sqrt{2\pi}}\int_{-\pi/2}^{\pi/4}d\theta\int_{\pi/4}^{\pi/2}d\psi\int_0^\infty\Big[\exp\Big\{-\frac{r^2}{2}(2\sin^2\psi+\cos^2\psi)\Big\}-\exp\Big\{-r^2(2+\sec^2\theta)\sin^2\psi-\frac{r^2}{2}\cos^2\psi\Big\}\Big]rdr=II_1-II_2$, where II_1 becomes $\frac{3}{128\pi^2\sqrt{\pi}}\tan^{-1}\frac{1}{\sqrt{2}}$, while $II_2=\frac{1}{16\pi^3\sqrt{2\pi}}\int_{-\pi/2}^{-\pi/4}\frac{\cot^{-1}\sqrt{2(2+\sec^2\theta)}}{\sqrt{2(2+\sec^2\theta)}}d\theta$ seems to be numerically integrated. Also $III=\frac{1}{16\pi^3\sqrt{2\pi}}\int_{-3\pi/4}^{-\pi/2}\frac{\cot^{-1}\sqrt{2(1+\sec^2\theta)}}{\sqrt{2(1+\sec^2\theta)}}d\theta$ requires a numerical integration.

Table IX Values of $E(t_{i|n}^2 t_{k|n})$ $(i \neq k)$

n	$t_i^2 t_k$	$t_{n-k-1} t_{n-i+1}^2$	$E(t_{i\|n}^2 t_{k\|n}) = -E(t_{n-k+1} t_{n-i+1}^2)$
2	$t_1^2 t_2$	$t_1 t_2^2$	$\frac{1}{2\sqrt{\pi}} = 0.28209\ 48$
3	$t_1^2 t_2$	$t_2 t_3^2$	$-\frac{3}{4\sqrt{\pi}} = -0.42314\ 22$
	$t_1^2 t_3$	$t_1 t_3^2$	$\frac{3}{2\sqrt{\pi}} = 0.84628\ 44$
	$t_2^2 t_3$	$t_1 t_2^2$	$\frac{3}{4\sqrt{\pi}} = 0.42314\ 22$
4	$t_1^2 t_2$	$t_3 t_4^2$	$-\frac{1}{\pi\sqrt{2\pi}} + \frac{3}{\sqrt{\pi}}\left[\frac{1}{2} - 5S(2)\right] = -0.93868\ 63$
	$t_1^2 t_3$	$t_2 t_4^2$	$-\frac{1}{\pi\sqrt{2\pi}} + \frac{3}{\sqrt{\pi}}[-1+6S(2)] = 0.17002\ 41$
	$t_1^2 t_4$	$t_1 t_4^2$	$\frac{3}{\pi\sqrt{2\pi}} + \frac{3}{\sqrt{\pi}}[1-2S(2)] = 1.41033\ 72$
	$t_2^2 t_3$	$t_2 t_3^2$	$\frac{3}{\pi\sqrt{2\pi}} + \frac{3}{\sqrt{\pi}}[-1+4S(2)] = 0.01477\ 98$
	$t_2^2 t_4$	$t_1 t_3^2$	$-\frac{5}{\pi\sqrt{2\pi}} + \frac{3}{\sqrt{\pi}}[1-2S(2)] = 0.39443\ 90$
	$t_3^2 t_4$	$t_1 t_2^2$	$\frac{1}{\pi\sqrt{2\pi}} + \frac{3}{\sqrt{\pi}}\left[\frac{1}{2} - S(2)\right] = 0.64167\ 50$
5	$t_1^2 t_2$	$t_4 t_5^2$	$-\frac{5}{2\pi\sqrt{2\pi}} + \frac{5}{\sqrt{\pi}}\left[\frac{1}{2} - \frac{9}{2}S(2)\right] = -1.39396\ 94$
	$t_1^2 t_3$	$t_3 t_5^2$	$-\frac{5}{2\pi\sqrt{2\pi}} = -0.31746\ 82$
	$t_1^2 t_4$	$t_2 t_5^2$	$\frac{15}{2\pi\sqrt{2\pi}} - \frac{15}{2\pi\sqrt{3\pi}} + \frac{5}{\sqrt{\pi}}[-1+6S(2)] = 0.66978\ 85$
	$t_1^2 t_5$	$t_1 t_5^2$	$\frac{15}{2\pi\sqrt{3\pi}} + \frac{5}{\sqrt{\pi}}[+1-3S(2)] = 1.94059\ 95$
	$t_2^2 t_3$	$t_3 t_4^2$	$\frac{5}{\pi\sqrt{2\pi}} + \frac{5}{\sqrt{\pi}}\left[0 - \frac{3}{2}S(2)\right] = -0.19405\ 54$
	$t_2^2 t_4$	$t_2 t_4^2$	$-\frac{15}{\pi\sqrt{2\pi}} + \frac{15}{\pi\sqrt{3\pi}} + \frac{5}{\sqrt{\pi}}[-1+6S(2)] = 0.14548\ 00$
	$t_2^2 t_5$	$t_1 t_4^2$	$\frac{15}{2\pi\sqrt{2\pi}} - \frac{15}{\pi\sqrt{3\pi}} + \frac{5}{\sqrt{\pi}}[1-3S(2)] = 0.56009\ 89$
	$t_3^2 t_4$	$t_2 t_3^2$	$\frac{10}{\pi\sqrt{2\pi}} - \frac{15}{2\pi\sqrt{3\pi}} + \frac{5}{\sqrt{\pi}}\left[-1+\frac{9}{2}S(2)\right] = 0.15826\ 49$
	$t_3^2 t_5$	$t_1 t_3^2$	$-\frac{25}{2\pi\sqrt{2\pi}} + \frac{15}{2\pi\sqrt{3\pi}} + \frac{5}{\sqrt{\pi}}[1-3S(2)] = 0.35325\ 86$
	$t_4^2 t_5$	$t_1 t_2^2$	$\frac{5}{2\pi\sqrt{2\pi}} + \frac{5}{\sqrt{\pi}}\left[\frac{1}{2} - \frac{3}{2}S(2)\right] = 0.89895\ 04$

Checks: E.g. $\sum_{k=1}^{3} E(t_{1|3}^2 t_{k|3}) = \frac{-3}{\sqrt{\pi}} = -1.6925688,\quad 2E(t_{1|3}) = -1.69256\ 88.$ (cf. Table III, Part I)

$$\sum_{k=1}^{4} E(t_{1|4}^2 t_{k|4}) = \frac{-6}{\sqrt{\pi}}[-2S(2)+1] = -2.05875\ 07,\quad 2E(t_{1|1}) = -2.05875\ 08.$$

$$\sum_{k=1}^{5} E(t_{1|5}^2 t_{k|5}) = 2\times\frac{5}{\sqrt{\pi}}[-1+3S(2) = -2.32592\ 89,\quad 2E(t_{1|5}) = -2.32592\ 90.$$

$$\sum_{k=1}^{5} E(t_{2|5}^2 t_{k|5}) = \frac{10}{\sqrt{\pi}}[1-6S(2) = -0.99003\ 84,\quad 2E(t_{2|5}) = -0.99003\ 80.$$

$$\sum_{k=1}^{5} E(t_{3|5}^2 t_{k|5}) = 0 = 0,\quad 2E(t_{3|5}) = 0.$$

Otherwise, e.g. $\sum_{k=1}^{5} E(t_{1|5} t_{k|5}^2) = 6E(t_{1|5}) = -\frac{30}{\sqrt{\pi}}[1-3S(2)] = -6.9777\ 870,$ &c.

Also $\sum_{k=2}^{n}\sum_{i=1}^{k-1} E(t_{i|n}^2 t_{k|n}) = \frac{n(n-1)}{4\sqrt{\pi}} = \frac{1}{2\sqrt{\pi}},\ \frac{3}{2\sqrt{\pi}},\ \frac{3}{\sqrt{\pi}},\ \frac{5}{\sqrt{\pi}}$ for $n=2, 3, 4, 5.$

Table X Values of $E(t^3_{i|n} t_{k|n})$ $(i \neq k)$

n	$t_j^3 t_k$	$t_{n-k+1} t^3_{n-i+1}$	$E(t^3_{i\|n} t_{k\|n}) = E(t_{n-i+1} t^3_{n-i+1})$
2	$t_1^3 t_2$	$t_1 t_2^3$	0
3	$t_1^3 t_2$	$t_2 t_3^3$	$\frac{5}{\sqrt{3}\pi} = 0.91888\ 15$
	$t_1^3 t_3$	$t_1 t_3^3$	$-\frac{7}{\sqrt{3}\pi} = -1.28643\ 41$
	$t_2^3 t_3$	$t_1 t_2^3$	$\frac{2}{\sqrt{3}\pi} = 0.36755\ 26$
4	$t_1^3 t_2$	$t_3 t_4^3$	$\frac{10}{\sqrt{3}\pi} = 1.83776\ 30$
	$t_1^3 t_3$	$t_2 t_4^3$	$-\frac{14}{\sqrt{3}\pi} + \frac{15}{2\pi} = -0.18554\ 40$
	$t_1^3 t_4$	$t_1 t_4^3$	$-\frac{15}{2\pi} = -2.38732\ 41$
	$t_2^3 t_3$	$t_2 t_3^3$	$\frac{14}{\sqrt{3}\pi} - \frac{15}{2\pi} = 0.18554\ 40$
	$t_2^3 t_4$	$t_1 t_3^3$	$-\frac{14}{\sqrt{3}\pi} + \frac{15}{2\pi} = -0.18554\ 40$
	$t_3^3 t_4$	$t_1 t_2^3$	$\frac{4}{\sqrt{3}\pi} = 0.73510\ 52$
5	$t_1^3 t_2$	$t_4 t_5^3$	$\frac{25}{\sqrt{3}\pi}[1-2S(3)] + \frac{\sqrt{5}}{4\pi^2} = 2.72337\ 51$
	$t_1^3 t_3$	$t_3 t_5^3$	$-\frac{35}{\sqrt{3}\pi}[1-2S(3)] + \frac{\sqrt{5}}{4\pi^2} + \frac{75}{2\pi}\left[\frac{1}{2} - S\left(\frac{1}{2}\right)\right] = 0.69368\ 31$
	$t_1^3 t_4$	$t_2 t_5^3$	$+\frac{\sqrt{5}}{4\pi^2} + \frac{75}{2\pi}\left[-\frac{1}{2} + 3S\left(\frac{1}{2}\right)\right] = -1.11815\ 35$
	$t_1^3 t_5$	$t_1 t_5^3$	$-\frac{\sqrt{5}}{\pi^2} - \frac{75}{\pi} S\left(\frac{1}{2}\right) = -3.42223\ 88$
	$t_2^3 t_3$	$t_3 t_4^3$	$\frac{10}{\sqrt{3}\pi}[1+8S(3)] - \frac{\sqrt{5}}{\pi^2} + \frac{75}{2\pi}\left[-\frac{1}{2} + S\left(\frac{1}{2}\right)\right] = 0.32500\ 67$
	$t_2^3 t_4$	$t_2 t_4^3$	$-\frac{140}{\sqrt{3}\pi} S(3) - \frac{\sqrt{5}}{\pi^2} + \frac{75}{\pi}\left[\frac{1}{2} - 2S\left(\frac{1}{2}\right)\right] = -0.07877\ 93$
	$t_2^3 t_5$	$t_1 t_4^3$	$+\frac{11\sqrt{5}}{4\pi^2} - \frac{75}{2\pi}\left[\frac{1}{2} - 3S\left(\frac{1}{2}\right)\right] = -0.55175\ 08$
	$t_3^3 t_4$	$t_2 t_3^3$	$\frac{5}{\sqrt{3}\pi}[5-2S(3)] + \frac{3\sqrt{5}}{2\pi^2} - \frac{75}{2\pi}\left[\frac{1}{2} - S\left(\frac{1}{2}\right)\right] = 0.17824\ 30$
	$t_3^3 t_5$	$t_1 t_3^3$	$-\frac{70}{\sqrt{3}\pi}\left[\frac{1}{2} - S(3)\right] - \frac{9\sqrt{5}}{4\pi^2} + \frac{75}{2\pi}\left[\frac{1}{2} - S\left(\frac{1}{2}\right)\right] = 0.12728\ 04$
	$t_4^3 t_5$	$t_1 t_2^3$	$\frac{10}{\sqrt{3}\pi}[1-2S(3)] + \frac{\sqrt{5}}{4\pi^2} = 1.12333\ 42$

Checks: E.g. $\sum_{k=1}^{3} E(t^3_{1|3} t_{k|3}) = 3\left(1 + \frac{\sqrt{3}}{2\pi}\right) = 3.82699\ 32 = 3E(t^2_{1|3})$, (cf. Table III, Part I)

$$\sum_{k=1}^{4} E(t^3_{1|4} t_{k|4}) = 3\left(1 + \frac{\sqrt{3}}{\pi}\right) = 4.65398\ 67 = 3E(t^2_{1|4}),$$

$$\sum_{k=1}^{4} E(t^3_{2|4} t_{k|4}) = 3\left(1 - \frac{\sqrt{3}}{\pi}\right) = 1.34601\ 33 = 3E(t^2_{2|4}),$$

$$\sum_{k=1}^{5} E(t^3_{1|5} t_{k|5}) = 3 + \frac{15\sqrt{3}}{2\pi}[1-2S(3)] = 5.40006\ 12 = 3E(t^2_{1|5}),$$

$$\sum_{k=1}^{5} E(t^3_{2|5} t_{k|5}) = 3 + \frac{15\sqrt{3}}{\pi}[-1+4S(3)] = 1.66968\ 81 = 3E(t^2_{2|5}),$$

otherwise $\sum_{k=1}^{5} E(t^3_{3|5} t_{k|5}) = 3 + \frac{15\sqrt{3}}{\pi}[1-6S(3)] = 0.86050\ 11 = 3E(t^2_{3|5}),$

$$\sum_{k=1}^{5} E(t_{1|5} t^3_{k|5}) = 3 + \frac{5\sqrt{3}}{2\pi}[1-2S(3)] = 2 + E(t^2_{1|5}) = 3.80002\ 04, \quad \&c,$$

Also $\sum_{k=2}^{n} \sum_{i=1}^{k-1} E(t^3_{i|n} t_{k|n}) = 0$ for $n = 2, 3, 4, 5.$

Table XI Values of $E(t^2_{i|n}\,t^2_{k|n})$ $(i<k)$

n	$t_i^2\,t_k^2$	$t^2_{n-k+1}\,t^2_{n-i+1}$	$E(t^2_{i\|n}\,t^2_{k\|n})=E(t^2_{n-k+1}\,t^2_{n-i+1})$
2	$t_1^2\,t_2^2$		1
3	$t_1^2\,t_2^2$	$t_2^2\,t_3^2$	$1-\frac{1}{\pi\sqrt{3}}=0.81622\ 37$
	$t_1^2\,t_3^2$		$1+\frac{2}{\pi\sqrt{3}}=1.36755\ 26$
4	$t_1^2\,t_2^2$	$t_3^2\,t_4^2$	$1+\frac{1}{\pi\sqrt{3}}=1.18377\ 63$
	$t_1^2\,t_3^2$	$t_2^2\,t_4^2$	$1-\frac{2}{\pi\sqrt{3}}=0.63244\ 74$
	$t_1^2\,t_4^2$		$1+\frac{6}{\pi\sqrt{3}}=2.10265\ 78$
	$t_2^2\,t_3^2$		$1-\frac{4}{\pi\sqrt{3}}=0.26489\ 48$
5	$t_1^2\,t_2^2$	$t_4^2\,t_5^2$	$1+\frac{5}{\pi\sqrt{3}}[-1+8S(3)]+\frac{\sqrt{5}}{4\pi^2}=1.67989\ 69$
	$t_1^2\,t_3^2$	$t_3^2\,t_5^2$	$1+\frac{5}{\pi\sqrt{3}}\left[\frac{7}{2}-19S(3)\right]+\frac{\sqrt{5}}{4\pi^2}=0.61014\ 74$
	$t_1^2\,t_4^2$	$t_2^2\,t_5^2$	$1+\frac{5}{\pi\sqrt{3}}\left[-\frac{3}{2}+9S(3)\right]-\frac{9\sqrt{5}}{4\pi^2}=0.84682\ 08$
	$t_1^2\,t_5^2$		$1+\frac{5}{\pi\sqrt{3}}[3-6S(3)]+\frac{3\sqrt{5}}{2\pi^2}=2.93988\ 25$
	$t_2^2\,t_3^2$	$t_3^2\,t_4^2$	$1+\frac{5}{\pi\sqrt{3}}\left[\frac{1}{2}-5S(3)\right]-\frac{\sqrt{5}}{\pi^2}=0.26904\ 34$
	$t_2^2\,t_4^2$		$1+\frac{5}{\pi\sqrt{3}}\left[-6+20S(3)+\right]\frac{4\sqrt{5}}{\pi^2}=0.24830\ 06$

Checks: E.g.

$$\sum_{k=1}^{4} E(t^2_{1|4}\,t^2_{k|4})=6\left[1+\frac{\sqrt{3}}{\pi}\right]=9.30797\ 35, \qquad 6E(t^2_{1|4})=9.30797\ 34.$$

$$\sum_{k=1}^{5} E(t^2_{1|5}\,t^2_{k|5})=\sum_{k=1}^{5} E(t^2_{k|5}\,t^2_{5|5})=7\left\{1+\frac{5\sqrt{3}}{\pi}\left[\frac{1}{2}-S(3)\right]\right\}=12.60014\ 33, \quad 7E(t^2_{1|5})=12.60014\ 28.$$

$$\sum_{k=1}^{5} E(t^2_{2|5}\,t^2_{k|5})=\sum_{k=1}^{5} E(t^2_{4|5}\,t^2_{k|5})=7\left\{1+\frac{15\sqrt{3}}{\pi}[-1+4S(3)]\right\}=3.89593\ 92, \quad 7E(t^2_{2|5})=3.89593\ 89.$$

$$\sum_{k=1}^{5} E(t^2_{3|5}\,t^2_{k|5})=7\left\{1+\frac{5\sqrt{3}}{\pi}[1-6S(3)]\right\}=2.00783\ 57, \qquad 7E(t^2_{3|5})=2.00783\ 59.$$

So

$$\sum_{k=1}^{5}\sum_{j=1}^{5} E(t^2_{j|5}\,t^2_{k|5})=35.00000\ 07\doteqdot 35, \qquad 7\sum_{i=1}^{5} E(t^2_{i|5})=34.99999\ 93\doteqdot 35.$$

Also

$$\sum_{k=2}^{n}\sum_{i=1}^{k-1} E(t^2_{i|n}\,t^2_{k|n})=\frac{1}{2}n(n-1)=1, 3, 6, 10 \qquad \text{for} \quad n=2, 3, 4, 5.$$

Table XII Values of $E(t_{i|n}\,t_{j|n}\,t_{k|n})$ $(i<j<k)$

n	$t_i\,t_j\,t_k$	$t_{n-k+1}\,t_{n-j+1}\,t_{n-i+1}$	$E(t_{i|n}\,t_{j|n}\,t_{k|n})=-E(t_{n-k+1|n}\,t_{n-j+1|n}\,t_{n-i+1|n})$
3	$t_1\,t_2\,t_3$	$t_2\,t_3\,t_4$	0
4	$t_1\,t_2\,t_3$	$t_1\,t_3\,t_4$	$\frac{-1}{\pi\sqrt{2\pi}}=-0.12698\ 73$
	$t_1\,t_2\,t_4$	$t_3\,t_4\,t_5$	$\frac{3}{\pi\sqrt{2\pi}}=0.38096\ 18$
5	$t_1\,t_2\,t_3$	$t_3\,t_4\,t_5$	$\frac{-5}{2\pi\sqrt{2\pi}}=-0.31746\ 82$
	$t_1\,t_2\,t_4$	$t_2\,t_4\,t_5$	$\frac{15}{2\pi\sqrt{2\pi}}-\frac{15}{2\pi\sqrt{3\pi}}=0.17476\ 95$
	$t_1\,t_2\,t_5$	$t_1\,t_4\,t_5$	$\frac{15}{2\pi\sqrt{3\pi}}=0.77763\ 50$
	$t_1\,t_3\,t_4$	$t_2\,t_3\,t_5$	$\frac{-15}{2\pi\sqrt{2\pi}}+\frac{15}{2\pi\sqrt{3\pi}}=-0.17476\ 95$
	$t_1\,t_3\,t_5$		0
	$t_2\,t_3\,t_4$		0

Checks: E.g.

$$\sum_{k=1}^{5} E(t_{1|4}\,t_{2|4}\,t_{k|4})=-\frac{12}{\sqrt{\pi}}S(2)=-1.32638\ 68, \quad E(t_{1|4})+E(t_{2|4})=-1.32638\ 68.$$

$$\sum_{k=1}^{5} E(t_{1|5}^2\,t_{k|5})=\frac{10}{\sqrt{\pi}}[-1+3S(2)]=-2.32592\ 89, \quad 2E(t_{1|5})=-2.32592\ 90;$$

$$\sum_{k=1}^{5} E(t_{1|5}\,t_{2|5}\,t_{k|5})=\frac{15}{\sqrt{\pi}}[0-3S(2)]=-1.6579\ 834, \quad E(t_{1|5})+E(t_{2|5})=-1.6579\ 835;$$

$$\sum_{k=1}^{5} E(t_{1|5}\,t_{3|5}\,t_{k|5})=\frac{5}{\sqrt{\pi}}[-1+3S(2)]=-1.16296\ 45, \quad E(t_{1|5})+E(t_{3|5})=-1.16296\ 45;$$

$$\sum_{k=1}^{5} E(t_{1|5}\,t_{4|5}\,t_{k|5})=\frac{5}{\sqrt{\pi}}[-2+9S(2)]=-0.66794\ 55, \quad E(t_{1|5})+E(t_{5|5})=-0.66794\ 55;$$

$$\sum_{k=1}^{5} E(t_{1|5}\,t_{k|5}\,t_{5|5})=0, \quad E(t_{1|5})+E(t_{5|5})=0.$$

So that

$$\sum_{j=k}^{5}\sum_{1=1}^{5} E(t_{1|5}\,t_{j|5}\,t_{k|5})=\frac{25}{\sqrt{\pi}}[-1+3S(2)]=-5.81482\ 24, \quad 5E(t_{1|5})=-5.81482\ 25.$$

Similarly

$$\sum_{j=1}^{5}\sum_{k=1}^{5} E(t_{2|5}\,t_{j|5}\,t_{k|5})=\frac{25}{\sqrt{\pi}}[1-6S(2)]=-2.47509\ 54, \quad 5E(t_{2|5})=-2.47509\ 50.$$

$$\sum_{j=1}^{5}\sum_{k=1}^{5} E(t_{3|5}\,t_{j|5}\,t_{k|5})=0, \quad 5E(t_{3|5})=0.$$

$$\sum_{j=1}^{5}\sum_{k=1}^{5} E(t_{4|5}\,t_{j|5}\,t_{k|5})=\frac{25}{\sqrt{\pi}}[-1+6S(2)]=2.47509\ 54, \quad 5E(t_{4|5})=-2.47509\ 50.$$

$$\sum_{j=1}^{5}\sum_{k=1}^{5} E(t_{j|5}\,t_{k|5}\,t_{5|5})=\frac{25}{\sqrt{\pi}}[1-3S(2)]=5.81482\ 24, \quad 5E(t_{5|5})=5.81482\ 25.$$

Therefore

$$\sum_{i=1}^{5}\sum_{j=1}^{5}\sum_{k=1}^{5} E(t_{i|5}\,t_{j|5}\,t_{k|5})=0, \quad 5\sum_{i=1}^{5} E(t_{i|5})=0.$$

Table XIII Values of $E(t_{i|n}^2 t_{j|n} t_{k|n})$ and $E(t_{i|n} t_{j|n} t_{k|n}^2)$ $(i<j<k)$

n	$t_i^2 t_j t_k$	$t_{k'} t_{j'} t_{i'}$	$E(t_{i\|n}^2 t_{j\|n} t_{k\|n})=E(t_{n-k+1\|n} t_{n-j+1\|n} t_{n-i+1}^2)$
3	$t_1^2 t_2 t_3$	$t_1 t_2 t_3^2$	$\frac{\sqrt{3}}{6\pi}=0.09188\ 81$
4	$t_1^2 t_2 t_3$	$t_2 t_3 t_4^2$	$\frac{1}{6\pi}[8\sqrt{3}-9]\ =0.25764\ 04$
	$t_1^2 t_2 t_4$	$t_1 t_3 t_4^2$	$\frac{1}{2\pi}[-4\sqrt{3}+3]=-0.62519\ 30$
	$t_1^2 t_3 t_4$	$t_1 t_2 t_4^2$	$\frac{\sqrt{3}}{\pi}\ =0.55132\ 89$
	$t_2^2 t_3 t_4$	$t_1 t_2 t_3^2$	$\frac{\sqrt{3}}{3\pi}\ =0.18377\ 63$
5	$t_1^2 t_2 t_3$	$t_3 t_4 t_5^2$	$\frac{5\sqrt{3}}{\pi}\left[\frac{1}{6}+\frac{5}{3}S(3)\right]+\frac{\sqrt{5}}{4\pi^2}+\frac{15}{\pi}\left[-\frac{1}{4}+\frac{1}{2}S\left(\frac{1}{2}\right)\right]=0.60582\ 30$
	$t_1^2 t_2 t_4$	$t_2 t_4 t_5^2$	$-\frac{20\sqrt{3}S(3)}{\pi}+\frac{\sqrt{5}}{4\pi^2}+\frac{15}{\pi}\left[\frac{3}{4}-\frac{5}{2}S\left(\frac{1}{2}\right)\right]\ =-0.27341\ 96$
	$t_1^2 t_2 t_5$	$t_1 t_4 t_5^2$	$-\frac{\sqrt{5}}{\pi^2}+\frac{15}{\pi}\left[-\frac{1}{2}+2S\left(\frac{1}{2}\right)\right]\ =-1.33561\ 41$
	$t_1^2 t_3 t_4$	$t_2 t_3 t_5^2$	$\frac{5\sqrt{3}}{\pi}\left[\frac{1}{2}+S(3)\right]+\frac{\sqrt{5}}{4\pi^2}+\frac{15}{\pi}\left[-\frac{1}{2}+S\left(\frac{1}{2}\right)\right]\ =0.26507\ 57$
	$t_1^2 t_3 t_5$	$t_1 t_3 t_5^2$	$\frac{5\sqrt{3}}{\pi}[-1+2S(3)]-\frac{\sqrt{5}}{\pi^2}+\frac{15}{\pi}\left[\frac{1}{2}-S\left(\frac{1}{2}\right)\right]\ =-0.07841\ 33$
	$t_1^2 t_4 t_5$	$t_1 t_2 t_5^2$	$\frac{5\sqrt{3}}{\pi}\left[\frac{1}{2}-S(3)\right]+\frac{3\sqrt{5}}{2\pi^2}\ =1.13986\ 20$
	$t_2^2 t_3 t_4$	$t_2 t_3 t_4^2$	$\frac{5\sqrt{3}}{\pi}\left[\frac{1}{2}-\frac{1}{3}S(3)\right]-\frac{\sqrt{5}}{\pi^2}+\frac{15}{\pi}\left[-\frac{1}{4}+\frac{1}{2}S\left(\frac{1}{2}\right)\right]=0.08489\ 96$
	$t_2^2 t_3 t_5$	$t_1 t_3 t_4^2$	$\frac{5\sqrt{3}}{\pi}[-1+2S(3)]+\frac{11\sqrt{5}}{4\pi^2}+\frac{15}{\pi}\left[\frac{1}{4}-\frac{1}{2}S\left(\frac{1}{2}\right)\right]=-0.10290\ 37$
	$t_2^2 t_4 t_5$	$t_1 t_2 t_4^2$	$\frac{5\sqrt{3}}{\pi}\left[\frac{1}{2}-S(3)\right]-\frac{9\sqrt{5}}{4\pi^2}\ =0.29025\ 81$
	$t_3^2 t_4 t_5$	$t_1 t_2 t_3^2$	$\frac{5\sqrt{3}}{\pi}\left[\frac{1}{6}-\frac{1}{3}S(3)\right]+\frac{\sqrt{5}}{4\pi^2}\ =0.32331\ 37$

Checks: E.g. $\sum_{j=1}^{4}\sum_{k=1}^{4} E(t_{1|4}^2 t_{j|4} t_{k|4})=6+\frac{4\sqrt{3}}{\pi}=8.20531\ 59,\quad 4E(t_{1|4})+2=8.20531\ 56;$

$$\sum_{j=1}^{4}\sum_{k=1}^{4} E(t_{2|4}^2 t_{j|4} t_{k|4})=6-\frac{4\sqrt{3}}{\pi}=3.79468\ 44,\quad 4E(t_{2|4})+2=3.79468\ 44;$$

$$\sum_{k=1}^{5} E(t_{1|5}^3 t_{k|5})=3+\frac{15}{2\pi}[1-2S(3)]=5.40006\ 14,\quad 3E(t_{1|5}^2)=5.40006\ 12;$$

$$\sum_{k=1}^{5} E(t_{1|5}^2 t_{2|5} t_{k|5})=1+\frac{15}{2\pi}\sqrt{3}[1-2S(3)]=3.40006\ 13,\quad E(t_{1|5}^2)+2E(t_{1|5} t_{2|5})=3.40006\ 12;$$

$$\sum_{k=1}^{5} E(t_{1|5}^2 t_{3|5} t_{k|5})=1+\frac{15\sqrt{3}}{2\nu}[-1+2S(3)]+\frac{15}{\pi}\left[1-2S\left(\frac{1}{2}\right)\right]=2.09631\ 59,\quad E(t_{1|5}^2)+2E(t_{1|5} t_{3|5})=2.09631\ 58;$$

$$\sum_{k=1}^{5} E(t_{1|5}^2 t_{4|5} t_{k|5})=1+\frac{5\sqrt{3}}{2\pi}[1-2S(3)]+\frac{15}{\pi}\left[-1+6S\left(\frac{1}{2}\right)\right]=0.86018\ 54,\quad E(t_{1|5}^2)+2E(t_{1|5} t_{4|5})=0.86018\ 54;$$

$$\sum_{k=1}^{5} E(t_{1|5}^2 t_{k|5} t_{5|5})=1+\frac{5\sqrt{3}}{2\pi}[1-2S(3)]-\frac{60}{\pi}S\left(\frac{1}{2}\right)=-0.75652\ 17,\quad E(t_{1|5}^2)+2E(t_{1|5} t_{5|5})=-0.75652\ 18.$$

Hence $\sum_{k=1}^{5}\sum_{j=1}^{5} E(t_{1|5}^2 t_{j|5} t_{k|5})=7+\frac{25}{2\pi}\sqrt{3}[1-2S(3)]=11.00010\ 23,\quad 5E(t_{1|5}^2)+2=11.00010\ 20.$

So also $\sum_{k=1}^{5}\sum_{j=1}^{5} E(t_{2|5}^3 t_{j|5} t_{k|5})=7+\frac{25\sqrt{3}}{\pi}[-1+4S(3)]=4.78281\ 38,\quad 5E(t_{2|5}^2)+2=4.78281\ 35.$

$$\sum_{k=1}^{5}\sum_{j=1}^{5} E(t_{3|5}^2 t_{j|5} t_{k|5})=7+\frac{25\sqrt{3}}{\pi}[1-6S(3)]=3.43416\ 79,\quad 5E(t_{3|5}^2)+2=3.43416\ 85,\quad \&c.$$

Also $\sum_{k=1}^{n}\sum_{j=1}^{k-1}\sum_{i=1}^{j-1} E(t_{i|n}^2 t_{j|n} t_{k|n})=\frac{n(n-1)(n-2)}{12\pi\sqrt{3}}=\frac{1}{2\pi\sqrt{3}},\ \frac{2}{\pi\sqrt{3}},\ \frac{5}{\pi\sqrt{3}}$ for $n=3, 4, 5$.

Table XIV Values of $E(t_{i|n}\, t^2_{j|n}\, t_{k|n})$ $(i<j<k)$

n	$t_i\, t_j^2\, t_k$	$t_{k'}\, t_{2^{j'}}\, t_{i'}$	$E(t_{i\|n}\, t^2_{j\|n}\, t_{k\|n})=E(t_{n-k+1}\, t^2_{n-j+1}\, t_{n-i+1})$
3	$t_1 t_2^2 t_3$		$-\dfrac{1}{\sqrt{3}\pi}=-0.18377\ 63$
4	$t_1 t_2^2 t_3$	$t_2 t_3^2 t_4$	$-\dfrac{2}{\pi\sqrt{3}}+\dfrac{3}{2\pi}=0.10991\ 22$
	$t_1 t_2^2 t_4$	$t_1 t_3^2 t_4$	$-\dfrac{3}{2\pi}=-0.47746\ 48$
5	$t_1 t_2^2 t_3$	$t_3 t_4^2 t_5$	$-\dfrac{5}{\pi\sqrt{3}}[1-2S(3)]+\dfrac{\sqrt{5}}{4\pi^2}+\dfrac{15}{2\pi}\left[\dfrac{1}{2}-S\left(\dfrac{1}{2}\right)\right]=0.39738\ 76$
	$t_1 t_2^2 t_4$	$t_2 t_4^2 t_5$	$\dfrac{\sqrt{5}}{4\pi^2}+\dfrac{15}{2\pi}\left[-\dfrac{1}{2}+3S\left(\dfrac{1}{2}\right)\right]=-0.17831\ 85$
	$t_1 t_2^2 t_5$	$t_1 t_4^2 t_5$	$-\dfrac{\sqrt{5}}{\pi^2}-\dfrac{15}{\pi}S\left(\dfrac{1}{2}\right)=-0.86569\ 66$
	$t_1 t_3^2 t_4$	$t_2 t_3^2 t_5$	$-\dfrac{9\sqrt{5}}{4\pi^2}+\dfrac{15}{2\pi}\left[-\dfrac{1}{2}+5S\left(\dfrac{1}{2}\right)\right]=-0.10558\ 56$
	$t_1 t_3^2 t_5$		$\dfrac{4\sqrt{5}}{\pi^2}-\dfrac{30}{\pi}S\left(\dfrac{1}{2}\right)=-0.37202\ 69$
	$t_2 t_3^2 t_4$		$-\dfrac{20}{\pi\sqrt{3}}S(3)+\dfrac{3\sqrt{5}}{2\pi^2}+\dfrac{15}{2\pi}\left[1-6S\left(\dfrac{1}{2}\right)\right]=0.03869\ 00$

Checks: E.g. $\sum\limits_{k=3}^{n}\sum\limits_{j=2}^{k-1}\sum\limits_{i=1}^{j-1} E(t_i\, t_j^2\, t_k)=\dfrac{-n(n-1)(n-2)}{6\pi\sqrt{3}}$:

$n=4.$ $\sum\limits_{k=3}^{4}\sum\limits_{j=2}^{k-1}\sum\limits_{i=1}^{j-1} E(t_i\, t_j^2\, t_k)=-\dfrac{4}{\pi\sqrt{3}}=-0.73510\ 52$

$n=5.$ $\sum\limits_{k=3}^{5}\sum\limits_{j=2}^{k-1}\sum\limits_{i=1}^{j-1} E(t_i\, t_j^2\, t_k)=-\dfrac{10}{\pi\sqrt{3}}=-1.83776\ 30.$

Otherwise

$$\sum_{k=1}^{5} E(t_{1|5}\, t^2_{2|5}\, t_{k|5})=1+\frac{10}{\pi}\sqrt{3}\, S(3)=E(t^2_{2|5})+2E(t_{1|5}\, t_{2|5}),\quad \text{(cf. Tables III, IV, Part I)}$$

$$\sum_{k=1}^{5} E(t_{1|5}\, t^2_{3|5}\, t_{k|5})=1+\frac{5\sqrt{3}}{\pi}[-1-2S(3)]+\frac{15}{\pi}\left[1-2S\left(\frac{1}{2}\right)\right]=E(t^3_{3|5})+2E(t_{1|5}\, t_{3|5}),$$

$$\sum_{k=1}^{5} E(t_{1|5}\, t^2_{4|5}\, t_{k|5})=1+\frac{5\sqrt{3}}{\pi}[-1+4S(3)]+\frac{15}{\pi}\left[-1+6S\left(\frac{1}{2}\right)\right]=E(t^2_{4|5})+2E(t_{1|5}\, t_{4|5}),$$

$$\sum_{k=1}^{5} E(t_{1|5}\, t^2_{5|5}\, t_{k|5})=1+\frac{5\sqrt{3}}{\pi}\left[\frac{1}{2}-S(3)\right]-\frac{15}{\pi}\times 4S\left(\frac{1}{2}\right)=E(t^2_{5|5})+2E(t_{1|5}\, t_{5|5}),$$

and

$$\sum_{k=1}^{5} E(t^3_{1|5}\, t_{k|5})=3+\frac{5\sqrt{3}}{\pi}\left[\frac{3}{2}-3S(3)\right]=3E(t^2_{1|5}).$$

Consequently we get

$$\sum_{k=1}^{5}\sum_{j=1}^{5} E(t_{1|5}\, t^2_{j|5}\, t_{k|5})=7 \qquad \text{in accordance with (16.7.2).}$$

Lastly we deal with the case of 4 arguments. Since we saved to construct general formulas, we should proceed to work up each value separately.

For $n=4$ we have only one moment:

$$(18.0)\qquad E(t_{1|4}t_{2|4}t_{3|4}t_{4|4})=\underline{|4}\int\varphi' dt\int^t\varphi_1' dt_1\int^{t_1}\varphi_2' dt_2\int^{t_2}\varphi_3' dt_3=0\,.$$

For, we have successively $\int^2\varphi_3' dt_3=\varphi_2$, $\int^1\varphi_2\varphi_2' dt_2=\frac{1}{2}\varphi_1^2$, $\frac{1}{2}\int^t\varphi_1^2\varphi_1' dt_1=\frac{1}{\underline{|3}}\varphi^3$ and $4\int\varphi^3\varphi' dt=\varphi^4\Big|_{-\infty}^{\infty}=0.$

Let $n=5$, and consider all possible 5 combinations. Firstly

$$(18.1)\quad E(t_{1|5}t_{2|5}t_{3|5}t_{4|5})=\underline{|5}\int(1-\Phi)\varphi' dt\int^t\varphi_1' dt_1\int^{t_1}\varphi_2' dt_2\int^{t_2}\varphi_3' dt_3=5c_5=\sqrt{5}/4\pi^2\,,$$

because, by a similar manner as above, we get

$$E=\underline{|5}\int(1-\Phi)\frac{1}{\underline{|3}}\varphi^3\varphi' dt=5\int\varphi^5 dt=5c_5\,.$$

Secondly

$$(18.2)\quad E(t_{1|5}t_{2|5}t_{3|5}t_{5|5})=\underline{|5}\int\varphi' dt\int^t(\Phi-\Phi_1)\varphi_1' dt_1\int^1\varphi_2' dt_2\int^2\varphi_3' dt_3=-20c_5=-\sqrt{5}/\pi^2,$$

because, as before $\int^1\varphi_2' dt_2\int^2\varphi_3' dt_3=\frac{1}{2}\varphi_1^2$ and consequently $\frac{1}{2}\int^t(\Phi-\Phi_1)\varphi_1^2\varphi_1' dt_1$ $=\frac{1}{6}\int^t\varphi_1^4 dt_1$ and $\frac{\underline{|5}}{6}\int\varphi' dt\int^t\varphi_1^4 dt_1=-20\int\varphi^5 dt=-20c_5\,.$

Thirdly

$$(18.3)\qquad E(t_{1|5}t_{2|5}t_{4|5}t_{5|5})=\underline{|5}\int\varphi' dt\int^t\varphi_1' dt\int^1(\Phi_1-\Phi_2)\varphi_2' dt_2\int^2\varphi_3' dt_3=30c_5\,,$$

since we get $\int^2\varphi_3' dt_3=\varphi_2$, $\int^1(\Phi_1-\Phi_2)\varphi_2\varphi_2' dt_2=\frac{1}{2}\int^1\varphi_2^3 dt_2$ and $\frac{1}{2}\int^t\varphi_1' dt_1\int^1\varphi_2^3 dt_2$ $=\frac{\varphi}{2}\int^t\varphi_1^3 dt_1-\frac{1}{2}\int^t\varphi_1^4 dt_1$, so that $E=\frac{\underline{|5}}{2}\int\varphi\varphi' dt\int^t\varphi_1^3 dt_1-\frac{\underline{|5}}{2}\int\varphi' dt\int^t\varphi_1^4 dt_1$ $=-30\int\varphi^5 dt+60\int\varphi^5 dt=30c_5=3\sqrt{5}/2\pi^2.$

The remaining two could be similarly obtained. However, by symmetry, we have immediately

$$(18.4)\qquad E(t_{1|5}t_{3|5}t_{4|5}t_{5|5})=E(t_{5|5}t_{3|5}t_{2|5}t_{1|5})=-20c_5\,,$$

$$(18.5)\qquad E(t_{2|5}t_{3|5}t_{4|5}t_{5|5})=E(t_{4|5}t_{3|5}t_{2|5}t_{1|5})=5c_5\,.$$

Thus, we obtain the following

Table XV Values of $E(t_{i|n}\, t_{j|n}\, t_{k|n}\, t_{l|n})$

n	$t_i\, t_j\, t_k\, t_l$	$t_{l'}\, t_{k'}\, t_{j'}\, t_{i'}$	$E(t_{i\|n}\, t_{j\|n}\, t_{k\|n}\, t_{l\|n})=E(t_{n-l+1}\, t_{n-k+1}\, t_{n-j+1}\, t_{n-i+1})$
4	$t_1\, t_2\, t_3\, t_4$		0
5	$t_1\, t_2\, t_3\, t_4$	$t_2\, t_3\, t_4\, t_5$	$\frac{\sqrt{5}}{4\pi^2}=0.05664\ 03$
	$t_1\, t_2\, t_3\, t_5$	$t_1\, t_3\, t_4\, t_5$	$-\frac{\sqrt{5}}{\pi^2}=-0.22656\ 11$
	$t_1\, t_2\, t_4\, t_5$		$\frac{3\sqrt{5}}{2\pi^2}=0.33984\ 16$

Checks E.g.: $\sum_{k=1}^{5} E(t_{1|5}\, t_{2|5}\, t_{3|5}\, t_{k|5})=\frac{10\sqrt{3}}{\pi}S(3)=\sum_{i\neq j=1,2,3} E(t_{i|5}\, t_{j|5})=1.1566\ 035.$

$$\sum_{k=1}^{5} E(t_{1|5}\, t_{2|5}\, t_{4|5}\, t_{k|5})=\frac{5\sqrt{3}}{\pi}\left[\frac{1}{2}-5S(3)\right]+\frac{15}{\pi}\left[\frac{1}{2}-S\left(\frac{1}{2}\right)\right]=\sum_{i\neq j=1,2,4} E(t_{i|5}\, t_{j|5})=0.3350\ 018.$$

Also $\sum_{l=4}^{n}\sum_{k=3}^{l-1}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1} E(t_i\, t_j\, t_k\, t_l)=0.$

§ **19. The Frequency Functions of $t_{i|n}$ and their Central Moments.** So far we have obtained the expectations of $t^k_{i|n}$ for $k=1, 2, 3, 4$, viz., the k-th moment $\nu_k(t_{i|n})$ about axis $t=0$ and especially mean $m=E(t_{i|n})$. Now we shall find their central moments, viz., $\mu_k(t_{i|n})$ by the well known formulas

(19.1) $\mu_k = E[(t_{i|n}-m)^k] = \nu_k-\binom{k}{1}m\nu_{k-1}+\binom{k}{2}m^2\nu_{k-2}-+\cdots+(-1)^{k-1}\left[\binom{k}{1}-1\right]m^k.$

In particular

(19.2) $\mu_0=1,\ \mu_1=0,\ \mu_2=\nu_2-m^2,\ \mu_3=\nu_3-3m\nu_2+2m^3,\ \mu_4=\nu_4-4m\nu_3+6m^2\nu_2-3m^4.$

Thus we obtain the following

Table XVI Central Moments of $t_{i|n}$

n	$t_{i\|n}$	$t_{i'\|n}$	μ_2	μ_3	μ_4	Skewness $\mu_3/\sqrt{\mu_2}^3$	Excess μ_4/μ_2^2-3
2	$t_{2\|2}$	$t_{1\|2}$	0.6816 901	$\pm$0.0770 828	1.4227 879	0.13695	0.06172
3	$t_{3\|3}$	$t_{1\|3}$	0.5594 671	$\pm$0.0891 995	4.5853 562	0.15944	7.95749
	$t_{2\|3}$		0.4486 711	0	0.6109 081	0	−0.96725
4	$t_{4\|4}$	$t_{1\|4}$	0.4917 152	$\pm$0.0912 069	8.0701 896	0.26452	30.37770
	$t_{3\|4}$	$t_{2\|4}$	0.3604 553	$\pm$0.0141 876	0.3493 598	0.06556	−0.31113
5	$t_{5\|5}$	$t_{1\|5}$	0.4475 340	$\pm$0.0905 873	13.3031 003	0.30257	63.42025
	$t_{4\|5}$	$t_{2\|5}$	0.3115 189	$\pm$0.0186 864	0.3646 656	0.10747	0.75774
	$t_{3\|5}$		0.2868 337	0	0.2494 541	0	0.03201

The skewness and excess being as in the above Table XVI, none of them distributes normally, what is a matter of course, since really they belong to Beta-distributions. However even when $n=5$, it appears already that the central part approaches toward N.D.

Incidentally we may here prove once more again, that the distribution of $\zeta = ct_{1|3} + (1-2c)t_{2|3} + ct_{3|3}$ in Cramér's example is not normal, i.e. its excess is not zero. Really

$$m = E(\zeta) = cE(t_{1|3}) + cE(t_{3|3}) + (1-2c)E(t_{2|3}),$$

$$\sigma^2 = D^2(\zeta) = E(\zeta^2) - E(\zeta)^2 = E[\{c(t_1+t_3)+(1-2c)t_2\}^2]$$
$$= c^2E(t_1+t_3)^2 + (1-2c)^2E(t_3^2) + 2c(1-2c)E(t_1t_2+t_2t_3),$$

$$E(\zeta^3) = c^3E[(t_1+t_3)^3] + 3c^2(1-2c)E[(t_1+t_3)^2t_2] + 3c(1-2c)^2E(t_1t_2^2+t_2^2t_3) + c^3E(t_3^3),$$

$$D(\zeta^4) = c^4E[(t_1+t_3)^4] + 4c^3(1-2c)E[(t_1+t_3)^3t_2] + 6c(1-2c)^2E[(t_1+t_3)^2t_2^2]$$
$$+ 4c(1-2c)^3E[(t_1+t_3)t_2^3] + (1-2c)^4E(t_2^4).$$

Excecuting these calculations actually and substituting those values of $E(t_i^p)$ for $p=1, 2, 3, 4$, and $E(t_i^pt_k^q)$ for $p+q=2, 3, 4$ and $E(t_i^pt_j^qt_k^r)$ for $p+q+r=3, 4$, given in our Tables III–XIV, we find

$$m = 0, \quad \sigma^2 = \frac{1}{3} + 3\left[2 - \frac{3\sqrt{3}}{\pi}\right]\left(c - \frac{1}{3}\right)^2.$$

Or, on putting $1-4c+6c^2=\gamma^2$, viz., $3\gamma^2-1=2(1-3c)^2$, it yields

$$\sigma^2 = \frac{1}{3}\left[1+\left(1-\frac{3\sqrt{3}}{2\pi}\right)(3\gamma^2-1)\right] = \left(1-\frac{3\sqrt{3}}{2\pi}\right)\gamma^2 + \frac{\sqrt{3}}{2\pi} = \gamma^2 + \frac{\sqrt{3}}{2\pi}(1-3\gamma^2),$$

which is the same as (6. 8) in §6, Part I. Furthermore

$$E(\zeta^3) = 0, \qquad \text{so that} \qquad \text{Skewness} = 0, \qquad \text{but}$$

$$E(\zeta^4) = \left(108 - \frac{567}{\pi\sqrt{3}}\right)c^4 - \left(144 + \frac{756}{\pi\sqrt{3}}\right)c^3 + \left(84 - \frac{426}{\pi\sqrt{3}}\right)c^2 - \left(24 - \frac{120}{\pi\sqrt{3}}\right)c + \left(3 - \frac{13}{\pi\sqrt{3}}\right),$$

while

$$3E(\zeta^2)^2 = \left(108 - \frac{972}{\pi\sqrt{3}} + \frac{729}{\pi^2}\right)c^4 - \left(144 - \frac{1296}{\pi\sqrt{3}} + \frac{972}{\pi^2}\right)c^3 + \left(84 - \frac{702}{\pi\sqrt{3}} + \frac{486}{\pi^2}\right)c^2$$
$$- \left(24 - \frac{180}{\pi\sqrt{3}} + \frac{108}{\pi^2}\right)c + \left(3 - \frac{18}{\pi\sqrt{3}} + \frac{9}{\pi^2}\right).$$

Hence, we get

$$E(\zeta^4) - 3E(\zeta^2)^2$$
$$= \left(\frac{405}{\pi\sqrt{3}} - \frac{729}{\pi^2}\right)c^4 - \left(\frac{540}{\pi\sqrt{3}} - \frac{972}{\pi^2}\right)c^3 + \left(\frac{270}{\pi\sqrt{3}} - \frac{486}{\pi^2}\right)c^2 - \left(\frac{60}{\pi\sqrt{3}} - \frac{108}{\pi^2}\right)c + \left(\frac{5}{\pi\sqrt{3}} - \frac{9}{\pi^2}\right)$$
$$= \left(\frac{5}{\pi\sqrt{3}} - \frac{9}{\pi^2}\right)[81c^4 - 108c^3 + 54c^2 - 12c + 1] = \frac{5\pi - 9\sqrt{3}}{\pi^2\sqrt{3}}(3c-1)^2.$$

Or, since $\gamma^2 = 1-4c+6c^2$, $3\gamma^2-1 = 2(3c-1)^4$,

$$\text{Excess} = \frac{E(\zeta^4)}{E(\zeta^2)^2} - 3 = \frac{(5\pi-9\sqrt{3})(3\gamma^2-1)^2}{4\pi^2\sqrt{3}} \Big/ \left[\gamma^2 + \frac{(1-3\gamma^2)\sqrt{3}}{2\pi}\right]^2 > 0, \text{ if } \gamma^2 \neq \frac{1}{3}.$$

Thus, the present result just coincides with (6. 10) obtained in § 6, Part I.

By a similar manner it may be shown for the cases $n=4$ or 5, that the distribution of $\zeta=\sum_{i=1}^{n} c_i t_i$ ($c_i \geqq 0$, $\sum c_i=1$), even when $c_{n-i+1}=c_i$ cannot be normal.

§ 20. The Asymptotic Feature of $t_{i|n}$'s Distribution as well as those of the Joint Distributions

$$f(t_{i|n})dt_{i|n}=n!\,\frac{\Phi^{i-1}}{(i-1)!}\,\frac{(1-\Phi)^{n-i}}{(n-i)!}d\Phi \equiv g(\Phi)d\Phi \quad (d\Phi=\varphi dt) \tag{20. 1}$$

being materially a Beta-distribution, it is already recognized e.g. in Cramér's treatise, loc. cit., p. 252, that the fr. f. of the standardized variable tends to the normal fr. f. However, as there is not given any detailed proof, let us now demonstrate it below. Of course, if i be fixed and n alone allowed to become ∞, then $g(\Phi)\to 0$. For, on making use of Stirling's formula, we get

$$\frac{n!}{(i-1)!\,(n-i)!} \simeq \frac{\sqrt{2\pi}\,n^{n+1/2}e^{-n}}{(i-1)!\,\sqrt{2\pi}(n-i)^{n-i+1/2}e^{-n+i}} = \frac{e^{-i}}{(i-1)!}\left(\frac{n}{n-i}\right)^{n+1/2}(n-i)^i \simeq \frac{n^i}{\Gamma(i)},$$

and thus

$$\lim_{n\to\infty} g(\Phi)=\frac{\Phi^{i-1}}{\Gamma(i)}\lim_{n\to\infty} n^i(1-\Phi)^{n-i}=0\,.$$

When, however, i as well as n considered to tend to infinity, the ordinal number i itself becomes meaningless; rather the percentage figure i/n comes into consideration. Hence, let $i=[np]=np-\alpha$ where p denotes a certain positive proper fraction: $0<p<1$ and $0\leqq\alpha<1$ though fluctuating, so that $n-i=n-np+\alpha=nq+\alpha$, where $q=1-p$ is also a fixed positive proper fraction. Under these assumptions, let us conceive

$$\lim_{n\to\infty} g(\Phi)d\Phi=\lim_{n\to\infty} \beta_{i|n}\Phi^{i-1}(1-\Phi)^{n-i}d\Phi\,. \tag{20. 2}$$

Now to make the variable Φ standardize, we require the mean and the variance. These are readily found to be

$$m=E(\Phi)=\beta_{i|n}\int_0^1 \Phi^{i-1}(1-\Phi)^{n-i}\Phi d\Phi=\beta_{i|n}B(i+1,\,n+i+1)=\frac{i}{n+1}\simeq p\,, \tag{20. 3}$$

and
$$E(\Phi^2)=\beta_{i|n}\int_0^1 \Phi^{i+1}(1-\Phi)^{n-i}d\Phi=\frac{i(i+1)}{(n+1)(n+2)}\simeq p^2\,,$$

so that

$$\sigma^2=D^2(\Phi)=E(\Phi^2)-E(\Phi)^2=\frac{i(n-i+1)}{(n+1)^2(n+2)}\simeq\frac{pq}{n}\,. \tag{20. 4}$$

Accordingly we put to make the variable standardize

$$\xi=(\Phi-m)/\sigma \quad \text{or} \quad \Phi=m+\sigma\xi\,, \tag{20. 5}$$

and transform the probability element so as

(20. 6) $$g(\Phi)d\Phi=\beta_{i|n}(\sigma\xi+m)^{i-1}(1-m-\sigma\xi)^{n-i}\sigma d\xi=h(\xi)d\xi\,,\quad\text{say,}$$

where $d\Phi=\sigma d\xi(=\varphi(t)dt)$ and the probability element expresses the probability that the variable t lies between some given t and $t+dt$, so that Φ lies between $\Phi=\int^{t}\varphi dt$ and $\Phi=\int^{t+dt}\varphi dt$ and vice versa. We have to find the asymptotic estimation of the frequency function

(20. 7) $$h(\xi)=\beta_{i|n}m^{i-1}(1-m)^{n-i}\sigma\left[1+\frac{\sigma}{m}\xi\right]^{i-1}\left[1-\frac{\sigma}{1-m}\xi\right]^{n-i},$$

where $i=np-\alpha$, $n-i=nq+\alpha$ and we are going to show that indeed $h(\xi)$ tends to the standard normal distribution.

By use of Stirling's asymptotic formula, we have

(20. 8) $$\beta_{i|n}=\frac{n!}{\Gamma(np-\alpha)\lfloor nq+\alpha}\simeq\sqrt{\frac{n}{2\pi}}(p^pq^q)^{-n}\left(\frac{p}{q}\right)^{\alpha+1/2}\left[1-\frac{\alpha}{np}\right]^{np}\left[1+\frac{\alpha}{nq}\right]^{nq}\left[\frac{(np-\alpha)q}{(nq+\alpha)p}\right]^{\alpha+1/2}$$
$$\simeq\sqrt{\frac{n}{2\pi}}(p^pq^q)^{-n}\left(\frac{p}{q}\right)^{\alpha+1/2},$$

since the remaining factor tends to unity as $n\to\infty$. On the other hand $m=\frac{i}{n+1}=\frac{np-\alpha}{n+1}$, $1-m=\frac{nq+1+\alpha}{n+1}$ and consequently

(20. 9) $$m^{i-1}(1-m)^{n-i}=\left(\frac{np-\alpha}{n+1}\right)^{np-\alpha-1}\left(\frac{nq+1+\alpha}{n+1}\right)^{nq+\alpha}$$
$$=p^{np}q^{nq}\left(\frac{q}{p}\right)^{\alpha}\frac{1}{p}\left(1+\frac{1}{n}\right)^{-n}\left[1-\frac{\alpha}{np}\right]^{np}\left[1+\frac{1+\alpha}{nq}\right]^{nq}$$
$$\simeq(p^pq^q)^n\left(\frac{q}{p}\right)^{\alpha}\frac{1}{p}.$$

Also

(20. 10) $$\sigma=\sqrt{\frac{(n-i+1)i}{(n+1)^2(n+2)}}\simeq\sqrt{\frac{pq}{n}}\left[1+O\left(\frac{1}{n}\right)\right].$$

so that

$$\frac{\sigma}{m}\simeq\sqrt{\frac{q}{pn}}\left[1+O\left(\frac{1}{n}\right)\right],\quad\frac{\sigma}{1-m}\simeq\sqrt{\frac{p}{qn}}\left[1+O\left(\frac{1}{n}\right)\right].$$

All these values being substituted in (20. 7), we have

$$h(\xi)\simeq\sqrt{\frac{n}{2\pi}}(p^pq^q)^{-n}\left(\frac{p}{q}\right)^{\alpha+1/2}\cdot(p^pq^q)^n\left(\frac{q}{p}\right)^{\alpha}\frac{1}{p}\cdot\sqrt{\frac{pq}{n}}\left[1+O\left(\frac{1}{n}\right)\right]$$
$$\times\left\{1+\sqrt{\frac{q}{pn}}\,\xi+O\left(\frac{1}{n^{3/2}}\right)\right\}^{np-\alpha-1}\left\{1-\sqrt{\frac{p}{qn}}\,\xi+O\left(\frac{1}{n^{3/2}}\right)\right\}^{nq+\alpha}$$
$$=\frac{1}{\sqrt{2\pi}}\exp\left\{(np-\alpha-1)\log\left[1+\sqrt{\frac{q}{pn}}\,\xi+O\left(\frac{1}{n^{3/2}}\right)\right]+(nq+\alpha)\log\left[1-\sqrt{\frac{p}{qn}}\,\xi+O\left(\frac{1}{n^{3/2}}\right)\right]\right\}$$

$$=\frac{1}{\sqrt{2\pi}}\exp\left\{(np-\alpha-1)\left[\sqrt{\frac{q}{pn}}\xi-\frac{1}{2n}\frac{q}{p}\xi^2+O\left(\frac{1}{n^{3/2}}\right)\right]\right.$$
$$\left.-(nq+\alpha)\left[\sqrt{\frac{p}{qn}}\xi+\frac{1}{2n}\frac{p}{q}\xi^2+O\left(\frac{1}{n^{3/2}}\right)\right]\right\}$$
$$=\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\xi^2+O\left(\frac{1}{\sqrt{n}}\right)\right\}.$$

Thus, our probability element $f(t_{i|n})dt_{i|n}=g(\Phi)d\Phi=h(\xi)d\xi$ for large value of n with $\frac{i}{n}\simeq p$ (fixed) approaches to $\frac{1}{\sqrt{2\pi}}e^{-\xi^2/2}d\xi$; namely the standardized variable $\xi=[\Phi-E(\Phi)]/\sigma(\Phi)$ tends to distribute normally.

Moreover, the alike conclusion can be said about the joint frequency function

$$(20.11)\qquad f(t_{i|n},\, t_{k|n})dt_{i|n}dt_{k|n}=\gamma_{i,k|n}(1-\Phi)^{n-k}(\Phi-\Phi_1)^{k-i-1}\Phi^{i-1}d\Phi d\Phi_1 \quad (0<\Phi_1<\Phi<1).$$

Here again we conceive certain fixed positive proper fractions p, p_1, such that

$$k=[np]=np-\alpha\,,\quad i=[np_1]=np_1-\alpha_1\,,\quad (0<p_1<p<1).$$

Or, letting $\Phi_1=\Phi\Psi$ and considering Φ as tempolarily fixed, we have

$$f_{ik}dt_{i|n}dt_{k|n}=\gamma_{i,k|n}(1-\Phi)^{n-k}\Phi^{k-1}d\Phi\cdot(1-\Psi)^{k-i-1}\Psi^{i-1}d\Psi\quad (0<\Psi<1,\ 0<\Phi<1).$$

Consequently Φ and Ψ can be considered as independent of each other, and accordingly the numerical coefficient may also be put as

$$\gamma_{i,k|n}=\frac{\lfloor n}{\lfloor n-k\lfloor i-1\lfloor k-i-1}=\frac{\lfloor n}{\lfloor n-k\lfloor k-1}\times\frac{\lfloor k-1}{\lfloor i-1\lfloor k-i-1}=\beta_{k|n}\cdot\beta_{i|k-1}.$$

Thus

$$f(t_{i|n},\, t_{k|n})dt_{i|n}dt_{k|n}=\beta_{k|n}(1-\Phi)^{n-k}\Phi^{k-1}d\Phi\cdot\beta_{i|k-1}(1-\Psi)^{k-i-1}\Psi^{i-1}d\Psi$$
$$=f(t_{i|n})dt_{i|n}\cdot f(t'_{i|k-1})dt'_{i|k-1},$$

So that t and t' distribute independently. But, by the foregoing proof, if we put

$$k=np-\alpha\quad(0\leqq\alpha<1)\qquad\text{and}\qquad i=(k-1)p_1-\alpha_1\quad(0\leqq\alpha_1<1),$$

then the variables

$$\xi=\left(\Phi-\frac{k}{n+1}\right)\Big/\frac{1}{n+1}\sqrt{\frac{(n+k+1)k}{n+2}}\quad\text{and}\quad\eta=\left(\Psi-\frac{i}{k}\right)\Big/\frac{1}{k}\sqrt{\frac{(k-i)i}{k+1}}$$

distribute asymptotically normaly: viz., so as with fr. f.

$$\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\xi^2\right\}\quad\text{and}\quad\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\eta^2\right\}\quad\text{respectively.}$$

Accordingly

$$f(_{i|n},\, t_{k|n})dt_{i|n}dt_{k|n}\simeq\frac{1}{2\pi}\exp\left\{-\frac{1}{2}(\xi^2+\eta^2)\right\}d\xi d\eta\,;$$

namely, the joint frequency function tends to become a bivariate non-singular normal distribution.

The general case of several arguments may be treated by induction. Let the joint frequency function be

$$(20.12)\quad f(t_{i_1|n}, t_{i_2|n}, \cdots, t_{i_k|n})dt_{i_1|n}dt_{i_2|n}\cdots dt_{i_k|n}\quad (1\leq i_1<i_2<\cdots<i_k\leq n)$$
$$=\underline{|n}\,\underline{(1-\Phi)^{n-i_k}}d\Phi\cdot\underline{(\Phi-\Phi_1)^{i_k-i_{k-1}-1}}d\Phi_1\cdot\cdots\underline{(\Phi_{k-2}-\Phi_{k-1})^{i_2-i_1-1}}\underline{\Phi_{k-1}^{i_1-1}}d\Phi_{k-1},$$

and conceive certain fixed positive proper fractions $0<p_{k-1}<p_{k-2}<\cdots<p_1<p<1$ and those arguments, such that

$$i_k=[np]=np-\alpha,\quad i_{k-1}=[np_1]=np_1-\alpha_1,\cdots,\quad i_1=[np_{k-1}]=np_{k-1}-\alpha_{k-1}$$

with all α_ν: $0\leq\alpha_\nu<1$. Putting $\Phi_1=\Phi\Psi$, $\Phi_\nu=\Phi\Psi_{\nu-1}$ $(\nu=2,3,\cdots,k-1)$, and $i_k=l$, we get

$$fdt_{i_1}dt_{i_2}\cdots dt_{i_k}=\underline{|n}\,\underline{(1-\Phi)^{n-l}}\underline{\Phi^{l-1}}d\Phi\cdot\underline{|l-1}\,\underline{(1-\Psi)^{l-1-i_{k-1}}}\underline{(\Psi-\Psi_1)^{i_{k-1}-i_{k-2}-1}}\cdots$$
$$\underline{(\Psi_{k-3}-\Psi_{k-2})^{i_2-i_1-1}}\underline{\Psi_{k-2}^{i_1-1}}d\Psi_1d\Psi_2\cdots d\Psi_{k-2}$$
$$=f(t_{l|n})dt_{l|n}\cdot f(t_{i_1|l-1}, t_{i_2|l-1}, \cdots, t_{i_{k-2}|l-1}, t_{i_{k-1}|l-1})dt_{i_1|l-1}dt_{i_2|l-1}\cdots dt_{i_{k-1}|l-1}.$$

But, assuming that the asymptotic behaviour toward N.D. as $n, l\to\infty$ up to the case with $k-1$ arguments is already established, viz.,

$$f(t_{i_1|l-1}, t_{i_2|l-1}, \cdots, t_{i_{k-1}|l-1})dt_{i_1|n}\cdots dt_{i_{k-1}}\simeq\frac{1}{\sqrt{2\pi}^{k-1}}\exp\left\{-\frac{1}{2}\sum_{\nu=1}^{k-1}\xi_\nu^2\right\}d\xi_1d\xi_2\cdots d\xi_{k-1}$$

and
$$f(t_{i_k|n})dt_{i_k|n}\simeq\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\xi_k^2\right\}d\xi_k,$$

it follows consequently that

$$f(t_{i_1|n}, t_{i_2|n}, \cdots, t_{i_k|n})dt_{i_1|n}\cdots dt_{i_k|n}\simeq\frac{1}{\sqrt{2\pi}^k}\exp\left\{-\frac{1}{2}\sum_{\nu=1}^{k}\xi_\nu^2\right\}d\xi_1d\xi_2\cdots d\xi_k,$$

namely, the assertion is also true for the case with k arguments, and our induction has been completed.

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