

SOME CONTRIBUTIONS TO ORDER STATISTICS (Continued)

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Introduction. We had endeavoured in Part I and II of the previous paper¹⁾ to obtain systematically the first and second moments of order statistics up to the sample-size $n=7$. In the present note we shall continue it to find the third and fourth moments, but now reducing to sizes $n \leq 5$, because of a pretty great number of various combinations.

We shall use the before made abbreviations, and besides, as a shorter notation of integral power $\Phi^i/i!$ simply Φ^i by omitting the factorial in denominator, because this facilitates the differentiation as $\frac{d^p}{d\Phi^p} \Phi^i = \Phi^{i-p}$; of course, it means zero, if the index i be a negative integer, since then $1/\Gamma(i+1)=0$. Thus, e.g.

$$E(t_{i|n}^p t_{k|n}^q) = n! \int (1-\Phi)^{n-k} \varphi t^q dt \int^t (\Phi - \Phi_1)^{k-i-1} \Phi^{i-1} \varphi_1 t_1^p dt_1,$$

$$E(t_{i|n} t_{j|n} t_{k|n}) = -n! \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi - \Phi_1)^{k-j-1} \varphi_1' dt_1 \int^{t_1} (\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' dt_2.$$

1) Y. Watanabe and others, Some Contributions to Order Statistics, this Journ. Vol. VIII (1957), pp. 41-90.

Our main task is firstly to make these expressions of moments free from t and t -derivatives by repeating integrations by parts (§§ 14-15); and next to write them in sums of auxiliary integrals

$$J_{\lambda}^{\alpha} = \int \Phi^{\lambda} \varphi^{\alpha} dt, \quad J_{\mu\nu}^{\alpha\beta} = \int \Phi^{\mu} \varphi^{\alpha} dt \int^t \Phi_1^{\nu} \varphi_1^{\beta} dt_1, \quad J_{\lambda\mu\nu}^{\alpha\beta\gamma} = \int \Phi^{\lambda} \varphi^{\alpha} dt \int^t \Phi_1^{\mu} \varphi_1^{\beta} dt_1 \int^{t_1} \Phi_2^{\nu} \varphi_2^{\gamma} dt_2,$$

which were already partly treated in Part II, however, now in § 17 shall be further supplemented. Consequently the required moments could be explicitly expressed and numerically evaluated (§ 18), for which some check formulas should be constructed (§ 16). Lastly the features of frequency functions $f(t_{i|n})$ and their asymptotic behaviours toward N.D. are discussed (§§ 19-20).

PART III

§ 13. Some Preliminary Formulas. We begin to set up some preliminary formulas which shall be useful in the subsequent paragraphs. The evident identity

$$\int F(\Phi) \varphi^{\alpha} \varphi' dt = \frac{1}{\alpha+1} \int F'(\Phi) \varphi^{\alpha+1} dt \quad (F(\Phi) \text{ being a polynomial of } \Phi)$$

is most often availed. Also, on differentiating the fundamental equality $\varphi' = -\varphi t$ about t , we have $\varphi'' = -\varphi' t - \varphi$, and whence

$$(13.0) \quad \varphi \varphi'' + \varphi^2 = -\varphi \varphi' t = \varphi'^2.$$

follows. By aid of this relation an integrand having φ'^2 as factor could be derived from that containing $\varphi \varphi''$. In fact, we have

$$(13.1) \quad A \equiv \int F(\Phi) \varphi^{\alpha} \varphi'' dt = \frac{1}{(\alpha+1)(\alpha+2)} \int F''(\Phi) \varphi^{\alpha+3} dt - \frac{\alpha}{\alpha+1} \int F(\Phi) \varphi^{\alpha+1} dt.$$

For, considering φ'' as $(\varphi')'$ and repeating integrations by parts, we obtain

$$\begin{aligned} A &= - \int F'(\Phi) \varphi^{\alpha+1} \varphi' dt - \alpha \int F(\Phi) \varphi^{\alpha-1} \varphi'^2 dt \\ &= \frac{1}{\alpha+2} \int F''(\Phi) \varphi^{\alpha+3} dt - \alpha \int F(\Phi) \varphi^{\alpha} \varphi'' dt - \alpha \int F(\Phi) \varphi^{\alpha+1} dt, \end{aligned}$$

because of (13.0). On transposing the integral last before ($= -\alpha A$) into the left side and dividing out by $\alpha+1$, we get (13.1). Moreover, by addition of $\int F(\Phi) \varphi^{\alpha+1} dt$ to both sides of (13.1), we get

$$(13.2) \quad \int F(\Phi) \varphi^{\alpha-1} \varphi'^2 dt = \frac{1}{(\alpha+1)(\alpha+2)} \int F''(\Phi) \varphi^{\alpha+3} dt + \frac{1}{\alpha+1} \int F(\Phi) \varphi^{\alpha+1} dt.$$

In particular, letting $\alpha=1, 2, 3, \dots$, we obtain

$$(13.2.1) \quad \int F(\Phi) \varphi'^2 dt = \frac{1}{6} \int F'' \varphi^4 dt + \frac{1}{2} \int F \varphi^2 dt,$$

$$(13.2.2) \quad \int F(\Phi) \varphi \varphi'^2 dt = \frac{1}{12} \int F'' \varphi^5 dt + \frac{1}{3} \int F \varphi^3 dt,$$

$$(13.2.3) \quad \int F(\Phi) \varphi^2 \varphi'^2 dt = \frac{1}{20} \int F'' \varphi^6 dt + \frac{1}{4} \int F \varphi^4 dt.$$

Example 1. Show that

$$(13.3) \quad \int F(\Phi) \varphi'^2 t dt = \frac{2}{3} \int F' \varphi^3 dt + \frac{1}{24} \int F''' \varphi^5 dt.$$

Observing that $\varphi'^2 t = -\varphi t^2 \varphi'$ and integrating by parts, we get

$$\begin{aligned} -\int F \varphi t^2 \varphi' dt &= \frac{2}{2} \int F \varphi^2 t dt + \frac{1}{2} \int F' \varphi^3 t^2 dt = -\int F \varphi \varphi' dt + \frac{1}{2} \int F' \varphi \varphi'^2 dt \\ &= \frac{1}{2} \int F' \varphi^3 dt + \frac{1}{24} \int F''' \varphi^5 dt + \frac{1}{6} \int F' \varphi^3 dt \quad (\text{by (13.2.2)}) \end{aligned}$$

which, however, nothing but the right side of (13.3), Q.E.D.

Example 2. Make $\int F(\Phi) \varphi'^3 dt$ free from t and t -derivatives.

Considering φ'^3 as $(\varphi^2 + \varphi \varphi'') \varphi' = \varphi^2 \varphi' + \frac{1}{2} \varphi (\varphi'^2)'$, we get

$$\int F \varphi'^3 dt = \int F \varphi^2 \varphi' dt + \frac{1}{2} \int F \varphi (\varphi'^2)' dt = -\frac{1}{3} \int F' \varphi^3 dt - \frac{1}{2} \int F' \varphi^2 \varphi'^2 dt - \frac{1}{2} \int F \varphi'^3 dt.$$

Transposing the last integral into left and applying (13.2.3) on the integral before last, and finally multiplying the whole by $\frac{2}{3}$, we find that

$$(13.4) \quad \int F(\Phi) \varphi'^3 dt = -\frac{11}{36} \int F'(\Phi) \varphi^4 dt - \frac{1}{60} \int F'''(\Phi) \varphi^6 dt, \quad \text{Q.E.I.}$$

Next, for a double integral containing besides $F(\Phi)$ also a polynomial of Φ, Φ_1 , say $G(\Phi, \Phi_1)$, holds

$$\begin{aligned} (13.5) \quad B \equiv \int F(\Phi) \varphi^\alpha \varphi'' dt \int^t G(\Phi, \Phi_1) \varphi_1^\beta dt_1 &= \frac{1}{(\alpha+1)(\alpha+2)} \left[\int F'' \varphi^{\alpha+3} dt \int^t G \varphi_1^\beta dt_1 \right. \\ &\quad \left. + 2 \int F' \varphi^{\alpha+3} dt \int^t \mathfrak{D}G \cdot \varphi_1^\beta dt_1 + \int F \varphi^{\alpha+3} dt \int \mathfrak{D}^2 G \cdot \varphi_1^\beta dt_1 + \int \{F'g + Fg'\} \varphi^{\alpha+\beta+2} dt \right] \\ &\quad + \frac{1}{(\alpha+1)(\alpha+\beta+1)} \int D(Fg) \varphi^{\alpha+\beta+2} dt - \frac{\alpha}{\alpha+1} \int F \varphi^{\alpha+1} dt \int^t G \varphi_1^\beta dt_1, \end{aligned}$$

where $D = \frac{d}{d\Phi}$, $\mathfrak{D} = \frac{\partial}{\partial \Phi}$, $\mathfrak{D}^2 = \frac{\partial^2}{\partial \Phi^2}$ and²⁾ $g = G(\Phi, \Phi)$, $g' = DG(\Phi, \Phi)$ and $\mathfrak{g}' = \mathfrak{D}G(\Phi, \Phi_1)|_{t_1=t}$.

To prove this, consider φ'' as $(\varphi')'$ and integrate by parts. We have

2) It should be noticed that $\mathfrak{g}' \neq g'$ i.e. $\mathfrak{D}G(\Phi, \Phi_1)|_{t_1=t} \neq DG(\Phi, \Phi)$, so that $F'g + F\mathfrak{g}' \neq D(Fg)$.

$$B = - \int F' \varphi^{\alpha+1} \varphi' dt \int^t G \varphi_1^\beta dt_1 - \alpha \int F \varphi^{\alpha-1} \varphi'^2 dt \int^t G \varphi_1^\beta dt_1 \\ - \int F \varphi^{\alpha+1} \varphi' dt \int^t \mathfrak{D} G \varphi_1^\beta dt_1 - \int F g \varphi^{\alpha+\beta} \varphi' dt,$$

whose second integral in view of (13.0) equals $-\alpha B - \alpha \int F \varphi^{\alpha+1} dt \int^t G \varphi_1^\beta dt_1$. On transposing this $-\alpha B$ into the left side and repeating integrations by parts and finally dividing out by $\alpha+1$, we obtain (13.5). Moreover, again in view of (13.0), by adding $\int F \varphi^{\alpha-1} \varphi'^2 dt \int^t G \varphi_1^\beta dt_1$ to both sides, we see that

$$(13.6) \quad \int F(\Phi) \varphi^{\alpha-1} \varphi'^2 dt \int^t G(\Phi, \Phi_1) \varphi_1^\beta dt_1 = \text{“the expression on the right side of (13.5) but having the last integral with the coefficient } \frac{1}{\alpha+1} \text{ (instead of } \frac{-\alpha}{\alpha+1} \text{)”}.$$

In particular, for $\alpha=1, 2, \dots$, we get

$$(13.6.1) \quad \int F(\Phi) \varphi'^2 dt \int^t G(\Phi, \Phi_1) \varphi_1^\beta dt_1 = \frac{1}{6} \left[\int F'' \varphi^4 dt \int^t G \varphi_1^\beta dt_1 + 2 \int F' \varphi^4 dt \int^t \mathfrak{D} G \varphi_1^\beta dt_1 \right. \\ \left. + \int F \varphi^4 dt \int^t \mathfrak{D}^2 G \varphi_1^\beta dt_1 + \int (F'g + Fg') \varphi^{\beta+3} dt \right] + \frac{1}{2(\beta+2)} \int D(Fg) \varphi^{\beta+3} dt + \frac{1}{2} \int F \varphi^2 dt \int^t G \varphi_1^\beta dt_1,$$

$$(13.6.2) \quad \int F(\Phi) \varphi \varphi'^2 dt \int^t G(\Phi, \Phi_1) \varphi_1^\beta dt_1 = \frac{1}{12} \left[\int F'' \varphi^5 dt \int^t G \varphi_1^\beta dt_1 + 2 \int F' \varphi^5 dt \int^t \mathfrak{D} G \varphi_1^\beta dt_1 \right. \\ \left. + \int F \varphi^5 dt \int^t \mathfrak{D}^2 G \varphi_1^\beta dt_1 + \int (F'g + Fg') \varphi^{\beta+4} dt \right] + \frac{1}{3(\beta+3)} \int D(Fg) \varphi^{\beta+4} dt + \frac{1}{3} \int F \varphi^3 dt \int^t G \varphi_1^\beta dt_1,$$

and so on.

Thirdly, let the double integral be of the form $\int F(\Phi) \varphi^\alpha dt \int^t G(\Phi, \Phi_1) \varphi_1^{\beta-1} \varphi_1'^2 dt_1$. To make the inner integral free from t and t -derivatives, we may again treat as above: An integration by parts yields

$$C \equiv \int^t G(\Phi, \Phi_1) \varphi_1^\beta \varphi_1'' dt_1 = G(\Phi, \Phi) \varphi^\beta \varphi' - \int \mathfrak{D}_1 G \cdot \varphi_1^{\beta+1} \varphi_1' dt_1 - \beta \int^t G \varphi_1^{\beta-1} \varphi_1'^2 dt_1,$$

where $\mathfrak{D}_1 = \frac{\partial}{\partial \Phi_1}$.

On substituting $\varphi_1'^2 = \varphi_1 \varphi_1'' + \varphi_1^2$ in the last integral and transposing thus obtained $-\beta C$ into the left side, and repeating integrations by parts, we obtain

$$(13.7) \quad \int^t G(\Phi, \Phi_1) \varphi_1^\beta \varphi_1'' dt_1 = \frac{1}{\beta+1} G(\Phi, \Phi) \varphi^\beta \varphi' - \frac{\beta}{\beta+1} \int^t G \varphi_1^{\beta+1} dt_1 \\ - \frac{1}{(\beta+1)(\beta+2)} \left[\mathfrak{D}_1 G(\Phi, \Phi_1) \Big|_{t_1=t} \varphi^{\beta+2} - \int^t \mathfrak{D}_1^2 G \cdot \varphi_1^{\beta+3} dt_1 \right].$$

Whence, again by aid of (13.0), we find that

(13.8) $\int^t G(\Phi, \Phi_1) \varphi_1^{\beta} \varphi_1'^2 dt_1 =$ "the same expression on the right side of (13.7), but now having the second integral with the coefficient $\frac{1}{\beta+1}$ instead of $\frac{-\beta}{\beta+1}$ ".

In particular, for $\beta=1, 2, 3, \dots$,

$$(13.8.1) \quad \int^t G(\Phi, \Phi_1) \varphi_1'^2 dt_1 = \frac{1}{2} G(\Phi, \Phi) \varphi \varphi' + \frac{1}{2} \int^t G(\Phi, \Phi_1) \varphi_1^2 dt_1 \\ - \frac{1}{6} \mathfrak{D}_1 G \Big|_{t_1=t} \varphi^3 + \frac{1}{6} \int^t \mathfrak{D}_1^2 G \cdot \varphi_1^4 dt_1,$$

$$(13.8.2) \quad \int^t G(\Phi, \Phi_1) \varphi_1 \varphi_1'^2 dt_1 = \frac{1}{3} G(\Phi, \Phi) \varphi^2 \varphi' + \frac{1}{3} \int^t G(\Phi, \Phi_1) \varphi_1^3 dt_1 \\ - \frac{1}{12} \mathfrak{D}_1 G \Big|_{t_1=t} \varphi^4 + \frac{1}{12} \int^t \mathfrak{D}_1^2 G \cdot \varphi_1^5 dt_1,$$

$$(13.8.3) \quad \int^t G(\Phi, \Phi_1) \varphi_1^2 \varphi_1'^2 dt_1 = \frac{1}{4} G(\Phi, \Phi) \varphi^3 \varphi' + \frac{1}{4} \int^t G(\Phi, \Phi_1) \varphi_1^4 dt_1 \\ - \frac{1}{20} \mathfrak{D}_1 G \Big|_{t_1=t} \varphi^5 + \frac{1}{20} \int^t \mathfrak{D}_1^2 G \cdot \varphi_1^6 dt_1, \quad \text{and so on.}$$

Remark. If the double integral be of the form

$$\int F(\Phi) \varphi^a \varphi'^2 dt \int^t G(\Phi, \Phi_1) \varphi_1^{\beta} \varphi_1'^2 dt_1,$$

we may first employ (13.8) and reduce it to the sum of forms (13.2), (13.6) and then make the whole integrals dashfree.

Example 3. Write the double integral

$$\int F(\Phi) \varphi^a dt \int^t G(\Phi, \Phi_1) t_1^3 \varphi_1 dt_1$$

in a form whose integrand is free from t_1 and t_1 -derivatives.

From the equality $\varphi_1'^2 = t_1^2 \varphi_1^2 = \varphi_1^2 + \varphi_1 \varphi_1''$, we get $t_1^2 \varphi_1 = \varphi_1 + \varphi_1''$. This yields on differentiation $t_1^3 \varphi_1' + 2t_1 \varphi_1 = \varphi_1' + \varphi_1'''$, i.e. $t_1^3 \varphi_1 = -3\varphi_1' - \varphi_1'''$. Hence

$$\int^t G(\Phi, \Phi_1) t_1^3 \varphi_1 dt_1 = -3 \int^t G(\Phi, \Phi_1) \varphi_1' dt_1 - \int^t G(\Phi, \Phi_1) (\varphi_1''') dt_1 \\ = -3G(\Phi, \Phi) \varphi + 3 \int^t \mathfrak{D}_1 G \cdot \varphi_1^2 dt_1 - \underline{G(\Phi, \Phi) \varphi''} + \int^t \mathfrak{D}_1 G \cdot \varphi_1 \varphi_1'' dt_1.$$

The last integral being of the form (13.7) with $\beta=1$, equals

$$\frac{1}{2} \mathfrak{D}_1 G \Big|_{t_1=t} \varphi \varphi' - \frac{1}{2} \int^t \mathfrak{D}_1 G \cdot \varphi_1^2 dt_1 - \frac{1}{6} \mathfrak{D}_1^2 G \Big|_{t_1=t} \varphi^3 + \frac{1}{6} \int^t \mathfrak{D}_1^3 G \cdot \varphi_1^4 dt_1.$$

Thus the given double integral attains the required form, if those corresponding to the underlined two:

$$- \int F(\Phi) G(\Phi, \Phi) \varphi^a \varphi'' dt \quad \text{and} \quad \frac{1}{2} \int F(\Phi) \mathfrak{D}_1 G \Big|_{t_1=t} \varphi^{a+1} \varphi' dt$$

do so. For the former it is so by (13.1), while for the latter it is clear. And thus we get

$$(13.9) \quad \int F(\Phi) \varphi^\alpha dt \int^t G(\Phi, \Phi_1) t_1^3 \varphi_1 dt_1 = -3 \int F(\Phi) G(\Phi, \Phi) \varphi^{\alpha+1} dt \\ - \frac{1}{(\alpha+1)(\alpha+2)} \int D^2[F(\Phi) G(\Phi, \Phi)] \varphi^{\alpha+3} dt + \frac{\alpha}{\alpha+1} \int F(\Phi) G(\Phi, \Phi) \varphi^{\alpha+1} dt \\ - \frac{1}{2(\alpha+2)} \int D[F(\Phi)] \cdot [\mathcal{D}_1 G]_{t_1=t} \varphi^{\alpha+3} dt - \frac{1}{6} \int F(\Phi) [\mathcal{D}_1^2 G]_{t_1=t} \varphi^{\alpha+3} dt \\ + \frac{5}{2} \int F(\Phi) \varphi^\alpha dt \int^t \mathcal{D}_1 G \cdot \varphi_1^2 dt_1 + \frac{1}{2} \int F(\Phi) \varphi^\alpha dt \int^t \mathcal{D}_1^3 G \cdot \varphi_1^4 dt_1, \quad \text{Q.E.I.}$$

(D) It requires frequently to be found those values of $G=G(\Phi, \Phi_1)$, $\mathcal{D}G$, $\mathcal{D}_1 G$ and $\mathcal{D}_1^2 G$, when $t_1=t$. In the following the actual expression of $G(\Phi, \Phi_1)$ is often of the form

$$G(\Phi, \Phi_1) = \frac{(\Phi - \Phi_1)^j \Phi_1^{i-1}}{j! (i-1)!} \equiv (\Phi - \Phi_1)^j \Phi_1^{i-1}.$$

Hence, if $j \geq 1$, $G(\Phi, \Phi) = 0$. Moreover, if $j > k \geq 1$, all $\mathcal{D}^l \mathcal{D}_1^{k-l} G(\Phi, \Phi_1)$ for $l=0, 1, \dots, k$ do vanish when $t_1=t$. However, if $j \leq k$, it is not necessarily so. In the most frequent case $j=1$, we have by Leibnitz's formula

$$(13.10) \quad \mathcal{D}_1^k [(\Phi - \Phi_1) \Phi_1^{i-1}]_{t_1=t} = -k \Phi^{i-k} \quad \text{for } k=0, 1, 2, \dots.$$

$$\text{E.g.} \quad \mathcal{D}_1 [(\Phi - \Phi_1) \Phi_1^{i-1}]_{t_1=t} = -\Phi^{i-1}, \quad \mathcal{D}_1^2 [(\Phi - \Phi_1) \Phi_1^{i-1}] = -2\Phi^{i-2}, \dots, \\ \mathcal{D}_1^i [(\Phi - \Phi_1) \Phi_1^{i-1}] = -i,$$

and if $k > i$, of course, $\mathcal{D}_1^k [(\Phi - \Phi_1) \Phi_1^{i-1}] = -k \Phi^{i-k} = 0$, because, then the denominator $\Gamma(i-k+1) = \infty$.

(E) Any formal expression obtained on the way should be reduced to a concrete one if possible. For example, it is really

$$(13.11) \quad \int (\Phi - \Phi_1)^{n-i-1} \Phi_1^{i-1} \varphi_1 dt_1 = \Phi^{k-1}.$$

For, upon putting $\Phi_1 = \Phi v$, $d\Phi_1 = \varphi_1 dt_1 = \Phi dv$, the integral reduces to

$$\frac{\Phi^{k-1}}{[k-i-1][i-1]} \int_0^1 (1-v)^{k-i-1} v^{i-1} dv = \frac{\Phi^{k-1}}{[k-1]} = \Phi^{k-1}.$$

In particular, if $k=i+1$, $\int \Phi_1^{i-1} \varphi_1 dt_1 = \Phi^i$; if $t=\infty$, $\Phi=1$ and $\int (1-\Phi_1)^{k-i-1} \times \Phi^{i-1} \varphi_1 dt_1 = 1$. It is evident that $\int F(\Phi) \varphi dt = \int_0^1 F(\Phi) d\Phi$. On denoting $\Phi^{i-1} (1-\Phi)^{n-i} \equiv F$ and $D = \frac{d}{d\Phi}$, we had in (1.3) (2.4) (2.6) (2.7) (2.8) of Part I:

$$E(t_{i|n}^0) = 1 = [n] \int F \varphi dt, \quad E(t_{i|n}) = [n] \int D F \cdot \varphi^2 dt, \quad E(t_{i|n}^2) = 1 + \frac{1}{2} [n] \int D^2 F \cdot \varphi^3 dt, \\ E(t_{i|n}^3) = \frac{5}{2} E(t_{i|n}) + \frac{[n]}{[3]} \int D^3 F \cdot \varphi^4 dt, \quad E(t_{i|n}^4) = \frac{13}{3} E(t_{i|n}^2) - \frac{4}{3} + \frac{[n]}{[4]} \int D^4 F \cdot \varphi^5 dt,$$

so that $\int D^k F \cdot \varphi^{k+1} dt$ ($k=0, 1, 2, \dots$) may be expressed in terms of $E(t_{i|n}^{k-j})$.

Example 4. Show that

$$1-n! \int (1-\Phi)^{\alpha} \Phi^{\beta+\gamma-1} \varphi^{\delta} dt = n! \int (1-\Phi)^{n-k} \varphi dt \int^t (\Phi-\Phi_1)^{k-i-1} \Phi_1^{i-1} \varphi_1 dt_1 \\ - n! \int (1-\Phi)^{\alpha} \varphi^{\delta} dt \int^t (\Phi-\Phi_1)^{\beta-1} \Phi_1^{\gamma-1} \varphi_1 dt_1.$$

This is evident: For, the first double integral on the right side is nothing but

$$\iint f(t_{i|n}, t_{k|n}) dt_{i|n} dt_{k|n} = 1,$$

while, the second reduces to the single integral on the left side because of (13.11).

§ 14. Expectations of Products $t_{i|n}^p t_{k|n}^q$ ($p+q=3$ or 4 and $1 \leq i < k \leq n$).

Practically it need not obtain the formal expression of moments, especially for small samples such as $n=2, 3$: We may solely proceed to calculate directly every value separately. However, for the sake of theoretical completeness and eventual use even in case of large sample, we have prepared the detailed formulas, which are free from t and t -derivatives. By symmetrical property obtained in (5.1): $E(t_{n-i+1}^p t_{n-k+1}^q) = (-)^{p+q} E(t_{i|n}^p t_{k|n}^q)$, we have only to consider those cases $p \geq q$.

$$1^\circ \quad E(t_{i|n}^2 t_{k|n}) = \underline{n} \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi-\Phi_1)^{k-i-1} \Phi_1^{i-1} \varphi_1' t_1 dt_1.$$

Firstly for $k=i+1$

$$E(t_{i|n}^2 t_{i+1|n}) = \underline{n} \int (1-\Phi)^{n-i-1} \varphi' dt \int^t \Phi_1^{i-1} t_1 \varphi_1' dt_1$$

By integrations by parts the inner integral reduces to

$$(14.1.0) \quad I(t) \equiv \int \Phi_1^{i-1} t_1 \varphi_1' dt_1 = -\Phi_i - \Phi_1^{i-1} \varphi' + \frac{1}{2} \Phi_1^{i-2} \varphi^2 - \frac{1}{2} \int \Phi_1^{i-3} \varphi_1^3 dt_1.$$

This being substituted in $E(t_{i|n}^2 t_{i+1|n}) = \underline{n} \int (1-\Phi)^{n-i-1} I(t) \varphi' dt$ and repeatedly integrated by parts, we find³⁾

$$(14.1.1) \quad E(t_{i|n}^2 t_{i+1|n}) = -\underline{n} \int (1-\Phi)^{n-i-2} \Phi_i \varphi^2 dt + \frac{\underline{n}}{2} \int (1-\Phi)^{n-i-1} \Phi_i \varphi^2 dt \\ - \frac{\underline{n}}{6} \int (1-\Phi)^{n-i-3} \Phi_1^{i-1} \varphi^4 dt + \frac{\underline{n}}{2} \int (1-\Phi)^{n-i-2} \Phi_i \varphi^4 dt \\ + \frac{\underline{n}}{6} \int (1-\Phi)^{n-i-1} \Phi_i \varphi^4 dt - \frac{\underline{n}}{2} \int (1-\Phi)^{n-i-2} \varphi^2 dt \int^t \Phi_1^{i-3} \varphi_1^3 dt_1,$$

where the first integral becomes $E(t_{i+1|n})$.

3) As check, we may examine the dimension of each integral to be n , e.g. that of the first integral is really $n-i-2+i+2=n$, &c.

Secondly, for $k \geq i+2$, the inner integral $I(t)$ becomes now

$$I(t) = \int_0^t (\Phi - \Phi_1) \Phi_1^{i-1} \varphi_1' t_1 dt_1 = -\Phi^{i+1} - \frac{1}{2} \Phi^{i-1} \varphi^2 - \frac{1}{2} \int_0^t \mathfrak{D}_1^2 [(\Phi - \Phi_1) \Phi_1^{i-1}] \varphi_1^3 dt_1,$$

and we obtain

$$(14.1.2) \quad E(t_{i|n}^2 t_{i+2|n}) = \underline{n} \int D[(1-\Phi)^{n-i-2} \Phi^{i+1}] \varphi^2 dt - \underline{n} \int (1-\Phi)^{n-i-2} \Phi^{i-2} \varphi^4 dt \\ + \frac{\underline{n}}{6} \int D[(1-\Phi)^{n-i-2} \Phi^{i-1}] \varphi^4 dt + \frac{\underline{n}}{2} \int (1-\Phi)^{n-i-2} \varphi^2 dt \int_0^t \Phi_1^{i-3} \varphi_1^3 dt_1 \\ - \frac{\underline{n}}{2} (1-\Phi)^{n-i-3} \varphi^2 dt \int_0^t \mathfrak{D}_1^2 [(\Phi - \Phi_1) \Phi_1^{i-1}] \varphi_1^3 dt_1.$$

where the first integral is $E(t_{i+2|n})$.

Thirdly for $k \geq i+3$, the inner integral becomes

$$I(t) = -\Phi^{k-1} - \frac{1}{2} \int_0^t \mathfrak{D}_1^2 [(\Phi - \Phi_1)^{k-i-1} \Phi^{i-1}] \varphi_1^3 dt_1$$

and we get

$$(14.1.3) \quad E(t_{i|n}^2 t_{k|n}) = \underline{n} \int D[(1-\Phi)^{n-k} \Phi^{k-1}] \varphi^2 dt - \frac{\underline{n}}{2} \int (1-\Phi)^{n-k-1} \varphi^2 dt \\ \times \int_0^t \mathfrak{D}_1^2 [(\Phi - \Phi_1)^{k-i-1} \Phi_1^{i-1}] \varphi_1^3 dt_1 + \frac{\underline{n}}{2} \int (1-\Phi)^{n-k} \varphi^2 dt \int_0^t \mathfrak{D}_1^2 [(\Phi - \Phi_1)^{k-i-2} \Phi_1^{i-1}] \varphi_1^3 dt_1 \\ + \delta_{i+3}^k \frac{\underline{n}}{2} \int (1-\Phi)^{n-k} \Phi^{i-1} \varphi^4 dt,$$

where the first integral is $E(t_{k|n})$ and δ_{i+3}^k denotes Kronecker's delta.

$$2^\circ \quad E(t_{i|n}^2 t_{k|n}^2) = \underline{n} \int (1-\Phi)^{n-k} \varphi' t dt \int_0^t (\Phi - \Phi_1)^{k-i-1} \Phi_1^{i-1} \varphi_1' t_1 dt_1.$$

Firstly for $k=i+1$,

$$E(t_{i|n}^2 t_{i+1|n}^2) = \underline{n} \int (1-\Phi)^{n-i-1} \varphi' t dt \int_0^t \Phi_1^{i-1} \varphi_1' t_1 dt_1,$$

so that the inner integral $I(t)$ is the same as in (14.1.0), and $I'(t) = \Phi^{i-1} \varphi' t$. Hence, if we write $\underline{n} (1-\Phi)^{n-i-1} = F(\Phi)$,

$$E(t_{i|n}^2 t_{i+1|n}^2) = \int F(\Phi) I(t) \varphi' t dt = - \int F(\Phi) I(t) \varphi dt - \int F'(\Phi) I(t) \varphi^2 t dt - \int F(\Phi) I'(t) \varphi t dt \\ = (i) + (ii) + (iii).$$

On, substituting the values of $I(t)$ and $I'(t)$, we obtain

$$(i) = 1 - \frac{1}{2} \int D(F \Phi^{i-1}) \varphi^3 dt - \frac{1}{2} \int F \Phi^{i-2} \varphi^3 dt + \frac{1}{2} \int F \varphi dt \int_0^t \Phi_1^{i-3} \varphi_1^3 dt_1 \\ (ii) = \frac{1}{2} \int D(F' \Phi^i) \varphi^3 dt - \int F' \Phi^{i-1} \varphi \varphi'^2 dt - \frac{1}{8} \int D(F' \Phi^{i-2}) \varphi^5 dt \\ + \frac{1}{4} \int F'' \varphi^3 dt \int_0^t \Phi_1^{i-3} \varphi_1^3 dt_1 + \frac{1}{4} \int F' \Phi^{i-3} \varphi^5 dt \quad \dots\dots\dots(a)$$

where for the second integral (13.2.2) may be employed⁴⁾. And by (13.3)

$$(iii) = \int F \underline{\Phi}^{i-1} \varphi'^2 dt = \frac{2}{3} \int D(F \underline{\Phi}^{i-1}) \varphi^3 dt + \frac{1}{24} \int D^3(F \underline{\Phi}^{i-1}) \varphi^5 dt.$$

Hence, we obtain

$$(14.2.1) \quad E(t_{i|n}^2 t_{i+1|n}^2) = 1 + |n| \int \left\{ \frac{1}{2} (\underline{1} - \underline{\Phi})^{n-i-3} \underline{\Phi}^i - \frac{1}{3} (\underline{1} - \underline{\Phi})^{n-i-2} \underline{\Phi}^{i-1} \right. \\ \left. - \frac{1}{3} (\underline{1} - \underline{\Phi})^{n-i-1} \underline{\Phi}^{i-2} \right\} \varphi^3 dt + \frac{|n|}{24} \int \{ (\underline{1} - \underline{\Phi})^{n-i-4} \underline{\Phi}^{i-1} - 4(\underline{1} - \underline{\Phi})^{n-i-3} \underline{\Phi}^{i-2} \\ - 4(\underline{1} - \underline{\Phi})^{n-i-2} \underline{\Phi}^{i-3} + (\underline{1} - \underline{\Phi})^{n-i-1} \underline{\Phi}^{i-4} \} \varphi^5 dt \\ + \frac{|n|}{2} \int (\underline{1} - \underline{\Phi})^{n-i-1} \varphi dt \int \underline{\Phi}_1^{i-3} \varphi_1^3 dt_1 + \frac{|n|}{4} \int (\underline{1} - \underline{\Phi})^{n-i-3} \varphi^3 dt \int \underline{\Phi}_1^{i-3} \varphi_1^3 dt_1.$$

Secondly for $k=i+2$, on treating similarly as in (2.1), but now with the inner integral of (1.2): $I(t) = -\underline{\Phi}^{i+1} - \frac{1}{2} \int \underline{\Phi}^{i-1} \varphi^2 - \frac{1}{2} \int \mathfrak{D}_1^2[(\underline{\Phi} - \underline{\Phi}_1) \underline{\Phi}_1^{i-1}] \varphi_1^3 dt_1$, so that $I'(t) = -\underline{\Phi}^i \varphi - \underline{\Phi}^{i-1} \varphi \varphi' + \frac{1}{2} \underline{\Phi}^{i-2} \varphi^3 - \frac{1}{2} \int \underline{\Phi}_1^{i-3} \varphi_1^3 dt_1$, we find

$$(14.2.2) \quad E(t_{i|n}^2 t_{k|n}^2) = 1 + |n| \int \left\{ \frac{1}{2} (\underline{1} - \underline{\Phi})^{n-i-4} \underline{\Phi}^{i+1} - [(\underline{1} - \underline{\Phi})^{n-i-3} \underline{\Phi}^i] \right. \\ \left. + \frac{2}{3} [(\underline{1} - \underline{\Phi})^{n-i-2} \underline{\Phi}^{i-1}] \right\} \varphi^3 dt + |n| \int \left\{ \frac{1}{24} (\underline{1} - \underline{\Phi})^{n-i-4} \underline{\Phi}^{i-1} + \frac{2}{3} (\underline{1} - \underline{\Phi})^{n-i-3} \underline{\Phi}^{i-2} \right. \\ \left. + \frac{1}{24} (\underline{1} - \underline{\Phi})^{n-i-2} \underline{\Phi}^{i-3} \right\} \varphi^5 dt - \frac{|n|}{2} \int (\underline{1} - \underline{\Phi})^{n-i-3} \varphi^3 dt \int \underline{\Phi}_1^{i-3} \varphi_1^3 dt_1 \\ + \frac{|n|}{2} \int (\underline{1} - \underline{\Phi})^{n-i-2} \varphi dt \int \mathfrak{D}_1^2[(\underline{\Phi} - \underline{\Phi}_1) \underline{\Phi}_1^{i-1}] \varphi_1^3 dt_1 \\ + \frac{|n|}{4} \int (\underline{1} - \underline{\Phi})^{n-i-4} \varphi^3 dt \int \mathfrak{D}_1^2[(\underline{\Phi} - \underline{\Phi}_1) \underline{\Phi}_1^{i-1}] \varphi_1^3 dt_1.$$

Thirdly for $k \geq i+3$, the inner integral is the same as in (1.3): $I(t) = \int^t (\underline{\Phi} - \underline{\Phi}_1)^{n-i-1} \underline{\Phi}_1^{i-1} \varphi_1' dt_1 = -\underline{\Phi}^{k-1} - \frac{1}{2} \int \mathfrak{D}_1^2[(\underline{\Phi} - \underline{\Phi}_1)^{k-i-1} \underline{\Phi}_1^{i-1}] \varphi_1^3 dt_1$, so that $I'(t) = -\underline{\Phi}^{k-2} \varphi - \frac{\varphi}{2} \int \mathfrak{D}_1^2[(\underline{\Phi} - \underline{\Phi}_1)^{k-i-2} \underline{\Phi}_1^{i-1}] \varphi_1^3 dt_1$.

4) Otherwise :

$$(ii) = \int F' I \varphi \varphi' dt = -\frac{1}{2} \int F'' I \varphi^3 dt - \frac{1}{2} \int F' I \varphi^2 dt = \frac{1}{2} \int F'' \underline{\Phi}^i \varphi^3 dt - \frac{1}{8} \int D[F'' \underline{\Phi}^{i-1}] \varphi^5 dt \\ - \frac{1}{4} \int F'' \underline{\Phi}^{i-2} \varphi^5 dt + \frac{1}{4} \int F'' \varphi^3 dt \int^t \underline{\Phi}_1^{i-3} \varphi_1^3 dt_1 + \frac{1}{2} \int F' \underline{\Phi}^{i-1} \varphi \varphi'^2 dt, \quad \dots (b)$$

which appears to differ from the before obtained result (a). However, this difference is only superficial, because, if (a) be equated to (b), we shall thereupon obtain $\frac{3}{2} \int F' \underline{\Phi}^{i-1} \varphi \varphi'^2 dt = \frac{1}{8} \int \left\{ F'' \underline{\Phi}^{i-1} + 2F'' \underline{\Phi}^{i-2} + F' \underline{\Phi}^{i-3} \right\} \varphi^5 dt + \frac{1}{2} \int F' \underline{\Phi}^{i-1} \varphi^3 dt$ which agrees materially with (13.2.2). Such alike occurs frequently.

And we get

$$(14.2.3) \quad E(t_{i|n}^2 t_{i+1|n}^2) \quad (k \geq i+3)$$

$$\begin{aligned} &= 1 + \frac{|n|}{2} \int (1-\Phi)^{n-k-2} \Phi^{k-1} \varphi^3 dt - |n| \int (1-\Phi)^{n-k-1} \Phi^{k-2} \varphi^3 dt + \frac{|n|}{2} \int (1-\Phi)^{n-k} \Phi^{k-3} \varphi^3 dt \\ &\quad + \frac{|n|}{2} \int (1-\Phi)^{n-k} \varphi dt \int^t \mathcal{D}_1^2[(\Phi-\Phi_1)^{k-i-1} \Phi_1^{i-1}] \varphi_1^3 dt_1 - \frac{|n|}{2} \int (1-\Phi)^{n-k-1} \varphi^3 dt \\ &\quad \times \int \mathcal{D}_1^2[(\Phi-\Phi_1)^{k-i-2} \Phi_1^{i-1}] \varphi_1^3 dt_1 + \frac{|n|}{4} \int (1-\Phi)^{n-k-2} \varphi^3 dt \int^t \mathcal{D}_1^2[(\Phi-\Phi_1)^{k-i-1} \Phi_1^{i-1}] \varphi_1^3 dt_1 \\ &\quad - \frac{|n|}{4} \int (1-\Phi)^{n-k} \varphi^3 dt \int^t \mathcal{D}_1^2[(\Phi-\Phi_1)^{k-i-3} \Phi_1^{i-1}] \varphi_1^3 dt_1 - \delta_{i+3}^k \frac{3|n|}{8} \int \{(1-\Phi)^{n-k} \Phi^{i-2} \\ &\quad + (1-\Phi)^{n-k-1} \Phi^{i-1}\} \varphi^5 dt + \delta_{k+4}^k \frac{|n|}{4} \int (1-\Phi)^{n-k} \Phi^{i-1} \varphi^5 dt. \end{aligned}$$

$$3^\circ \quad E(t_{i|n}^3 t_{i+1|n}) = |n| \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi-\Phi_1)^{k-i-1} \Phi_1^{i-1} \varphi_1' t_1^2 dt_1.$$

Firstly for $k=i+1$

$$E(t_{i|n}^3 t_{i+1|n}) = |n| \int (1-\Phi)^{n-i-1} \varphi' dt \int^t \Phi_1^{i-1} \varphi_1' t_1^2 dt_1 = |n| \int (1-\Phi)^{n-i-1} \varphi' I(t) dt.$$

Here the inner integral being

$$\begin{aligned} I(t) &= \int^t \Phi_1^{i-1} \varphi_1' t_1^2 dt_1 = -\Phi^{i-1} \varphi' t - \int^t \Phi_1^{i-2} \varphi_1'^2 dt_1 - 2 \int \Phi_1^{i-1} \varphi_1' dt_1, \\ E(t_{i|n}^3 t_{i+1|n}) &= -|n| \int (1-\Phi)^{n-i-1} \Phi^{i-1} \varphi'^2 t dt - |n| \int (1-\Phi)^{n-i-1} \varphi' dt \int^t \Phi_1^{i-1} \varphi_1'^2 dt_1 \\ &\quad + 2|n| \int (1-\Phi)^{n-i-1} \varphi' dt \int^t \Phi_1^{i-1} \varphi_1' dt_1. \end{aligned}$$

Hence, making use of (13.3), (13.8.1) (13.2.3) &c., we find⁵⁾

$$\begin{aligned} (14.3.1) \quad E(t_{i|n}^3 t_{i+1|n}) &= |n| \int \left\{ \frac{2}{3} (1-\Phi)^{n-i-1} \Phi^{i-2} + \frac{5}{3} (1-\Phi)^{n-i-2} \Phi^{i-1} \right\} \varphi^3 dt \\ &\quad + \frac{|n|}{24} \int [(1-\Phi)^{n-i-4} \Phi^{i-1} - 4(1-\Phi)^{n-i-3} \Phi^{i-2} + 6(1-\Phi)^{n-i-2} \Phi^{i-3} + (1-\Phi)^{n-i-1} \Phi_1^{i-4}] \varphi^5 dt \\ &\quad - \frac{5}{2} |n| \int (1-\Phi)^{n-i-2} \varphi^2 dt \int^t \Phi_1^{i-2} \varphi_1^2 dt_1 - \frac{|n|}{6} \int (1-\Phi)^{n-i-2} \varphi^2 dt \int^t \Phi_1^{i-4} \varphi_1^4 dt_1. \end{aligned}$$

Secondly for $k=i+2$, $E(t_{i|n}^3 t_{i+2|n}) = |n| \int (1-\Phi)^{n-i-2} \varphi' dt \int^t (\Phi-\Phi_1) \Phi_1^{i-1} \varphi_1' t_1^2 dt_1$,
and we find the inner integral to be

5) Otherwise, we may utilize e.g. (13.9) directly and obtain a result that is apparently different from (14.3.1) in the integrand containing φ^5 as follows:

$$\begin{aligned} &|n| \int \left[\frac{1}{24} D^3 \{ (1-\Phi)^{n-i-1} \Phi^{i-2} \} + \frac{1}{12} D^2 \{ (1-\Phi)^{n-i-2} \Phi^{i-1} \} + \frac{1}{8} D \{ (1-\Phi)^{n-i-2} \Phi^{i-2} \} \right. \\ &\quad \left. + \frac{1}{6} (1-\Phi)^{n-i-2} \Phi^{i-3} \right] \varphi^5 dt. \end{aligned}$$

However, they are really equal, since it is readily seen that the expressions under both square brackets are the same polynomial: $\frac{1}{8} (1-\Phi)^{n-i-2} \Phi^{i-3}$.

$$\begin{aligned}
I(t) &= -\int^t \mathfrak{D}_1 G \cdot \varphi_1'^2 dt_1 - 2 \int^t \mathfrak{D}_1 G \cdot \varphi_1^2 dt_1 \quad (\text{if } G = G(\Phi, \Phi_1) = (\Phi - \Phi_1) \Phi_1^{i-1}) \\
&= \frac{1}{2} \Phi^{i-1} \varphi \varphi' - \frac{1}{3} \Phi^{i-2} \varphi^3 - \frac{5}{2} \int^t \mathfrak{D}_1 G \cdot \varphi_1^2 dt_1 - \frac{1}{6} \int^t \mathfrak{D}_1^3 G \cdot \varphi_1^4 dt_1
\end{aligned}$$

on applying (13.8.1). Hence, we get

$$\begin{aligned}
(14.3.2) \quad E(t_{i|n}^3 t_{i+2|n}) &= |n| \int (1-\Phi)^{n-i-2} I(t) \varphi' dt \\
&= -\frac{7}{3} |n| \int (1-\Phi)^{n-i-2} \Phi^{i-1} \varphi^3 dt + \frac{|n|}{24} \int [(1-\Phi)^{n-i-4} \Phi^{i-1} - 4(1-\Phi)^{n-i-3} \Phi^{i-2} \\
&\quad - 9(1-\Phi)^{n-i-2} \Phi^{i-3}] \varphi^5 dt + \frac{5}{2} |n| \int (1-\Phi)^{n-i-2} \varphi^2 dt \int^t \Phi_1^{i-2} \varphi_1^2 dt_1 \\
&\quad - \frac{5}{2} |n| \int (1-\Phi)^{n-i-3} \varphi^2 dt \int^t \mathfrak{D}_1 [(\Phi - \Phi_1) \Phi_1^{i-1}] \varphi_1^2 dt_1 + \frac{1}{6} |n| \int (1-\Phi)^{n-i-2} \varphi^2 dt \int^t \Phi_1^{i-4} \varphi_1^4 dt_1 \\
&\quad - \frac{1}{6} |n| \int (1-\Phi)^{n-i-3} \varphi^2 dt \int^t \mathfrak{D}_1^3 [(\Phi - \Phi_1) \Phi_1^{i-1}] \varphi_1^4 dt_1.
\end{aligned}$$

Thirdly for $k \geq i+3$, we have the integral $I(t) = \int^t (\Phi - \Phi_1)^{k-i-1} \Phi_1^{i-1} \varphi_1' t_1^2 dt_1 =$
 $-\frac{5}{2} \int^t \mathfrak{D}_1 [(\Phi - \Phi_1)^{k-i-1} \Phi_1^{i-1}] \varphi_1^2 dt_1 - \frac{1}{6} \int^t \mathfrak{D}_1^3 [(\Phi - \Phi_1)^{k-i-1} \Phi_1^{i-1}] \varphi_1^4 dt_1 + \delta_{i+3}^k \frac{1}{6} \Phi^{i-1} \varphi^3.$

Hence, we get

$$\begin{aligned}
(14.3.3) \quad E(t_{i|n}^3 t_{k|n}) &= |n| \int (1-\Phi)^{n-k} \varphi' I(t) dt \quad (k \geq i+3) \\
&= \frac{5}{2} |n| \int (1-\Phi)^{n-k} \varphi^2 dt \int^t \mathfrak{D}_1 [(\Phi - \Phi_1)^{k-i-2} \Phi_1^{i-1}] \varphi_1^2 dt_1 - \frac{5}{2} |n| \int (1-\Phi)^{n-k-1} \varphi^2 dt \\
&\quad \times \int^t \mathfrak{D}_1 [(\Phi - \Phi_1)^{k-i-1} \Phi_1^{i-1}] \varphi_1^2 dt_1 + \frac{1}{6} |n| \int (1-\Phi)^{n-k} \varphi^2 dt \int^t \mathfrak{D}_1^3 [(\Phi - \Phi_1)^{k-i-2} \Phi_1^{i-1}] \varphi_1^4 dt_1 \\
&\quad - \frac{1}{6} |n| \int (1-\Phi)^{n-k-1} \varphi^2 dt \int^t \mathfrak{D}_1^3 [(\Phi - \Phi_1)^{k-i-1} \Phi_1^{i-1}] \varphi_1^4 dt_1 \\
&\quad + \delta_{i+3}^k \frac{|n|}{24} \int \{11(1-\Phi)^{n-k} \Phi^{i-2} + (1-\Phi)^{n-k-1} \Phi^{i-1}\} \varphi^5 dt - \delta_{i+4}^k \frac{|n|}{6} \int (1-\Phi)^{n-k} \Phi^{i-1} \varphi^5 dt.
\end{aligned}$$

§ 15. Expectation of Products $t_{i|n}^p t_{j|n}^q t_{k|n}^r$ for $p+q+r=3$ or 4 and $1 \leq i < j < k \leq n$.

1° $E(t_{i|n} t_{j|n} t_{k|n})$

$$= -|n| \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi - \Phi_1)^{k-j-1} \varphi_1' dt_1 \int^{t_1} (\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' dt_2.$$

Firstly, let $j=i+1$, $k=i+2$:

$$E(t_{i|n} t_{i+1|n} t_{i+2|n}) = -|n| \int (1-\Phi)^{n-i-2} \varphi' dt \int^t \varphi_1' dt_1 \int^{t_1} \Phi_2^{i-1} \varphi_2' dt_2.$$

Hence, the inner integrals are

$$I_1(t_1) = \int^{t_1} \Phi_2^{i-1} \varphi_2' dt_2 = \Phi_1^{i-1} \varphi_1 - \int^{t_1} \Phi_2^{i-2} \varphi_2^2 dt_2,$$

and

$$I(t) = \int^t I_1(t_1) \varphi_1' dt_1 = \frac{1}{2} \underline{\Phi}^{i-1} \varphi^2 - \varphi \int^t \underline{\Phi}_1^{i-2} \varphi_1^2 dt_1 + \frac{1}{2} \int^t \underline{\Phi}_1^{i-2} \varphi_1^3 dt_1.$$

Therefore

$$\begin{aligned} (15.1.1) \quad E(t_{i|n} t_{i+1|n} t_{i+2|n}) &= -|n \int (1-\Phi)^{n-i-2} \varphi' I(t) dt \\ &= \frac{|n}{6} \int D[(1-\Phi)^{n-i-2} \underline{\Phi}^{i-1}] \varphi^4 dt - \frac{|n}{2} \int (1-\Phi)^{n-i-3} \varphi^2 dt \int^t \underline{\Phi}_1^{i-2} \varphi_1^3 dt_1 \\ &\quad + \frac{|n}{2} \int (1-\Phi)^{n-i-3} \varphi^3 dt \int^t \underline{\Phi}_1^{i-2} \varphi_1^2 dt_1. \end{aligned}$$

Secondly, let still $j=i+1$ but $k=i+3$; then the innermost integral $I_1(t_1)$ is the same as before. However, now

$$\begin{aligned} I(t) &= \int^t (\Phi - \Phi_1) \varphi_1' I_1(t_1) dt_1 = \int^t \underline{\Phi}_1^{i-1} \varphi_1^3 dt_1 - \int^t \varphi_1^2 dt_1 \int^{t_1} \underline{\Phi}_2^{i-2} \varphi_2^2 dt_2 \\ &\quad + \frac{1}{2} \int^t \mathfrak{D}_1[(\Phi - \Phi_1) \underline{\Phi}_1^{i-1}] \varphi_1^3 dt_1 \end{aligned}$$

Therefore

$$\begin{aligned} (15.1.2) \quad E(t_{i|n} t_{i+1|n} t_{i+3|n}) &= \frac{|n}{2} \int (1-\Phi)^{n-i-3} \underline{\Phi}^{i-1} \varphi^4 dt - |n \int (1-\Phi)^{n-i-4} \varphi^2 dt \\ &\quad \times \int^t \underline{\Phi}_1^{i-1} \varphi_1^3 dt_1 + \frac{|n}{2} \int (1-\Phi)^{n-i-3} \varphi^2 dt \int^t \underline{\Phi}_1^{i-2} \varphi_1^3 dt_1 - |n \int (1-\Phi)^{n-i-3} \varphi^3 dt \int^t \underline{\Phi}_1^{i-2} \varphi_1^2 dt_1 \\ &\quad - \frac{|n}{2} \int (1-\Phi)^{n-i-4} \varphi^2 dt \int^t \mathfrak{D}_1[(\Phi - \Phi_1) \underline{\Phi}_1^{i-1}] \varphi_1^3 dt_1 \\ &\quad + |n \int (1-\Phi)^{n-i-4} \varphi^2 dt \int^t \varphi_1^2 dt_1 \int^{t_1} \underline{\Phi}_2^{i-2} \varphi_2^2 dt_2. \end{aligned}$$

Thirdly, let $j=i+2$ but $k \geq i+4$. The innermost integral is still the same as before, but now

$$\begin{aligned} I(t) &= \int^t (\Phi - \Phi_1)^{k-i-2} I_1(t_1) \varphi_1' dt_1 \quad (I_1(t_1) = \underline{\Phi}_1^{i-1} \varphi_1 - \int^{t_1} \underline{\Phi}_2^{i-2} \varphi_2^2 dt_2) \\ &= -\frac{1}{2} \int^t \mathfrak{D}_1[(\Phi - \Phi_1)^{k-i-2} \underline{\Phi}_1^{i-1}] \varphi_1^3 dt_1 - \int^t (\Phi - \Phi_1)^{k-i-3} \varphi_1^2 dt_1 \int^t \underline{\Phi}_2^{i-2} \varphi_2^2 dt_2 \\ &\quad + \int^t (\Phi - \Phi_1)^{k-i-2} \underline{\Phi}_1^{i-2} \varphi_1^3 dt_1 \end{aligned}$$

and $I'(t) = \varphi \int^t (\Phi - \Phi_1)^{k-i-3} I_1(t_1) \varphi_1' dt_1;$

so that $I'(t)$ is obtained by replacing⁶⁾ k by $k-1$ in the expression of $I(t)$, and multiplying by φ . Hence

6) If the power index obtained by this replacement becomes negative, of course, that term vanishes and therefore need not be written down. This remark shall be used so frequently below.

$$\begin{aligned}
E(t_{i|n} t_{i+1|n} t_{k|n}) &= -|n \int (1-\Phi)^{n-k} \varphi' I(t) dt = -|n \int (1-\Phi)^{n-k-1} \varphi^2 I(t) dt \\
&+ |n \int (1-\Phi)^{n-k} \varphi I'(t) dt = -|n \int (1-\Phi)^{n-k-1} \varphi^2 I(t) dt + |n \int (1-\Phi)^{n-(k-1)-1} \varphi I'(t) dt \\
&= (i) + (ii) .
\end{aligned}$$

Therefore, (ii) is obtainable by simply writing $k-1$ for k in the resulting expression of (i), then changing the sign, and thus the labour is reduced by half.

(15.1.3) $E(t_{i|n} t_{i+1|n} t_{k|n}) \quad (k \geq i+4)$

$$\begin{aligned}
&= -\frac{|n}{2} \int (1-\Phi)^{n-k-1} \varphi^2 dt \int^t \{(\Phi-\Phi_1)^{k-i-2} \Phi_1^{i-2} + (\Phi-\Phi_1)^{k-i-3} \Phi_1^{i-1}\} \varphi_1^3 dt_1 \\
&+ \frac{|n}{2} \int (1-\Phi)^{n-k} \varphi^2 dt \int^t \{(\Phi-\Phi_1)^{k-i-3} \Phi_1^{i-2} + (\Phi-\Phi_1)^{k-i-4} \Phi_1^{i-1}\} \varphi_1^3 dt_1 \\
&+ |n \int (1-\Phi)^{n-k-1} \varphi^2 dt \int^t (\Phi-\Phi_1)^{k-i-3} \varphi_1^2 dt_1 \int^{t_1} \Phi_2^{i-2} \varphi_2^2 dt_2 \\
&- |n \int (1-\Phi)^{n-k} \varphi^2 dt \int^t (\Phi-\Phi_1)^{k-i-4} \varphi_1^2 dt_1 \int^{t_1} \Phi_2^{i-2} \varphi_2^2 dt_2 .
\end{aligned}$$

Fourthly we proceed to the case that $j=i+2$. At first letting $k=i+3$,

$$E(t_{i|n} t_{i+2|n} t_{i+3|n}) = -|n \int (1-\Phi)^{n-i-3} \varphi' dt \int^t \varphi_1' dt_1 \int^{t_1} (\Phi_1-\Phi_2) \Phi_2^{i-1} \varphi_2' dt_2 .$$

Here $I_1(t_1) = \int^{t_1} (\Phi_1-\Phi_2) \Phi_2^{i-1} \varphi_2' dt_2 = -\int^{t_1} \mathfrak{D}_2[(\Phi_1-\Phi_2) \Phi_2^{i-1}] \varphi_2^2 dt_2$

and $I_1'(t_1) = \varphi_1 \int^{t_1} \Phi_2^{i-1} \varphi_2' dt_2 = \Phi_1^{i-1} \varphi_1^2 - \varphi_1 \int^{t_1} \Phi_2^{i-2} \varphi_2^2 dt_2 .$

So that

$$I(t) = \int^t \varphi_1' I_1(t_1) dt_1 = -\varphi \int^t \mathfrak{D}_1[(\Phi-\Phi_1) \Phi_1^{i-1}] \varphi_2^2 dt_2 - \int^t \Phi_1^{i-1} \varphi_1^3 dt_1 + \int^t \varphi_1^2 dt_1 \int^{t_1} \Phi_2^{i-2} \varphi_2^2 dt_2 ,$$

and $I'(t) = \varphi' I_1(t) = -\varphi' \int^t \mathfrak{D}_1[(\Phi-\Phi_1) \Phi_1^{i-1}] \varphi_1^2 dt_1 .$

These being substituted in

$$\begin{aligned}
E(t_{i|n} t_{i+2|n} t_{i+3|n}) &= -|n \int (1-\Phi)^{n-i-3} \varphi' I(t) dt = -|n \int (1-\Phi)^{n-i-4} \varphi^2 I(t) dt \\
&+ |n \int (1-\Phi)^{n-i-3} \varphi I'(t) dt ,
\end{aligned}$$

we get

$$\begin{aligned}
(15.1.4) \quad E(t_{i|n} t_{i+2|n} t_{i+3|n}) &= -\frac{|n}{2} \int (1-\Phi)^{n-i-3} \Phi_1^{i-1} \varphi^4 dt \\
&+ |n \int (1-\Phi)^{n-i-4} \varphi^2 dt \int^t \Phi_1^{i-1} \varphi_1^3 dt_1 + \frac{|n}{2} \int (1-\Phi)^{n-i-3} \varphi^3 dt \int^t \Phi_1^{i-2} \varphi_1^2 dt_1 \\
&+ \frac{|n}{2} \int (1-\Phi)^{n-i-4} \varphi^3 dt \int^t \mathfrak{D}_1[(\Phi-\Phi_1) \Phi_1^{i-1}] \varphi_1^2 dt_1 \\
&- |n \int (1-\Phi)^{n-i-4} \varphi^2 dt \int^t \varphi_1^2 dt_1 \int^{t_1} \Phi_2^{i-2} \varphi_2^2 dt_2 .
\end{aligned}$$

Fifthly, letting $j=i+2$, $k \geq i+4$

$$E(t_{i|n}t_{i+2|n}t_{k|n}) = -|n \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi-\Phi_1)^{k-i-3} \varphi_1' dt_1 \int^{t_1} (\Phi_1-\Phi_2) \Phi_2^{i-1} \varphi_2' dt_2.$$

The innermost integral is the same as before, i.e. $I_1(t_1) = -\int^{t_1} \mathcal{D}_2[(\Phi_1-\Phi_2)\Phi_2^{i-1}] \varphi_2^2 dt_2$, but

$$\begin{aligned} I(t) &= \int^t (\Phi-\Phi_1)^{k-i-3} I_1(t_1) \varphi_1' dt_1 \quad (k-i-3 \geq 1) \\ &= -\int^t (\Phi-\Phi_1)^{k-i-4} \varphi_1^2 dt_1 \int^{t_1} \mathcal{D}_2[(\Phi_1-\Phi_2)\Phi_2^{i-1}] \varphi_2^2 dt_2 - \int^t (\Phi-\Phi_1)^{k-i-3} \Phi_1^{i-1} \varphi_1^3 dt_1 \\ &\quad + \int^t (\Phi-\Phi_1)^{k-i-3} \varphi_1^2 dt_1 \int^{t_1} \Phi_2^{i-2} \varphi_2^2 dt_2 \end{aligned}$$

with $I'(t) = \varphi \int^t (\Phi-\Phi_1)^{k-i-4} I_1(t_1) \varphi_1' dt_1$. Hence, to obtain $I'(t)$, we have to replace k in the integrand of $I(t)$ by $k-1$, multiply by φ , and write down the eventual additional term $\delta_{i+4}^k I_1(t) \varphi^2$. Thus

$$\begin{aligned} I'(t) &= -\varphi \int^t (\Phi-\Phi_1)^{k-i-5} \varphi_1^2 dt_1 \int^{t_1} \mathcal{D}_2[(\Phi_1-\Phi_2)\Phi_2^{i-1}] \varphi_2^2 dt_2 - \varphi \int^t (\Phi-\Phi_1)^{k-i-4} \Phi_1^{i-1} \varphi_1^3 dt_1 \\ &\quad + \varphi \int^t (\Phi-\Phi_1)^{k-i-4} \varphi_1^2 dt_1 \int^{t_1} \Phi_2^{i-2} \varphi_2^2 dt_2 - \delta_{i+4}^k \int^t \mathcal{D}_1[(\Phi-\Phi_1)\Phi_1^{i-1}] \varphi_1^2 dt_1. \end{aligned}$$

Therefore

$$\begin{aligned} E(t_{i|n}t_{i+2|n}t_{k|n}) &= -|n \int (1-\Phi)^{n-k} \varphi' I(t) dt = -|n \int (1-\Phi)^{n-k-1} \varphi^2 I(t) dt \\ &\quad + |n \int (1-\Phi)^{n-k} \varphi I'(t) dt = \text{(i)} + \text{(ii)}, \end{aligned}$$

where (ii) is obtained simply by replacing k in (i) by $k-1$ and changing the sign, yet with the additional Kronecker's term. Thus we find

(15.1.5) $E(t_{i|n}t_{i+2|n}t_{k|n})$ ($k \geq i+4$)

$$\begin{aligned} &= |n \int (1-\Phi)^{n-k-1} \varphi^2 dt \int^t (\Phi-\Phi_1)^{k-i-3} \Phi_1^{i-1} \varphi_1^3 dt_1 - |n \int (1-\Phi)^{n-k} \varphi^2 dt \\ &\quad \times \int^t (\Phi-\Phi_1)^{k-i-4} \Phi_1^{i-1} \varphi_1^3 dt_1 - |n \int (1-\Phi)^{n-k-1} \varphi^2 dt \left[\int^t (\Phi-\Phi_1)^{k-i-3} \varphi_1^2 dt_1 \right. \\ &\quad \times \left. \int^{t_1} \Phi_2^{i-2} \varphi_2^2 dt_2 - \int^t (\Phi-\Phi_1)^{k-i-4} \varphi_1^2 dt_1 \int^{t_1} \mathcal{D}_2[(\Phi_1-\Phi_2)\Phi_2^{i-1}] \varphi_2^2 dt_2 \right] \\ &\quad + |n \int (1-\Phi)^{n-k} \varphi^2 dt \left[\int^t (\Phi-\Phi_1)^{k-i-4} \varphi_1^2 dt_1 \int^{t_1} \Phi_2^{i-2} \varphi_2^2 dt_2 - \int^t (\Phi-\Phi_1)^{k-i-5} \varphi_1^2 dt_1 \right. \\ &\quad \times \left. \int^{t_1} \mathcal{D}_2[(\Phi_1-\Phi_2)\Phi_2^{i-1}] \varphi_2^2 dt_2 \right] - \delta_{i+4}^k |n \int (1-\Phi)^{n-k} \varphi^3 dt \int^t \mathcal{D}_1[(\Phi-\Phi_1)\Phi_1^{i-1}] \varphi_1^2 dt_1. \end{aligned}$$

Sixthly, if $j \geq i+3$, $k=j+1$, we have

$$E(t_{i|n}t_{j|n}t_{j+1|n}) = -|n \int (1-\Phi)^{n-j-1} \varphi' dt \int^t \varphi_1' dt_1 \int^{t_1} (\Phi_1-\Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' dt_2.$$

The inner integral are

$$I_1(t_1) = \int^{t_1} (\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' dt_2 = - \int^{t_1} \mathfrak{D}_2 [(\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1}] \varphi_2^2 dt_2 \quad \text{with}$$

$$I_1'(t_1) = -\varphi_1 \int^{t_1} \mathfrak{D}_2 [(\Phi_1 - \Phi_2)^{j-i-2} \Phi_2^{i-1}] \varphi_2^2 dt_2,$$

$$\begin{aligned} \text{and} \quad I(t) &= \int^t \varphi_1' I_1(t_1) dt_1 = -\varphi \int^t \mathfrak{D}_1 [(\Phi - \Phi_1)^{j-i-1} \Phi_1^{i-1}] \varphi_1^2 dt_2 + \int^t \varphi_1^2 dt_1 \\ &\quad \times \int^{t_1} \mathfrak{D}_2 [(\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1}] \varphi_2^2 dt_2 \end{aligned}$$

$$\text{with} \quad I'(t) = \varphi' I_1(t) = -\varphi' \int^t \mathfrak{D}_1 [(\Phi - \Phi_1)^{j-i-1} \Phi_1^{i-1}] \varphi^2 dt_1.$$

Therefore

$$\begin{aligned} (15.1.6) \quad E(t_{i|n} t_{j|n} t_{j+1|n}) \quad (j \geq i+3) \\ = \frac{|n|}{2} \int (1-\Phi)^{n-j-2} \varphi^3 dt \int^t \mathfrak{D}_1 [(\Phi - \Phi_1)^{j-i-1} \Phi_1^{i-1}] \varphi_1^2 dt_1 + \frac{|n|}{2} \int (1-\Phi)^{n-j-1} \varphi^3 dt \\ \times \int^t \mathfrak{D}_1 [(\Phi - \Phi_1)^{j-i-2} \Phi_1^{i-1}] \varphi_1^2 dt_1 - |n| \int (1-\Phi)^{n-j-2} \varphi^2 dt \int^t \varphi_2^2 dt_2 \\ \times \int^{t_1} \mathfrak{D}_2 [(\Phi_1 - \Phi_2)^{j-i-2} \Phi_2^{i-1}] \varphi_2^2 dt_2. \end{aligned}$$

Seventhly, if $j \geq i+3$ and $k \geq j+2$, we have

$$E(t_{i|n} t_{j|n} t_{k|n}) = -|n| \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi - \Phi_1)^{k-j-1} \varphi_1' dt_1 \int^{t_1} (\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' dt_2.$$

Here the innermost integral is

$$I_1(t_1) = - \int^{t_1} \mathfrak{D}_2 [(\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1}] \varphi_2^2 dt_2 \quad \text{with} \quad I_1'(t_1) = -\varphi_1 \int^{t_1} \mathfrak{D}_2 [(\Phi_1 - \Phi_2)^{j-i-2} \Phi_2^{i-1}] \varphi_2^2 dt_2.$$

Therefore

$$\begin{aligned} I(t) &= \int^t (\Phi_1 - \Phi_2)^{k-j-1} \varphi_1' I_1(t_1) dt_1 = - \int^t (\Phi - \Phi_1)^{k-j-2} \varphi_1^2 dt_1 \int^{t_1} \mathfrak{D}_2 [(\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1}] \varphi_2^2 dt_2 \\ &\quad + \int^t (\Phi - \Phi_1)^{k-j-1} \varphi_1^2 dt_1 \int^{t_1} \mathfrak{D}_2 [(\Phi_1 - \Phi_2)^{j-i-2} \Phi_2^{i-1}] \varphi_2^2 dt_2, \end{aligned}$$

and

$$\begin{aligned} I'(t) &= \varphi \int^t (\Phi - \Phi_1)^{k-j-2} \varphi_1' I_1(t_1) dt_1 \\ &= \varphi \int^t (\Phi - \Phi_1)^{k-j-3} \varphi_1^2 I_1(t_1) dt_1 - \varphi \int^t (\Phi - \Phi_1)^{k-j-2} \varphi_1 I_1'(t_1) dt_1 + \delta_{j+2}^k \varphi^2 I_1(t) \\ &= -\varphi \int (\Phi - \Phi_1)^{k-j-3} \varphi_1^2 dt_1 \int^t \mathfrak{D}_2 [(\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1}] \varphi_2^2 dt_2 + \varphi \int (\Phi - \Phi_1)^{k-j-2} \varphi_1^2 dt_1 \\ &\quad \times \int^{t_1} \mathfrak{D}_2 [(\Phi_1 - \Phi_2)^{j-i-2} \Phi_2^{i-1}] \varphi_2^2 dt_2 - \delta_{j+2}^k \varphi^2 \int^t \mathfrak{D}_1 [(\Phi - \Phi_1)^{j-i-1} \Phi_1^{i-1}] \varphi_1^2 dt_1. \end{aligned}$$

And we have to find

$$\begin{aligned}
E(t_{i|n}t_{j|n}t_{k|n}) &= -|n \int (1-\Phi)^{n-k} \varphi' I(t) dt = -|n \int (1-\Phi)^{n-k-1} \varphi^2 I(t) dt \\
&\quad + |n \int (1-\Phi)^{n-k} \varphi I'(t) dt.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
(15.1.7) \quad E(t_{i|n}t_{j|n}t_{k|n}) \quad (j \geq i+3, k \geq j+2) \\
&= |n \int (1-\Phi)^{n-k-1} \varphi^2 dt \int^t (\Phi-\Phi_1)^{k-j-2} \varphi_1^2 dt_1 \int^{t_1} \mathcal{D}_2[(\Phi_1-\Phi_2)^{j-i-1} \Phi_2^{i-1}] \varphi_2^2 dt_2 \\
&\quad - |n \int (1-\Phi)^{n-k} \varphi^2 dt \int^t (\Phi-\Phi_1)^{k-j-3} \varphi_1^2 dt_1 \int^{t_1} \mathcal{D}_2[(\Phi_1-\Phi_2)^{j-i-1} \Phi_2^{i-1}] \varphi_2^2 dt_2 \\
&\quad - |n \int (1-\Phi)^{n-k-1} \varphi^2 dt \int^t (\Phi-\Phi_1)^{k-j-1} \varphi_1^2 dt_1 \int^{t_1} \mathcal{D}_2[(\Phi_1-\Phi_2)^{j-i-2} \Phi_2^{i-1}] \varphi_2^2 dt_2 \\
&\quad + |n \int (1-\phi)^{n-k} \varphi^2 dt \int^t (\Phi-\Phi_1)^{k-j-2} \varphi_1^2 dt_1 \int^{t_1} \mathcal{D}_2[(\Phi_1-\Phi_2)^{j-i-2} \Phi_2^{i-1}] \varphi_2^2 dt_2 \\
&\quad - \delta_{j+2}^k |n \int (1-\phi)^{n-k} \varphi^3 dt \int \mathcal{D}_1[(\Phi-\Phi_1)^{j-i-1} \Phi_1^{i-1}] \varphi_1^2 dt_1.
\end{aligned}$$

$$\begin{aligned}
2^\circ \quad E(t_{i|n}^2 t_{j|n} t_{k|n}) \\
&= -|n \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi-\Phi_1)^{k-j-1} \varphi_1' dt_1 \int^{t_1} (\Phi-\Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' t_2 dt_2.
\end{aligned}$$

Firstly for $j=i+1, k=i+2$,

$$E(t_{i|n}^2 t_{i+1|n} t_{i+2|n}) = -|n \int (1-\Phi)^{n-i-2} \varphi' dt \int^t \varphi_1' dt_1 \int^{t_1} \Phi_2^{i-1} \varphi_2' t_2 dt_2.$$

The innermost integral is, as in (14.1.1),

$$I_1(t_1) = \int^{t_1} \Phi_2^{i-1} \varphi_2' t_2 dt_2 = -\Phi_1^i - \Phi_2^{i-1} \varphi_1' + \frac{1}{2} \Phi_1^{i-2} \varphi_1^2 - \frac{1}{2} \int^{t_1} \Phi_2^{i-3} \varphi_2^3 dt_2$$

and

$$I_1'(t_1) = \Phi_1^{i-1} \varphi_1' t_1.$$

Hence, making use of (13.8.1), the next inner integral is found to be

$$\begin{aligned}
I(t) &= \int^t I_1(t_1) \varphi_1' dt_1 \\
&= -\Phi^i \varphi - \frac{1}{2} \Phi^{i-1} \varphi \varphi' + \frac{1}{3} \Phi^{i-2} \varphi^3 + \frac{1}{2} \int^t \Phi_1^{i-1} \varphi_1^2 dt_1 - \frac{\varphi}{2} \int^t \Phi_1^{i-3} \varphi_1^3 dt_1 + \frac{1}{6} \int^t \Phi_1^{i-3} \varphi_1^4 dt_1.
\end{aligned}$$

Substituting this in

$$E(t_{i|n}^2 t_{i+1|n} t_{i+2|n}) = -|n \int (1-\Phi)^{n-i-2} I(t) \varphi' dt$$

and integrating by parts (on the way formula (13.8.1), (13.2.2) &c. may be employed), we find

$$\begin{aligned}
(15.2.1) \quad E(t_{i|n}^2 t_{i+1|n} t_{i+2|n}) &= |n| \left\{ \frac{1}{6} (1-\Phi)^{n-i-2} \Phi^{i-1} + \frac{1}{2} (1-\Phi)^{n-i-3} \Phi^i \right\} \varphi^3 dt \\
&+ \frac{|n|}{24} \left\{ (1-\Phi)^{n-i-4} \Phi^{i-1} - 4(1-\Phi)^{n-i-3} \Phi^{i-2} + (1-\Phi)^{n-i-2} \Phi^{i-3} \right\} \varphi^5 dt \\
&- \frac{|n|}{2} \int (1-\Phi)^{n-i-3} \varphi^2 dt \int^t \Phi_1^{i-1} \varphi_1^2 dt_2 - \frac{|n|}{6} \int (1-\Phi)^{n-i-3} \varphi^2 dt \int^t \Phi_1^{i-3} \varphi_1^4 dt_1 \\
&+ \frac{|n|}{4} \int (1-\Phi)^{n-i-3} \varphi^3 dt \int^t \Phi_1^{i-3} \varphi_1^3 dt_1.
\end{aligned}$$

Secondly, for $j=i+1$, $k=i+3$,

$$E(t_{i|n}^2 t_{i+1|n} t_{i+3|n}) = -|n| \int (1-\Phi)^{n-i-3} \varphi' dt \int^t (\Phi - \Phi_1) \varphi_1' dt_1 \int^{t_1} \Phi_2^{i-1} \varphi_2' t_2 dt_2.$$

The innermost integral is the same as before, but now

$$\begin{aligned}
I(t) &= \int^t (\Phi - \Phi_1) \varphi_1' I_1(t_1) dt_1 = \int^t \varphi_1^2 I_1(t_1) dt_1 - \int^t (\Phi - \Phi_1) \varphi I_1'(t_1) dt_1 \\
&= - \int^t \Phi_1^{i-1} \varphi_1^2 dt_1 - \frac{1}{6} \Phi^{i-1} \varphi^3 + \frac{5}{6} \int^t \Phi_1^{i-2} \varphi_1^4 dt_1 - \frac{1}{2} \int^t \varphi_1^2 dt_1 \int^{t_1} \Phi_2^{i-3} \varphi_2^3 dt_2 \\
&+ \frac{1}{2} \int^t (\Phi - \Phi_1) \Phi_1^{i-1} \varphi_1^2 dt_1 + \frac{1}{6} \int \mathcal{D}_1^2 [(\Phi - \Phi_1) \Phi_1^{i-1}] \varphi_1^4 dt_1,
\end{aligned}$$

in which (13.8.1) has been used. Hence, we have

$$\begin{aligned}
(15.2.2) \quad E(t_{i|n}^2 t_{i+1|n} t_{i+3|n}) &= -|n| \int (1-\Phi)^{n-i-3} \Phi^{i-1} \varphi^3 dt + \frac{|n|}{24} \int (1-\Phi)^{n-i-4} \Phi^{i-1} \varphi^5 dt \\
&+ \frac{11}{24} |n| \int (1-\Phi)^{n-i-3} \Phi^{i-2} \varphi^5 dt + \frac{|n|}{2} \int (1-\Phi)^{n-i-3} \varphi^2 dt \int^t \Phi_1^{i-1} \varphi_1^2 dt_1 \\
&+ |n| \int (1-\Phi)^{n-i-4} \varphi^2 dt \int^t \Phi_1^i \varphi_1^2 dt_1 - \frac{|n|}{2} \int (1-\Phi)^{n-i-4} \varphi^2 dt \int^t (\Phi - \Phi_1) \Phi_1^{i-1} \varphi_1^2 dt_1 \\
&+ \frac{|n|}{6} \int (1-\Phi)^{n-i-3} \varphi^2 dt \int^t \Phi_1^{i-3} \varphi_1^4 dt_1 - \frac{5}{6} |n| \int (1-\Phi)^{n-i-4} \varphi^2 dt \int^t \Phi_1^{i-2} \varphi_1^4 dt_1 \\
&- \frac{|n|}{6} \int (1-\Phi)^{n-i-4} \varphi^2 dt \int^t \mathcal{D}_1^2 [(\Phi - \Phi_1) \Phi_1^{i-1}] \varphi_1^4 dt_1 - \frac{|n|}{2} \int (1-\Phi)^{n-i-3} \varphi^3 dt \\
&\times \int^t \Phi_1^{i-3} \varphi_1^3 dt_1 + \frac{|n|}{2} \int (1-\Phi)^{n-i-4} \varphi^2 dt \int^t \varphi_1^2 dt_1 \int^{t_1} \Phi_2^{i-3} \varphi_2^3 dt_2.
\end{aligned}$$

Thirdly for $j=i+1$, $k \geq i+4$,

$$E(t_{i|n}^2 t_{i+1|n} t_{k|n}) = -|n| \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi - \Phi_1)^{k-i-2} \varphi_1' dt_1 \int^{t_1} \Phi_2^{i-1} \varphi_2' t_2 dt_2.$$

Here $I_1(t_1)$ has still the same form as before. However

$$\begin{aligned}
I(t) &= \int^t (\Phi - \Phi_1)^{k-i-2} \varphi_1' I_1(t_1) dt_1 = \int^t (\Phi - \Phi_1)^{k-i-3} \varphi_1^2 I_1(t_1) dt_1 - \int^t (\Phi - \Phi_1)^{k-i-2} \varphi_1 I_1'(t_1) dt_1 \\
&= - \int^t (\Phi - \Phi_1)^{k-i-3} \Phi_1^i \varphi_1^2 dt_1 + \frac{1}{3} \int^t \mathcal{D} [(\Phi - \Phi_1)^{k-i-3} \Phi_1^{i-1}] \varphi_1^4 dt_1
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \int^t (\Phi - \Phi_1)^{k-i-3} \Phi_1^{i-2} \varphi_1^4 dt_1 - \frac{1}{2} \int^t (\Phi - \Phi_1)^{k-i-3} \varphi^2 dt \int^t \Phi_2^{i-3} \varphi_2^2 dt_2 \\
& + \frac{1}{2} \int^t (\Phi - \Phi_1)^{k-i-2} \Phi_1^{i-1} \varphi_1^2 dt_1 + \frac{1}{6} \int^t \mathfrak{D}[(\Phi - \Phi_1)^{k-i-2} \Phi_1^{i-1}] \varphi_1^4 dt_1.
\end{aligned}$$

And $I'(t) = \varphi \int^t (\Phi - \Phi_1)^{k-i-3} \varphi_1' I_1(t_1) dt_1$ has the integral which shall be obtained by replacing k by $k-1$ in $I(t)$. Thus, if $I(t) = H(t, k)$, we get $I'(t) = \varphi H(t, k-1)$. Hence

$$\begin{aligned}
E(t_{i|n}^2 t_{i+1|n} t_{k|n}) \\
& = -|n \int (1-\Phi)^{n-k} \varphi' I(t) dt = -|n \int (1-\Phi)^{n-k-1} \varphi^2 I(t) dt + |n \int (1-\Phi)^{n-k} \varphi I'(t) dt \\
& = -|n \int (1-\Phi)^{n-k-1} \varphi^2 H(t, k) dt + |n \int (1-\Phi)^{n-(k-1)-1} \varphi^2 H(t, k-1) dt = (i) + (ii).
\end{aligned}$$

Therefore if (i) be obtained, we get immediately (ii). However, those terms that ceases to vanish when k is replaced by $k-1$, should be examined with particular caution. Thus we obtain

$$\begin{aligned}
(15.2.3) \quad E(t_{i|n}^2 t_{i+1|n} t_{k|n}) \quad (k \geq i+4) \\
& = |n \int (1-\Phi)^{n-k-1} \varphi^2 dt \left[\int^t (\Phi - \Phi_1)^{k-i-3} \Phi_1^i \varphi_1^2 dt_1 - \frac{1}{2} \int^t (\Phi - \Phi_1)^{k-i-2} \Phi_1^{i-1} \varphi_1^2 dt_1 \right. \\
& \quad - \frac{1}{2} \int^t (\Phi - \Phi_1)^{k-i-3} \Phi_1^{i-2} \varphi_1^4 dt_1 - \frac{1}{3} \int^t \mathfrak{D}_1[(\Phi - \Phi_1)^{k-i-3} \Phi_1^{i-1}] \varphi_1^4 dt_1 \\
& \quad \left. - \frac{1}{6} \int^t \mathfrak{D}_1^2[(\Phi - \Phi_1)^{k-i-2} \Phi_1^{i-1}] \varphi_1^4 dt_1 + \frac{1}{2} \int^t (\Phi - \Phi_1)^{k-i-3} \varphi_1^2 dt_1 \int^t \Phi_2^{i-3} \varphi_2^3 dt_2 \right] \\
& - \text{“the expression that would be obtained by replacing } k \text{ by } k-1 \text{ in the above”} \\
& - \delta_{i+4}^k \frac{|n}{6} \int (1-\Phi)^{n-k} \Phi^{i-1} \varphi^5 dt,
\end{aligned}$$

because, although on applying (13.8.1) to the inner integral

$$\int^t G(\Phi, \Phi_1) \varphi_1'^2 dt_1 \equiv \int (\Phi - \Phi_1)^{k-i-2} \Phi_1^{i-1} \varphi_1'^2 dt_1$$

in (i) for $k \geq i+4$, really $G(\Phi, \Phi_1)$ as well as $\mathfrak{D}_1 G$ do vanish when $t_1 = t$, it is not so in (ii), since the corresponding $G(\Phi, \Phi_1)$ in (ii) being $(\Phi - \Phi_1)^{k-i-3} \Phi_1^{i-1}$, itself still vanishes when $t_1 = t$, but now $\mathfrak{D}_1 G|_{t_1=t}$ for $k = i+4$ becomes $-\Phi^{i-1}$ instead of vanishing, and thus there yields $\delta_{i+4}^k \frac{1}{6} \Phi^i \varphi^3$ from the inner integral, so that after integration we must add $\delta_{i+4}^k \frac{|n}{6} \int (1-\Phi)^{n-k} \Phi^i \varphi^5 dt$ with the sign changed.

Further, let now $j = i+2$ and $k > j$; then we have

$$E(t_{i|n}^2 t_{i+2|n} t_{k|n}) = -|n \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi - \Phi_1)^{k-i-3} \varphi_1' dt_1 \int^{t_1} (\Phi_1 - \Phi_2) \Phi_2^{i-1} \varphi_2' t_2 dt_2.$$

Here the innermost integral being

$$\begin{aligned} I_1(t_1) &= \int^{t_1} (\Phi_1 - \Phi_2) \Phi_2^{i-1} \varphi_2' t_2 dt_2 = \int^{t_1} \mathfrak{D}_2 [(\Phi_1 - \Phi_2) \Phi_2^{i-1}] \varphi_2 \varphi_2' dt_2 - \int^{t_1} (\Phi_1 - \Phi_2) \Phi_2^{i-1} \varphi_2 dt_2 \\ &= -\Phi_1^{i+1} - \frac{1}{2} \Phi_1^{i-1} \varphi_1^2 - \frac{1}{2} \int^{t_1} \mathfrak{D}_2^2 [(\Phi_1 - \Phi_2) \Phi_2^{i-1}] \varphi_2^3 dt_2 \end{aligned}$$

with

$$I_1'(t_1) = -\Phi_1^i \varphi_1 - \Phi_1^{i-1} \varphi_1 \varphi_1' + \frac{1}{2} \Phi_1^{i-2} \varphi_1^3 - \frac{1}{2} \varphi_1 \int^{t_1} \Phi_2^{i-3} \varphi_2^3 dt_2,$$

the next inner integral becomes

$$\begin{aligned} I(t) &= \int^t (\Phi - \Phi_1)^{k-i-3} \varphi_1' I_1(t_1) dt_1 = \delta_{i+3}^k \varphi I_1(t) + \int^t (\Phi - \Phi_1)^{k-i-4} \Phi_1^i I_1(t_1) dt_1 \\ &\quad - \int^t (\Phi - \Phi_1)^{k-i-3} \varphi_1 I_1'(t_1) dt_1 \\ &= \delta_{i+3}^k \left[-\Phi^{i+1} \varphi - \frac{1}{2} \Phi^{i-1} \varphi^3 - \frac{1}{2} \varphi \int^t \mathfrak{D}_1 [(\Phi - \Phi_1) \Phi_1^{i-1}] \varphi_1^3 dt_1 \right] - \int^t (\Phi - \Phi_1)^{k-i-4} \Phi_1^{i+1} \varphi_1^2 dt_1 \\ &\quad - \frac{1}{2} \int^t (\Phi - \Phi_1)^{k-i-4} \Phi_1^{i-1} \varphi_1^4 dt_1 - \frac{1}{2} \int^t (\Phi - \Phi_1)^{k-i-4} \varphi_1^2 dt_1 \int^{t_1} \mathfrak{D}_2 [(\Phi_1 - \Phi_2) \Phi_2^{i-1}] \varphi_2^3 dt_2 \\ &\quad + \int^t (\Phi - \Phi_1)^{k-i-3} \Phi_1^i \varphi_1^2 dt_1 - \frac{1}{2} \int^t (\Phi - \Phi_1)^{k-i-3} \Phi_1^{i-2} \varphi_1^4 dt_1 + \frac{1}{2} \int^t (\Phi - \Phi_1)^{k-i-3} \varphi_1^2 dt_1 \\ &\quad \times \int^{t_1} \Phi_2^{i-3} \varphi_2^3 dt_2 - \frac{1}{3} \int^t \mathfrak{D}_1 [(\Phi - \Phi_1)^{k-i-3} \Phi_1^{i-1}] \varphi_1^4 dt_1 + \frac{1}{3} \delta_{i+3}^k \Phi^{i-1} \varphi^3 \end{aligned}$$

with

$$I'(t) = \delta_{i+3}^k \varphi' I_1(t) + \varphi \int^t (\Phi - \Phi_1)^{k-i-4} \varphi_1' I_1(t_1) dt_1.$$

These expressions being substituted in

$$\begin{aligned} E(t_{i|n}^2 t_{i+2|n} t_{i+3|n}) &= -|n| \int (1-\Phi)^{n-k} \varphi' I(t) dt \\ &= -|n| \int (1-\Phi)^{n-k-1} \varphi^2 I(t) dt + |n| \int (1-\Phi)^{n-k} \varphi I'(t) dt = (i) + (ii), \end{aligned}$$

(i) attains a form that is already free from t and t -derivatives, while we need to continue the integrations for (ii).

Now *fourthly*, let $k = j + 1 = i + 3$. Then, since those with negative exponent $k - i - 4 = -1$ vanish because of their denominator's ∞ , both of $I(t)$ and $I'(t)$ become somewhat simpler: Especially $I'(t) = \varphi' I_1(t)$, and (ii) $= |n| \int (1-\Phi)^{n-i-3} I_1(t) \varphi \varphi' dt$ can be completed by once integration by parts. Hence we obtain

$$\begin{aligned} (15.2.4) \quad E(t_{i|n}^2 t_{i+2|n} t_{i+3|n}) &= \frac{|n|}{2} \int (1-\Phi)^{n-i-3} \Phi^i \varphi^3 dt + \frac{|n|}{2} \int (1-\Phi)^{n-i-4} \Phi^{i+1} \varphi^3 dt \\ &\quad - \frac{3|n|}{8} \int (1-\Phi)^{n-i-3} \Phi^{i-2} \varphi^5 dt + \frac{1}{24} |n| \int (1-\Phi)^{n-i-4} \Phi^{i-1} \varphi^5 dt \end{aligned}$$

$$\begin{aligned}
& -|n| \int (1-\Phi)^{n-i-4} \varphi^2 dt \int^t \Phi_1^i \varphi_1^2 dt_1 + \frac{5}{6} |n| \int (1-\Phi)^{n-i-4} \varphi^2 dt \int^t \Phi_1^{i-2} \varphi_1^4 dt_1 \\
& + \frac{|n|}{4} \int (1-\Phi)^{n-i-3} \varphi^3 dt \int^t \Phi_1^{i-3} \varphi_1^3 dt_1 + \frac{|n|}{4} \int (1-\Phi)^{n-i-4} \varphi^3 dt \\
& \times \int^t \mathfrak{D}_1^2[(\Phi-\Phi_1) \Phi_1^{i-1}] \varphi_1^3 dt_1 - \frac{|n|}{4} (1-\Phi)^{n-i-4} \varphi^2 dt \int^t \varphi_1^2 dt_1 \int^{t_1} \Phi_2^{i-3} \varphi dt_2.
\end{aligned}$$

Fifthly, let $j=i+2$ and $k \geq i+4$. Then the foregoing expressions for $I(t)$ and $I'(t)$ both drop those terms with δ_{i+3}^k , and especially

$$I'(t) = \varphi \int^t (\Phi - \Phi_1)^{k-i-4} \varphi_1' I_1(t_1) dt_1$$

is nothing but which is obtainable from

$$I(t) = \int^t (\Phi - \Phi_1)^{k-i-3} \varphi_1' I_1(t_1) dt_1$$

by replacing k by $k-1$, and multiplying by φ , with caution that those vanishing terms in $I(t)$ might eventually become non-vanishing in $I'(t)$. Hence, we obtain

$$\begin{aligned}
(12.2.5.1) \quad E(t_{i|n}^2 t_{i+2|n} t_{k|n}) &= -|n| \int (1-\Phi)^{n-k} \varphi' I(t) dt \\
&= -|n| \int (1-\Phi)^{n-k-1} \varphi^2 I(t) dt + |n| \int (1-\Phi)^{n-k} \varphi I'(t) dt = (i) + (ii),
\end{aligned}$$

where

$$\begin{aligned}
(i) &= |n| \int (1-\Phi)^{n-k-1} \varphi^2 dt \int^t (\Phi - \Phi_1)^{k-i-4} \left\{ \Phi_1^{i+1} + \frac{1}{6} \Phi_1^{i-1} \varphi_1^2 \right. \\
&\quad + \frac{1}{2} \int^{t_1} \mathfrak{D}_1^2[(\Phi_1 - \Phi_2) \Phi_1^{i-1}] \varphi_2^3 dt_2 \left. \right\} \varphi_1^2 dt_1 - |n| \int (1-\Phi)^{n-k-1} \varphi^2 dt \int^t (\Phi - \Phi_1)^{k-i-3} \left\{ \Phi_1^i \right. \\
&\quad \left. - \frac{5}{6} \Phi_1^{i-2} \varphi_1^2 + \frac{1}{2} \int^{t_1} \Phi_2^{i-3} \varphi_2^3 dt_2 \right\} \varphi_1^2 dt_1,
\end{aligned}$$

(ii) = "the integrals with integrands to be obtained by replacing k by $k-1$ in those of (i) and multiplying by $-\varphi$ "

$$-\delta_{i+4}^k |n| \int (1-\Phi)^{n-k} \left\{ \Phi_1^{i+1} + \frac{1}{6} \Phi_1^{i-1} \varphi^2 + \frac{1}{2} \int^t \mathfrak{D}_1^2[(\Phi - \Phi_1) \Phi_1^{i-1}] \varphi_1^3 dt_1 \right\} \varphi^3 dt.$$

In particular, for $k=i+4$,

$$\begin{aligned}
(15.2.5) \quad E(t_{i|n}^2 t_{i+2|n} t_{i+4|n}) &= -|n| \int (1-\Phi)^{n-i-4} \Phi_1^{i+1} \varphi^3 dt - \frac{|n|}{6} \int (1-\Phi)^{n-i-4} \Phi_1^{i-1} \varphi^5 dt \\
&\quad + |n| \int (1-\Phi)^{n-i-4} \varphi^2 dt \int^t \Phi_1^i \varphi_1^2 dt_1 + |n| \int (1-\Phi)^{n-i-5} \varphi^2 dt \int^t \Phi_1^{i+1} \varphi_1^2 dt_1 \\
&\quad - |n| \int (1-\Phi)^{n-i-5} \varphi^2 dt \int^t (\Phi - \Phi_1) \Phi_1^i \varphi_1^2 dt_1 - \frac{5}{6} |n| \int (1-\Phi)^{n-i-4} \varphi^2 dt \\
&\quad \times \int^t \Phi_1^{i-2} \varphi_1^4 dt_1 + \frac{|n|}{6} \int (1-\Phi)^{n-i-5} \varphi^2 dt \int^t \Phi_1^{i-1} \varphi_1^4 dt_1 + \frac{5}{6} |n| \int (1-\Phi)^{n-i-5} \varphi^2 dt
\end{aligned}$$

$$\begin{aligned}
& \times \int^t (\Phi - \Phi_1) \underline{\Phi}_1^{i-2} \varphi_1^4 dt_1 - \frac{|n|}{2} \int (1 - \Phi)^{n-i-4} \varphi^3 dt \int^t \mathfrak{D}_1^2 [(\Phi - \Phi_1) \underline{\Phi}_1^{i-1}] \varphi_1^3 dt_1 \\
& + \frac{|n|}{2} \int (1 - \Phi)^{n-i-4} \varphi^2 dt_1 \int^t \varphi_1^2 dt_1 \int^{t_1} \underline{\Phi}_2^{i-3} \varphi_2^3 dt_2 - \frac{|n|}{2} \int (1 - \Phi)^{n-i-5} \varphi^2 dt \\
& \times \int^t (\Phi - \Phi_1) \varphi_1^2 dt_1 \int^{t_1} \underline{\Phi}_2^{i-3} \varphi_2^3 dt_2 + \frac{|n|}{2} \int (1 - \Phi)^{n-i-5} \varphi^2 dt \int^t \varphi_1^2 dt_1 \\
& \times \int^{t_1} \mathfrak{D}_2^2 [(\Phi_1 - \Phi_2) \underline{\Phi}_2^{i-1}] \varphi_2^3 dt_2.
\end{aligned}$$

Sixthly, for $j=i+3$, $k=i+4$

$$E(t_{i|n}^2 t_{i+3|n} t_{i+4|n}) = -|n| \int (1 - \Phi)^{n-i-4} \varphi' dt \int^t \varphi_1' dt_1 \int^{t_1} (\Phi_1 - \Phi_2)^2 \underline{\Phi}_2^{i-1} \varphi_2' t_2 dt_2.$$

Here the innermost integral becomes

$$I_1(t_1) = \int^{t_1} (\Phi_1 - \Phi_2)^2 \underline{\Phi}_2^{i-1} \varphi_2' t_2 dt_2 = -\underline{\Phi}_1^{i+2} - \frac{1}{2} \int^{t_1} \mathfrak{D}_2^2 [(\Phi_1 - \Phi_2)^2 \underline{\Phi}_2^{i-1}] \varphi_2^3 dt_2$$

with

$$I'(t_1) = -\underline{\Phi}_1^{i+1} \varphi_1 - \frac{1}{2} \underline{\Phi}_1^{i-1} \varphi_1^3 - \frac{1}{2} \varphi_1 \int^{t_1} \mathfrak{D}_2^2 [(\Phi_1 - \Phi_2) \underline{\Phi}_2^{i-1}] \varphi_2^3 dt_2.$$

So that

$$\begin{aligned}
I(t) &= \int^t \varphi_1' I_1(t_1) dt_1 = \varphi I_1(t) - \int^t I_1'(t_1) \varphi_1 dt_1 \\
&= -\underline{\Phi}^{i+2} \varphi + \int^t \underline{\Phi}_1^{i+1} \varphi_1^2 dt_1 + \frac{1}{2} \int^t \underline{\Phi}_1^{i-1} \varphi_1^4 dt_1 - \frac{1}{2} \varphi \int^t \mathfrak{D}_1^2 [(\Phi - \Phi_1)^2 \underline{\Phi}_1^{i-1}] \varphi_1^3 dt_1 \\
&\quad + \frac{1}{2} \int^t \varphi_1^2 dt_1 \int^{t_1} \mathfrak{D}_2^2 [(\Phi_1 - \Phi_2)^2 \underline{\Phi}_2^{i-1}] \varphi_2^3 dt_2
\end{aligned}$$

and

$$I'(t) = \varphi' I_1(t) = -\underline{\Phi}^{i-2} \varphi' - \frac{1}{2} \varphi' \int^t \mathfrak{D}_1^2 [(\Phi - \Phi_1)^2 \underline{\Phi}_1^{i-1}] \varphi_1^3 dt_1.$$

Hence, we obtain

$$\begin{aligned}
(15.2.6) \quad E(t_{i|n}^2 t_{i+3|n} t_{i+4|n}) &= \frac{|n|}{2} \int (1 - \Phi)^{n-i-5} \underline{\Phi}^{i+2} \varphi^3 dt + \frac{|n|}{2} \int (1 - \Phi)^{n-i-4} \underline{\Phi}^{i+1} \varphi^3 dt \\
&\quad + \frac{|n|}{4} \int (1 - \Phi)^{n-i-4} \underline{\Phi}^{i-1} \varphi^5 dt - |n| \int (1 - \Phi)^{n-i-5} \varphi^2 dt \int^t \underline{\Phi}_1^{i+1} \varphi_1^2 dt_1 \\
&\quad - \frac{|n|}{2} \int (1 - \Phi)^{n-i-5} \varphi^2 dt \int^t \underline{\Phi}_1^{i-1} \varphi_1^4 dt_1 + \frac{|n|}{4} \int (1 - \Phi)^{n-i-5} \varphi^3 dt \\
&\quad \times \int^t \mathfrak{D}_1^2 [(\Phi - \Phi_1)^2 \underline{\Phi}_1^{i-1}] \varphi_1^3 dt_1 + \frac{|n|}{4} \int (1 - \Phi)^{n-i-4} \varphi^3 dt \int^t \mathfrak{D}_1^2 [(\Phi - \Phi_1) \underline{\Phi}_1^{i-1}] \varphi_1^3 dt_1 \\
&\quad - \frac{|n|}{2} \int (1 - \Phi)^{n-i-5} \varphi^2 dt \int^t \varphi_1^2 dt_1 \int^{t_1} \mathfrak{D}_2^2 [(\Phi_1 - \Phi_2) \underline{\Phi}_2^{i-1}] \varphi_2^3 dt_2.
\end{aligned}$$

Seventhly, for $j=i+3$, $k \geq i+5$,

$$E(t_{i|n}^2 t_{i+3|n} t_{k|n}) = -|n| \int (1 - \Phi)^{n-k} \varphi' dt \int^t (\Phi - \Phi_1)^{k-i-3} \varphi_1' dt_1 \int^{t_1} (\Phi_1 - \Phi_3)^2 \underline{\Phi}_2^{i-1} \varphi_2' t_2 dt_2.$$

Still $I_1(t_1)$, $I_1'(t_1)$ are the same as in the foregoing, and

$$\begin{aligned} I(t) &= \int^t (\Phi - \Phi_1)^{k-i-4} \varphi_1' I_1(t_1) \quad (k \geq i+5) \\ &= \int^t (\Phi - \Phi_1)^{k-i-5} \varphi_1^2 I_1(t_1) dt_1 - \int^t (\Phi - \Phi_1)^{k-i-4} \varphi_1 I_1'(t_1) dt_1 \end{aligned}$$

with
$$I'(t) = \varphi \int^t (\Phi - \Phi_1)^{k-i-5} \varphi_1' I_1(t_1) dt_1.$$

Hence $I'(t)$ is obtained by replacing k by $k-1$ in $I(t) \equiv H(t, k)$ and multiplying by φ , eventually with the additional term when $k=i+5$. Thus we obtain

$$\begin{aligned} (15.2.7) \quad E(t_{2|n}^i t_{i+3|n} t_{k|n}) &= -|n| \int (1-\Phi)^{n-k} \varphi' I(t) dt \quad (k \geq i+5) \\ &= -|n| \int (1-\Phi)^{n-k-1} \varphi^2 H(t, k) dt + |n| \int (1-\Phi)^{n-k} \varphi^2 H(t, k-1) dt \\ &\quad + \delta_{i+5}^k |n| \int (1-\Phi)^{n-k} \varphi^2 \varphi' I_1(t) dt = (i) + (ii) + (iii). \end{aligned}$$

where

$$\begin{aligned} (i) &= |n| \int (1-\Phi)^{n-k-1} \varphi^2 dt \int^t (\Phi - \Phi_1)^{k-i-5} \left[\Phi_1^{i+2} + \frac{1}{2} \int^t \mathfrak{D}_2^2[(\Phi_1 - \Phi_2)^2 \Phi_2^{i-1}] \varphi_2^3 dt_2 \right] \varphi_1^2 dt_1 \\ &\quad - |n| \int (1-\Phi)^{n-k-1} \varphi^2 dt \int^t (\Phi - \Phi_1)^{k-i-4} \left[\Phi_1^{i+1} + \frac{1}{2} \Phi_1^{i-1} \varphi_1^2 \right. \\ &\quad \left. + \frac{1}{2} \int^t \mathfrak{D}_2^2[(\Phi_1 - \Phi_2) \Phi_2^{i-1}] \varphi_2^3 dt_2 \right] \varphi_1^2 dt_1, \end{aligned}$$

(ii) = - "the integrals to be obtained by replacing k by $k-1$ in those of (i),"

$$\begin{aligned} (iii) &= \delta_{i+5}^k \left\{ \frac{|n|}{3} \int D[(1-\Phi)^{n-k} \Phi_1^{i+2}] \varphi^4 dt - \frac{|n|}{6} \int (1-\Phi)^{n-k-1} \varphi^4 dt \int^t \mathfrak{D}_1^2[(\Phi - \Phi_1)^2 \Phi_1^{i-1}] \varphi_1^3 dt_1 \right. \\ &\quad \left. + \frac{|n|}{6} \int (1-\Phi)^{n-k} \Phi_1^{i-1} \varphi^6 dt + \frac{|n|}{6} \int (1-\Phi)^{n-k} \varphi^4 dt \int^t \mathfrak{D}_1^2[(\Phi - \Phi_1) \Phi_1^{i-1}] \varphi_1^3 dt \right\}. \end{aligned}$$

Eighthly, for $j \geq i+4$, $k \geq j+1$,

$$E(t_{i|n}^2 t_{j|n} t_{k|n}) = -|n| \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi - \Phi_1)^{k-j-1} \varphi_1' dt_1 \int^{t_1} (\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' t_2 dt_2.$$

Now

$$\begin{aligned} I_1(t_1) &= \int^{t_1} (\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' t_2 dt_2 \quad (j-i-1 \geq 3) \\ &= -\Phi_1^{j-1} - \frac{1}{2} \int^{t_1} \mathfrak{D}_2^2[(\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1}] \varphi_2^3 dt_2 \end{aligned}$$

and

$$I_1'(t_1) = -\Phi_1^{j-2} \varphi_1 - \frac{1}{2} \varphi_1 \int^{t_1} \mathfrak{D}_2^2[(\Phi - \Phi_2)^{j-i-2} \Phi_2^{i-1}] \varphi_2^3 dt_2.$$

Hence

$$\begin{aligned}
I(t) &= \int^t (\Phi - \Phi_1)^{k-j-1} \varphi_1' I_1(t_1) dt_1 \\
&= \delta_{j+1}^k I_1(t) \varphi + \int^t (\Phi - \Phi_1)^{k-j-2} \varphi_1^2 I_1(t_1) dt_1 - \int^t (\Phi - \Phi_1)^{k-j-1} \varphi_1 I_1'(t_1) dt_1,
\end{aligned}$$

and

$$I'(t) = \varphi \int^t (\Phi - \Phi_1)^{k-j-2} \varphi_1' I_1(t_1) dt_1 - \delta_{j+1}^j I_1(t) \varphi'.$$

Thus $I'(t)$ is obtained by replacing k by $k-1$ in $I(t)$ and multiplying by φ with the further additional terms. Hence

$$\begin{aligned}
(15.2.8) \quad E(t_{i|n}^2 t_{j|n} t_{k|n}) &= -|n \int (1-\Phi)^{n-k} \varphi' I(t) dt \quad (j \geq i+4, k \geq j+1) \\
&= -|n \int (1-\Phi)^{n-k-1} \varphi^2 I(t) dt + |n \int (1-\Phi)^{n-k} \varphi I'(t) dt = (i) + (ii),
\end{aligned}$$

where

$$\begin{aligned}
(i) &= |n \int (1-\Phi)^{n-k-1} \varphi^2 dt \int^t (\Phi - \Phi_1)^{k-j-2} \left\{ \Phi_1^{j-1} + \frac{1}{2} \int^{t_1} \mathfrak{D}_2^2 [(\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1}] \varphi_2^3 dt_2 \right\} \varphi_1^2 dt_1 \\
&\quad - |n \int (1-\Phi)^{n-k-1} \varphi^2 dt \int^t (\Phi - \Phi_1)^{k-j-1} \left\{ \Phi_1^{j-2} + \frac{1}{2} \int^{t_1} \mathfrak{D}_2^2 [(\Phi_1 - \Phi_2)^{j-i-2} \Phi_2^{i-1}] \varphi_2^3 dt_2 \right\} \varphi_1^2 dt_1 \\
&\quad + \delta_{j+1}^k |n \int (1-\Phi)^{n-k-1} \left\{ \Phi_1^{j-1} + \frac{1}{2} \int^{t_1} \mathfrak{D}_1^2 [(\Phi - \Phi_1)^{j-i-1} \Phi_1^{i-1}] \varphi_1^3 dt_1 \right\} \varphi^3 dt,
\end{aligned}$$

and accordingly

$$\begin{aligned}
(ii) &= -|n \int (1-\Phi)^{n-k} \varphi^2 dt \int^t (\Phi - \Phi_1)^{k-j-3} \left\{ \Phi_1^{j-1} + \frac{1}{2} \int^{t_1} \mathfrak{D}_2^2 [(\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1}] \varphi_2^3 dt_2 \right\} \varphi_1^2 dt_1 \\
&\quad + |n \int (1-\Phi)^{n-k} \varphi^2 dt \int^t (\Phi - \Phi_1)^{k-j-2} \left\{ \Phi_1^{j-2} + \frac{1}{2} \int^{t_1} \mathfrak{D}_2^2 [(\Phi_1 - \Phi_2)^{j-i-2} \Phi_2^{i-1}] \varphi_2^3 dt_2 \right\} \varphi_1^2 dt_1 \\
&\quad - \delta_{j+1}^{k-1} |n \int (1-\Phi)^{n-k} \left\{ \Phi_1^{j-1} + \frac{1}{2} \int^{t_1} \mathfrak{D}_1^2 [(\Phi - \Phi_1)^{j-i-1} \Phi_1^{i-1}] \varphi_1^3 dt_1 \right\} \varphi^3 dt \\
&\quad - \delta_{j+1}^k |n \int (1-\Phi)^{n-k} I_1(t) \varphi \varphi' dt.
\end{aligned}$$

The last term having $-\delta_{j+1}^k$ as factor becomes, after integration, in details

$$\begin{aligned}
&= \delta_{j+1}^k \frac{|n}{2} \left[\int (1-\Phi)^{n-k-1} \varphi^3 I_1(t) dt - \int (1-\Phi)^{n-k} \varphi^2 I_1'(t) dt \right] \\
&= \delta_{j+1}^k \frac{|n}{2} \left[\int (1-\Phi)^{n-k-1} \Phi_1^{j-1} \varphi^3 dt + \frac{1}{2} \int (1-\Phi)^{n-k-1} \varphi^3 dt \int^t \mathfrak{D}_1^2 [(\Phi - \Phi_1)^{j-i-1} \Phi_1^{i-1}] \varphi_1^3 dt_1 \right. \\
&\quad \left. - \int (1-\Phi)^{n-k} \Phi_1^{j-2} \varphi^3 dt - \frac{1}{2} \int (1-\Phi)^{n-k} \varphi^3 dt \int^t \mathfrak{D}_1^2 [(\Phi - \Phi_1)^{j-i-2} \Phi_1^{i-1}] \varphi_1^3 dt_1 \right].
\end{aligned}$$

$$3^\circ \quad E(t_{i|n} t_{j|n}^2 t_{k|n})$$

$$= -|n \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi - \Phi_1)^{k-j-1} \varphi_1' t_1 dt_1 \int^{t_1} (\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' dt_2$$

Firstly

$$E(t_{i|n}t_{i+1|n}^2t_{i+2|n}) = -|n \int (1-\Phi)^{n-i-2}\varphi' dt \int \varphi_1' t_1 dt_1 \int^{t_1} \Phi_2^{i-1}\varphi_2' dt_2.$$

Here the innermost integral is the same as in (15.1.1): i.e.

$$I_1(t_1) = \Phi_1^{i-1}\varphi_1 - \int^t \Phi_2^{i-2}\varphi_2 dt_2, \quad I_1'(t_1) = \Phi_1^{i-1}\varphi_1'.$$

But

$$I(t) = \int^t I_1(t)\varphi_1' t_1 dt_1 = -I_1(t)\varphi' - \int^t I_1(t)\varphi_1 dt_1 + \int^t I_1'(t_1)\varphi_1' dt_1.$$

On substituting $I_1(t)$, $I_1'(t_1)$ and integrating by parts, on the way applying e.g. (13.8.1), we get

$$\begin{aligned} I(t) = & -\frac{1}{2}\Phi_1^{i-1}\varphi\varphi' + \varphi' \int^t \Phi_1^{i-2}\varphi_1^2 dt_1 - \frac{1}{6}\Phi_1^{i-2}\varphi^3 + \frac{1}{6} \int^t \Phi_1^{i-3}\varphi_1^4 dt_1 - \frac{1}{2} \int^t \Phi_1^{i-1}\varphi_1^2 dt_1 \\ & + \int^t \varphi_1 dt_1 \int^{t_1} \Phi_2^{i-2}\varphi_2^2 dt_2, \end{aligned}$$

whose all terms, except the first two, are already free from t -derivative, and

$$I_1'(t) = I_1(t)\varphi't = -\Phi_1^{i-1}\varphi'^2 - \varphi't \int^t \Phi_1^{i-2}\varphi_1^2 dt_1.$$

Hence, on applying (13.3.2) and (13.6.1), we obtain

$$\begin{aligned} (15.3.1) \quad E(t_{i|n}t_{i+1|n}^2t_{i+2|n}) = & -|n \int (1-\Phi)^{n-i-3}\varphi^2 I(t) dt + |n \int (1-\Phi)^{n-i-2}\varphi I(t) dt \\ = & -\frac{|n}{3} \int (1-\Phi)^{n-i-2}\Phi_1^{i-1}\varphi^3 dt + \frac{|n}{24} \int (1-\Phi)^{n-i-4}\Phi_1^{i-1}\varphi^5 dt + \frac{|n}{4} \int (1-\Phi)^{n-i-3}\Phi_1^{i-2}\varphi^5 dt \\ & + \frac{|n}{24} \int (1-\Phi)^{n-i-2}\Phi_1^{i-3}\varphi^5 dt + \frac{|n}{2} \int (1-\Phi)^{n-i-2}\varphi^2 dt \int^t \Phi_1^{i-2}\varphi_1^2 dt_1 \\ & + \frac{|n}{2} \int (1-\Phi)^{n-i-3}\varphi^2 dt \int^t \Phi_1^{i-1}\varphi_1^2 dt_1 - \frac{|n}{6} \int (1-\Phi)^{n-i-3}\varphi^2 dt \int^t \Phi_1^{i-3}\varphi_1^4 dt_1 \\ & - \frac{|n}{6} \int (1-\Phi)^{n-i-4}\varphi^4 dt \int^t \Phi_1^{i-2}\varphi_1^2 dt_1 - |n \int (1-\Phi)^{n-i-3}\varphi^2 dt \int^t \varphi_1 dt_1 \int^{t_1} \Phi_2^{i-2}\varphi_2^2 dt_2. \end{aligned}$$

$$\text{Secondly, for } E(t_{i|n}t_{i+1|n}^2t_{i+3|n}) = -|n \int (1-\Phi)^{n-i-3}\varphi dt \int^t (\Phi-\Phi_1)\varphi_1' t_1 dt_1 \int^{t_1} \Phi_2^{i-1}\varphi_2' dt_2,$$

the innermost integral being again $I_1(t_1) = \Phi_1^{i-1}\varphi_1 - \int^{t_1} \Phi_2^{i-2}\varphi_2^2 dt_2$, $I_1'(t_1) = \Phi_1^{i-1}\varphi_1'$, we have

$$\begin{aligned} I(t) = & \int^t (\Phi-\Phi_1)I_1(t_1)\varphi_1' t_1 dt_1 = - \int^t I_1(t_1)\varphi_1\varphi_1' dt_1 - \int^t (\Phi-\Phi_1)I_1(t_1)\varphi_1 dt_1 \\ & + \int^t (\Phi-\Phi_1)\Phi_1^{i-1}\varphi'^2 dt_1, \end{aligned}$$

whose last integral by (13.8.1) splits up into

$$\frac{1}{2} \int^t (\Phi-\Phi_1)\Phi_1^{i-1}\varphi_1^2 dt_1 + \frac{1}{6}\Phi_1^{i-1}\varphi^3 + \frac{1}{6} \int^t \mathfrak{D}_1^2[(\Phi-\Phi_1)\Phi_1^{i-1}]\varphi_1^4 dt_1.$$

Also $I(t) = \varphi \int^t I_1(t_1) \varphi_1' t_1 dt_1$ and accordingly

$$E(t_{i|n} t_{i+1|n}^2 t_{i+2|n}) = -|n \int (1-\Phi)^{n-i-3} I(t) \varphi' dt = -|n \int (1-\Phi)^{n-i-4} \varphi^2 I(t) dt + |n \int (1-\Phi)^{n-i-3} \varphi I'(t) dt.$$

On substituting $I(t)$ and $I'(t)$ obtained above and integrating, we get

$$\begin{aligned} (15.3.2) \quad E(t_{i|n} t_{i+1|n}^2 t_{i+3|n}) &= \frac{|n}{24} \int (1-\Phi)^{n-i-4} \Phi^{i-1} \varphi^5 dt - \frac{3|n}{8} \int (1-\Phi)^{n-i-3} \Phi^{i-2} \varphi^5 dt \\ &- \frac{|n}{2} \int (1-\Phi)^{n-i-3} \varphi^2 dt \int^t \Phi_1^{i-1} \varphi_1^2 dt_1 + \frac{|n}{2} \int (1-\Phi)^{n-i-4} \varphi^2 dt \int^t (\Phi - \Phi_1) \Phi^{i-1} \varphi_1^2 dt_1 \\ &+ \frac{|n}{6} \int (1-\Phi)^{n-i-3} \varphi^2 dt \int^t \Phi_1^{i-3} \varphi_1^4 dt_1 + \frac{|n}{6} \int (1-\Phi)^{n-i-4} \varphi^2 dt \int^t \Phi_1^{i-2} \varphi_1^4 dt_1 \\ &- \frac{|n}{6} \int (1-\Phi)^{n-i-4} \varphi^2 dt \int^t \mathfrak{D}_1^2[(\Phi - \Phi_1) \Phi_1^{i-1}] \varphi_1^4 dt_1 - \frac{|n}{6} \int (1-\Phi)^{n-i-4} \varphi^4 dt \int \Phi_1^{i-2} \varphi_1^2 dt_1 \\ &+ |n \int (1-\Phi)^{n-i-3} \varphi^2 dt \int^t \varphi_1 dt_1 \int^{t_1} \Phi_2^{i-2} \varphi_2^2 dt_2 - |n \int (1-\Phi)^{n-i-4} \varphi^2 dt \int^t (\Phi - \Phi_1) \varphi_1 dt_1 \\ &\times \int^{t_1} \Phi_2^{i-2} \varphi_2^2 dt_2. \end{aligned}$$

Thirdly, to find $E(t_{i|n} t_{i+1|n}^2 t_{k|n})$ for $k \geq i+4$, $I_1(t_1)$ and $I_1'(t_1)$ have still the same values as before. But now

$$\begin{aligned} I(t) &= \int^t (\Phi - \Phi_1)^{k-i-2} I_1(t) \varphi_1' t_1 dt_1 \\ &= - \int^t (\Phi - \Phi_1)^{k-i-3} I_1(t) \varphi_1 \varphi_1' dt_1 - \int^t (\Phi_1 - \Phi_1)^{k-i-2} I_1(t) \varphi_1 dt_1 \\ &\quad + \int^t (\Phi - \Phi_1)^{k-i-2} \Phi_1^{i-1} \varphi_1'^2 dt_1 \end{aligned}$$

with $I'(t) = \varphi \int^t (\Phi - \Phi_1)^{k-i-3} I_1(t) \varphi_1' t_1 dt_1$

and

$$E(t_{i|n} t_{i+1|n}^2 t_{k|n}) = -|n \int (1-\Phi)^{n-k-1} \varphi^2 I(t) dt + |n \int (1-\Phi)^{n-k} \varphi I'(t) dt = (i) + (ii).$$

So that $I'(t)$ and accordingly (ii) could be obtained by replacing k in $I(t)$ or (i) by $k-1$ and simply changing the sign (the factor φ^2 being already present in both of (i) and (ii)). However, the eventually arising Kronecker's term should be considered, and we obtain

$$\begin{aligned} (15.3.3) \quad E(t_{i|n} t_{i+1|n}^2 t_{k|n}) \quad (k \geq i+4) \\ &= \frac{|n}{2} \int (1-\Phi)^{n-k-1} \varphi^2 dt \int^t (\Phi - \Phi_1)^{k-i-2} \Phi_1^{i-1} \varphi dt_1 + \frac{|n}{6} \int (1-\Phi)^{n-k-1} \varphi^2 dt \\ &\times \int^t \{(\Phi - \Phi_1)^{k-i-4} \Phi_1^{i-1} + 3(\Phi - \Phi_1)^{k-i-3} \Phi_1^{i-2} - (\Phi - \Phi_1)^{k-i-2} \Phi_1^{i-3}\} \varphi_1^4 dt_1 \\ &- |n \int (1-\Phi)^{n-k-1} \varphi^2 dt \left\{ \int^t (\Phi - \Phi_1)^{k-i-2} \varphi_1 dt_1 \int^{t_1} \Phi_2^{i-2} \varphi_2^2 dt_2 \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \int^t (\Phi - \Phi_1)^{k-i-4} \varphi_1^3 dt_1 \int^t \Phi_2^{i-2} \varphi_2^2 dt_2 \Big\} \\
& - \text{“the expression to be obtained by replacing } k \text{ in the above by } k-1\text{”} \\
& - \delta_{i+4}^k \frac{|n|}{6} \int (1-\Phi)^{n-k} \Phi^{i-1} \varphi^5 dt.
\end{aligned}$$

Fourthly, for $j=i+2$, and at first $k=i+3$,

$$E(t_{i|n} t_{i+2|n}^2 t_{i+3|n}) = -|n| \int (1-\Phi)^{n-i-3} \varphi' dt \int^t \varphi_1' t_1 dt_1 \int^{t_1} (\Phi_1 - \Phi_2) \Phi_2^{i-1} \varphi_2' k t_2.$$

Now the innermost integral is that given in 1° also fourthly:

$$I_1(t_1) = - \int^{t_1} \mathfrak{D}_2[(\Phi_1 - \Phi_2) \Phi_2^{i-1}] \varphi_2^2 dt_2 \quad \text{with} \quad I_1'(t_1) = \Phi_1^{i-1} \varphi_1^2 - \varphi_1 \int^{t_1} \Phi_2^{i-2} \varphi_2^2 dt_2.$$

But now

$$I(t) = \int^t I_1(t_1) \varphi_1' t_1 dt_1 = \frac{1}{3} \Phi^{i-1} \varphi^3 - \frac{\varphi^2}{2} \int^t \Phi_1^{i-2} \varphi_1^2 dt_1 + \frac{1}{6} \int^t \Phi_1^{i-2} \varphi_1^4 dt_1$$

and

$$I'(t) = -\varphi' t \int^t \mathfrak{D}_1[(\Phi - \Phi_1) \Phi_1^{i-1}] \varphi_1^2 dt_1.$$

Hence

$$\begin{aligned}
(15.3.4) \quad E(t_{i|n} t_{i+2|n}^2 t_{i+3|n}) &= -\frac{3}{8} |n| \int (1-\Phi)^{n-i-4} \Phi^{i-1} \varphi^5 dt + \frac{1}{24} |n| \int (1-\Phi)^{n-i-3} \Phi^{i-2} \varphi^5 dt \\
&+ \frac{|n|}{2} \int (1-\Phi)^{n-i-3} \varphi^2 dt \int^t \mathfrak{D}_1[(\Phi - \Phi_1) \Phi_1^{i-1}] \varphi_1^2 dt_1 - \frac{|n|}{6} \int (1-\Phi)^{n-i-4} \varphi^2 dt \int^t \Phi_1^{i-2} \varphi_1^4 dt_1 \\
&+ \frac{|n|}{2} \int (1-\Phi)^{n-i-4} \varphi^4 dt \int^t \Phi_1^{i-2} \varphi_1^2 dt_1 - \frac{|n|}{6} \int (1-\Phi)^{n-i-5} \varphi^4 dt \int^t \mathfrak{D}_1[(\Phi - \Phi_1) \Phi_1^{i-1}] \varphi_1^2 dt_1 \\
&- |n| \int (1-\Phi)^{n-i-4} \varphi^2 dt \int^t \varphi_1 dt_1 \int^{t_1} \mathfrak{D}_2[(\Phi_1 - \Phi_2) \Phi_2^{i-1}] \varphi_2^2 dt_2.
\end{aligned}$$

Fifthly, for $E(t_{i|n} t_{i+2|n}^2 t_{i+4|n})$ still with the same $I_1(t)$ as in the above

$$\begin{aligned}
I(t) &= \int^t (\Phi - \Phi_1) I_1(t_1) \varphi_1' t_1 dt_1 \\
&= \int^t I_1(t_1) \varphi_1^2 t_1 dt_1 - \int^t (\Phi - \Phi_1) I_1(t_1) \varphi_1 dt_1 + \int^t I_1'(t_1) (\Phi - \Phi_1) \varphi_1' dt_1 \\
&= \frac{1}{2} \int^t \Phi^{i-1} \varphi_1^4 dt_1 + \frac{1}{2} \int^t (\Phi - \Phi_1) \Phi_1^{i-2} \varphi_1^4 dt_1 + \frac{1}{2} \varphi^2 \int^t \mathfrak{D}_1[(\Phi - \Phi_1) \Phi_1^{i-1}] \varphi_1^2 dt_1 \\
&- \frac{1}{3} \int^t \mathfrak{D}_1[(\Phi - \Phi_1) \Phi_1^{i-1}] \varphi_1^4 dt_1 - \int^t \varphi_1^3 dt_1 \int^{t_1} \Phi_2^{i-2} \varphi_2^2 dt_2 + \int^t (\Phi - \Phi_1) \varphi_1 dt_1 \\
&\times \int^{t_1} \mathfrak{D}_2[(\Phi_1 - \Phi_2) \Phi_2^{i-1}] \varphi_2^2 dt_2.
\end{aligned}$$

And we have

$$\begin{aligned}
(15.3.5) \quad E(t_{i|n} t_{i+2|n}^2 t_{i+4|n}) &= \frac{2n}{3} \int (1-\Phi)^{n-i-4} \Phi^{i-1} \varphi^5 dt \\
&- \frac{1}{2} |n| \int (1-\Phi)^{n-i-5} \varphi^2 dt \int^t \Phi^{i-1} \varphi^4 dt - \frac{1}{2} |n| \int (1-\Phi)^{n-i-5} \varphi^2 dt \int^t (\Phi - \Phi_1) \Phi_1^{i-2} \varphi_1^4 dt_1 \\
&+ \frac{1}{3} |n| \int (1-\Phi)^{n-i-5} \varphi^2 dt \int^t \mathcal{D}_1[(\Phi - \Phi_1) \Phi_1^{i-1}] \varphi_1^4 dt_1 \\
&+ \frac{|n|}{6} \int (1-\Phi)^{n-i-4} \varphi^2 dt \int^t \Phi^{i-2} \varphi_1^4 dt_1 - \frac{5|n|}{6} \int (1-\Phi)^{n-i-4} \varphi^4 dt \int^t \Phi_1^{i-2} \varphi_1^2 dt_1 \\
&- \frac{|n|}{6} \int (1-\Phi)^{n-i-5} \varphi^4 dt \int^t \mathcal{D}_1[(\Phi - \Phi_1) \Phi_1^{i-1}] \varphi_1^2 dt_1 + |n| \int (1-\Phi)^{n-i-5} \varphi^2 dt \int^t \varphi_1^3 dt_1 \\
&\times \int^{t_1} \Phi_2^{i-2} \varphi_2^2 dt_2 - |n| \int (1-\Phi)^{n-i-5} \varphi^2 dt \int^t (\Phi - \Phi_1) \varphi_1 dt_1 \int^{t_1} \mathcal{D}_2[(\Phi_1 - \Phi_2) \Phi_2^{i-1}] \varphi_2^2 dt_2 \\
&+ |n| \int (1-\Phi)^{n-i-4} \varphi^2 \int^t \varphi_1 dt_1 \int^{t_1} \mathcal{D}_2[(\Phi_1 - \Phi_2) \Phi_2^{i-1}] \varphi_2^2 dt_2.
\end{aligned}$$

Sixthly, for $j=i+2$, $k \geq i+5$,

$$E(t_{i|n} t_{i+2|n}^2 t_{k|n}) = -|n| \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi - \Phi_1)^{n-i-3} \varphi_1' t_1 dt_1 \int^{t_1} (\Phi_1 - \Phi_2) \Phi_1^{i-1} \varphi_2' dt_2.$$

The inner integral

$$\begin{aligned}
I(t) &= \int^t (\Phi - \Phi_1)^{k-i-3} I_1(t_1) \varphi_1' t_1 dt_1 \quad (I_1(t_1) \text{ being still the same as before}) \\
&= \int^t (\Phi - \Phi_1)^{k-i-4} \varphi_1 \varphi_1' dt_1 \int^{t_1} \mathcal{D}_2[(\Phi_1 - \Phi_2) \Phi_2^{i-1}] \varphi_2^2 dt_2 + \int^t (\Phi - \Phi_1)^{k-i-3} \Phi_1^{i-1} \varphi_1^2 \varphi_1' dt_1 \\
&- \int^t (\Phi - \Phi_1)^{k-i-3} \varphi_1 dt_1 \int^{t_1} \mathcal{D}_2[(\Phi_1 - \Phi_2) \Phi_2^{i-1}] \varphi_2^2 dt_2 - \int^t (\Phi - \Phi_1)^{k-i-3} \varphi_1 \varphi_1' dt_1 \\
&\times \int^{t_1} \Phi_2^{i-2} \varphi_2^2 dt_2
\end{aligned}$$

with

$$I'(t) = \varphi \int^t (\Phi - \Phi_1)^{k-i-4} I_1(t_1) \varphi_1' t_1 dt_1.$$

Therefore $I'(t)$ is obtained by replacing k by $k-1$ in $I(t)$ but with some additional term, which is eventually non-vanishing as it does vanish for k in $I(t)$. Thus

$$\begin{aligned}
(15.3.6) \quad E(t_{i|n} t_{i+2|n}^2 t_{k|n}) &= -|n| \int (1-\Phi)^{n-k} \varphi' I(t) dt \\
&= -|n| \int (1-\Phi)^{n-k-1} \varphi^2 I(t) dt + |n| \int (1-\Phi)^{n-k} \varphi I'(t) dt = (i) + (ii),
\end{aligned}$$

where

$$\begin{aligned}
(i) &= |n| \int (1-\Phi)^{n-k-1} \varphi^2 dt \left[\int^t \left\{ -\frac{1}{6} (\Phi - \Phi_1)^{k-i-3} \Phi_1^{i-2} - \frac{5}{6} (\Phi - \Phi_1)^{k-i-4} \Phi_1^{i-1} \right\} \varphi_1^4 dt_1 \right. \\
&+ \int^t (\Phi - \Phi_1)^{k-i-4} \varphi_1^3 dt \int^{t_1} \Phi_2^{i-2} \varphi_2^2 dt_2 + \int^t (\Phi - \Phi_1)^{k-i-3} \varphi_1 dt_1 \int^{t_1} \mathcal{D}_2[(\Phi_1 - \Phi_2) \Phi_2^{i-1}] \varphi_2^2 dt_2 \\
&\left. - \frac{1}{2} \int^t (\Phi - \Phi_1)^{k-i-5} \varphi_1^3 dt_1 \int^{t_1} \mathcal{D}_2[(\Phi_1 - \Phi_2) \Phi_2^{i-1}] \varphi_2^2 dt_2 \right],
\end{aligned}$$

and (ii) is obtained by replacing k by $k-1$ in (i), changing the sign and further adding

$$\delta_{i+5}^k \frac{|n|}{2} (1-\Phi)^{n-k} \varphi^4 dt \int^t \mathcal{D}_1[(\Phi-\Phi_1)\Phi_1^{i-1}] \varphi_1^2 dt_1.$$

Seventhly for $j \geq i+3$, $k=j+1$

$$E(t_{i|n} t_{j|n}^2 t_{i+1|n}) = -|n| \int (1-\Phi)^{n-j-1} \varphi' dt \int^t \varphi_1' t_1 dt_1 \int^{t_1} (\Phi_1-\Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' dt_2.$$

Here the inner integral are

$$I_1(t_1) = \int^{t_1} (\Phi_1-\Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' dt_2 = - \int^{t_1} \mathcal{D}_2[(\Phi_1-\Phi_2)^{j-i-1} \Phi_2^{i-1}] \varphi_2^2 dt_2,$$

with

$$I'(t_1) = \varphi_1 \int^{t_1} (\Phi_1-\Phi_2)^{j-i-2} \Phi_2^{i-1} \varphi_2' dt_2 = -\varphi_1 \int^{t_1} \mathcal{D}_2[(\Phi_1-\Phi_2)^{j-i-2} \Phi_2^{i-1}] \varphi_2^2 dt_2$$

and

$$\begin{aligned} I(t) &= \int^t I_1(t_1) \varphi_1' t_1 dt_1 = -\varphi' \int^t \mathcal{D}_1 G(\Phi_1, \Phi_1) \varphi_1^2 dt_1 + \int^t \varphi_1 dt_1 \int^{t_1} \mathcal{D}_2 G(\Phi_1, \Phi_2) \varphi_2^2 dt_2 \\ &\quad - \int^t \varphi_1 \varphi_1' dt_1 \int^{t_1} \mathcal{D}_1 \mathcal{D}_2 G(\Phi_1, \Phi_2) \varphi_2^2 dt_2, \end{aligned}$$

where $G'(\Phi_1, \Phi_2) = (\Phi_1-\Phi_2)^{j-i-1} \Phi_2^{i-1}$, and the last double integral splits into

$$- \frac{1}{2} \int^t \mathcal{D}_1[\mathcal{D}G(\Phi, \Phi_1)] \varphi_1^2 dt_1 + \frac{1}{2} \int^t \varphi_1^3 dt_1 \int^{t_1} \mathcal{D}_2 \mathcal{D}_1 G(\Phi_1, \Phi_2) \varphi_2^2 dt_2 - \frac{1}{2} \delta_{i+3}^j \int^t \Phi_1^{i-1} \varphi_1^4 dt_1.$$

Therefore

$$\begin{aligned} (15.3.7) \quad & E(i|_n t_{j|n}^2 t_{j+1|n}) \quad (j \geq i+3) \\ &= \frac{|n|}{2} \int (1-\Phi)^{n-j-1} \varphi^2 dt \int^t \mathcal{D}_1[(\Phi-\Phi_1)^{i-j-1} \Phi_1^{i-1}] \varphi_1^2 dt_1 + \frac{|n|}{2} \int (1-\Phi)^{n-j-2} \varphi^4 dt \\ &\quad \times \int^t \mathcal{D}_1[(\Phi-\Phi_1)^{j-i-2} \Phi_1^{i-1}] \varphi_1^2 dt_1 - \frac{|n|}{6} \int (1-\Phi)^{n-j-3} \varphi^4 dt \int^t \mathcal{D}_1[(\Phi-\Phi_1)^{j-i-1} \Phi_1^{i-1}] \varphi_1^2 dt_1 \\ &\quad + \frac{|n|}{6} \int (1-\Phi)^{n-j-1} \varphi^4 dt \int^t \mathcal{D}_1[(\Phi-\Phi_1)^{j-i-3} \Phi_1^{i-1}] \varphi_1^2 dt_1 - |n| \int (1-\Phi)^{n-j-2} \varphi^2 dt \int^t \varphi_1 dt_1 \\ &\quad \times \int^{t_1} \mathcal{D}_2[(\Phi_1-\Phi_2)^{j-i-1} \Phi_2^{i-1}] \varphi_2^2 dt_2 + \frac{|n|}{2} \int (1-\Phi)^{n-j-2} \varphi^2 dt \int^t \varphi_1^3 dt_1 \int^{t_1} \mathcal{D}_2[(\Phi_1-\Phi_2)^{j-i-3} \Phi_2^{i-1}] \varphi_2^2 dt_2 \\ &\quad + \delta_{i+3}^j \frac{|n|}{2} \left[\int (1-\Phi)^{n-j-2} \varphi^2 dt \int^t \Phi_1^{i-1} \varphi_1^4 dt_1 - \int (1-\Phi)^{n-j-1} \Phi_1^{i-1} \varphi^5 dt \right]. \end{aligned}$$

Eighthly, for $j \geq i+3$, $k \geq j+2$,

$$E(t_{i|n} t_{j|n} t_{k|n}) = -|n| \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi-\Phi_1)^{k-j-1} \varphi_1' t_1 dt_1 \int^{t_1} (\Phi_1-\Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' dt_2.$$

Here the innermost integral being

$$I_1(t_1) = - \int^{t_1} \mathcal{D}_2[(\Phi_1-\Phi_2)^{j-i-1} \Phi_2^{i-1}] \varphi_2^2 dt_2$$

with

$$I_1'(t_1) = -\varphi_1 \int^{t_1} \mathfrak{D}_2[(\Phi_1 - \Phi_2)^{j-i-2} \Phi_2^{i-1}] \varphi_2^2 dt_2$$

and

$$I''(t_1) = \delta_{i+3}^j \Phi_1^{i-1} \varphi_1^3 - \varphi_1' \int^{t_1} \mathfrak{D}_2[(\Phi_1 - \Phi_2)^{j-i-2} \Phi_2^{i-1}] \varphi_2^2 dt_2 + \varphi_1 \int^{t_1} \mathfrak{D}_2[(\Phi_1 - \Phi_2)^{j-i-3} \Phi_2^{i-1}] \varphi_2^2 dt_2,$$

the inner integral becomes

$$\begin{aligned} I(t) &= \int^t (\Phi - \Phi_1)^{k-j-1} I_1(t_1) \varphi_1' t_1 dt_1 \\ &= - \int^t (\Phi - \Phi_1)^{k-j-2} I_1(t_1) \varphi_1 \varphi_1' dt_1 - \int^t (\Phi - \Phi_1)^{k-j-1} I_1(t_1) \varphi_1 dt_1 + \int^t (\Phi - \Phi_1)^{k-j-1} I_1'(t_1) \varphi_1' dt_1 \\ &= \delta_{j+2}^k \frac{\varphi^2}{2} \int^t \mathfrak{D}_1[(\Phi - \Phi_1)^{j-i-1} \Phi_1^{i-1}] \varphi_1^2 dt_1 + \frac{1}{2} \int^t (\Phi - \Phi_1)^{k-j-3} \varphi_1^3 dt_1 \\ &\quad \times \int^{t_1} \mathfrak{D}_2[(\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1}] \varphi_2^2 dt_2 - \frac{3}{2} \int^t (\Phi - \Phi_1)^{k-j-2} \varphi_1^3 dt_1 \int^{t_1} \mathfrak{D}_2[(\Phi_1 - \Phi_2)^{j-i-2} \Phi_2^{i-1}] \varphi_2^2 dt_2 \\ &\quad + \int^t (\Phi - \Phi_1)^{k-j-1} \varphi_1 dt_1 \int^{t_1} \mathfrak{D}_2[(\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1}] \varphi_2^2 dt_2 - \int^t (\Phi - \Phi_1)^{k-j-1} \varphi_1^4 dt_1 \\ &\quad \times \int^{t_1} \mathfrak{D}_2[(\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1}] \varphi_2^2 dt_2 - \delta_{i+3}^j \int^t (\Phi - \Phi_1)^{k-j-1} \Phi_1^{i-1} \varphi_1^4 dt_1 \\ &\quad + \int^t (\Phi - \Phi_1)^{k-j-1} \varphi_1 \varphi_1' dt_1 \int^t [(\Phi_1 - \Phi_2)^{j-i-2} \Phi_2^{i-1}] \varphi_2^2 dt_2, \end{aligned}$$

where the latest double integral again splits up into

$$\begin{aligned} &\frac{1}{2} \int^t (\Phi - \Phi_1)^{k-j-2} \varphi_1^3 dt_1 \int^t \mathfrak{D}_2[(\Phi_1 - \Phi_2)^{j-i-2} \Phi_2^{i-1}] \varphi_2^2 dt_2 - \frac{1}{2} \int^t (\Phi - \Phi_1)^{k-j-1} \varphi_1^3 dt_1 \\ &\quad \times \int^t \mathfrak{D}_2[(\Phi_1 - \Phi_2)^{j-i-3} \Phi_2^{i-1}] \varphi_2^2 dt_2 + \frac{1}{2} \delta_{i+3}^j \int^t (\Phi - \Phi_1)^{k-j-1} \Phi_1^{i-1} \varphi_1^4 dt_1 \\ &\quad \text{(this becomes } -\frac{1}{2} \delta_{i+3}^j \text{ if combined with the before obtained } -\delta_{i+3}^j). \end{aligned}$$

Hence

$$\begin{aligned} (15.3.8) \quad E(t_{i|n} t_{j|n}^2 t_{k|n}) &= -|n| \int (1 - \Phi)^{n-k} I(t) \varphi' dt \\ &= -|n| \int (1 - \Phi)^{n-k-1} \varphi^2 I(t) dt + |n| \int (1 - \Phi)^{n-k} \varphi I'(t) dt = (i) + (ii). \end{aligned}$$

The sum (i) is already free from dash, while (ii) is obtainable by replacing k by $k-1$ in (i) and changing the sign, so that $-\delta_{j+2}^k$ standing at head in (i) should be replaced by $+\delta_{j+2}^{k-1}$ namely

$$\delta_{j+3}^k \frac{|n|}{2} \int (1 - \Phi)^{n-k-1} \varphi^4 dt \int^t \mathfrak{D}_1[(\Phi - \Phi_1)^{j-i-1} \Phi_1^{i-1}] \varphi_1^2 dt_1,$$

and δ_{i+3}^j standing at tail in (i) by

$$-\delta_{i+3}^j \frac{|n|}{2} \int (1 - \Phi)^{n-k} \varphi^2 dt \int^t (\Phi - \Phi_1)^{k-j-2} \Phi_1^{i-1} \varphi_1^4 dt_1.$$

So far we have exhausted all the cases in regard to the third and fourth moments up to with 3 arguments. However, we omit general formulas for the fourth moment with 4 arguments, i.e. those about

$$E(t_{i|n}t_{j|n}t_{k|n}t_{l|n}) = |n| \int (1-\Phi)^{n-l} \varphi' dt \int^t (\Phi-\Phi_1)^{l-k-1} \varphi_1' dt_1 \int^{t_1} (\Phi_1-\Phi_2)^{k-j-1} \varphi_2' dt_2 \\ \times \int^{t_2} (\Phi_2-\Phi_3)^{j-i-1} \Phi_3^{i-1} \varphi_3' dt_3$$

to make it dashfree, since it is rather simple to calculate them directly for each case that i, j, k, l are numerically given, as long as n is rather small (cf. the end of § 18).

§ 16. Check Formulas. By use of the formulas in the preceding and subsequent paragraphs (§§ 14, 15 and 17), we can obtain the explicit forms or numerical values of moments. However, to ascertain their correctness, we should verify it by means of some check formulas. Some of them were already described in § 5 Part I, among which (5.10) i.e. $\sum_{k=1}^n E(t_{i|n}t_{k|n})=1$ was especially eminent. Therefore we would extend it somehow. Now by means of identities (3.2) in Part I, we can finish all work to find e.g. $E(t_{i|n}^2)$ by half: namely, we have only to compute them for $i=1, 2, \dots, \left[\frac{n+1}{2}\right]$. Also we get identity (3.1): $\sum_{k=1}^n E(t_{i|n}^p)=0$ or $1 \cdot 3 \cdots (n-1)p$ according as p is odd or even, and hence this for even p suffices as check. However, if p be odd, with our determination $E(t_{n-i+1|n}^p)=-E(t_{i|n}^p)$, naturally the whole sum becomes zero, which, however, of course, gives no check. Therefore, to verify e.g. $E(t_{i|n}^3)$, we require other check formulas, which would be found below.

1° Firstly, treating the case of 2 arguments, we shall prove the following identity:

$$(16.1) \quad \sum_{k=1}^n E(t_{i|n}^p t_{k|n}) = p E(t_{i|n}^{p-1}) \quad \text{for any fixed } i (=1, 2, \dots, n).$$

Proof. Putting

$$\sum_{k=1}^n = \sum_{k=1}^{i-1} + E(t_{i|n}^{p+1}) + \sum_{k=i+1}^n = (i) + (ii) + (iii),$$

we have

$$(i) = - \sum_{k=1}^{i-1} |n| \int (1-\Phi)^{n-i} \varphi t^p dt \int^t (\Phi-\Phi_1)^{i-k-1} \Phi_1 \varphi_1' dt_1 \\ = - |n| \int (1-\Phi)^{n-i} \varphi t^p dt \int^t \sum_{k=1}^{i-1} (\Phi-\Phi_1)^{i-k-1} \Phi_1^{k-1} \varphi_1' dt_1 \\ = - |n| \int (1-\Phi)^{n-i} \Phi^{i-2} \varphi^2 t^p dt = - |n| \int (1-\Phi)^{n-i} D(\Phi^{i-1}) \varphi^2 t^p dt; \\ (ii) = - |n| \int (1-\Phi)^{n-i} \Phi^{i-1} \varphi' t^p dt \\ = |n| \int D[(1-\Phi)^{n-i} \Phi^{i-1}] \varphi^2 t^p dt + |n| p \int (1-\Phi)^{n-i} \Phi^{i-1} \varphi t^{p-1} dt,$$

whose last integral is evidently $pE(t_{i|n}^{p-1})$; and lastly

$$\begin{aligned} \text{(iii)} &= - \sum_{k=i+1}^n |n| \int (1-\Phi)^{n-k} \varphi' dt \int_t^t (\Phi - \Phi_1)^{k-i-1} \Phi_1^{i-1} \varphi_1 t_1^i dt_1 \\ &= -|n| \int \Phi_1^{i-1} \varphi_1 t_1^i dt_1 \int_{t_1}^n \sum_{k=i+1}^n (1-\Phi)^{n-k} (\Phi - \Phi_1)^{k-i-1} \varphi' dt \\ &= |n| \int (1-\Phi)^{n-i-1} \Phi^{i-1} \varphi^2 t^i dt = -|n| \int D(1-\Phi)^{n-i} \cdot \Phi^{i-1} \varphi^2 t^i dt. \end{aligned}$$

By adding (i) and (iii) together it amounts to $-|n| \int D[(1-\Phi)^{n-i} \Phi^{i-1}] \varphi^2 t^i dt$ and therefore the total sum (i)+(ii)+(iii) gives $pE(t_{i|n}^{p-1})$, as asserted, Q.E.D.

Against the foregoing, the evaluation of $\sum_{k=1}^n E(t_{i|n} t_{k|n}^q)$ becomes somewhat complicate, so that we treat only those actually needed, i.e. the cases $q=2$ and 3.

$$(16.2) \quad \sum_{k=1}^n E(t_{i|n} t_{k|n}^2) = (n+1)E(t_{i|n}).$$

Proof. As before, letting $\sum_{k=1}^n = \sum_{k=1}^{i-1} + E(t_{i|n}^3) + \sum_{k=i+1}^n = \text{(i)} + \text{(ii)} + \text{(iii)}$, we have

$$\begin{aligned} \text{(i)} &= |n| \int (1-\Phi)^{n-i} \varphi' dt \int \sum_{k=1}^{i-1} (\Phi - \Phi_1)^{i-k-1} \Phi_1^{k-1} \varphi_1' t_1 dt_1 \\ &= |n| \int (1-\Phi)^{n-i} \Phi^{i-2} \varphi' dt \int \varphi_1' t_1 dt_1 (= \varphi t - \Phi) \\ &= -|n| \int (1-\Phi)^{n-i} \Phi^{i-2} \varphi'^2 dt + (i-1)|n| \int (1-\Phi)^{n-i} \Phi^{i-1} \varphi t dt (= (i-1)E(t_{i|n})); \\ \text{(ii)} &= -|n| \int (1-\Phi)^{n-i} \Phi^{i-1} \varphi' t^2 dt \\ &= |n| \int D(1-\Phi)^{n-i} \Phi^{i-1} \varphi^2 t^2 dt + 2|n| \int (1-\Phi)^{n-i} \Phi^{i-1} \varphi t dt \\ &= |n| \int D(1-\Phi)^{n-i} \Phi^{i-1} \varphi'^2 dt + 2E(t_{i|n}); \\ \text{(iii)} &= \sum_{k=i+1}^n |n| \int (1-\Phi)^{n-k} \varphi' t dt \int_t^t (\Phi - \Phi_1)^{k-i-1} \Phi_1^{i-1} \varphi_1' dt_1 \\ &= |n| \int \Phi_1^{i-1} \varphi_1' dt_1 \int_{t_1}^n \sum_{k=i+1}^n (1-\Phi)^{n-k} (\Phi - \Phi_1)^{k-i-1} \varphi' t dt \\ &= |n| \int \Phi_1^{i-1} (1-\Phi_1)^{n-i-1} \varphi_1' dt_1 \int_{t_1} \varphi' t dt (= \varphi_1' - (1-\Phi_1)) \\ &= -|n| \int \Phi^{i-1} D(1-\Phi)^{n-i} \varphi'^2 dt + (n-i)E(t_{i|n}). \end{aligned}$$

Therefore

$$\text{(i)} + \text{(ii)} + \text{(iii)} = (n+1) E(t_{i|n}), \quad \text{Q.E.D.}$$

Also

$$(16.3) \quad \sum_{k=1}^n E(t_{i|n} t_{k|n}^3) = 2 + E(t_{i|n}^2).$$

Proof. We put again $\sum_{k=1}^n = \sum_{k=1}^{i-1} + E(t_{i|n}^4) + \sum_{k=i+1}^n$ (i)+(ii)+(iii), where

$$\begin{aligned} (i) &= |n| \int (1-\Phi)^{n-i} \varphi' dt \int \sum_{k=1}^{i-1} (\Phi - \Phi_1)^{i-k-1} \Phi_1^{k-1} \varphi_1' t_1^2 dt_1 \\ &= |n| \int (1-\Phi)^{n-i} \Phi^{i-2} \varphi' dt \int \varphi_1' t_1^2 dt_1; \end{aligned}$$

but, since the inner integral becomes $\varphi t^2 - 2 \int \varphi t dt = -\varphi' t + 2\varphi$,

$$(i) = |n| \int (1-\Phi)^{n-i} D(\Phi^{i-1}) \varphi \varphi' t^2 dt + 2|n| \int (1-\Phi)^{n-i} D[\Phi^{i-1}] \varphi \varphi' dt;$$

and similarly

$$\begin{aligned} (iii) &= |n| \int \Phi_1^{i-1} \varphi_1' dt_1 \int_{t_1}^n \sum_{k=i+1}^n (1-\Phi)^{n-k} (\Phi - \Phi_1)^{k-i-1} \varphi' t^2 dt \\ &= |n| \int \Phi_1^{i-1} (1-\Phi_1)^{n-i-1} [-\varphi_1 t^2 - 2\varphi_1] \varphi_1' dt_1 \\ &= |n| \int \Phi^{i-1} D[(1-\Phi)^{n-i}] \varphi \varphi' t^2 dt + 2|n| \int \Phi^{i-1} D[(1-\Phi)^{n-i}] \varphi \varphi' dt. \end{aligned}$$

Lastly

$$\begin{aligned} (ii) &= |n| \int (1-\Phi)^{n-i} \Phi^{i-1} (-\varphi' t^3) dt \\ &= -|n| \int D[(1-\Phi)^{n-i} \Phi^{i-1}] \varphi \varphi' t^2 dt + 3|n| \int (1-\Phi)^{n-i} \Phi^{i-1} \varphi t^2 dt \quad (= 3E(t_{1|n}^2)). \end{aligned}$$

But

$$\begin{aligned} E(t_{i|n}^2) &= -|n| \int (1-\Phi)^{n-i} \Phi^{i-1} \varphi' t dt \\ &= |n| \int (1-\Phi)^{n-i} \Phi^{i-1} \varphi dt + |n| \int D[(1-\Phi)^{n-i} \Phi^{i-1}] \varphi^2 t dt \\ &= 1 - |n| \int D[(1-\Phi)^{n-i} \Phi^{i-1}] \varphi \varphi' dt, \end{aligned}$$

that is

$$|n| \int D[(1-\Phi)^{n-i} \Phi^{i-1}] \varphi \varphi' dt = 1 - E(t_{i|n}^2).$$

Hence, the total sum (i)+(ii)+(iii) yields the right hand side of (16.3), Q.E.D.

Further

$$(16.4) \quad \sum_{k=1}^n E(t_{i|n}^2 t_{k|n}^2) = (n+2)E(t_{i|n}^2), \quad \sum_{i=1}^n \sum_{k=1}^n E(t_{i|n}^2 t_{k|n}^2) = n(n+2)$$

Proof. Letting $\sum_{k=1}^n = \sum_{k=1}^{i-1} + E(t_{i|n}^4) + \sum_{k=i+1}^n = (i)+(ii)+(iii)$,

$$\begin{aligned} (i) &= |n| \int (1-\Phi)^{n-i} \Phi^{i-2} (\varphi t - \Phi) \varphi' t dt \\ &= -|n| \int (1-\Phi)^{n-i} D[\Phi^{i-1}] \varphi'^2 t dt + (i-1)E(t_{i|n}^2); \end{aligned}$$

$$\begin{aligned}
\text{(iii)} &= |n \int (1-\Phi)^{n-i-1} \Phi^{i-2} [-\varphi t - (1-\Phi)] \varphi' t dt \\
&= -|n \int \Phi^{i-1} D[(1-\Phi)^{n-i}] \varphi'^2 t dt + (n-i) \cdot E(t_{i|n}^2) .
\end{aligned}$$

But

$$\begin{aligned}
\text{(ii)} &= E(t_{i|n}^4) = |n \int (1-\Phi)^{n-i} \Phi^{i-1} (-\varphi' t^3) dt \\
&= |n \int D[(1-\Phi)^{n-i} \Phi^{i-1}] \varphi'^2 t dt + 3E(t_{i|n}^2) .
\end{aligned}$$

Hence their sum (i)+(ii)+(iii) gives $(n+2)E(t_{i|n}^2)$, as asserted.

Remark. We may here recapitulate another kind of check formulæ, which was already described at (5.3), Part I :

$$(16.5) \quad \sum_{k=2}^n \sum_{i=1}^{k-1} E(t_{i|n}^p t_{k|n}^q) = \frac{n(n-1)}{2\pi} 2^{\frac{p+q}{2}} \Gamma\left(\frac{p+q}{2}+1\right) \int_{\pi/4}^{5\pi/4} \cos^p \theta \sin^q \theta d\theta ,$$

so that, in particular,

$$(16.5.1) \quad \sum_{k=1}^n \sum_{i=1}^{k-1} E(t_{i|n}^2 t_{k|n}) = \frac{n(n-1)}{4\sqrt{\pi}} \quad \text{and} \quad \sum_{k=2}^n \sum_{i=1}^{k-1} E(t_{i|n} t_{n|n}^2) = -\frac{n(n-1)}{4\sqrt{\pi}} .$$

$$(16.5.2) \quad \sum_{k=2}^n \sum_{i=1}^{k-1} E(t_{i|n}^3 t_{k|n}) = \sum_{k=2}^n \sum_{i=1}^{k-1} E(t_{i|n} t_{k|n}^3) = 0 .$$

$$(16.5.3) \quad \sum_{k=2}^n \sum_{i=1}^{k-1} E(t_{i|n}^2 t_{k|n}^2) = \frac{1}{2} n(n-1) .$$

2° Next, we consider the case of 3 arguments : we shall prove the identity

$$(16.6) \quad \sum_{k=1}^n E(t_{i|n} t_{j|n} t_{k|n}) = E(t_{i|n}) + E(t_{j|n}) .$$

Proof. When $i=j$, (16.6) reduces to (16.1) for $p=2$. Hence, without the loss of generality we may assume that $i < j$. We put

$$\sum_{k=1}^n E(t_{i|n} t_{j|n} t_{k|n}) = \sum_{k=1}^{i-1} + E(t_{i|n}^2 t_{j|n}) + \sum_{k=i+1}^{j-1} + E(t_{i|n} t_{j|n}^2) + \sum_{k=j+1}^n = \text{(i)} + \text{(ii)} + \text{(iii)} + \text{(iv)} + \text{(v)},$$

where (iii) becomes 0, if $i+1=j$. Upon integrating by parts, the successive terms by the assumption $j-i-1 > 0$ yield

$$\begin{aligned}
\text{(i)} &= -|n \int (1-\Phi)^{n-j} \varphi' dt \int^t (\Phi-\Phi_1)^{j-i-1} \varphi_1' dt_1 \int^t \sum_{k=1}^{i-1} (\Phi_1-\Phi_2)^{i-k-1} \Phi_2^{k-1} \varphi_2' dt_2 \\
&= -|n \int (1-\Phi)^{n-j} \varphi' dt \int^t (\Phi-\Phi_1)^{j-i-1} \Phi_1^{i-2} \varphi_1 \varphi_1' dt_1 ; \\
\text{(iii)} &= - \sum_{k=i+1}^{j-1} |n \int (1-\Phi)^{n-j} \varphi' dt \int^t (\Phi-\Phi_1)^{j-k-1} \varphi_1' dt_1 \int^{t_1} (\Phi_1-\Phi_2)^{k-i-1} \Phi_1^{i-1} \varphi_2' dt_2 \\
&= -|n \int (1-\Phi)^{n-j} \varphi' dt \int^t \Phi_2^{i-1} \varphi_2' dt_2 \int_{t_2}^{t_1} \sum_{k=i+1}^{j-1} (\Phi_1-\Phi_2)^{j-k-1} (\Phi-\Phi_1)^{k-i-1} \varphi_1' dt_1 \\
&= -|n \int (1-\Phi)^{n-j} \varphi \varphi' dt \int^t (\Phi-\Phi_1)^{j-i-2} \Phi_1^{i-1} \varphi_1' dt_1
\end{aligned}$$

$$\begin{aligned}
& + |n \int (1-\Phi)^{n-j} \varphi' dt \int^t (\Phi-\Phi_1)^{j-i-2} \Phi_1^{i-1} \varphi_1 \varphi_1' dt_1 (=0, \text{ if } i+1=j); \\
(\text{v}) &= - \sum_{k=i+1}^n |n \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi-\Phi_1)^{k-j-1} \varphi_1' dt_1 \int^{t_1} (\Phi_1-\Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' dt_2 \\
&= - |n \int \varphi_1' dt_1 \int_{t_1}^n \sum_{k=j+1}^n (1-\Phi)^{n-k} (\Phi-\Phi_1)^{k-j-1} \varphi' dt \int^{t_1} (\Phi_1-\Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' dt_2 \\
&= |n \int (1-\Phi)^{n-j-1} \varphi \varphi' dt \int^t (\Phi-\Phi_1)^{j-i-1} \Phi_1^{i-1} \varphi_1' dt_1.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(\text{ii}) &= |n \int (1-\Phi)^{n-j} \varphi' dt \int^t (\Phi-\Phi_1)^{j-i-1} \varphi_1' t_1 dt_1 \\
&= |n \int (1-\Phi)^{n-j} \varphi' dt \int^t D_1[(\Phi-\Phi_1)^{j-i-1} \Phi_1^{i-1}] \varphi_1 \varphi_1' dt_1 \\
&\quad - |n \int (1-\Phi)^{n-j} \varphi' dt \int^t (\Phi-\Phi_1)^{j-i-1} \Phi_1^{i-1} \varphi_1 dt_1; \\
(\text{iv}) &= |n \int (1-\Phi)^{n-j} \varphi' t dt \int^t (\Phi-\Phi_1)^{j-i-1} \Phi_1^{i-1} \varphi_1' dt_1 \\
&= - |n \int (1-\Phi)^{n-j-1} \varphi \varphi' dt \int^t (\Phi-\Phi_1)^{j-i-1} \Phi_1^{i-1} \varphi_1' dt_1 \\
&\quad - |n \int (1-\Phi)^{n-j} \varphi dt \int^t (\Phi-\Phi_1)^{j-i-1} \Phi_1^{i-1} \varphi_1' dt_1 \\
&\quad + |n \int (1-\Phi)^{n-j} \varphi \varphi' dt \int^t (\Phi-\Phi_1)^{j-i-2} \Phi_1^{i-1} \varphi_1' dt_1.
\end{aligned}$$

Therefore, on summing up all of them, they almost cancel two by two, except those underlined two integrals. But, the first underlined integral becomes, on writing $\Phi_1 = \Phi v$,

$$- |n \int (1-\Phi)^{n-j} \varphi' dt \int_0^1 \Phi^{i-1} (1-v)^{j-i-1} v^{i-1} dv = |n \int (1-\Phi)^{n-j} \Phi^{j-1} \varphi t dt = E(t_{j|n}),$$

while the second, on interchanging the order of integrations and then writing $\Phi - \Phi_1 = (1 - \Phi)v$,

$$- |n \int \Phi_1^{i-1} \varphi_1' dt_1 \int (1-\Phi)^{n-j} (\Phi-\Phi_1)^{j-i-1} \varphi dt = - |n \int \Phi^{i-1} (1-\Phi)^{n-j} \varphi' dt = E(t_{i|n}).$$

Thus, (16.6) is established.

Adding (16.6) for $j=1, 2, \dots, n$ and applying (3.1), Part I, $\sum_j E(t_{j|n})=0$, we get

$$(16.6.1) \quad \sum_{j=1}^n \sum_{k=1}^n E(t_{i|n} t_{j|n} t_{k|n}) = n E(t_{i|n}).$$

Further, we have

$$(16.7) \quad \sum_{k=1}^n E(t_{i|n}^2 t_{j|n} t_{k|n}) = E(t_{i|n}^2) + 2E(t_{i|n} t_{j|n}).$$

Proof. First assuming that $i+1 < j$, we put, as before,

$$\sum_{k=1}^n = \sum_{k=1}^{i-1} + E(t_{i|n}^3 t_{j|n}) + \sum_{k=i+1}^{j-1} + E(t_{i|n}^2 t_{j|n}^2) + \sum_{k=j+1}^n = \text{(i)} + \text{(ii)} + \text{(iii)} + \text{(iv)} + \text{(v)},$$

where

$$\begin{aligned} \text{(i)} &= -|n \int (1-\Phi)^{n-j} \varphi' dt \int^t (\Phi-\Phi_1)^{j-i-1} \varphi_1' t_1 dt_1 \int^t \sum_{k=1}^{i-1} (\Phi_1-\Phi_2)^{i-k-1} \Phi_2^{k-1} \varphi_2' dt_2 \\ &= -|n \int (1-\Phi)^{n-j} \varphi' dt \int^t (\Phi-\Phi_1)^{j-i-1} \Phi_1^{i-2} \varphi_1' t_1 dt_1; \\ \text{(iii)} &= -|n \int \sum_{k=i+1}^{j-1} (1-\Phi)^{n-j} \varphi' dt \int^t (\Phi-\Phi_1)^{j-k-1} \varphi_1' dt \int^t (\Phi_1-\Phi_2)^{k-i-1} \Phi_2^{i-1} \varphi_2' t_2 dt_2 \\ &= -|n \int (1-\Phi)^{n-j} \varphi' dt \int^t \Phi_2^{i-1} \varphi_2' t_2 dt_2 \int^t \sum_{k=i+1}^{j-1} (\Phi-\Phi_1)^{k-i-1} \varphi_1' dt_1 \\ &= -|n \int (1-\Phi)^{n-j} \varphi \varphi' dt \int^t (\Phi-\Phi_1)^{j-i-2} \Phi_1^{i-1} \varphi_1' t_1 dt_1 \\ &\quad + |n \int (1-\Phi)^{n-j} \varphi' dt \int^t (\Phi-\Phi_1)^{j-i-2} \Phi_1^{i-1} \varphi_1' t_1 dt_1; \\ \text{(v)} &= -|n \int \sum_{k=j+1}^n (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi-\Phi_1)^{k-j-1} \varphi_1' dt_1 \int^t (\Phi_1-\Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' t_2 dt_2 \\ &= -|n \int \varphi_1' dt_1 \int_{t_1}^n \sum_{k=j+1}^n (1-\Phi)^{n-k} (\Phi-\Phi_1)^{k-j-1} \varphi' dt \int^t (\Phi_1-\Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' t_2 dt_2 \\ &= |n \int (1-\Phi)^{n-j-1} \varphi \varphi' dt \int^t (\Phi-\Phi_1)^{j-i-1} \Phi_1^{i-1} \varphi_1' t_1 dt_1; \\ \text{(ii)} &= |n \int (1-\Phi)^{n-j} \varphi' dt \int^t (\Phi-\Phi_1)^{j-i-1} \Phi_1^{i-1} \varphi_1' t_1^2 dt_1 \\ &= |n \int (1-\Phi)^{n-j} \varphi' dt \int^t D_i[(\Phi-\Phi_1)^{j-i-1} \Phi_1^{i-1}] \varphi_1 \varphi_1' t_1 dt_1 \\ &\quad + 2|n \int (1-\Phi)^{n-j} \varphi' dt \int^t (\Phi-\Phi_1)^{j-i-1} \Phi_1^{i-1} \varphi_1' t_1 dt_1; \\ \text{(iv)} &= |n \int (1-\Phi)^{n-j} \varphi' t dt \int^t (\Phi-\Phi_1)^{j-i-1} \Phi_1^{i-1} \varphi_1' t_1 dt_1 \\ &= -|n \int (1-\Phi)^{n-j-1} \varphi \varphi' dt \int^t (\Phi-\Phi_1)^{j-i-1} \Phi_1^{i-1} \varphi_1' t_1 dt_1 \\ &\quad - |n \int (1-\Phi)^{n-j} \varphi dt \int^t (\Phi-\Phi_1)^{j-i-1} \Phi_1^{i-1} \varphi_1' t_1 dt_1 \\ &\quad + |n \int (1-\Phi)^{n-j} \varphi \varphi' dt \int^t (\Phi-\Phi_1)^{j-i-2} \Phi_1^{i-1} \varphi_1' t_1 dt_1. \end{aligned}$$

On summing up all of them, they almost cancel, except those two underlined integrals, one of which is clearly $2E(t_{j|n} t_{i|n})$, while the other is

$$\begin{aligned} &|n \int (1-\Phi)^{n-j} \varphi dt \int^t (\Phi-\Phi_1)^{j-i-1} \Phi_1^{i-1} \varphi_1 t_1^2 dt_1 \\ &= |n \int \Phi_1^{i-1} \varphi_1 t_1^2 dt_1 \int_{t_1}^n (1-\Phi)^{n-j} (\Phi-\Phi_1)^{j-i-1} \varphi dt; \end{aligned}$$

or, on putting $1-\Phi=(1-\Phi_1)v$, it reduces to $\int \Phi^{i-1}(1-\Phi)^{n-i} \varphi t^2 dt = E(t_{i|n}^2)$,
Q.E.D.

The case $j=i+1$ as well as $j < i$ could be quite similarly proved⁷⁾. Thus we see that two summations

$$\begin{aligned} \sum_{k=1}^n E(t_{i|n}^2 t_{j|n} t_{k|n}) &= E(t_{i|n}^2) + 2E(t_{i|n} t_{j|n}), \quad (i < j) \\ \sum_{k=1}^n E(t_{i|n} t_{j|n}^2 t_{k|n}) &= E(t_{j|n}^2) + 2E(t_{i|n} t_{j|n}) \quad (i < j) \end{aligned}$$

either has the same form. Consequently, we obtain

$$(16.7.1) \quad \sum_{j=1}^n \sum_{k=1}^n E(t_{i|n}^2 t_{j|n} t_{k|n}) = nE(t_{i|n}^2) + 2, \quad \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n E(t_{i|n}^2 t_{j|n} t_{k|n}) = n(n+2).$$

$$(16.7.2) \quad \sum_{j=1}^n \sum_{k=1}^n E(t_{i|n} t_{j|n}^2 t_{k|n}) = n+2.$$

For,

$$\begin{aligned} \sum_{j=1}^n \sum_{k=1}^n E(t_{i|n}^2 t_{j|n} t_{k|n}) &= \sum_{j=1}^{i-1} \sum_{k=1}^n E(t_j t_i^2 t_k) + \sum_{k=1}^n E(t_i^3 t_k) + \sum_{j=i+1}^n \sum_{k=1}^n E(t_i^2 t_j t_k) \\ &= \sum_{j=1}^{i-1} [E(t_{j|n}^2) + 2E(t_i t_j)] + 3E(t_{i|n}^3) + \sum_{j=i+1}^n [E(t_{i|n}^2) + 2E(t_i t_j)] \\ &= nE(t_{i|n}^2) + 2 \sum_{j=1}^n E(t_{i|n} t_{j|n}) = nE(t_{i|n}^2) + 2 \end{aligned}$$

in view of (5.10), Part I. Also, by reason of (3.1) for $p=2$, and (5.10)

$$\begin{aligned} \sum_{i=1}^n \sum_{k=1}^n E(t_{i|n} t_{j|n}^2 t_{k|n}) &= \sum_{j=1}^{i-1} E(t_{i|n} t_{j|n}^2 t_{k|n}) + \sum_{k=1}^n E(t_{i|n}^3 t_{k|n}) + \sum_{j=i+1}^n \sum_{k=1}^n E(t_{i|n} t_{j|n}^2 t_{k|n}) \\ &= \sum_{j=1}^{i-1} [E(t_{j|n}^2) + 2E(t_i t_j)] + 3E(t_{i|n}^3) + \sum_{j=i+1}^n [E(t_{j|n}^2) + 2E(t_i t_j)] \\ &= \sum_{j=1}^n E(t_{j|n}^2) + 2 \sum_{j=1}^n E(t_{i|n} t_{j|n}) = n+2. \end{aligned}$$

Remark. We may again recapitulate the formulas (5.6) in Part I:

$$(16.8) \quad \left\{ \begin{aligned} \sum \sum \sum E(t_i^2 t_j t_k) &= \sum \sum \sum E(t_i t_j t_k^2) = \frac{n(n-1)(n-2)}{12\pi\sqrt{3}}, \\ \sum \sum \sum E(t_i t_j^2 t_k) &= -\frac{n(n-1)(n-2)}{6\pi\sqrt{3}}, \end{aligned} \right.$$

where the summation $\sum \sum \sum$ denotes $\sum_{k=3}^n \sum_{j=2}^{k-1} \sum_{i=1}^{j-1}$. These may be conveniently employed, because, although the number of summands becomes somewhat larger, here all the summands are, so to speak, pure i.e. only $E(t_{i|n}^2 t_{j|n} t_{k|n})$ or $E(t_{i|n} t_{j|n}^2 t_{k|n})$ come forth without mixing those degenerated kinds e.g. $E(t_i^2 t_j^2)$.

3° Furthermore, we consider the case of 4 arguments. We have

$$(16.9) \quad \sum_{l=1}^n E(t_{i|n} t_{j|n} t_{k|n} t_{l|n}) = E(t_{i|n} t_{j|n}) + E(t_{i|n} t_{k|n}) + E(t_{j|n} t_{k|n}).$$

7) The case $j=i$ reduces to (16.1) with $p=3$, so that (16.7) still holds formally.

Proof. By reason of symmetry we may consider only the case $i < j < k$.
Let

$$\begin{aligned} \sum_{l=1}^n &= \sum_{l=1}^{i-1} + E(t_i^2 t_j t_k) + \sum_{l=i+1}^{j-1} + E(t_i t_j^2 t_k) + \sum_{l=j+1}^{k-1} + E(t_i t_k t_j^2) + \sum_{l=k+1}^n \\ &= (i) + (ii) + (iii) + (iv) + (v) + (vi) + (vii) . \end{aligned}$$

we obtain by summation each by each (details being omitted)

$$(i) = \underline{n} \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi - \Phi_1)^{k-j-1} \varphi_1' dt_1 \int^{t_1} (\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-2} \varphi_2' dt_2 \dots (\alpha)$$

$$(iii) = \underline{n} \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi - \Phi_1)^{k-j-1} \varphi_1' dt_1 \int^{t_1} (\Phi_1 - \Phi_2)^{j-i-2} \Phi_2^{i-1} \varphi_2' dt_2 \dots (\beta)$$

$$- \underline{n} \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi - \Phi_1)^{k-j-1} \varphi_1' dt_1 \int^{t_1} (\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' dt_2 \dots (\alpha)$$

$$(v) = - \underline{n} \int (1-\Phi)^{n-k} \varphi \varphi' dt \int^t (\Phi - \Phi_1)^{k-j-2} \varphi_1' dt_1 \int^{t_1} (\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' dt_2 \dots (\gamma)$$

$$- \underline{n} \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi - \Phi_1)^{k-j-2} \varphi_1 \varphi_1' dt_1 \int^{t_1} (\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' dt_2 \dots (\delta)$$

$$(vii) = - \underline{n} \int (1-\Phi)^{n-k-1} \varphi \varphi' dt \int^t (\Phi - \Phi_1)^{k-j-1} \varphi_1' dt_1 \int^{t_1} (\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' dt_2 \dots (\varepsilon)$$

On the otherhand,

$$(ii) = - \underline{n} \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi - \Phi_1)^{k-j-1} \varphi_1' dt_1 \int^{t_1} \mathcal{D}_2[(\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1}] \varphi_2' dt_2 \dots (\alpha)$$

$$+ \underline{n} \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi - \Phi_1)^{k-j-1} \varphi_1' dt_1 \int^{t_1} (\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' dt_2 \dots (\xi)$$

$$(iv) = \underline{n} \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi - \Phi_1)^{k-j-2} \varphi_1 \varphi_1' dt_1 \int^{t_1} (\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' dt_2 \dots (\delta)$$

$$+ \underline{n} \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi - \Phi_1)^{k-j-1} \varphi_1 dt_1 \int^{t_1} (\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' dt_2 \dots (\eta)$$

$$- \underline{n} \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi - \Phi_1)^{k-j-1} \varphi_1 \varphi_1' dt_1 \int^{t_1} (\Phi_1 - \Phi_2)^{j-i-2} \Phi_2^{i-1} \varphi_2' dt_2 \dots (\beta)$$

$$(vi) = \underline{n} \int (1-\Phi)^{n-k-1} \varphi \varphi' dt \int^t (\Phi - \Phi_1)^{k-j-1} \varphi_1' dt_1 \int^{t_1} (\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' dt_2 \dots (\varepsilon)$$

$$+ \underline{n} \int (1-\Phi)^{n-k} \varphi dt \int^t (\Phi - \Phi_1)^{k-j-1} \varphi_1' dt_1 \int^{t_1} (\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' dt_2 \dots (\zeta)$$

$$- \underline{n} \int (1-\Phi)^{n-k} \varphi \varphi' dt \int^t (\Phi - \Phi_1)^{k-j-2} \varphi_1' dt_1 \int^{t_1} (\Phi_1 - \Phi_2)^{j-i-1} \Phi_2^{i-1} \varphi_2' dt_2 \dots (\gamma)$$

Since those samely named (α) (β) (γ) (δ) (ε) cancel each other⁸⁾ there remain only underlined 2 integrals (ξ) (η) (ζ) . But

8) If $j = i+1$, or $k = j+1$, (β) or (γ) and (δ) reduce to naught, respectively, which, however, does not hinder our result.

$$\begin{aligned}
 (\xi) &= \underline{n} \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi-\Phi_1)^{k-j-1} \varphi_1' dt_1 \int_0^1 \Phi_1^{j-1} (1-v)^{j-i-1} \underline{v}^{i-1} dv \quad (\text{if } \Phi_2 = \Phi_1 v) \\
 &= \underline{n} \int (1-\Phi)^{n-k} \varphi' dt \int^t (\Phi-\Phi_1)^{k-j-1} \Phi_1^{j-1} \varphi_1 dt_1 = E(t_{j|n} t_{k|n}).
 \end{aligned}$$

Also, on interchanging the order of integrations, we have

$$(\eta) = \underline{n} \int (1-\Phi)^{n-k} \varphi' dt \int^t \Phi_2^{j-1} \varphi_2' dt_2 \int_{t_2}^t (\Phi-\Phi_1)^{k-j-1} (\Phi_1-\Phi_2)^{j-i-1} \varphi_1 dt_1$$

and

$$(\zeta) = \underline{n} \int \varphi_1' dt_1 \int_{t_1}^t (1-\Phi)^{n-k} (\Phi-\Phi_1)^{k-j-1} \varphi dt \int^{t_1} (\Phi_1-\Phi_2)^{j-i-1} \Phi_2^{j-1} \varphi_2' dt_2.$$

Hence, if we put $\Phi-\Phi_1=(\Phi-\Phi_2)v$ in the former, and $1-\Phi=(1-\Phi_1)v$ in the latter, they reduce to $E(t_{i|n} t_{k|n})$ and $E(t_{j|n} t_{k|n})$, respectively, Q.E.D.

From the foregoing and (5.10), Part I, it follows that

$$(16.9.1) \quad \sum_{k=1}^n \sum_{l=1}^n E(t_{i|n} t_{j|n} t_{k|n} t_{l|n}) = n E(t_{i|n} t_{j|n}) + 2,$$

and

$$(16.9.2) \quad \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n E(t_{i|n} t_{j|n} t_{k|n} t_{l|n}) = 3n.$$

Remark. If we concern about the fourth moments with genuine 4 arguments, as seen at (5.9), Part I, we know that

$$(16.10) \quad \sum_{l=4}^n \sum_{k=3}^{l-1} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} E(t_{i|n} t_{j|n} t_{k|n} t_{l|n}) = 0.$$

4° Lastly we ought to extend the symmetric property (5.1) : $E(t_{n-i+1|n}^p t_{n-k+1|n}^q) = (-1)^{p+q} E(t_{i|n}^p t_{k|n}^q)$ in order to save the labour to calculate $E(t_{i|n}^p t_{j|n}^q t_{k|n}^r)$ for all combinations of i, j, k . In fact, we obtain a similar formula⁹⁾:

$$(16.11) \quad E(t_{n-i+1|n}^p t_{n-j+1|n}^q t_{n-k+1|n}^r) = (-1)^{p+q+r} E(t_{i|n}^p t_{j|n}^q t_{k|n}^r).$$

Proof. On assuming $i < j < k$ and considering Φ, Φ_1, Φ_2 as independent variables, and t, t_1, t_2 as functions of Φ, Φ_1, Φ_2 respectively, all being monotonely increasing, we have

$$\begin{aligned}
 E(t_{i|n}^p t_{j|n}^q t_{k|n}^r) &= \delta_{ijk|n} \int_0^1 t^r (1-\Phi)^{n-k} d\Phi \int_0^\Phi t_1^q (\Phi-\Phi_1)^{k-j-1} d\Phi_1 \int_0^{\Phi_1} t_2^p (\Phi_1-\Phi_2)^{j-i-1} \Phi_2^{i-1} d\Phi_2 \\
 &\quad \dots\dots(i)
 \end{aligned}$$

where $\delta_{ijk|n} = n!/(n-k)!(k-j-1)!(j-i-1)!(i-1)!$. But, for $i' = n-i+1$, $j' = n-j+1$, $k' = n-k+1$, it follows that $k' < j' < i'$ and $n-i' = i-1$, $j'-k' = k-j$, $i'-j' = j-i$, $k'-1 = n-k$ and therefore $\delta_{k'j'i'|n} = \delta_{ijk|n}$. And

9) Of course, if $n=3$ and $p=q=r$, we have only one $E(t_{1|3}^p t_{2|3}^p t_{3|3}^p)$, because here the problem concerns with combination but not permutation, so that $E(t_3^p t_2^p t_1^p)$ &c., is nothing else $E(t_1^p t_2^p t_3^p)$, and (16.11) is of no use.

$$\begin{aligned}
E(t_{i' | n}^p t_{j' | n}^p t_{k' | n}^r) &= \delta_{i' j' i' | n} \int_0^1 t^p (1-\Phi)^{n-i'} d\Phi \int_0^{\Phi} t_1^q (\Phi - \Phi_1)^{i'-j'-1} d\Phi_1 \\
&\quad \times \int_0^{\Phi_1} t_2^r (\Phi_1 - \Phi_2)^{j'-k'-1} \Phi_2^{k'-1} d\Phi_2 \\
&= \delta_{i j k | n} \int_0^1 t^p (1-\Phi)^{i-1} d\Phi \int_0^{\Phi} t_1^q (\Phi - \Phi_1)^{j-i-1} d\Phi_1 \int_0^{\Phi_1} t_2^r (\Phi_1 - \Phi_2)^{k-i-1} \Phi_2^{n-k} d\Phi_2. \\
&\quad \dots\dots (ii)
\end{aligned}$$

Hence we have to prove that the above two integrals (i) and (ii) are equal. Now, on interchanging the order of integrations in (ii), it yields

$$\begin{aligned}
&\int_0^1 t_1^q d\Phi_1 \int_{\Phi_1}^1 t^p (1-\Phi)^{i-1} (\Phi - \Phi_1)^{j-i-1} d\Phi \int_1^{\Phi_1} t_2^r (\Phi_1 - \Phi_2)^{k-j-1} \Phi_2^{n-k} d\Phi_2 \\
&= \int_0^1 t_1^q d\Phi_1 \int_0^{\Phi_1} t_2^r (\Phi_1 - \Phi_2)^{k-j-1} \Phi_2^{n-k} d\Phi_2 \int_{\Phi_1}^1 t^p (1-\Phi)^{i-1} (\Phi - \Phi_1)^{j-i-1} d\Phi \\
&= \int_0^1 t_2^r \Phi_2^{n-k} d\Phi_2 \int_{\Phi_2}^1 t_1^q (\Phi_1 - \Phi_2)^{k-j-1} d\Phi_1 \int_1^{\Phi_1} t^p (\Phi - \Phi_1)^{j-i-1} (1-\Phi)^{i-1} d\Phi.
\end{aligned}$$

Instead of the old independent variables Φ, Φ_1, Φ_2 with $\Phi_2 < \Phi_1 < \Phi$, we take, as new variables $\Psi_2 = 1 - \Phi, \Psi_1 = 1 - \Phi_1, \Psi = 1 - \Phi_2$ with the order $\Psi_2 < \Psi_1 < \Psi$. Every function $t = t(\Phi), t_1 = t_1(\Phi_1), t_2 = t_2(\Phi_2)$ being each monotonely increasing the functions $T_2 = -t, T_1 = -t_1, T_2 = -t_2$ are also all monotonely decreasing, in regard to Φ, Φ_1, Φ_2 respectively; or they are all monotonely increasing in regard to new independent variables Ψ_2, Ψ_1, Ψ and vice versa. Hence, the last triple integral becomes

$$(-1)^{p+q+r} \int_0^1 T^r (1-\Psi)^{n-k} d\Psi \int_0^{\Psi} T_1^q (\Psi - \Psi_1)^{k-j-1} d\Psi_1 \int_0^{\Psi_1} T_2^r (\Psi_1 - \Psi_2)^{j-i-1} \Psi_2^{i-1} d\Psi_2,$$

which integral is equal to (i). Therefore, we have, as asserted,

$$E(t_{i' | n}^p t_{j' | n}^q t_{k' | n}^r) = (-1)^{p+q+r} E(t_{i | n}^p t_{j | n}^q t_{k | n}^r), \quad \text{Q.E.D.}$$

§ 17. Computations of $J_{\lambda}^{\alpha}, J_{\mu\nu}^{\alpha\beta}$ and $J_{\lambda\mu\nu}^{\alpha\beta\gamma}$. We had already computed J_{λ}^{α} and $J_{\mu\nu}^{\alpha\beta}$ generally in Part II, so that now we have only to tabulate those necessary for calculations of the third and fourth moments.

1° It is required to express

$$E(t_{i | n}^3) = \frac{5}{2} E(t_{i | n}) + \frac{|n|}{6} \int D^3 [\Phi^{i-1} (1-\Phi)^{n-i}] \varphi^4 dt, \quad (2.7)$$

$$E(t_{i | n}^4) = \frac{13}{3} E(t_{i | n}^2) - \frac{4}{3} + \frac{|n|}{24} \int D^4 [\Phi^{i-1} (1-\Phi)^{n-i}] \varphi^5 dt \quad (2.8)$$

by $J_{\lambda}^{\alpha} = \int \Phi^{\lambda} \varphi^{\alpha} dt$. We had already obtained (7.4) (7.5) (7.10) and (7.11) in Part II: i.e.

$$J_0^{\alpha} = c_{\alpha} = \frac{1}{\sqrt{\alpha} \sqrt{2\pi}^{\alpha-1}}, J_1^{\alpha} = \frac{1}{2} c_{\alpha}, J_2^{\alpha} = \frac{c_{\alpha}}{2} [1 - 2S(\alpha)], J_3^{\alpha} = \frac{c_{\alpha}}{2} [1 - 3S(\alpha)], \text{ \&c.}$$

These values are given in the following

Table VI Values of J_λ^α and Allied Constants

$\alpha \backslash \lambda$	$\lambda=0$	$\lambda=1$	$\lambda=2$	$\lambda=3$	Allied constants
$\alpha=1$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$S(2)=0.19591\ 328$
$\alpha=2$	$\frac{1}{2\sqrt{\pi}}=0.28209\ 47918$	$\frac{1}{4\sqrt{\pi}}$	$\frac{1}{4\sqrt{\pi}}[1-2S(2)]$	$\frac{1}{4\sqrt{\pi}}[1-3S(2)]$	$S(3)=0.20978\ 469$
$\alpha=3$	$\frac{1}{2\pi\sqrt{3}}=0.09188\ 81492$	$\frac{1}{4\pi\sqrt{3}}$	$\frac{1}{4\pi\sqrt{3}}[1-2S(3)]$	$\frac{1}{4\pi\sqrt{3}}[1-3S(3)]$	$S(4)=0.21795\ 288$
$\alpha=4$	$\frac{1}{4\pi\sqrt{2\pi}}=0.03174\ 68180$	$\frac{1}{8\pi\sqrt{2\pi}}$	$\frac{1}{8\pi\sqrt{2\pi}}[1-2S(4)]$	$\frac{1}{8\pi\sqrt{2\pi}}[1-3S(4)]$	$S(5)=0.22334\ 979$
$\alpha=5$	$\frac{1}{4\pi^2\sqrt{5}}=0.01132\ 80527$	$\frac{1}{8\pi^2\sqrt{5}}$	$\frac{1}{8\pi^2\sqrt{5}}[1-2S(5)]$	$\frac{1}{8\pi^2\sqrt{5}}[1-3S(5)]$	

2° To compute $E(t_{i|n}^p t_{k|n}^q)$ for $p+q=3$ or 4 and $n \leq 5$, it needs to know the values of the following

$$(17.1) \quad J_{00}^{\alpha\beta} = \frac{1}{2} c_\alpha c_\beta \quad \text{with} \quad c_\alpha = \frac{1}{\sqrt{\alpha} \sqrt{2\pi^{\alpha-1}}} \quad \text{by (8.6) in Part II;}$$

$$(17.2) \quad J_{10}^{\alpha\beta} = J_{00}^{\alpha\beta} [1 - S(p-2) - S(q)] \quad \text{by (8.9), also cf. (17.7) below;}$$

$$(17.3) \quad J_{01}^{\alpha\beta} = J_{00}^{\alpha\beta} - J_{10}^{\beta\alpha} \quad \text{by (8.10);}$$

$$(17.4) \quad J_{11}^{\alpha\beta} = \frac{1}{4} J_{00}^{\alpha\beta} = \frac{1}{8} c_\alpha c_\beta = J_{11}^{\beta\alpha} \quad \text{by (8.12);}$$

$$(17.5) \quad J_{20}^{\alpha\beta} = J_{10}^{\alpha\beta} - J_{00}^{\alpha\beta} S(\alpha) \quad \text{by (8.14) and (8.9);}$$

$$(17.6) \quad J_{02}^{\alpha\beta} = J_{01}^{\alpha\beta} - J_{00}^{\alpha\beta} S(\beta) \quad \text{by (8.16) and (8.10).}$$

In fact we have by (8.9)

$$\begin{aligned} J_{10}^{\alpha\beta} &= \frac{1}{2} c_\alpha c_\beta \left[\frac{1}{2} + \frac{1}{\pi} \sin^{-1} \sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}} - \frac{1}{\pi} \sin^{-1} \sqrt{\frac{\alpha}{(\alpha+\beta)(\beta+1)}} \right] \\ &= J_{00}^{\alpha\beta} \left[\frac{1}{2} + S\left(\frac{2(\alpha+\beta+1)}{\alpha\beta - (\alpha+\beta+1)}\right) - S\left(\frac{2\alpha}{(\alpha+\beta+1)\beta - \alpha}\right) \right]. \end{aligned}$$

However, the second term in the last brackets becomes of a negative argument $-p$, because $\alpha, \beta \geq 1$, $\alpha+\beta+1 > \alpha\beta$, if one of α, β be 1 or if $\alpha=\beta \leq 2$, so that we must contrive to get rid of this inconvenience: Really, since

$$S(-p) + S(p-2) = \frac{1}{2\pi} [\sec^{-1}(1-p) + \sec^{-1}(p-1)] = \frac{1}{2},$$

$$\text{i.e.} \quad \frac{1}{2} + S(-p) = 1 - S(p-2),$$

we may write the above $J_{00}^{\alpha\beta}$ as¹⁰⁾

10) If $\alpha, \beta = 2, 3$, then $\alpha+\beta+1-\alpha\beta = 0$ and $p = \frac{2(\alpha+\beta+1)}{\alpha+\beta+1-\alpha\beta}$ becomes ∞ , so that $S(\infty) = \frac{1}{4}$ (cf. p. 61, Part. I). We get therefore $J_{1,0}^{2,3} = J_{00}^{2,3} \left[\frac{3}{4} - S\left(\frac{1}{4}\right) \right]$ and $J_{10}^{3,2} = J_{00}^{3,2} \left[\frac{3}{4} - S\left(\frac{2}{3}\right) \right]$. But $S\left(\frac{1}{4}\right) = \frac{1}{\pi} \sin^{-1} \frac{1}{\sqrt{10}}$ and $S\left(\frac{2}{3}\right) = \frac{1}{\pi} \sin^{-1} \frac{1}{\sqrt{5}} = \frac{1}{4} - S\left(\frac{1}{4}\right)$. Hence $J_{10}^{3,2} = J_{00}^{3,2} \left[\frac{1}{2} + S\left(\frac{1}{4}\right) \right]$ (Table VII).

$$(17.7) \quad J_{10}^{\alpha\beta} = J_{06}^{\alpha\beta} [1 - S(p-2) - S(q)], \text{ where } p = \frac{2(\alpha+\beta+1)}{\alpha+\beta+1-\alpha\beta}, \quad q = \frac{2\alpha}{(\alpha+\beta+1)\beta-\alpha}.$$

Furthermore, by reasons of complementary angles, we have $S\left(\frac{2}{7}\right) = \frac{1}{2} - 2S(2)$, $S\left(\frac{1}{7}\right) = \frac{1}{2} - 2S(3)$, and $S(8) = \left(\frac{1}{2}\right) - 2S\left(\frac{1}{2}\right)$. Using these relations, we get the following

Table VII Values of $J_{\mu\nu}^{\alpha\beta}$

α	β	$J_{00}^{\alpha\beta}$	$J_{10}^{\alpha\beta}$	$J_{01}^{\alpha\beta}$	$J_{11}^{\alpha\beta}$	$J_{20}^{\alpha\beta}$	$J_{02}^{\alpha\beta}$
1	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{12}$
1	2	$\frac{1}{4\sqrt{\pi}}$	$\frac{1}{4\sqrt{\pi}} \left[\frac{1}{2} + S(2) \right]$	$\frac{1}{2\sqrt{\pi}} S(2)$	$\frac{1}{16\sqrt{\pi}}$	$\frac{1}{4\sqrt{\pi}} \left[\frac{1}{3} + S(2) \right]$	$\frac{1}{4\sqrt{\pi}} S(2)$
2	1		$\frac{1}{4\sqrt{\pi}} [1 - 2S(2)]$	$\frac{1}{4\sqrt{\pi}} \left[\frac{1}{2} - S(2) \right]$		$\frac{1}{4\sqrt{\pi}} [1 - 3S(2)]$	$\frac{1}{4\sqrt{\pi}} \left[\frac{1}{3} - S(2) \right]$
2	2	$\frac{1}{8\pi}$	$\frac{1}{8\pi} \left[\frac{1}{2} + S\left(\frac{1}{2}\right) \right]$	$\frac{1}{8\pi} \left[\frac{1}{2} - S\left(\frac{1}{2}\right) \right]$	$\frac{1}{32\pi}$	$\frac{1}{8\pi} \left[\frac{1}{2} + S\left(\frac{1}{2}\right) - S(2) \right]$	$\frac{1}{8\pi} \left[\frac{1}{2} - S\left(\frac{1}{2}\right) - S(2) \right]$
1	3	$\frac{1}{4\pi\sqrt{3}}$	$\frac{1}{4\pi\sqrt{3}} \left[\frac{1}{2} + S(3) \right]$	$\frac{1}{2\pi\sqrt{3}} S(3)$	$\frac{1}{16\pi\sqrt{3}}$	$\frac{1}{4\pi\sqrt{3}} \left[\frac{1}{3} + S(3) \right]$	$\frac{1}{4\pi\sqrt{3}} S(3)$
3	1		$\frac{1}{4\pi\sqrt{3}} [1 - 2S(3)]$	$\frac{1}{4\pi\sqrt{3}} \left[\frac{1}{2} - S(3) \right]$		$\frac{1}{4\pi\sqrt{3}} [1 - 3S(3)]$	$\frac{1}{4\pi\sqrt{3}} \left[\frac{1}{3} - S(3) \right]$
2	3	$\frac{1}{8\pi\sqrt{3\pi}}$	$\frac{1}{8\pi\sqrt{3\pi}} \left[\frac{3}{4} - S\left(\frac{1}{4}\right) \right]$	$\frac{1}{8\pi\sqrt{3\pi}} \left[\frac{1}{2} - S\left(\frac{1}{4}\right) \right]$	$\frac{1}{32\pi\sqrt{3\pi}}$	and so on.	
3	2		$\frac{1}{8\pi\sqrt{3\pi}} \left[\frac{1}{2} + S\left(\frac{1}{4}\right) \right]$	$\frac{1}{8\pi\sqrt{3\pi}} \left[\frac{1}{4} + S\left(\frac{1}{4}\right) \right]$			

3° Finally we touch the triple integral

$$(17.8) \quad J_{\lambda\mu\nu}^{\alpha\beta\gamma} = \int \Phi^\lambda \varphi^\alpha dt \int^t \Phi_1^\mu \varphi_1^\beta dt_1 \int^{t_1} \Phi_2^\nu \varphi_2^\gamma dt_2.$$

Or, on putting $U = \int_0^t \varphi dt$ &c.,

$$\begin{aligned} J_{\lambda\mu\nu}^{\alpha\beta\gamma} &= \int \left(U + \frac{1}{2} \right)^\lambda \varphi^\alpha dt \int^t \left(U_1 + \frac{1}{2} \right)^\mu \varphi_1^\beta dt_1 \int^{t_1} \left(U_2 + \frac{1}{2} \right)^\nu \varphi_2^\gamma dt_2 \\ &= \sum_{l=0}^{\lambda} \sum_{m=0}^{\mu} \sum_{n=0}^{\nu} \binom{\lambda}{l} \binom{\mu}{m} \binom{\nu}{n} \binom{1}{2}^{\lambda+\mu+\nu-l-m-n} K_{lmn}^{\alpha\beta\gamma}, \end{aligned}$$

where

$$(17.9) \quad K_{lmn}^{\alpha\beta\gamma} = \int U^l \varphi^\alpha dt \int^t U_1^m \varphi_1^\beta dt_1 \int^{t_1} U_2^n \varphi_2^\gamma dt_2.$$

Thus, to compute $J_{\lambda\mu\nu}^{\alpha\beta\gamma}$ it is sufficient to know $K_{lmn}^{\alpha\beta\gamma}$ for $l \leq \lambda$, $m \leq \mu$, $n \leq \nu$.

However, the said integrals, especially with positive λ , μ , ν , seldom take place for smaller values of n , so that we shall consider only the special case

$$(17.10) \quad K_{000}^{\alpha\beta\gamma} = J_{000}^{\alpha\beta\gamma} = \int \varphi^\alpha dt \int^t \varphi_1^\beta dt_1 \int^{t_1} \varphi_2^\gamma dt_2.$$

We express it as the sum of three triple integrals I, II, III corresponding to (i) $t > 0$, $t_1 < 0$, (ii) $t > 0$, $t_1 > 0$ and (iii) $t < 0$, respectively. Using polar coordinates,

$$\begin{aligned}
 \text{I} &= \int_0^\infty \varphi^\alpha dt \int_{-\infty}^0 \varphi_1^\beta dt_1 \int^{t_1} \varphi_2^\gamma dt_2 \\
 &= \frac{1}{\sqrt{2\pi}^{\alpha+\beta+\gamma}} \int_0^\infty \exp\left\{-\frac{\alpha}{2}t^2\right\} dt \int_{-\pi/4}^{-\pi/2} d\theta \int_0^\infty \exp\left\{-\frac{\rho^2}{2}(\beta \cos^2 \theta + \gamma \sin^2 \theta)\right\} \rho d\rho \\
 &= \frac{1}{\sqrt{2\pi}^{\alpha+\beta+\gamma}} \frac{\sqrt{2\pi}}{2\sqrt{\alpha}} \int_{-\pi/4}^{-\pi/2} \frac{d\theta}{\beta \cos^2 \theta + \gamma \sin^2 \theta} = \frac{c_\alpha c_\beta c_\gamma}{4\pi} \int_1^\infty \frac{d\tau}{\beta + \gamma \tau^2} = \frac{c_\alpha c_\beta c_\gamma}{4\pi} \left[\frac{\pi}{2} - \tan^{-1} \sqrt{\frac{\gamma}{\beta}} \right]; \\
 \text{II} &= \int_0^\infty \varphi^\alpha dt \int_0^t \varphi_1^\beta dt_1 \int^{t_1} \varphi_2^\gamma dt_2 \\
 &= \frac{1}{\sqrt{2\pi}^{\alpha+\beta+\gamma}} \int_0^\infty \exp\left\{-\frac{\alpha}{2}t^2\right\} dt \int_{-\pi/2}^{\pi/4} d\theta \int_0^{t \sec \theta} \exp\left\{-\frac{\rho^2}{2}(\beta \cos^2 \theta + \gamma \sin^2 \theta)\right\} \rho d\rho \\
 &= \frac{c_\alpha c_\beta c_\gamma \sqrt{\alpha\beta\gamma}}{\sqrt{2\pi}^3} \int_0^\infty \exp\left\{-\frac{\alpha}{2}t^2\right\} dt \int_{-\pi/2}^{\pi/4} \frac{d\theta}{\beta \cos^2 \theta + \gamma \sin^2 \theta} \left[1 - \exp\left\{-\frac{t^2}{2}(\beta + \gamma \tan^2 \theta)\right\} \right] \\
 &= \frac{c_\alpha c_\beta c_\gamma \sqrt{\alpha\beta\gamma}}{4\pi} \left[\frac{1}{\sqrt{\alpha}} \int_{-\infty}^1 \frac{d\tau}{\beta + \gamma \tau^2} - \int_{-\infty}^1 \frac{d\tau}{(\beta + \gamma \tau^2) \sqrt{\alpha + \beta + \gamma \tau^2}} \right] \quad (\tau = \tan \theta) \\
 &= \frac{c_\alpha c_\beta c_\gamma}{4\pi} \left[\tan^{-1} \sqrt{\frac{\gamma}{\beta}} + \frac{\pi}{2} - \sin^{-1} \frac{\sqrt{\alpha\gamma\tau}}{\sqrt{(\beta + \gamma \tau^2)(\beta + \gamma \tau^2)}} \Big|_{-\infty}^1 \right] \\
 &= \frac{c_\alpha c_\beta c_\gamma}{4\pi} \left[\tan^{-1} \sqrt{\frac{\gamma}{\beta}} + \frac{\pi}{2} - \sin^{-1} \sqrt{\frac{\alpha\gamma}{(\alpha + \beta)(\beta + \gamma)}} - \sin^{-1} \sqrt{\frac{\alpha}{\alpha + \beta}} \right]; \\
 \text{III} &= \int_{-\infty}^0 \varphi^\alpha dt \int^t \varphi_1^\beta dt_1 \int_0^{t_1} \varphi_2^\gamma dt_2 \\
 &= \frac{1}{\sqrt{2\pi}^{\alpha+\beta+\gamma}} \int_{-\infty}^0 \exp\left\{-\frac{\alpha}{2}t^2\right\} dt \int_{-\pi/4}^{-\pi/2} d\theta \int_{t \sec \theta}^\infty \exp\left\{-\frac{\rho^2}{2}(\beta \cos^2 \theta + \gamma \sin^2 \theta)\right\} \rho d\rho \\
 &= \frac{c_\alpha c_\beta c_\gamma \sqrt{\alpha\beta\gamma}}{\sqrt{2\pi}^3} \int_{-\infty}^0 \exp\left\{-\frac{\alpha}{2}t^2\right\} dt \int_{-\pi/4}^{-\pi/2} \frac{d\theta}{\beta \cos^2 \theta + \gamma \sin^2 \theta} \cdot \exp\left\{-\frac{t^2}{2}(\beta + \gamma \tan^2 \theta)\right\} \\
 &= \frac{c_\alpha c_\beta c_\gamma \sqrt{\alpha\beta\gamma}}{4\pi} \int_1^\infty \frac{d\tau}{(\beta + \gamma \tau^2) \sqrt{\alpha + \beta + \gamma \tau^2}} \quad (\tau = \tan \theta) \\
 &= \frac{c_\alpha c_\beta c_\gamma}{4\pi} \sin^{-1} \sqrt{\frac{\alpha\gamma\tau^2}{(\alpha + \beta)(\beta + \gamma \tau^2)}} \Big|_1^\infty = \frac{c_\alpha c_\beta c_\gamma}{4\pi} \left[\sin^{-1} \sqrt{\frac{\alpha}{\alpha + \beta}} - \sin^{-1} \sqrt{\frac{\alpha\gamma}{(\alpha + \beta)(\beta + \gamma)}} \right].
 \end{aligned}$$

Therefore

$$\text{I} + \text{II} + \text{III} = \frac{c_\alpha c_\beta c_\gamma}{4\pi} \left[\pi - 2 \sin^{-1} \sqrt{\frac{\alpha\gamma}{(\alpha + \beta)(\beta + \gamma)}} \right].$$

Thus we obtain

$$(17.11) \quad J_{000}^{\alpha\beta\gamma} = \frac{1}{2} c_\alpha c_\beta c_\gamma \left[\frac{1}{2} - S\left(\frac{2\alpha\gamma}{(\alpha + \beta + \gamma)\beta - \alpha\gamma}\right) \right].$$

In particular

$$(17.11.1) \quad J_{000}^{\alpha\alpha\alpha} = \frac{1}{2} c_\alpha^3 \left[\frac{1}{2} - S(1) \right] = \frac{1}{6} c_\alpha^3.$$

E.g. if $\alpha=\beta=\gamma=1$, $J_{000}^{111}=\frac{1}{6}$ which agrees with the fundamental relation $3! \int \varphi dt \int^t \varphi_1 dt_1 \int^{t_1} \varphi_2 dt_2 = 1$.¹¹⁾

§ 18. Calculations of the Third and Fourth Moments. By making use of the formulas for $E(t_{i|n}^p)$, $E(t_{i|n}^p t_{k|n}^q)$ and $E(t_{i|n}^p t_{j|n}^q t_{k|n}^r)$ and the foregoing values of J_λ^α , $J_{\mu\nu}^{\alpha\beta}$ and $J_{\lambda\mu\nu}^{\alpha\beta\gamma}$, we have obtained the third and fourth moments for $n \leq 5$, as in the following Tables VIII–XIV:

Table VIII Values of $E(t_{i|n}^3)$ and $E(t_{i|n}^4)$

n	i	$n-i+1$	$E(t_{i n}^3) = -E(t_{n-i+1 n}^3)$	$E(t_{i n}^4) = E(t_{n-i+1 n}^4)$
2	2	1	$\frac{5}{2\sqrt{\pi}} = 1.4104\ 774$	3
3	3	1	$\frac{15}{4\sqrt{\pi}} = 2.1157\ 109$	$3 + \frac{13}{2\sqrt{3}\pi} = 4.1945\ 459$
	2	2	0	$3 - \frac{13}{\sqrt{3}\pi} = 0.6109\ 081$
4	4	1	$\frac{15}{2\sqrt{\pi}}[1 - 2S(2)] + \frac{1}{\pi\sqrt{2\pi}} = 2.7004\ 257$	$3 + \frac{13}{\sqrt{3}\pi} = 5.3890\ 919$
	3	2	$\frac{15}{2\sqrt{\pi}}[-1 + 6S(2)] - \frac{3}{\pi\sqrt{2\pi}} = 0.3615\ 667$	$3 - \frac{13}{\sqrt{3}\pi} = 0.6109\ 081$
5	5	1	$\frac{25}{2\sqrt{\pi}}[1 - 3S(2)] + \frac{5}{2\pi\sqrt{2\pi}} = 3.2248\ 794$	$3 + \frac{65}{2\sqrt{3}\pi}[1 - 2S(3)] + \frac{\sqrt{5}}{4\pi^2} = 6.5233\ 955$
	4	2	$\frac{25}{2\sqrt{\pi}}[-1 + 6S(2)] - \frac{5}{\pi\sqrt{2\pi}} = 0.6026\ 111$	$3 + \frac{65}{\sqrt{3}\pi}[-1 + 4S(3)] - \frac{\sqrt{5}}{\pi^2} = 0.8518\ 775$
	3	3	0	$3 + \frac{65}{\sqrt{3}\pi}[1 - 6S(3)] + \frac{3\sqrt{5}}{2\pi^2} = 0.2494\ 541$

Checks: E.g. $\sum_{i=1}^5 E(t_{i|5}^4) = 15 (= 15.00000\ 01 \text{ as sum of numerical values})$.

11) Although for the presently restricted case that $n \leq 5$ it suffices with all the above formulas, if e.g. $n=7$, $i=2$ in (15.1.2), we need to calculate $7! \int (1-\Phi) \varphi^2 dt \int_{t_0}^t \varphi_1^2 dt_1 \int_{t_0}^{t_1} \varphi_2^2 dt_2$, whose integration is somewhat intricate. Really to find $K_{100}^{222} = \int U \varphi^2 dt \int^t \varphi_1^2 dt_1 \int^{t_1} \varphi_2^2 dt_2$, treating similarly as in (17.10), we get easily $I = \frac{1}{128\pi^2\sqrt{\pi}} \tan^{-1} \frac{1}{\sqrt{2}}$. However $II = \frac{1}{8\pi^3\sqrt{2\pi}} \int_0^\infty e^{-t^2} dt \int_0^t e^{-\tau^2/2} d\tau \int_{-\pi/2}^{\pi/4} d\theta \int_0^{\sec \theta} e^{-\rho^2} \rho d\rho$
 $= \frac{1}{16\pi^3\sqrt{2\pi}} \int_{-\pi/2}^{\pi/4} d\theta \int_{\pi/4}^{\pi/2} d\psi \int_0^\infty \left[\exp\left\{-\frac{r^2}{2}(2\sin^2\psi + \cos^2\psi)\right\} - \exp\left\{-r^2(2 + \sec^2\theta)\sin^2\psi - \frac{r^2}{2}\cos^2\psi\right\} \right] r dr = II_1 - II_2$,
 where II_1 becomes $\frac{3}{128\pi^2\sqrt{\pi}} \tan^{-1} \frac{1}{\sqrt{2}}$, while $II_2 = \frac{1}{16\pi^3\sqrt{2\pi}} \int_{-\pi/2}^{-\pi/4} \frac{\cot^{-1}\sqrt{2(2+\sec^2\theta)}}{\sqrt{2(2+\sec^2\theta)}} d\theta$ seems to be numerically integrated. Also $III = \frac{1}{16\pi^3\sqrt{2\pi}} \int_{-\pi/4}^{-3\pi/4} \frac{\cot^{-1}\sqrt{2(1+\sec^2\theta)}}{\sqrt{2(1+\sec^2\theta)}} d\theta$ requires a numerical integration.

Table IX Values of $E(t_{i|n}^2 t_{k|n})$ ($i \neq k$)

n	$t_i^2 t_k$	$t_{n-k-1} t_{n-i+1}^2$	$E(t_{i n}^2 t_{k n}) = -E(t_{n-k+1} t_{n-i+1}^2)$
2	$t_1^2 t_2$	$t_1 t_2^2$	$\frac{1}{2\sqrt{\pi}} = 0.28209 \ 48$
3	$t_1^2 t_2$	$t_2 t_3^2$	$-\frac{3}{4\sqrt{\pi}} = -0.42314 \ 22$
	$t_1^2 t_3$	$t_1 t_3^2$	$\frac{3}{2\sqrt{\pi}} = 0.84628 \ 44$
	$t_2^2 t_3$	$t_1 t_2^2$	$\frac{3}{4\sqrt{\pi}} = 0.42314 \ 22$
4	$t_1^2 t_2$	$t_3 t_4^2$	$-\frac{1}{\pi\sqrt{2\pi}} + \frac{3}{\sqrt{\pi}} \left[\frac{1}{2} - 5S(2) \right] = -0.93868 \ 63$
	$t_1^2 t_3$	$t_2 t_4^2$	$-\frac{1}{\pi\sqrt{2\pi}} + \frac{3}{\sqrt{\pi}} [-1 + 6S(2)] = 0.17002 \ 41$
	$t_1^2 t_4$	$t_1 t_4^2$	$\frac{3}{\pi\sqrt{2\pi}} + \frac{3}{\sqrt{\pi}} [1 - 2S(2)] = 1.41033 \ 72$
	$t_2^2 t_3$	$t_2 t_3^2$	$\frac{3}{\pi\sqrt{2\pi}} + \frac{3}{\sqrt{\pi}} [-1 + 4S(2)] = 0.01477 \ 98$
	$t_2^2 t_4$	$t_1 t_3^2$	$-\frac{5}{\pi\sqrt{2\pi}} + \frac{3}{\sqrt{\pi}} [1 - 2S(2)] = 0.39443 \ 90$
	$t_3^2 t_4$	$t_1 t_2^2$	$\frac{1}{\pi\sqrt{2\pi}} + \frac{3}{\sqrt{\pi}} \left[\frac{1}{2} - S(2) \right] = 0.64167 \ 50$
5	$t_1^2 t_2$	$t_4 t_5^2$	$-\frac{5}{2\pi\sqrt{2\pi}} + \frac{5}{\sqrt{\pi}} \left[\frac{1}{2} - \frac{9}{2} S(2) \right] = -1.39396 \ 94$
	$t_1^2 t_3$	$t_3 t_5^2$	$-\frac{5}{2\pi\sqrt{2\pi}} = -0.31746 \ 82$
	$t_1^2 t_4$	$t_2 t_5^2$	$\frac{15}{2\pi\sqrt{2\pi}} - \frac{15}{2\pi\sqrt{3\pi}} + \frac{5}{\sqrt{\pi}} [-1 + 6S(2)] = 0.66978 \ 85$
	$t_1^2 t_5$	$t_1 t_5^2$	$\frac{15}{2\pi\sqrt{3\pi}} + \frac{5}{\sqrt{\pi}} [1 + 3S(2)] = 1.94059 \ 95$
	$t_2^2 t_3$	$t_3 t_4^2$	$\frac{5}{\pi\sqrt{2\pi}} + \frac{5}{\sqrt{\pi}} \left[0 - \frac{3}{2} S(2) \right] = -0.19405 \ 54$
	$t_2^2 t_4$	$t_2 t_4^2$	$-\frac{15}{\pi\sqrt{2\pi}} + \frac{15}{\pi\sqrt{3\pi}} + \frac{5}{\sqrt{\pi}} [-1 + 6S(2)] = 0.14548 \ 00$
	$t_2^2 t_5$	$t_1 t_4^2$	$\frac{15}{2\pi\sqrt{2\pi}} - \frac{15}{\pi\sqrt{3\pi}} + \frac{5}{\sqrt{\pi}} [1 - 3S(2)] = 0.56009 \ 89$
	$t_3^2 t_4$	$t_2 t_3^2$	$\frac{10}{\pi\sqrt{2\pi}} - \frac{15}{2\pi\sqrt{3\pi}} + \frac{5}{\sqrt{\pi}} \left[-1 + \frac{9}{2} S(2) \right] = 0.15826 \ 49$
	$t_3^2 t_5$	$t_1 t_3^2$	$-\frac{25}{2\pi\sqrt{2\pi}} + \frac{15}{2\pi\sqrt{3\pi}} + \frac{5}{\sqrt{\pi}} [1 - 3S(2)] = 0.35325 \ 86$
	$t_4^2 t_5$	$t_1 t_2^2$	$\frac{5}{2\pi\sqrt{2\pi}} + \frac{5}{\sqrt{\pi}} \left[\frac{1}{2} - \frac{3}{2} S(2) \right] = 0.89895 \ 04$

Checks: E.g. $\sum_{k=1}^3 E(t_{1|3}^2 t_{k|3}) = \frac{-3}{\sqrt{\pi}} = -1.6925688$, $2E(t_{1|3}) = -1.69256 \ 88$. (cf. Table III, Part I)

$$\sum_{k=1}^4 E(t_{1|4}^2 t_{k|4}) = \frac{-6}{\sqrt{\pi}} [-2S(2) + 1] = -2.05875 \ 07, \quad 2E(t_{1|4}) = -2.05875 \ 08.$$

$$\sum_{k=1}^5 E(t_{1|5}^2 t_{k|5}) = 2 \times \frac{5}{\sqrt{\pi}} [-1 + 3S(2)] = -2.32592 \ 89, \quad 2E(t_{1|5}) = -2.32592 \ 90.$$

$$\sum_{k=1}^5 E(t_{2|5}^2 t_{k|5}) = \frac{10}{\sqrt{\pi}} [1 - 6S(2)] = -0.99003 \ 84, \quad 2E(t_{2|5}) = -0.99003 \ 80.$$

$$\sum_{k=1}^5 E(t_{3|5}^2 t_{k|5}) = 0 = 0, \quad 2E(t_{3|5}) = 0.$$

Otherwise, e.g. $\sum_{k=1}^5 E(t_{1|5} t_{k|5}^2) = 6E(t_{1|5}) = -\frac{30}{\sqrt{\pi}} [1 - 3S(2)] = -6.9777 \ 870$, &c.

Also $\sum_{k=2}^n \sum_{i=1}^{k-1} E(t_{i|n}^2 t_{k|n}) = \frac{n(n-1)}{4\sqrt{\pi}} = \frac{1}{2\sqrt{\pi}}, \frac{3}{2\sqrt{\pi}}, \frac{3}{\sqrt{\pi}}, \frac{5}{\sqrt{\pi}}$ for $n=2, 3, 4, 5$.

Table X Values of $E(t_{i|n}^3 t_{k|n})$ ($i \neq k$)

n	$t_j^3 t_k$	$t_{n-k+1} t_{n-i+1}^3$	$E(t_{i n}^3 t_{k n}) = E(t_{n-i+1} t_{n-k+1}^3)$
2	$t_1^3 t_2$	$t_1 t_2^3$	0
3	$t_1^3 t_2$	$t_2 t_3^3$	$\frac{5}{\sqrt{3}\pi} = 0.91888 \quad 15$
	$t_1^3 t_3$	$t_1 t_3^3$	$-\frac{7}{\sqrt{3}\pi} = -1.28643 \quad 41$
	$t_2^3 t_3$	$t_1 t_2^3$	$\frac{2}{\sqrt{3}\pi} = 0.36755 \quad 26$
4	$t_1^3 t_2$	$t_3 t_4^3$	$\frac{10}{\sqrt{3}\pi} = 1.83776 \quad 30$
	$t_1^3 t_3$	$t_2 t_4^3$	$-\frac{14}{\sqrt{3}\pi} + \frac{15}{2\pi} = -0.18554 \quad 40$
	$t_1^3 t_4$	$t_1 t_4^3$	$-\frac{15}{2\pi} = -2.38732 \quad 41$
	$t_2^3 t_3$	$t_2 t_3^3$	$\frac{14}{\sqrt{3}\pi} - \frac{15}{2\pi} = 0.18554 \quad 40$
	$t_2^3 t_4$	$t_1 t_3^3$	$-\frac{14}{\sqrt{3}\pi} + \frac{15}{2\pi} = -0.18554 \quad 40$
	$t_3^3 t_4$	$t_1 t_2^3$	$\frac{4}{\sqrt{3}\pi} = 0.73510 \quad 52$
5	$t_1^3 t_2$	$t_4 t_5^3$	$\frac{25}{\sqrt{3}\pi} [1 - 2S(3)] + \frac{\sqrt{5}}{4\pi^2} = 2.72337 \quad 51$
	$t_1^3 t_3$	$t_3 t_5^3$	$-\frac{35}{\sqrt{3}\pi} [1 - 2S(3)] + \frac{\sqrt{5}}{4\pi^2} + \frac{75}{2\pi} \left[\frac{1}{2} - S\left(\frac{1}{2}\right) \right] = 0.69368 \quad 31$
	$t_1^3 t_4$	$t_2 t_5^3$	$+\frac{\sqrt{5}}{4\pi^2} + \frac{75}{2\pi} \left[-\frac{1}{2} + 3S\left(\frac{1}{2}\right) \right] = -1.11815 \quad 35$
	$t_1^3 t_5$	$t_1 t_5^3$	$-\frac{\sqrt{5}}{\pi^2} - \frac{75}{\pi} S\left(\frac{1}{2}\right) = -3.42223 \quad 88$
	$t_2^3 t_3$	$t_3 t_4^3$	$\frac{10}{\sqrt{3}\pi} [1 + 8S(3)] - \frac{\sqrt{5}}{\pi^2} + \frac{75}{2\pi} \left[-\frac{1}{2} + S\left(\frac{1}{2}\right) \right] = 0.32500 \quad 67$
	$t_2^3 t_4$	$t_2 t_4^3$	$-\frac{140}{\sqrt{3}\pi} S(3) - \frac{\sqrt{5}}{\pi^2} + \frac{75}{\pi} \left[\frac{1}{2} - 2S\left(\frac{1}{2}\right) \right] = -0.07877 \quad 93$
	$t_2^3 t_5$	$t_1 t_4^3$	$+\frac{11\sqrt{5}}{4\pi^2} - \frac{75}{2\pi} \left[\frac{1}{2} - 3S\left(\frac{1}{2}\right) \right] = -0.55175 \quad 08$
	$t_3^3 t_4$	$t_2 t_3^3$	$\frac{5}{\sqrt{3}\pi} [5 - 2S(3)] + \frac{3\sqrt{5}}{2\pi^2} - \frac{75}{2\pi} \left[\frac{1}{2} - S\left(\frac{1}{2}\right) \right] = 0.17824 \quad 30$
	$t_3^3 t_5$	$t_1 t_3^3$	$-\frac{70}{\sqrt{3}\pi} \left[\frac{1}{2} - S(3) \right] - \frac{9\sqrt{5}}{4\pi^2} + \frac{75}{2\pi} \left[\frac{1}{2} - S\left(\frac{1}{2}\right) \right] = 0.12728 \quad 04$
	$t_4^3 t_5$	$t_1 t_2^3$	$\frac{10}{\sqrt{3}\pi} [1 - 2S(3)] + \frac{\sqrt{5}}{4\pi^2} = 1.12333 \quad 42$

Checks: E.g. $\sum_{k=1}^3 E(t_{1|3}^3 t_{k|3}) = 3 \left(1 + \frac{\sqrt{3}}{2\pi} \right) = 3.82699 \quad 32 = 3E(t_{1|3}^2)$, (cf. Table III, Part I)

$$\sum_{k=1}^4 E(t_{1|4}^3 t_{k|4}) = 3 \left(1 + \frac{\sqrt{3}}{\pi} \right) = 4.65398 \quad 67 = 3E(t_{1|4}^2),$$

$$\sum_{k=1}^4 E(t_{2|4}^3 t_{k|4}) = 3 \left(1 - \frac{\sqrt{3}}{\pi} \right) = 1.34601 \quad 33 = 3E(t_{2|4}^2),$$

$$\sum_{k=1}^5 E(t_{1|5}^3 t_{k|5}) = 3 + \frac{15\sqrt{3}}{2\pi} [1 - 2S(3)] = 5.40006 \quad 12 = 3E(t_{1|5}^2),$$

$$\sum_{k=1}^5 E(t_{2|5}^3 t_{k|5}) = 3 + \frac{15\sqrt{3}}{\pi} [-1 + 4S(3)] = 1.66968 \quad 81 = 3E(t_{2|5}^2),$$

otherwise $\sum_{k=1}^5 E(t_{3|5}^3 t_{k|5}) = 3 + \frac{15\sqrt{3}}{\pi} [-1 - 6S(3)] = 0.86050 \quad 11 = 3E(t_{3|5}^2),$

$$\sum_{k=1}^5 E(t_{1|5} t_{k|5}^3) = 3 + \frac{5\sqrt{3}}{2\pi} [1 - 2S(3)] = 2 + E(t_{1|5}^2) = 3.80002 \quad 04, \quad \&c.$$

Also $\sum_{k=2}^n \sum_{i=1}^{k-1} E(t_{i|n}^3 t_{k|n}) = 0 \quad \text{for } n=2, 3, 4, 5.$

Table XI Values of $E(t_{i|n}^2 t_{k|n}^2)$ ($i < k$)

n	$t_i^2 t_k^2$	$t_{n-k+1}^2 t_{n-i+1}^2$	$E(t_{i n}^2 t_{k n}^2) = E(t_{n-k+1}^2 t_{n-i+1}^2)$
2	$t_1^2 t_2^2$		1
3	$t_1^2 t_2^2$	$t_2^2 t_3^2$	$1 - \frac{1}{\pi\sqrt{3}} = 0.81622 \quad 37$
	$t_1^2 t_3^2$		$1 + \frac{2}{\pi\sqrt{3}} = 1.36755 \quad 26$
4	$t_1^2 t_2^2$	$t_3^2 t_4^2$	$1 + \frac{1}{\pi\sqrt{3}} = 1.18377 \quad 63$
	$t_1^2 t_3^2$	$t_2^2 t_4^2$	$1 - \frac{2}{\pi\sqrt{3}} = 0.63244 \quad 74$
	$t_1^2 t_4^2$		$1 + \frac{6}{\pi\sqrt{3}} = 2.10265 \quad 78$
	$t_2^2 t_3^2$		$1 - \frac{4}{\pi\sqrt{3}} = 0.26489 \quad 48$
5	$t_1^2 t_2^2$	$t_4^2 t_5^2$	$1 + \frac{5}{\pi\sqrt{3}} \left[-1 + 8S(3) \right] + \frac{\sqrt{5}}{4\pi^2} = 1.67989 \quad 69$
	$t_1^2 t_3^2$	$t_3^2 t_5^2$	$1 + \frac{5}{\pi\sqrt{3}} \left[\frac{7}{2} - 19S(3) \right] + \frac{\sqrt{5}}{4\pi^2} = 0.61014 \quad 74$
	$t_1^2 t_4^2$	$t_2^2 t_5^2$	$1 + \frac{5}{\pi\sqrt{3}} \left[-\frac{3}{2} + 9S(3) \right] - \frac{9\sqrt{5}}{4\pi^2} = 0.84682 \quad 08$
	$t_1^2 t_5^2$		$1 + \frac{5}{\pi\sqrt{3}} \left[3 - 6S(3) \right] + \frac{3\sqrt{5}}{2\pi^2} = 2.93988 \quad 25$
	$t_2^2 t_3^2$	$t_3^2 t_4^2$	$1 + \frac{5}{\pi\sqrt{3}} \left[\frac{1}{2} - 5S(3) \right] - \frac{\sqrt{5}}{\pi^2} = 0.26904 \quad 34$
	$t_2^2 t_4^2$		$1 + \frac{5}{\pi\sqrt{3}} \left[-6 + 20S(3) \right] + \frac{4\sqrt{5}}{\pi^2} = 0.24830 \quad 06$

Checks: E.g.

$$\sum_{k=1}^4 E(t_{1|4}^2 t_{k|4}^2) = 6 \left[1 + \frac{\sqrt{3}}{\pi} \right] = 9.30797 \quad 35, \quad 6E(t_{1|4}^2) = 9.30797 \quad 34.$$

$$\sum_{k=1}^5 E(t_{1|5}^2 t_{k|5}^2) = \sum_{k=1}^5 E(t_{k|5}^2 t_{5|5}^2) = 7 \left\{ 1 + \frac{5\sqrt{3}}{\pi} \left[\frac{1}{2} - S(3) \right] \right\} = 12.60014 \quad 33, \quad 7E(t_{1|5}^2) = 12.60014 \quad 28.$$

$$\sum_{k=1}^5 E(t_{2|5}^2 t_{k|5}^2) = \sum_{k=1}^5 E(t_{k|5}^2 t_{2|5}^2) = 7 \left\{ 1 + \frac{15\sqrt{3}}{\pi} \left[-1 + 4S(3) \right] \right\} = 3.89593 \quad 92, \quad 7E(t_{2|5}^2) = 3.89593 \quad 89.$$

$$\sum_{k=1}^5 E(t_{3|5}^2 t_{k|5}^2) = 7 \left\{ 1 + \frac{5\sqrt{3}}{\pi} \left[1 - 6S(3) \right] \right\} = 2.00783 \quad 57, \quad 7E(t_{3|5}^2) = 2.00783 \quad 59.$$

So

$$\sum_{k=1}^5 \sum_{j=1}^5 E(t_{j|5}^2 t_{k|5}^2) = 35.00000 \quad 07 \approx 35, \quad 7 \sum_{i=1}^5 E(t_{i|5}^2) = 34.99999 \quad 93 \approx 35.$$

Also

$$\sum_{k=2}^n \sum_{i=1}^{k-1} E(t_{i|n}^2 t_{k|n}^2) = \frac{1}{2} n(n-1) = 1, 3, 6, 10 \quad \text{for } n=2, 3, 4, 5.$$

Table XII Values of $E(t_{i|n} t_{j|n} t_{k|n})$ ($i < j < k$)

n	$t_i t_j t_k$	$t_{n-k+1} t_{n-j+1} t_{n-i+1}$	$E(t_{i n} t_{j n} t_{k n}) = -E(t_{n-k+1 n} t_{n-j+1 n} t_{n-i+1 n})$
3	$t_1 t_2 t_3$	$t_2 t_3 t_4$	0
4	$t_1 t_2 t_3$	$t_1 t_3 t_4$	$\frac{-1}{\pi\sqrt{2\pi}} = -0.12698 \ 73$
	$t_1 t_2 t_4$	$t_3 t_4 t_5$	$\frac{3}{\pi\sqrt{2\pi}} = 0.38096 \ 18$
5	$t_1 t_2 t_3$	$t_3 t_4 t_5$	$\frac{-5}{2\pi\sqrt{2\pi}} = -0.31746 \ 82$
	$t_1 t_2 t_4$	$t_2 t_4 t_5$	$\frac{15}{2\pi\sqrt{2\pi}} - \frac{15}{2\pi\sqrt{3\pi}} = 0.17476 \ 95$
	$t_1 t_2 t_5$	$t_1 t_4 t_5$	$\frac{15}{2\pi\sqrt{3\pi}} = 0.77763 \ 50$
	$t_1 t_3 t_4$	$t_2 t_3 t_5$	$\frac{-15}{2\pi\sqrt{2\pi}} + \frac{15}{2\pi\sqrt{3\pi}} = -0.17476 \ 95$
	$t_1 t_3 t_5$		0
	$t_2 t_3 t_4$		0

Checks: E.g.

$$\begin{aligned} \sum_{k=1}^5 E(t_{1|4} t_{2|4} t_{k|4}) &= -\frac{12}{\sqrt{\pi}} S(2) = -1.32638 \ 68, \quad E(t_{1|4}) + E(t_{2|4}) = -1.32638 \ 68. \\ \sum_{k=1}^5 E(t_{1|5}^2 t_{k|5}) &= \frac{10}{\sqrt{\pi}} [-1 + 3S(2)] = -2.32592 \ 89, \quad 2E(t_{1|5}) = -2.32592 \ 90; \\ \sum_{k=1}^5 E(t_{1|5} t_{2|5} t_{k|5}) &= \frac{15}{\sqrt{\pi}} [0 - 3S(2)] = -1.6579 \ 834, \quad E(t_{1|5}) + E(t_{2|5}) = -1.6579 \ 835; \\ \sum_{k=1}^5 E(t_{1|5} t_{3|5} t_{k|5}) &= \frac{5}{\sqrt{\pi}} [-1 + 3S(2)] = -1.16296 \ 45, \quad E(t_{1|5}) + E(t_{3|5}) = -1.16296 \ 45; \\ \sum_{k=1}^5 E(t_{1|5} t_{4|5} t_{k|5}) &= \frac{5}{\sqrt{\pi}} [-2 + 9S(2)] = -0.66794 \ 55, \quad E(t_{1|5}) + E(t_{5|5}) = -0.66794 \ 55; \\ \sum_{k=1}^5 E(t_{1|5} t_{k|5} t_{5|5}) &= 0, \quad E(t_{1|5}) + E(t_{5|5}) = 0. \end{aligned}$$

So that

$$\sum_{j=k}^5 \sum_{l=1}^5 E(t_{1|5} t_{j|5} t_{k|5}) = \frac{25}{\sqrt{\pi}} [-1 + 3S(2)] = -5.81482 \ 24, \quad 5E(t_{1|5}) = -5.81482 \ 25.$$

Similarly

$$\begin{aligned} \sum_{j=1}^5 \sum_{k=1}^5 E(t_{2|5} t_{j|5} t_{k|5}) &= \frac{25}{\sqrt{\pi}} [1 - 6S(2)] = -2.47509 \ 54, \quad 5E(t_{2|5}) = -2.47509 \ 50. \\ \sum_{j=1}^5 \sum_{k=1}^5 E(t_{3|5} t_{j|5} t_{k|5}) &= 0, \quad 5E(t_{3|5}) = 0. \\ \sum_{j=1}^5 \sum_{k=1}^5 E(t_{4|5} t_{j|5} t_{k|5}) &= \frac{25}{\sqrt{\pi}} [-1 + 6S(2)] = 2.47509 \ 54, \quad 5E(t_{4|5}) = 2.47509 \ 50. \\ \sum_{j=1}^5 \sum_{k=1}^5 E(t_{j|5} t_{k|5} t_{5|5}) &= \frac{25}{\sqrt{\pi}} [1 - 3S(2)] = 5.81482 \ 24, \quad 5E(t_{5|5}) = 5.81482 \ 25. \end{aligned}$$

Therefore

$$\sum_{i=1}^5 \sum_{j=1}^5 \sum_{k=1}^5 E(t_{i|5} t_{j|5} t_{k|5}) = 0, \quad 5 \sum_{i=1}^5 E(t_{i|5}) = 0.$$

Table XIII Values of $E(t_{i|n}^2 t_{j|n} t_{k|n})$ and $E(t_{i|n} t_{j|n} t_{k|n}^2)$ ($i < j < k$)

n	$t_i^2 t_j t_k$	$t_k t_j t_i$	$E(t_{i n}^2 t_{j n} t_{k n}) = E(t_{n-k+1 n} t_{n-j+1 n} t_{n-i+1}^2)$
3	$t_1^2 t_2 t_3$	$t_1 t_2 t_3^2$	$\frac{\sqrt{3}}{6\pi} = 0.09188 \ 81$
4	$t_1^2 t_2 t_3$	$t_2 t_3 t_4^2$	$\frac{1}{6\pi} [8\sqrt{3} - 9] = 0.25764 \ 04$
	$t_1^2 t_2 t_4$	$t_1 t_3 t_4^2$	$\frac{1}{2\pi} [-4\sqrt{3} + 3] = -0.62519 \ 30$
	$t_1^2 t_3 t_4$	$t_1 t_2 t_4^2$	$\frac{\sqrt{3}}{\pi} = 0.55132 \ 89$
	$t_2^2 t_3 t_4$	$t_1 t_2 t_3^2$	$\frac{\sqrt{3}}{3\pi} = 0.18377 \ 63$
5	$t_1^2 t_2 t_3$	$t_3 t_4 t_5^2$	$\frac{5\sqrt{3}}{\pi} \left[\frac{1}{6} + \frac{5}{3} S(3) \right] + \frac{\sqrt{5}}{4\pi^2} + \frac{15}{\pi} \left[-\frac{1}{4} + \frac{1}{2} S\left(\frac{1}{2}\right) \right] = 0.60582 \ 30$
	$t_1^2 t_2 t_4$	$t_2 t_4 t_5^2$	$-\frac{20\sqrt{3} S(3)}{\pi} + \frac{\sqrt{5}}{4\pi^2} + \frac{15}{\pi} \left[\frac{3}{4} - \frac{5}{2} S\left(\frac{1}{2}\right) \right] = -0.27341 \ 96$
	$t_1^2 t_2 t_5$	$t_1 t_4 t_5^2$	$-\frac{\sqrt{5}}{\pi^2} + \frac{15}{\pi} \left[-\frac{1}{2} + 2S\left(\frac{1}{2}\right) \right] = -1.33561 \ 41$
	$t_1^2 t_3 t_4$	$t_2 t_3 t_5^2$	$\frac{5\sqrt{3}}{\pi} \left[\frac{1}{2} + S(3) \right] + \frac{\sqrt{5}}{4\pi^2} + \frac{15}{\pi} \left[-\frac{1}{2} + S\left(\frac{1}{2}\right) \right] = 0.26507 \ 57$
	$t_1^2 t_3 t_5$	$t_1 t_3 t_5^2$	$\frac{5\sqrt{3}}{\pi} [-1 + 2S(3)] - \frac{\sqrt{5}}{\pi^2} + \frac{15}{\pi} \left[\frac{1}{2} - S\left(\frac{1}{2}\right) \right] = -0.07841 \ 33$
	$t_1^2 t_4 t_5$	$t_1 t_2 t_5^2$	$\frac{5\sqrt{3}}{\pi} \left[\frac{1}{2} - S(3) \right] + \frac{3\sqrt{5}}{2\pi^2} = 1.13986 \ 20$
	$t_2^2 t_3 t_4$	$t_2 t_3 t_4^2$	$\frac{5\sqrt{3}}{\pi} \left[\frac{1}{2} - \frac{1}{3} S(3) \right] - \frac{\sqrt{5}}{\pi^2} + \frac{15}{\pi} \left[-\frac{1}{4} + \frac{1}{2} S\left(\frac{1}{2}\right) \right] = 0.08489 \ 96$
	$t_2^2 t_3 t_5$	$t_1 t_3 t_4^2$	$\frac{5\sqrt{3}}{\pi} [-1 + 2S(3)] + \frac{11\sqrt{5}}{4\pi^2} + \frac{15}{\pi} \left[\frac{1}{4} - \frac{1}{2} S\left(\frac{1}{2}\right) \right] = -0.10290 \ 37$
	$t_2^2 t_4 t_5$	$t_1 t_2 t_4^2$	$\frac{5\sqrt{3}}{\pi} \left[\frac{1}{2} - S(3) \right] - \frac{9\sqrt{5}}{4\pi^2} = 0.29025 \ 81$
	$t_3^2 t_4 t_5$	$t_1 t_2 t_3^2$	$\frac{5\sqrt{3}}{\pi} \left[\frac{1}{6} - \frac{1}{3} S(3) \right] + \frac{\sqrt{5}}{4\pi^2} = 0.32331 \ 37$

Checks: E.g. $\sum_{j=1}^4 \sum_{k=1}^4 E(t_{1|4}^2 t_{j|4} t_{k|4}) = 6 + \frac{4\sqrt{3}}{\pi} = 8.20531 \ 59$, $4E(t_{1|4}) + 2 = 8.20531 \ 56$;

$\sum_{j=1}^4 \sum_{k=1}^4 E(t_{2|4}^2 t_{j|4} t_{k|4}) = 6 - \frac{4\sqrt{3}}{\pi} = 3.79468 \ 44$, $4E(t_{2|4}) + 2 = 3.79468 \ 44$;

$\sum_{k=1}^5 E(t_{1|5}^3 t_{k|5}) = 3 + \frac{15}{2\pi} [1 - 2S(3)] = 5.40006 \ 14$, $3E(t_{1|5}^2) = 5.40006 \ 12$;

$\sum_{k=1}^5 E(t_{1|5}^2 t_{2|5} t_{k|5}) = 1 + \frac{15}{2\pi} \sqrt{3} [1 - 2S(3)] = 3.40006 \ 13$, $E(t_{1|5}^2) + 2E(t_{1|5} t_{2|5}) = 3.40006 \ 12$;

$\sum_{k=1}^5 E(t_{1|5}^2 t_{3|5} t_{k|5}) = 1 + \frac{15\sqrt{3}}{2\pi} [-1 + 2S(3)] + \frac{15}{\pi} \left[1 - 2S\left(\frac{1}{2}\right) \right] = 2.09631 \ 59$, $E(t_{1|5}^2) + 2E(t_{1|5} t_{3|5}) = 2.09631 \ 58$;

$\sum_{k=1}^5 E(t_{1|5}^2 t_{4|5} t_{k|5}) = 1 + \frac{5\sqrt{3}}{2\pi} [1 - 2S(3)] + \frac{15}{\pi} \left[-1 + 6S\left(\frac{1}{2}\right) \right] = 0.86018 \ 54$, $E(t_{1|5}^2) + 2E(t_{1|5} t_{4|5}) = 0.86018 \ 54$;

$\sum_{k=1}^5 E(t_{1|5}^2 t_{5|5} t_{k|5}) = 1 + \frac{5\sqrt{3}}{2\pi} [1 - 2S(3)] - \frac{60}{\pi} S\left(\frac{1}{2}\right) = -0.75652 \ 17$, $E(t_{1|5}^2) + 2E(t_{1|5} t_{5|5}) = -0.75652 \ 18$.

Hence $\sum_{k=1}^5 \sum_{j=1}^5 E(t_{1|5}^2 t_{j|5} t_{k|5}) = 7 + \frac{25}{2\pi} \sqrt{3} [1 - 2S(3)] = 11.00010 \ 23$, $5E(t_{1|5}^2) + 2 = 11.00010 \ 20$.

So also $\sum_{k=1}^5 \sum_{j=1}^5 E(t_{2|5}^2 t_{j|5} t_{k|5}) = 7 + \frac{25\sqrt{3}}{\pi} [-1 + 4S(3)] = 4.78281 \ 38$, $5E(t_{2|5}^2) + 2 = 4.78281 \ 35$.

$\sum_{k=1}^5 \sum_{j=1}^5 E(t_{3|5}^2 t_{j|5} t_{k|5}) = 7 + \frac{25\sqrt{3}}{\pi} [1 - 6S(3)] = 3.43416 \ 79$, $5E(t_{3|5}^2) + 2 = 3.43416 \ 85$, &c.

Also $\sum_{k=1}^n \sum_{j=1}^{k-1} \sum_{i=1}^{j-1} E(t_{i|n}^2 t_{j|n} t_{k|n}) = \frac{n(n-1)(n-2)}{12\pi\sqrt{3}} = \frac{1}{2\pi\sqrt{3}}$, $\frac{2}{\pi\sqrt{3}}$, $\frac{5}{\pi\sqrt{3}}$ for $n=3, 4, 5$.

Table XIV Values of $E(t_{i|n} t_{j|n}^2 t_{k|n})$ ($i < j < k$)

n	$t_i t_j^2 t_k$	$t_{k'} t_{2j'} t_{i'}$	$E(t_{i n} t_{j n}^2 t_{k n}) = E(t_{n-k+1} t_{n-j+1}^2 t_{n-i+1})$
3	$t_1 t_2^2 t_3$		$-\frac{1}{\sqrt{3}\pi} = -0.18377 \ 63$
4	$t_1 t_2^2 t_3$	$t_2 t_3^2 t_4$	$-\frac{2}{\pi\sqrt{3}} + \frac{3}{2\pi} = 0.10991 \ 22$
	$t_1 t_2^2 t_4$	$t_1 t_3^2 t_4$	$-\frac{3}{2\pi} = -0.47746 \ 48$
5	$t_1 t_2^2 t_3$	$t_3 t_4^2 t_5$	$-\frac{5}{\pi\sqrt{3}}[1-2S(3)] + \frac{\sqrt{5}}{4\pi^2} + \frac{15}{2\pi}\left[\frac{1}{2} - S\left(\frac{1}{2}\right)\right] = 0.39738 \ 76$
	$t_1 t_2^2 t_4$	$t_2 t_4^2 t_5$	$\frac{\sqrt{5}}{4\pi^2} + \frac{15}{2\pi}\left[-\frac{1}{2} + 3S\left(\frac{1}{2}\right)\right] = -0.17831 \ 85$
	$t_1 t_2^2 t_5$	$t_1 t_4^2 t_5$	$-\frac{\sqrt{5}}{\pi^2} - \frac{15}{\pi}S\left(\frac{1}{2}\right) = -0.86569 \ 66$
	$t_1 t_3^2 t_4$	$t_2 t_3^2 t_5$	$-\frac{9\sqrt{5}}{4\pi^2} + \frac{15}{2\pi}\left[-\frac{1}{2} + 5S\left(\frac{1}{2}\right)\right] = -0.10558 \ 56$
	$t_1 t_3^2 t_5$		$\frac{4\sqrt{5}}{\pi^2} - \frac{30}{\pi}S\left(\frac{1}{2}\right) = -0.37202 \ 69$
	$t_2 t_3^2 t_4$		$-\frac{20}{\pi\sqrt{3}}S(3) + \frac{3\sqrt{5}}{2\pi^2} + \frac{15}{2\pi}\left[1 - 6S\left(\frac{1}{2}\right)\right] = 0.03869 \ 00$

Checks: E.g. $\sum_{k=3}^n \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} E(t_i t_j^2 t_k) = \frac{-n(n-1)(n-2)}{6\pi\sqrt{3}};$

$$n=4. \quad \sum_{k=3}^4 \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} E(t_i t_j^2 t_k) = -\frac{4}{\pi\sqrt{3}} = -0.73510 \ 52$$

$$n=5. \quad \sum_{k=3}^5 \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} E(t_i t_j^2 t_k) = -\frac{10}{\pi\sqrt{3}} = -1.83776 \ 30.$$

Otherwise

$$\sum_{k=1}^5 E(t_{1|5} t_{2|5}^2 t_{k|5}) = 1 + \frac{10}{\pi}\sqrt{3}S(3) = E(t_{2|5}^2) + 2E(t_{1|5} t_{2|5}), \quad (\text{cf. Tables III, IV, Part I})$$

$$\sum_{k=1}^5 E(t_{1|5} t_{3|5}^2 t_{k|5}) = 1 + \frac{5\sqrt{3}}{\pi}[-1-2S(3)] + \frac{15}{\pi}\left[1-2S\left(\frac{1}{2}\right)\right] = E(t_{3|5}^2) + 2E(t_{1|5} t_{3|5}),$$

$$\sum_{k=1}^5 E(t_{1|5} t_{4|5}^2 t_{k|5}) = 1 + \frac{5\sqrt{3}}{\pi}[-1+4S(3)] + \frac{15}{\pi}\left[-1+6S\left(\frac{1}{2}\right)\right] = E(t_{4|5}^2) + 2E(t_{1|5} t_{4|5}),$$

$$\sum_{k=1}^5 E(t_{1|5} t_{5|5}^2 t_{k|5}) = 1 + \frac{5\sqrt{3}}{\pi}\left[\frac{1}{2} - S(3)\right] - \frac{15}{\pi} \times 4S\left(\frac{1}{2}\right) = E(t_{5|5}^2) + 2E(t_{1|5} t_{5|5}),$$

and

$$\sum_{k=1}^5 E(t_{1|5}^3 t_{k|5}) = 3 + \frac{5\sqrt{3}}{\pi}\left[\frac{3}{2} - 3S(3)\right] = 3E(t_{1|5}^2).$$

Consequently we get

$$\sum_{k=1}^5 \sum_{j=1}^5 E(t_{1|5} t_{j|5}^2 t_{k|5}) = 7 \quad \text{in accordance with (16.7.2).}$$

Lastly we deal with the case of 4 arguments. Since we saved to construct general formulas, we should proceed to work up each value separately.

For $n=4$ we have only one moment :

$$(18.0) \quad E(t_{1|4}t_{2|4}t_{3|4}t_{4|4}) = \underline{4} \int \varphi' dt \int^t \varphi_1' dt_1 \int^{t_1} \varphi_2' dt_2 \int^{t_2} \varphi_3' dt_3 = 0.$$

For, we have successively $\int^2 \varphi_3' dt_3 = \varphi_2$, $\int^1 \varphi_2 \varphi_2' dt_2 = \frac{1}{2} \varphi_1^2$, $\frac{1}{2} \int^t \varphi_1^2 \varphi_1' dt_1 = \frac{1}{\underline{3}} \varphi^3$
and $4 \int \varphi^3 \varphi' dt = \varphi^4 \Big|_{-\infty}^{\infty} = 0$.

Let $n=5$, and consider all possible 5 combinations. Firstly

$$(18.1) \quad E(t_{1|5}t_{2|5}t_{3|5}t_{4|5}) = \underline{5} \int (1-\Phi) \varphi' dt \int^t \varphi_1' dt_1 \int^{t_1} \varphi_2' dt_2 \int^{t_2} \varphi_3' dt_3 = 5c_5 = \sqrt{5}/4\pi^2,$$

because, by a similar manner as above, we get

$$E = \underline{5} \int (1-\Phi) \frac{1}{\underline{3}} \varphi^3 \varphi' dt = 5 \int \varphi^5 dt = 5c_5.$$

Secondly

$$(18.2) \quad E(t_{1|5}t_{2|5}t_{3|5}t_{5|5}) = \underline{5} \int \varphi' dt \int^t (\Phi - \Phi_1) \varphi_1' dt_1 \int^1 \varphi_2' dt_2 \int^2 \varphi_3' dt_3 = -20c_5 = -\sqrt{5}/\pi^2,$$

because, as before $\int^1 \varphi_2' dt_2 \int^2 \varphi_3' dt_3 = \frac{1}{2} \varphi_1^2$ and consequently $\frac{1}{2} \int^t (\Phi - \Phi_1) \varphi_1^2 \varphi_1' dt_1$
 $= \frac{1}{6} \int^t \varphi_1^4 dt_1$ and $\frac{\underline{5}}{6} \int \varphi' dt \int^t \varphi_1^4 dt_1 = -20 \int \varphi^5 dt = -20c_5$.

Thirdly

$$(18.3) \quad E(t_{1|5}t_{2|5}t_{4|5}t_{5|5}) = \underline{5} \int \varphi' dt \int^t \varphi_1' dt \int^1 (\Phi_1 - \Phi_2) \varphi_2' dt_2 \int^2 \varphi_3' dt_3 = 30c_5,$$

since we get $\int^2 \varphi_3' dt_3 = \varphi_2$, $\int^1 (\Phi_1 - \Phi_2) \varphi_2 \varphi_2' dt_2 = \frac{1}{2} \int^1 \varphi_2^3 dt_2$ and $\frac{1}{2} \int^t \varphi_1' dt_1 \int^1 \varphi_2^3 dt_2$
 $= \frac{\varphi}{2} \int^t \varphi_1^3 dt_1 - \frac{1}{2} \int^t \varphi_1^4 dt_1$, so that $E = \frac{\underline{5}}{2} \int \varphi \varphi' dt \int^t \varphi_1^3 dt_1 - \frac{\underline{5}}{2} \int \varphi' dt \int^t \varphi_1^4 dt_1$
 $= -30 \int \varphi^5 dt + 60 \int \varphi^5 dt = 30c_5 = 3\sqrt{5}/2\pi^2$.

The remaining two could be similarly obtained. However, by symmetry, we have immediately

$$(18.4) \quad E(t_{1|5}t_{3|5}t_{4|5}t_{5|5}) = E(t_{5|5}t_{3|5}t_{2|5}t_{1|5}) = -20c_5,$$

$$(18.5) \quad E(t_{2|5}t_{3|5}t_{4|5}t_{5|5}) = E(t_{4|5}t_{3|5}t_{2|5}t_{1|5}) = 5c_5.$$

Thus, we obtain the following

Table XV Values of $E(t_{i|n} t_{j|n} t_{k|n} t_{l|n})$

n	$t_i t_j t_k t_l$	$t_{i'} t_{j'} t_{k'} t_{l'}$	$E(t_{i n} t_{j n} t_{k n} t_{l n}) = E(t_{n-i+1} t_{n-k+1} t_{n-j+1} t_{n-l+1})$
4	$t_1 t_2 t_3 t_4$		0
5	$t_1 t_2 t_3 t_4$	$t_2 t_3 t_4 t_5$	$\frac{\sqrt{5}}{4\pi^2} = 0.05664 \ 03$
	$t_1 t_2 t_3 t_5$	$t_1 t_3 t_4 t_5$	$-\frac{\sqrt{5}}{\pi^2} = -0.22656 \ 11$
	$t_1 t_2 t_4 t_5$		$\frac{3\sqrt{5}}{2\pi^2} = 0.33984 \ 16$

Checks E.g.: $\sum_{k=1}^5 E(t_{1|5} t_{2|5} t_{3|5} t_{k|5}) = \frac{10\sqrt{3}}{\pi} S(3) = \sum_{i \neq j=1,2,3} E(t_{i|5} t_{j|5}) = 1.1566 \ 035$.

$$\sum_{k=1}^5 E(t_{1|5} t_{2|5} t_{4|5} t_{k|5}) = \frac{5\sqrt{3}}{\pi} \left[\frac{1}{2} - 5S(3) \right] + \frac{15}{\pi} \left[\frac{1}{2} - S\left(\frac{1}{2}\right) \right] = \sum_{i \neq j=1,2,4} E(t_{i|5} t_{j|5}) = 0.3350 \ 018.$$

Also $\sum_{l=4}^n \sum_{k=3}^{l-1} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} E(t_i t_j t_k t_l) = 0$.

§19. The Frequency Functions of $t_{i|n}$ and their Central Moments. So far we have obtained the expectations of $t_{i|n}^k$ for $k=1, 2, 3, 4$, viz., the k -th moment $\nu_k(t_{i|n})$ about axis $t=0$ and especially mean $m=E(t_{i|n})$. Now we shall find their central moments, viz., $\mu_k(t_{i|n})$ by the well known formulas

$$(19.1) \quad \mu_k = E[(t_{i|n} - m)^k] = \nu_k - \binom{k}{1} m \nu_{k-1} + \binom{k}{2} m^2 \nu_{k-2} - \dots + (-1)^{k-1} \left[\binom{k}{1} - 1 \right] m^k.$$

In particular

$$(19.2) \quad \mu_0 = 1, \mu_1 = 0, \mu_2 = \nu_2 - m^2, \mu_3 = \nu_3 - 3m\nu_2 + 2m^3, \mu_4 = \nu_4 - 4m\nu_3 + 6m^2\nu_2 - 3m^4.$$

Thus we obtain the following

Table XVI Central Moments of $t_{i|n}$

n	$t_{i n}$	$t_{i' n}$	μ_2	μ_3	μ_4	Skewness $\mu_3/\sqrt{\mu_2}^3$	Excess $\mu_4/\mu_2^2 - 3$
2	$t_{2 2}$	$t_{1 2}$	0.6816 901	$\pm 0.0770 \ 828$	1.4227 879	0.13695	0.06172
3	$t_{3 3}$	$t_{1 3}$	0.5594 671	$\pm 0.0891 \ 995$	4.5853 562	0.15944	7.95749
	$t_{2 3}$		0.4486 711	0	0.6109 081	0	-0.96725
4	$t_{4 4}$	$t_{1 4}$	0.4917 152	$\pm 0.0912 \ 069$	8.0701 896	0.26452	30.37770
	$t_{3 4}$	$t_{2 4}$	0.3604 553	$\pm 0.0141 \ 876$	0.3493 598	0.06556	-0.31113
5	$t_{5 5}$	$t_{1 5}$	0.4475 340	$\pm 0.0905 \ 873$	13.3031 003	0.30257	63.42025
	$t_{4 5}$	$t_{2 5}$	0.3115 189	$\pm 0.0186 \ 864$	0.3646 656	0.10747	0.75774
	$t_{3 5}$		0.2868 337	0	0.2494 541	0	0.03201

The skewness and excess being as in the above Table XVI, none of them distributes normally, what is a matter of course, since really they belong to Beta-distributions. However even when $n=5$, it appears already that the central part approaches toward N.D.

Incidentally we may here prove once more again, that the distribution of $\zeta = ct_{1|3} + (1-2c)t_{2|3} + ct_{3|3}$ in Cramér's example is not normal, i.e. its excess is not zero. Really

$$\begin{aligned} m &= E(\zeta) = cE(t_{1|3}) + cE(t_{3|3}) + (1-2c)E(t_{2|3}), \\ \sigma^2 &= D^2(\zeta) = E(\zeta^2) - E(\zeta)^2 = E[\{c(t_1 + t_3) + (1-2c)t_2\}^2] \\ &= c^2E(t_1 + t_3)^2 + (1-2c)^2E(t_2^2) + 2c(1-2c)E(t_1t_2 + t_2t_3), \\ E(\zeta^3) &= c^3E[(t_1 + t_3)^3] + 3c^2(1-2c)E[(t_1 + t_3)^2t_2] + 3c(1-2c)^2E(t_1t_2^2 + t_2^2t_3) + c^3E(t_3^3), \\ D(\zeta^4) &= c^4E[(t_1 + t_3)^4] + 4c^3(1-2c)E[(t_1 + t_3)^3t_2] + 6c(1-2c)^2E[(t_1 + t_3)^2t_2^2] \\ &\quad + 4c(1-2c)^3E[(t_1 + t_3)t_2^3] + (1-2c)^4E(t_2^4). \end{aligned}$$

Excecuting these calculations actually and substituting those values of $E(t_i^p)$ for $p=1, 2, 3, 4$, and $E(t_i^p t_k^q)$ for $p+q=2, 3, 4$ and $E(t_i^p t_j^q t_k^r)$ for $p+q+r=3, 4$, given in our Tables III–XIV, we find

$$m = 0, \quad \sigma^2 = \frac{1}{3} + 3 \left[2 - \frac{3\sqrt{3}}{\pi} \right] \left(c - \frac{1}{3} \right)^2.$$

Or, on putting $1-4c+6c^2=\gamma^2$, viz., $3\gamma^2-1=2(1-3c)^2$, it yields

$$\sigma^2 = \frac{1}{3} \left[1 + \left(1 - \frac{3\sqrt{3}}{2\pi} \right) (3\gamma^2 - 1) \right] = \left(1 - \frac{3\sqrt{3}}{2\pi} \right) \gamma^2 + \frac{\sqrt{3}}{2\pi} = \gamma^2 + \frac{\sqrt{3}}{2\pi} (1 - 3\gamma^2),$$

which is the same as (6.8) in §6, Part I. Furthermore

$$\begin{aligned} E(\zeta^3) &= 0, \quad \text{so that} \quad \text{Skewness} = 0, \quad \text{but} \\ E(\zeta^4) &= \left(108 - \frac{567}{\pi\sqrt{3}} \right) c^4 - \left(144 + \frac{756}{\pi\sqrt{3}} \right) c^3 + \left(84 - \frac{426}{\pi\sqrt{3}} \right) c^2 - \left(24 - \frac{120}{\pi\sqrt{3}} \right) c + \left(3 - \frac{13}{\pi\sqrt{3}} \right), \end{aligned}$$

while

$$\begin{aligned} 3E(\zeta^2)^2 &= \left(108 - \frac{972}{\pi\sqrt{3}} + \frac{729}{\pi^2} \right) c^4 - \left(144 - \frac{1296}{\pi\sqrt{3}} + \frac{972}{\pi^2} \right) c^3 + \left(84 - \frac{702}{\pi\sqrt{3}} + \frac{486}{\pi^2} \right) c^2 \\ &\quad - \left(24 - \frac{180}{\pi\sqrt{3}} + \frac{108}{\pi^2} \right) c + \left(3 - \frac{18}{\pi\sqrt{3}} + \frac{9}{\pi^2} \right). \end{aligned}$$

Hence, we get

$$\begin{aligned} E(\zeta^4) - 3E(\zeta^2)^2 &= \left(\frac{405}{\pi\sqrt{3}} - \frac{729}{\pi^2} \right) c^4 - \left(\frac{540}{\pi\sqrt{3}} - \frac{972}{\pi^2} \right) c^3 + \left(\frac{270}{\pi\sqrt{3}} - \frac{486}{\pi^2} \right) c^2 - \left(\frac{60}{\pi\sqrt{3}} - \frac{108}{\pi^2} \right) c + \left(\frac{5}{\pi\sqrt{3}} - \frac{9}{\pi^2} \right) \\ &= \left(\frac{5}{\pi\sqrt{3}} - \frac{9}{\pi^2} \right) [81c^4 - 108c^3 + 54c^2 - 12c + 1] = \frac{5\pi - 9\sqrt{3}}{\pi^2\sqrt{3}} (3c - 1)^2. \end{aligned}$$

Or, since $\gamma^2 = 1 - 4c + 6c^2$, $3\gamma^2 - 1 = 2(3c - 1)^4$,

$$\text{Excess} = \frac{E(\zeta^4)}{E(\zeta^2)^2} - 3 = \frac{(5\pi - 9\sqrt{3})(3\gamma^2 - 1)^2}{4\pi^2\sqrt{3}} \left/ \left[\gamma^2 + \frac{(1 - 3\gamma^2)\sqrt{3}}{2\pi} \right]^2 \right. > 0, \text{ if } \gamma^2 \neq \frac{1}{3}.$$

Thus, the present result just coincides with (6.10) obtained in §6, Part I.

By a similar manner it may be shown for the cases $n=4$ or 5 , that the distribution of $\xi = \sum_{i=1}^n c_i t_i$ ($c_i \geq 0$, $\sum c_i = 1$), even when $c_{n-i+1} = c_i$ cannot be normal.

§ 20. The Asymptotic Feature of $t_{i|n}$'s Distribution as well as those of the Joint Distributions

$$(20.1) \quad f(t_{i|n}) dt_{i|n} = n! \frac{\Phi^{i-1} (1-\Phi)^{n-i}}{(i-1)! (n-i)!} d\Phi \equiv g(\Phi) d\Phi \quad (d\Phi = \varphi dt)$$

being materially a Beta-distribution, it is already recognized e.g. in Cramér's treatise, loc. cit., p. 252, that the fr. f. of the standardized variable tends to the normal fr. f. However, as there is not given any detailed proof, let us now demonstrate it below. Of course, if i be fixed and n alone allowed to become ∞ , then $g(\Phi) \rightarrow 0$. For, on making use of Stirling's formula, we get

$$\frac{n!}{(i-1)! (n-i)!} \simeq \frac{\sqrt{2\pi} n^{n+1/2} e^{-n}}{(i-1)! \sqrt{2\pi} (n-i)^{n-i+1/2} e^{-(n-i)}} = \frac{e^{-i}}{(i-1)!} \left(\frac{n}{n-i} \right)^{n+1/2} (n-i)^i \simeq \frac{n^i}{\Gamma(i)},$$

and thus

$$\lim_{n \rightarrow \infty} g(\Phi) = \frac{\Phi^{i-1}}{\Gamma(i)} \lim_{n \rightarrow \infty} n^i (1-\Phi)^{n-i} = 0.$$

When, however, i as well as n considered to tend to infinity, the ordinal number i itself becomes meaningless; rather the percentage figure i/n comes into consideration. Hence, let $i = [np] = np - \alpha$ where p denotes a certain positive proper fraction: $0 < p < 1$ and $0 \leq \alpha < 1$ though fluctuating, so that $n-i = n - np + \alpha = nq + \alpha$, where $q = 1-p$ is also a fixed positive proper fraction. Under these assumptions, let us conceive

$$(20.2) \quad \lim_{n \rightarrow \infty} g(\Phi) d\Phi = \lim_{n \rightarrow \infty} \beta_{i|n} \Phi^{i-1} (1-\Phi)^{n-i} d\Phi.$$

Now to make the variable Φ standardize, we require the mean and the variance. These are readily found to be

$$(20.3) \quad m = E(\Phi) = \beta_{i|n} \int_0^1 \Phi^{i-1} (1-\Phi)^{n-i} \Phi d\Phi = \beta_{i|n} B(i+1, n+i+1) = \frac{i}{n+1} \simeq p,$$

$$\text{and} \quad E(\Phi^2) = \beta_{i|n} \int_0^1 \Phi^{i+1} (1-\Phi)^{n-i} d\Phi = \frac{i(i+1)}{(n+1)(n+2)} \simeq p^2,$$

so that

$$(20.4) \quad \sigma^2 = D^2(\Phi) = E(\Phi^2) - E(\Phi)^2 = \frac{i(n-i+1)}{(n+1)^2 (n+2)} \simeq \frac{pq}{n}.$$

Accordingly we put to make the variable standardize

$$(20.5) \quad \xi = (\Phi - m)/\sigma \quad \text{or} \quad \Phi = m + \sigma \xi,$$

and transform the probability element so as

$$(20.6) \quad g(\Phi)d\Phi = \beta_{i|n}(\sigma\xi + m)^{i-1}(1-m-\sigma\xi)^{n-i}\sigma d\xi = h(\xi)d\xi, \quad \text{say,}$$

where $d\Phi = \sigma d\xi (= \varphi(t)dt)$ and the probability element expresses the probability that the variable t lies between some given t and $t+dt$, so that Φ lies between $\Phi = \int^t \varphi dt$ and $\Phi = \int^{t+dt} \varphi dt$ and vice versa. We have to find the asymptotic estimation of the frequency function

$$(20.7) \quad h(\xi) = \beta_{i|n} m^{i-1} (1-m)^{n-i} \sigma \left[1 + \frac{\sigma}{m} \xi \right]^{i-1} \left[1 - \frac{\sigma}{1-m} \xi \right]^{n-i},$$

where $i = np - \alpha$, $n-i = nq + \alpha$ and we are going to show that indeed $h(\xi)$ tends to the standard normal distribution.

By use of Stirling's asymptotic formula, we have

$$(20.8) \quad \beta_{i|n} = \frac{n!}{\Gamma(np-\alpha)\Gamma(nq+\alpha)} \simeq \sqrt{\frac{n}{2\pi}} (p^p q^q)^{-n} \left(\frac{p}{q}\right)^{\alpha+1/2} \left[1 - \frac{\alpha}{np}\right]^{np} \left[1 + \frac{\alpha}{nq}\right]^{nq} \left[\frac{(np-\alpha)q}{(nq+\alpha)p}\right]^{\alpha+1/2} \\ \simeq \sqrt{\frac{n}{2\pi}} (p^p q^q)^{-n} \left(\frac{p}{q}\right)^{\alpha+1/2},$$

since the remaining factor tends to unity as $n \rightarrow \infty$. On the other hand $m = \frac{i}{n+1} = \frac{np-\alpha}{n+1}$, $1-m = \frac{nq+1+\alpha}{n+1}$ and consequently

$$(20.9) \quad m^{i-1} (1-m)^{n-i} = \left(\frac{np-\alpha}{n+1}\right)^{np-\alpha-1} \left(\frac{nq+1+\alpha}{n+1}\right)^{nq+\alpha} \\ = p^{np} q^{nq} \left(\frac{q}{p}\right)^{\alpha} \frac{1}{p} \left(1 + \frac{1}{n}\right)^{-n} \left[1 - \frac{\alpha}{np}\right]^{np} \left[1 + \frac{1+\alpha}{nq}\right]^{nq} \\ \simeq (p^p q^q)^n \left(\frac{q}{p}\right)^{\alpha} \frac{1}{p}.$$

Also

$$(20.10) \quad \sigma = \sqrt{\frac{(n-i+1)i}{(n+1)^2(n+2)}} \simeq \sqrt{\frac{pq}{n}} \left[1 + O\left(\frac{1}{n}\right)\right].$$

so that

$$\frac{\sigma}{m} \simeq \sqrt{\frac{q}{pn}} \left[1 + O\left(\frac{1}{n}\right)\right], \quad \frac{\sigma}{1-m} \simeq \sqrt{\frac{p}{qn}} \left[1 + O\left(\frac{1}{n}\right)\right].$$

All these values being substituted in (20.7), we have

$$h(\xi) \simeq \sqrt{\frac{n}{2\pi}} (p^p q^q)^{-n} \left(\frac{p}{q}\right)^{\alpha+1/2} \cdot (p^p q^q)^n \left(\frac{q}{p}\right)^{\alpha} \frac{1}{p} \cdot \sqrt{\frac{pq}{n}} \left[1 + O\left(\frac{1}{n}\right)\right] \\ \times \left\{1 + \sqrt{\frac{q}{pn}} \xi + O\left(\frac{1}{n^{3/2}}\right)\right\}^{np-\alpha-1} \left\{1 - \sqrt{\frac{p}{qn}} \xi + O\left(\frac{1}{n^{3/2}}\right)\right\}^{nq+\alpha} \\ = \frac{1}{\sqrt{2\pi}} \exp \left\{ (np-\alpha-1) \log \left[1 + \sqrt{\frac{q}{pn}} \xi + O\left(\frac{1}{n^{3/2}}\right)\right] + (nq+\alpha) \log \left[1 - \sqrt{\frac{p}{qn}} \xi + O\left(\frac{1}{n^{3/2}}\right)\right] \right\}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \exp \left\{ (np - \alpha - 1) \left[\sqrt{\frac{q}{pn}} \xi - \frac{1}{2n} \frac{q}{p} \xi^2 + O\left(\frac{1}{n^{3/2}}\right) \right] \right. \\
&\quad \left. - (nq + \alpha) \left[\sqrt{\frac{p}{qn}} \xi + \frac{1}{2n} \frac{p}{q} \xi^2 + O\left(\frac{1}{n^{3/2}}\right) \right] \right\} \\
&= \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \xi^2 + O\left(\frac{1}{\sqrt{n}}\right) \right\}.
\end{aligned}$$

Thus, our probability element $f(t_{i|n})dt_{i|n} = g(\Phi)d\Phi = h(\xi)d\xi$ for large value of n with $\frac{i}{n} \cong p$ (fixed) approaches to $\frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} d\xi$; namely the standardized variable $\xi = [\Phi - E(\Phi)]/\sigma(\Phi)$ tends to distribute normally.

Moreover, the alike conclusion can be said about the joint frequency function

$$(20.11) \quad f(t_{i|n}, t_{k|n})dt_{i|n}dt_{k|n} = \gamma_{i,k|n} (1-\Phi)^{n-k} (\Phi - \Phi_1)^{k-i-1} \Phi^{i-1} d\Phi d\Phi_1 \\ (0 < \Phi_1 < \Phi < 1).$$

Here again we conceive certain fixed positive proper fractions p, p_1 , such that

$$k = [np] = np - \alpha, \quad i = [np_1] = np_1 - \alpha_1, \quad (0 < p_1 < p < 1).$$

Or, letting $\Phi_1 = \Phi\Psi$ and considering Φ as temporarily fixed, we have

$$f_{ik} dt_{i|n} dt_{k|n} = \gamma_{i,k|n} (1-\Phi)^{n-k} \Phi^{k-1} d\Phi \cdot (1-\Psi)^{k-i-1} \Psi^{i-1} d\Psi \quad (0 < \Psi < 1, 0 < \Phi < 1).$$

Consequently Φ and Ψ can be considered as independent of each other, and accordingly the numerical coefficient may also be put as

$$\gamma_{i,k|n} = \frac{|n|}{|n-k| |i-1| |k-i-1|} = \frac{|n|}{|n-k| |k-1|} \times \frac{|k-1|}{|i-1| |k-i-1|} = \beta_{k|n} \cdot \beta_{i|k-1}.$$

Thus

$$\begin{aligned}
f(t_{i|n}, t_{k|n})dt_{i|n}dt_{k|n} &= \beta_{k|n} (1-\Phi)^{n-k} \Phi^{k-1} d\Phi \cdot \beta_{i|k-1} (1-\Psi)^{k-i-1} \Psi^{i-1} d\Psi \\
&= f(t_{i|n})dt_{i|n} \cdot f(t'_{i|k-1})dt'_{i|k-1},
\end{aligned}$$

So that t and t' distribute independently. But, by the foregoing proof, if we put

$$k = np - \alpha \quad (0 \leq \alpha < 1) \quad \text{and} \quad i = (k-1)p_1 - \alpha_1 \quad (0 \leq \alpha_1 < 1),$$

then the variables

$$\xi = \left(\Phi - \frac{k}{n+1} \right) / \frac{1}{n+1} \sqrt{\frac{(n+k+1)k}{n+2}} \quad \text{and} \quad \eta = \left(\Psi - \frac{i}{k} \right) / \frac{1}{k} \sqrt{\frac{(k-i)i}{k+1}}$$

distribute asymptotically normally: viz., so as with fr. f.

$$\frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \xi^2 \right\} \quad \text{and} \quad \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \eta^2 \right\} \quad \text{respectively.}$$

Accordingly

$$f(t_{i|n}, t_{k|n})dt_{i|n}dt_{k|n} \cong \frac{1}{2\pi} \exp \left\{ -\frac{1}{2} (\xi^2 + \eta^2) \right\} d\xi d\eta;$$

namely, the joint frequency function tends to become a bivariate non-singular normal distribution.

The general case of several arguments may be treated by induction. Let the joint frequency function be

$$(20.12) \quad f(t_{i_1|n}, t_{i_2|n}, \dots, t_{i_k|n}) dt_{i_1|n} dt_{i_2|n} \dots dt_{i_k|n} \quad (1 \leq i_1 < i_2 < \dots < i_k \leq n) \\ = |n| (1-\Phi)^{n-i_k} d\Phi \cdot (\Phi-\Phi_1)^{i_k-i_{k-1}-1} d\Phi_1 \cdot \dots \cdot (\Phi_{k-2}-\Phi_{k-1})^{i_2-i_1-1} \Phi_{k-1}^{i_1-1} d\Phi_{k-1},$$

and conceive certain fixed positive proper fractions $0 < p_{k-1} < p_{k-2} < \dots < p_1 < p < 1$ and those arguments, such that

$$i_k = [np] = np - \alpha, \quad i_{k-1} = [np_1] = np_1 - \alpha_1, \dots, \quad i_1 = [np_{k-1}] = np_{k-1} - \alpha_{k-1}$$

with all $\alpha_v: 0 \leq \alpha_v < 1$. Putting $\Phi_1 = \Phi\Psi$, $\Phi_v = \Phi\Psi_{v-1}$ ($v = 2, 3, \dots, k-1$), and $i_k = l$, we get

$$f dt_{i_1} dt_{i_2} \dots dt_{i_k} = |n| (1-\Phi)^{n-l} \Phi^{l-1} d\Phi \cdot |l-1| (1-\Psi)^{l-1-i_{k-1}} (\Psi-\Psi_1)^{i_{k-1}-i_{k-2}-1} \dots \\ (\Psi_{k-3}-\Psi_{k-2})^{i_2-i_1-1} \Psi_{k-2}^{i_1-1} d\Psi_1 d\Psi_2 \dots d\Psi_{k-2} \\ = f(t_{l|n}) dt_{l|n} \cdot f(t_{i_1|l-1}, t_{i_2|l-1}, \dots, t_{i_{k-2}|l-1}, t_{i_{k-1}|l-1}) dt_{i_1|l-1} dt_{i_2|l-1} \dots dt_{i_{k-1}|l-1}.$$

But, assuming that the asymptotic behaviour toward N.D. as $n, l \rightarrow \infty$ up to the case with $k-1$ arguments is already established, viz.,

$$f(t_{i_1|l-1}, t_{i_2|l-1}, \dots, t_{i_{k-1}|l-1}) dt_{i_1|l-1} \dots dt_{i_{k-1}|l-1} \cong \frac{1}{\sqrt{2\pi}^{k-1}} \exp \left\{ -\frac{1}{2} \sum_{v=1}^{k-1} \xi_v^2 \right\} d\xi_1 d\xi_2 \dots d\xi_{k-1}$$

and

$$f(t_{i_k|n}) dt_{i_k|n} \cong \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \xi_k^2 \right\} d\xi_k,$$

it follows consequently that

$$f(t_{i_1|n}, t_{i_2|n}, \dots, t_{i_k|n}) dt_{i_1|n} \dots dt_{i_k|n} \cong \frac{1}{\sqrt{2\pi}^k} \exp \left\{ -\frac{1}{2} \sum_{v=1}^k \xi_v^2 \right\} d\xi_1 d\xi_2 \dots d\xi_k,$$

namely, the assertion is also true for the case with k arguments, and our induction has been completed.