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ON SOME PROPERTIES OF PLANE CURVES IN RIEMANN SPACES

By

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§1. Consider in an m-dimensional Riemann space V_m a curve denoted by

$$y^\lambda = y^\lambda(s) \qquad (\lambda, \mu, \nu = 1, 2, \cdots, m),$$

where we denote by s arc length of this curve.
Then we see by Frenet's formula

$$\frac{dy^\lambda}{ds} = \underset{(1)}{\xi^\lambda}, \quad \frac{\delta \underset{(1)}{\xi^\lambda}}{ds} = \kappa_1 \underset{(2)}{\xi^\lambda}, \quad \frac{\delta \underset{(2)}{\xi^\lambda}}{ds} = -\kappa_1 \underset{(1)}{\xi^\lambda} + \kappa_2 \underset{(3)}{\xi^\lambda}, \cdots, \frac{\delta \underset{(m)}{\xi^\lambda}}{ds} = -\kappa_{m-1} \underset{(m-1)}{\xi^\lambda}.$$

Now we define a plane curve as a curve satisfying

$$\frac{\delta \underset{(2)}{\xi^\lambda}}{ds} = -\kappa_1 \underset{(1)}{\xi^\lambda}. \tag{1, 1}$$

Then from the relation

$$-\kappa_1 \underset{(1)}{\xi^\lambda} = \frac{\delta}{ds}\left(\frac{1}{\kappa_1} \frac{\delta \underset{(1)}{\xi^\lambda}}{ds}\right),$$

the equation (1, 1) has a form

$$\frac{\delta^3 y^\lambda}{ds^3} + \kappa_1 \frac{d\left(\frac{1}{\kappa_1}\right)}{ds} \frac{\delta^2 y^\lambda}{ds^2} + \kappa_1^2 \frac{\delta y^\lambda}{ds} = 0. \tag{1, 2}$$

Putting

$$\begin{cases} q = \kappa_1^2 = g_{\lambda\mu} \dfrac{\delta^2 y^\lambda}{ds^2} \dfrac{\delta^2 y^\mu}{ds^2}, \\ p = \dfrac{d}{ds}\left(\log \dfrac{1}{\sqrt{q}}\right) = \kappa_1 \dfrac{d\left(\frac{1}{\kappa_1}\right)}{ds}, \end{cases} \tag{1, 3}$$

we can rewrite (1, 2) in

$$\frac{\delta^3 y^\lambda}{ds^3} + p \frac{\delta^2 y^\lambda}{ds^2} + q \frac{dy^\lambda}{ds} = 0. \tag{1, 4}$$

§2. Consider in V_m an n-dimensional subspace V_n whose current point is given by a system of coordinate (x^i)[1]. If the curve $y^\lambda = y^\lambda(s)$ is contained in V_n, then we must have

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1) Hereafter we shall denote by $\alpha, \beta, \gamma, \delta, \mu, \nu$, the suffices which take the value $1, 2, \cdots m$; by a, b, c, i, j, k, those which take the value $1, 2, \cdots, n$ and P, Q, R, those which take the value $\dot{n}+\dot{1}, \dot{n}+\dot{2}, , \cdots, \dot{m}$.

$$
(2,1)\qquad
\begin{cases}
\dfrac{dy^\lambda}{ds}=B_i^{\cdot\lambda}\dfrac{dx^i}{ds}\,,\\[2mm]
\dfrac{\delta^2 y^\lambda}{ds^2}=H^\lambda_{ij}\dfrac{dx^i}{ds}\dfrac{dx^j}{ds}+B_i^{\cdot\lambda}\dfrac{\delta^2 x^i}{ds^2}\\[2mm]
\dfrac{\delta^3 y^\lambda}{ds^3}=H^\lambda_{ij;k}\dfrac{dx^i}{ds}\dfrac{dx^j}{ds}\dfrac{dx^k}{ds}+3H^\lambda_{ij}\dfrac{\delta^2 x^i}{ds^2}\dfrac{dx^j}{ds}+B_i^{\cdot\lambda}\dfrac{\delta^3 x^i}{ds^3}\,,
\end{cases}
$$

where we put

$$
B_i^{\cdot\lambda}=\frac{\partial y^\lambda}{\partial x^i}\,,\quad H^\lambda_{ij}=B^{\cdot\lambda}_{i;j}=\sum_P H^P_{ij}\xi^\lambda_P\,,
$$

and $(m-n)$ normal vectors of V_n in V_m ξ^α_P must be satisfied

$$
\xi^\alpha_{P;k}=-g^{ij}B_i^{\cdot\alpha}H^P_{jk}+\sum_Q L_{PQ|k}\xi^\alpha_Q\,.
$$

Hence we easily obtain the relations

$$
H^\alpha_{ij;k}=\sum_P H^P_{ij;k}\xi^\alpha_P-\sum_P(H^P_{ij}H^P_{bk})g^{ab}B_a^{\cdot\alpha}+\sum_P\sum_Q L_{PQ|k}\xi^\alpha_Q H^P_{ij}\,,
$$

and

$$
(2,2)\qquad \frac{\delta^3 y^\lambda}{ds^3}=\left[\frac{\delta^3 x^a}{ds^3}-\sum_P H^P_{ij}H^P_{bk}g^{ba}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}\right]B_a^{\cdot\lambda}
$$
$$
+\sum_P\left[H^P_{ijk}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}+3H^P_{ij}\frac{\delta^2 x^i}{ds^2}\frac{dx^j}{ds}+\sum_Q H^Q_{ij}L_{PQ|k}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}\right]\xi^\lambda_P\,.
$$

The quantities H^P_{ijk} are components of a tensor given by

$$
H^P_{ijk}=H^P_{(ij;k)}
$$

Putting (2, 2) into (1, 4) we obtain the equation of a plane curve contained in V_n immersed in V_m as

$$
(2,3)\qquad
\begin{cases}
\dfrac{\delta^3 x^h}{ds^3}-\sum_P H^P_{ij}\dfrac{dx^i}{ds}\dfrac{dx^j}{ds}H^P_{bk}\dfrac{dx^k}{ds}g^{hb}+p\dfrac{\delta^2 x^h}{ds^2}+q\dfrac{dx^h}{ds}=0\,,\\[2mm]
pH^P_{ij}\dfrac{dx^i}{ds}\dfrac{dx^j}{ds}+H^P_{ijk}\dfrac{dx^i}{ds}\dfrac{dx^j}{ds}\dfrac{dx^k}{ds}+3H^P_{ij}\dfrac{\delta^2 x^i}{ds^2}\dfrac{dx^j}{ds}+\sum_Q L_{PQ|k}H^Q_{ij}\dfrac{dx^i}{ds}\dfrac{dx^j}{ds}\dfrac{dx^k}{ds}=0\,.
\end{cases}
$$

The curves, however, whose equations are given by

$$
(2,4)\qquad H^P_{ijk}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}+3H^P_{ij}\frac{\delta^2 x^i}{ds^2}\frac{dx^j}{ds}+\sum_Q L_{PQ|k}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}=0
$$

are called[2)] Darboux lines of the third kind contained in V_n immersed in V_m (in brief Darboux lines in V_n in V_m) and plane curves such as at each point $p=0$, that is $\kappa_1=\text{const.}$, $\kappa_2=0$, are called geodesic circles or Riemann circles.[3)]

2) M. Prvanovitch; Ligne de Darboux dans l'espace riemannien. (Bull. Sci. Math. (2), 78, 1954, p.p. 9–14).

Y. Ichijô; On Darboux lines contained in a Riemannian space. (Journ. Gakugei, Tokushima Univ. Japan Vol. VIII, 1957, p.p. 27–32).

3) K. Yano; Concircular geometry I (Proc. Imp. Acad. Japan 16, 1940, p.p. 195–200).

Hence we conclude, by (2, 3) *that if plane curves in V_n in V_m are Darboux lines in V_n in V_m, then they are asymptotic curves or geodesic circles.*

§ **3.** As V_n is an n-dimensional Riemann space, along the curve in V_n in V_m we shall consider $\bar{p}$ and $\bar{q}$ in this Riemann space V_n.
By (2, 1) and $q = g_{\lambda\mu} \frac{\delta^2 y^\lambda}{ds^2} \frac{\delta^2 y^\mu}{ds^2}$
we see

$$(3, 1) \qquad q = \bar{q} + \sum_P \left(H^P_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} \right)^2$$

$$(3, 2) \qquad p = \bar{p} - \frac{\bar{p} \cdot \sum_P \left(H^P_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} \right)^2 + \sum_P \left\{ \left(H^P_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} \right) \cdot \frac{d}{ds} \left(H^P_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} \right) \right\}}{\bar{q} + \sum_P \left(H^P_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} \right)^2},$$

where we put

$$\bar{q} = g_{ij} \frac{\delta^2 x^i}{ds^2} \frac{\delta^2 x^j}{ds^2}, \quad \text{and} \quad \bar{p} = \frac{d}{ds} \log \frac{1}{\sqrt{\bar{q}}}.$$

Putting (3, 1) and (3, 2) we obtain the equations of plane curves

$$(3, 3) \qquad \left\{ \begin{aligned} & \frac{\delta^3 x^h}{ds^3} + \bar{p} \frac{\delta^2 x^h}{ds^2} + \bar{q} \frac{dx^h}{ds} - \sum_P A^P H^P_{ij} g^{ih} \frac{dx^k}{ds} \\ & \qquad - \frac{\bar{p} \sum_P (A^P)^2 + \sum_P A^P \frac{dA^P}{ds}}{\bar{q} + \sum_P (A^P)^2} \cdot \frac{\delta^2 x^h}{ds^2} + \sum_P (A^P)^2 \frac{dx^h}{ds} = 0, \\ & H^P_{ijk} \frac{dx^i}{ds} \frac{dx^j}{ds} \frac{dx^k}{ds} + 3 H^P_{ij} \frac{\delta^2 x^i}{ds^2} \frac{dx^j}{ds} + \sum_Q L_{PQ|k} A^Q \frac{dx^k}{ds} \\ & \qquad + \left(\bar{p} - \frac{\bar{p} (\sum_P (A^P)^2) + \sum_P A^P \frac{dA^P}{ds}}{\bar{q} + \sum_P (A^P)^2} \right) A^P = 0, \end{aligned} \right.$$

where we put $A^P = H^P_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds}$.

Hence we have that *if V_n is totally geodesic subspace in V_m, then the plane curve in V_m is a plane curve in V_n and, at the same time, is a Darboux lime in V_n in V_m.*

We have also by (3, 3) that *when a plane curve in V_m is contained in V_m, if it is a plane curve in V_n, then the following equation must be satisfied*

$$(3, 4) \qquad \sum_P A^P H^P_{ij} g^{ih} \frac{dx^k}{ds} + \frac{\bar{p} \sum_P (A^P)^2 + \sum_P A^P (A^P)'}{\bar{q} + \sum_P (A^P)^2} \frac{\delta^2 x^h}{ds^2} - \sum_P (A^P)^2 \frac{dx^h}{ds} = 0.$$

§ **4.** In this paragraph we consider the case $m = n+1$.
The equations of plane curves (3, 3) are written as

$$(4,1)\quad \begin{cases} \dfrac{\delta^3 x^h}{ds^3}+\bar{p}\dfrac{\delta^2 x^h}{ds^2}+\bar{q}\dfrac{dx^h}{ds}-AH_{ik}g^{hi}\dfrac{dx^k}{ds}-\dfrac{\bar{p}A^2+A\dfrac{dA}{ds}}{\bar{q}+A^2}\dfrac{\delta^2 x^h}{ds^2}+A^2\dfrac{dx^h}{ds}=0\,, \\ H_{ijk}\dfrac{dx^i}{ds}\dfrac{dx^j}{ds}\dfrac{dx^k}{ds}+3H_{ij}\dfrac{\delta^2 x^i}{ds^2}\dfrac{dx^j}{ds}+\left(\bar{p}-\dfrac{\bar{p}A^2+A\dfrac{dA}{ds}}{\bar{q}+A^2}\right)A=0\,. \end{cases}$$

Hence we have: *the necessary and sufficient condition that a plane curve in V_{n+1} be a plane curve in V_n is either this curve be an asymptotic curve or satisfy the relation*

$$(4,2)\qquad H_{ik}g^{ih}\frac{dx^k}{ds}+\frac{\bar{p}A+\dfrac{dA}{ds}}{\bar{q}+A^2}\frac{\delta^2 x^h}{ds^2}-A\frac{dx^h}{ds}=0\,.$$

In the case where V_n is perfectly totally umbilic hypersurface in V_{n+1}, that is $H_{ij}=\rho g_{ij}$ ($\rho=$const.), the second member of the equations (4, 1) has the following form

$$\bar{p}=\frac{\bar{p}\rho^2}{q+\rho^2}\,,$$

and moreover $H_{ijk}=0$.

Hence we have $\bar{p}=0$, that is, $\bar{q}=$const..

By K. Yano's theorem[4] the geodesic circle in V_n is, at the same time, geodesic circle in V_{n+1}.

Hence we have *if a plane curve of V_{n+1} is contained in a perfectly totally umbilic hypersurface, then it is a geodesic circle of V_{n+1}, and at the same time, of this hypersurface.*

By (4, 1) we easily see a sufficient condition that every plane curve in V_{n+1} is a plane curve in V_n when it is contained in V_n, is that V_n is a totally geodesic hypersuface in V_{n+1}.

This conclusion, however, is not necessary. For example, when we consider a perfectly totally umbilic hypersurface in V_{n+1}, the equations (4, 2) have the following form

$$\frac{\bar{p}A}{\bar{q}+\rho^2}=0\,.$$

Therefore we must have $q=$const..

However we must have seen that if a plane curve in V_{n+1} is contained in a perfectly umbilic hypersurface, it must satisfy $q=$const..

Hence we have: *a sufficient condition that every plane curve in V_{n+1} be plane curve in V_n when it is contained in V_n, is either V_n be totally geodesic or perfectly totally umbilic hypersurface in V_{n+1}.*

4) K. Yano; Concircular Geometry III. (Proc. of Imp. Acad. Japan. 16 1940 p.p. 447.)