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NOTES ON GENERAL ANALYSIS (VII)

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In this note, a theorem of complex valued functions is extended to the case of complex Banach spaces. Let E_1 and E_2 be complex Banach spaces.

Theorem. *Let the family of functions* $\{f(x)\}$ *from* E_1 *to* E_2 *satisfy following conditions*: (1) *each function* $f(x)$ *is analytic in* $||x||<1$ *in* E_1 *and is a one-to-one mapping to a domain* D_f *in* E_2 *and its inverse function* $f^{-1}(x)$ *is also analytic in* D_f, (2) $\{f(x)\}$ *are bounded, that is,* $||f(x)||\leqq M$, (3) *the norms of linear parts* $\{g_1(x)\}$ *of* $\{f^{-1}(x)\}$ *are bounded, that is,* $||g_1||\leqq K$, (4) $f(\theta)=\theta$, *then each domain* D_f *includes the sphere whose radius is constant.*

Proof. Since $f(\theta)=\theta$, we have

$$f^{-1}(x)=\sum_{n=1}^{\infty} g_n(x) \text{ and } f(x)=\sum_{n=1}^{\infty} f_n(x),$$

where $f_n(x)$ and $g_n(x)$ are homogeneous polynomials of degree n, for $n=1, 2, 3, \cdots$. Then,

$$\begin{aligned} x &= f^{-1}(f(x)) \\ &= \sum_{n=1}^{\infty} g_n(f(x)) \\ &= \sum_{n=1}^{\infty} g_1(f_n(x))+\sum_{n=2}^{\infty} g_n(f(x)) \\ &= g_1(f_1(x))+\sum_{n=2}^{\infty} g_1(f_n(x))+\sum_{n=2}^{\infty} g_n(f(x)). \end{aligned}$$

For an arbitrarily fixed x and an arbitrary complex number α, we have

$$\alpha x=\alpha g_1(f_1(x))+\sum_{n=2}^{\infty}\alpha^n g_1(f_n(x))+\sum_{n=2}^{\infty}\alpha^n g_n(f_1(x)+\sum_{m=2}^{\infty}\alpha^{m-1}f_m(x)).$$

Dividing each terms by α,

$$x=g_1(f_1(x))+\alpha\{\sum_{n=2}^{\infty}\alpha^{n-2}g_1(f_n(x))+\sum_{n=2}^{\infty}\alpha^{n-2}g_n(f_1(x)+\sum_{m=2}^{\infty}\alpha^{m-1}f_m(x))\}.$$

Put $\alpha=0$ and we have $x=g_1(f_1(x))$. Since x is arbitrary, $g_1(x)$ is an inverse function of $f_1(x)$. That is, $f_1(x)$ has the continuous inverse function $g_1(x)$. Since $||x||=||g_1(f_1(x))||\leqq||g_1||\cdot||f_1(x)||$, we have

$$||f_1(x)||\geqq\frac{1}{||g_1||}||x||\geqq\frac{1}{K}||x||,$$

from the assumption (3).

On the other hand, $f_n(x)=\dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(\alpha x)}{\alpha^{n+1}}d\alpha,\ n=1, 2, 3, \cdots$, where C is a circle with radius r satisfying $r\|x\|\leqq\delta<1$ and $r>1$.
From the assumption (2), $\|f(\alpha x)\|\leqq M$, when $\|\alpha x\|\leqq\delta$.

Thus we see that $$\|f_n(x)\|\leqq\frac{M}{r^n} \quad \text{for } n=1, 2, \cdots .$$

Then $$\|\sum_{n=2}^{\infty}f_n(x)\|\leqq\sum_{n=2}^{\infty}\frac{M}{r^n}=\frac{M}{r(r-1)}.$$

Taking a positive number δ_1 such that
$$0<\delta_1<\delta<1,\ r\delta_1=\delta \text{ and } KM\delta_1^2<\delta^3(\delta-\delta_1),$$
we have $$\frac{M}{r(r-1)}=\frac{M\delta_1}{\delta(\delta-\delta_1)}<\frac{\delta^2}{K}.$$

For an arbitrary y in $\|y\|\leqq\delta$, there exist x and α such that $y=\alpha x$, $|\alpha|\leqq r$, $\|x\|=\delta_1$ and $\|\alpha x\|\leqq\delta$.
$$\|\sum_{n=2}^{\infty}f_n(\alpha x)\|=|\alpha|^2\cdot\|\sum_{n=2}^{\infty}f_n(x)\alpha^{n-2}\|,$$
then $$\frac{\|\sum_{n=2}^{\infty}f_n(\alpha x)\|}{|\alpha|^2}=\|\sum_{n=2}^{\infty}f_n(x)\alpha^{n-2}\|.$$

Since $\sum_{n=2}^{\infty}f_n(x)\alpha^{n-2}$ is an analytic function of α, its norm takes the maximum on the boundary $|\alpha|=r$ and so we have
$$\frac{\|\sum_{n=2}^{\infty}f_n(\alpha x)\|}{|\alpha|^2}\leqq\frac{1}{r^2}\cdot\frac{M_1}{r(r-1)}.$$
Putting $\alpha x=y$, $\|\sum_{n=2}^{\infty}f_n(y)\|\leqq\dfrac{M}{r^2\cdot r(r-1)}|\alpha|^2\leqq\dfrac{M}{r(r-1)}\cdot\dfrac{\|y\|^2}{r^2\|x\|^2}=\dfrac{M}{r(r-1)\delta^2}\|y\|^2$.

Thus we see that
$$\begin{aligned}\|f(x)\|&\geqq\|f_1(x)\|-\|\sum_{n=2}^{\infty}f_n(x)\|\\&\geqq\frac{1}{K}\|x\|-\frac{M}{r(r-1)\delta^2}\|x\|^2,\end{aligned}$$
for $\|x\|\leqq\delta_1$. Letting $\|x\|=\delta_1$, we have
$$\begin{aligned}\|f(x)\|&\geqq\frac{\delta_1}{K}-\frac{M}{r(r-1)\delta^2}\cdot\delta_1^2\\&=\delta_1\left(\frac{1}{K}-\frac{M\delta_1}{r(r-1)\delta^2}\right),\end{aligned}$$

which is a positive number, since $0<\delta_1<1$ and $\dfrac{1}{K}>\dfrac{M}{r(r-1)\delta^2}>\dfrac{M\delta_1}{r(r-1)\delta^2}$. Put $\delta_1\left(\dfrac{1}{K}-\dfrac{M\delta_1}{r(r-1)\delta^2}\right)=\rho$, then we see that
$$D_f\supset U(\rho),$$
which is the sphere with radius ρ. This completes the proof.