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CHARACTERIZATION OF CERTAIN ADDITIVE SEMIGROUPS BY DISTRIBUTIVE MULTIPLICATIONS

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§ 1. Introduction.

Let S_+ be a semigroup with addition $+$ defined in a set S. We introduce another operation "multiplication" into the same set S, which is symbolized as $S_\times$, not necessarily associative, such that

$$(1) \qquad x(a+b)=xa+xb$$
$$(2) \qquad (x+y)a=xa+ya \qquad \text{for every } x, y, a, b \in S.$$

Then we say that S_+ has a multiplication (multiplicative system) $S_\times$. In the previous paper [1] we proved the two theorems:

Theorem (A). A right singular or left singular[1] semigroup S_+ has all arbitrary multiplications.

Theorem (B). Let S_+ be a semigroup defined as $x+y=0$ for all $x, y \in S$. S_+ has a multiplication $S_\times$ if and only if $S_\times$ has 0 as the two-sided zero.

In the present note, we shall prove that Theorem (A) characterizes a right or left singular semigroup S_+, but Theorem (B) does not characterize the semigroup S_+ defined as $x+y=0$, and we shall have the following Theorems 1, 2 under weaker conditions. Hereafter, by "a semigroup S has a multiplication (multiplicative system) $S_\times$" we mean "(1) holds i. e. the multiplication $S_\times$ is distributive to the addition S_+ with respect to the only one-side." Of course we assume that S is non-trivial, i. e. it contains two elements at least.

Theorem 1. *If a semigroup S_+ has all arbitrary multiplications, then S_+ is a right or left singular semigroup.*

Theorem 2. *If a semigroup S_+ has all multiplicative systems $S_\times$ with a right zero 0 and has nothing but such multiplications, then S_+ is either a semigroup with $x+y=0$ for all $x, y \in S$ or a group of order 2.*

§ 2. Proof of Theorem 1.

1. *$x_0 \neq y_0$ implies either $x_0+y_0=x_0$ or $x_0+y_0=y_0$.*

Proof. We shall prove that $x_0 \neq y_0$ and $x_0+y_0 \neq x_0$ imply $x_0+y_0=y_0$. Let

1) S is called a right singular or left singular if S is defined as $x+y=y$ or $x+y=x$ for any x, y respectively.

$z_0 = x_0 + y_0 \neq x_0$, and corresponding to x_0, y_0 and z_0, consider a multiplication which satisfies

$$x_0^2 = x_0, \quad x_0 y_0 = x_0 z_0 = y_0 .$$

Then we get $x_0 + y_0 = x_0^2 + x_0 y_0 = x_0(x_0 + y_0) = x_0 z_0 = y_0$.

2. *If $x_0 \neq y_0$ and $x_0 + y_0 = y_0$ for some x_0, y_0, then $x+y=y$ for all x, y.*

Proof. Take any x and y. Consider a multiplication satisfying

$$x_0^2 = x \text{ and } x_0 y_0 = y ,$$

then we get

$$x + y = x_0^2 + x_0 y_0 = x_0(x_0 + y_0) = x_0 y_0 = y .$$

Similarly we obtain

3. *If $x_0 \neq y_0$ and $x_0 + y_0 = x_0$ for some x_0, y_0, then $x+y=x$ for all x, y.*

Gathering together 1, 2, and 3, it has been proved that non-trivial S_+ is right or left singular.

§ 3. Proof of Theorem 2.

1. $0+0=0$.

Proof. For a special multiplication $S_\times$ with two-sided zero 0,

$$0+0 = 0^2 + 0^2 = 0(0+0) = 0 .$$

2. *Either $0+x=x$ for all x, or $0+x=0$ for all x.*

Proof. We shall prove that $0+x=0$ for all x if $0+x_0 \neq x_0$ for some x_0. From $0+x_0 \neq x_0$, we see easily $x_0 \neq 0$ by **1**. Let x be any element of S, let $u_0 = 0 + x_0$, and consider a multiplication satisfying

$$x_0^2 = x, \quad x_0 u_0 = 0, \quad x0 = 0 \text{ for all } x .$$

Then we have $0+x = x_0 0 + x_0^2 = x_0(0+x_0) = x_0 u_0 = 0$.
Hence $0+x_0 \neq x_0$ implies $0+x=0$ for all x.

Similarly we can prove

3. *Either $x+0=x$ for all x, or $x+0=0$ for all x.*

4. *Either $0+x=x+0=x$ for all x or $0+x=x+0=0$ for all x.*

Proof. Suppose both $0+x=x$ for all x and $x+0=0$ for all x. Then

$$x+y = x+(0+y) = (x+0)+y = 0+y = y$$

for every x, y, which concludes that S_+ is a right singular semigroup. This contradicts the assumption of this theorem because of Theorem 1. Similarly it is false that both $0+x=0$ for all x and $x+0=x$ for all x.

From **5** to **7** we assume that S contains three elements at least.

5. *$x_0 \neq 0$, $y_0 \neq 0$, $x_0 \neq y_0$ imply $x_0 + y_0 \neq x_0$ and $x_0 + y_0 \neq y_0$.*

Proof. Suppose $x_0 + y_0 = x_0$. In case $0+x=x+0=x$ for all x, considering a multiplication which satisfies for x_0, y_0

(3) $$x_0^2 = 0,\quad x_0 y_0 = x_0,\quad z0 = 0 \text{ for all } z.$$

we get $$0 + x_0 = x_0^2 + x_0 y_0 = x_0(x_0 + y_0) = x_0^2 = 0,$$

which contradicts the above assumption.
In case $0+x=x+0=0$ for all x, a multiplication satisfying

(4) $$x_0^2 = x_0,\quad x_0 y_0 = 0,\quad z0 = 0 \text{ for all } z$$

leads to $$x_0 + 0 = x_0^2 + x_0 y_0 = x_0(x_0 + y_0) = x_0^2 = x_0,$$

contradicting the assumption. Hence it has been proved that $x_0+y_0 \neq x_0$. Similarly we can prove $x_0+y_0 \neq y_0$. Under the supposition of $x_0+y_0=y_0$ we may use (4) in case $0+x=x+0=x$ for all x, (3) in case $0+x=x+0=0$ for all x.

6. $x_0 \neq 0$, $y_0 \neq 0$, $x_0 \neq y_0$ *imply* $x_0+y_0=0$.

Proof. Suppose $u_0 = x_0 + y_0 \neq 0$. We can consider a multiplication which fulfils

$$x_0^2 = x_0 y_0 = 0,\quad x_0 u_0 \neq 0,\quad z0 = 0 \text{ for all } z.$$

Possibility of such a multiplication follows from 5, i.e. $x_0+y_0 \neq x_0$ and $x_0+y_0 \neq y_0$.

Then $$0 \neq x_0 u_0 = x_0(x_0+y_0) = x_0^2 + x_0 y_0 = 0 + 0 = 0,$$

arriving at contradiction. Therefore $x_0+y_0=0$.

7. *If S_+ contains three elements at least, then S_+ is given as $x+y=0$ for all x, y.*

Proof. It is sufficient to prove only $0+x=x+0=0$ for all x. By the assumption, there are x_0, y_0 such that $x_0 \neq 0$, $y_0 \neq 0$, $x_0 \neq y_0$. For x, x_0, y_0, consider a multiplication fulfilling

$$x x_0 = 0,\quad x y_0 = x,\quad z0 = 0 \text{ for all } z.$$

Then $$0 + x = x x_0 + x y_0 = x(x_0 + y_0) = x0 = 0$$

$$x + 0 = x y_0 + x x_0 = x(y_0 + x_0) = x0 = 0 \quad \text{by 6.}$$

As consequence of 7, we have

8. *If $0+x=x+0=x$ for all x, then non-trivial S_+ is of order 2.*

Accordingly the type of S_+ is either a group of order 2

	0	a
0	0	a
a	a	0

or a semilattice

(5)

	0	a
0	0	a
a	a	a

9. *The addition S_+ of order 2 which satisfies the assumption of this theorem is nothing but a group of order 2.*

Proof. At first, the semilattice S_+ (5) of order 2 has a multiplication $S_\times$ defined as

$$\begin{array}{c|cc} & 0 & a \\ \hline 0 & a & a \\ a & a & a \end{array}$$

Because $x(y+z)=a$, $xy+xz=a+a=a$ whenever x, y, z are 0 or a. Hence the required S_+ is a group. Conversely if S_+ is a group of order 2, all the multiplications which S_+ has are proved to be

$$\begin{array}{c|cc} & 0 & a \\ \hline 0 & 0 & 0 \\ a & 0 & 0 \end{array} \qquad\qquad \begin{array}{c|cc} & 0 & a \\ \hline 0 & 0 & 0 \\ a & 0 & a \end{array}$$

by Theorem 1 of the previous paper [1].

References

[1] T. Tamura etc.: Distributive multiplications to semigroup operations, Jour. of Gakugei, Tokushima Univ., Vol VIII, 1957, 91-101.