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NOTES ON THE FUNCTIONS OF TWO COMPLEX VARIABLES

By

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In this note,*) we want to try some extension of the theorem of Hartogs in connection with the problem which was presented by Professor Hukuhara long ago.

Hartogs' Theorem: If a function $f(x, y)$ is regular in each variable, then it is regular with respect to (x, y).

Professor Hukuhara asked if a function $f(x, y)$ was yet regular or not when $f(x, y)$ was regular with respect to y for only x_n which converged to an inner point x_0.

He extended Osgood's Theorem as follows[1]: *If a function $f(x, y)$ is regular with respect to x for a fixed y and regular with respect to y for x_n which tends to x_0 in D and moreover bounded, then $f(x, y)$ is regular.*

Let x and y be complex variables lying in the domain D and D' respectively. (D, D') shoes the multiple domain of D and D', which are simply connected.

Theorem 1. *If a complex valued function $f(x, y)$ defined in (D, D') is regular with respect to x in D for any fixed y in D' and regular with respect to y in D' for fixed x_m $(m=1, 2, 3, \cdots\cdots)$ which converges to x_0 in D', then there exist regular points dencely in (D, D').*

Proof. Let D_1 and D_1' be arbitrary bounded domains whose closures lying in D and D' respectively. Put

$$M(y) = \underset{x \in D_1}{\text{Max.}} |f(x, y)|, \qquad \text{for a fixed } y,$$

and

$$S_k = \underset{y}{E}[M(y) \leqq K],$$

that is, S_k is a set of points y satisfying $M(y) \leqq K$. Suppose that a sequence of points $\{y_n\}$ converging to y_0 are included in S_k. Put $f(x, y_n) = f_n(x)$, then $f_n(x)$ is regular in D_1 and satisfies $|f_n(x)| \leqq K$. This shoes that $\{f_n(x)\}$ is a normal familly in D_1. On the other hand, for a fixed x_m, $f(x_m, y)$ is regular with respect to y. Then we see that $\lim_{n\to\infty} f(x_m, y_n) = f(x_m, y_0)$, that is, $\lim_{n\to\infty} f_n(x_m) = f(x_m, y_0)$, where $m=1, 2, 3, \cdots\cdots$.

Appealing to the theorem of Vitali, $f_n(x)$ converges uniformly to $f_0(x)$, which is regular in D_1. Since y_0 is an inner point of D_1, $f(x, y_0)$ is regular with respect to x. Moreover, $f_0(x_m) = f(x_m, y_0)$ for $m=1, 2, 3, \cdots\cdots$. Since x_m tends to x_0 which is an inner point of D_1, we see that $f(x) \equiv f(x, y_0)$. Since $|f(x, y_n)| \leqq K$,

$$|f(x, y_0)| = \lim_{n\to\infty} |f_n(x)| \leqq K \quad \text{in} \quad D_1 .$$

The regularity of $f(x, y_0)$ on D shows that

$$|f(x, y_0)| \leqq K \quad \text{on} \quad D_1 .$$

That is, y_0 is included in S_k and we see that S_k is a closed set. Now, we must show that at least a set S_k of $\{S_k\}$ include a circle. Let an any one of $\{S_k\}$ does not include a circle. Then there exists a closed circle C_1 in D_1 such that the intersection of C_1 and S_1 is the null set. If such circle C does not exist, then there exists at least a point of S_1 in an arbitrary neighbourhood of y, which is an arbitrary point in D_1. y must lie in S_1, because S_1 is a closed set. That is, $S_1 \supset D_1$ and we see that clearly S_1 include a closed circle, contradicting to the assumption that S_1 does not include a closed circle. Thus, we see that there exist a closed circle C_1 such that $C_1 \cdot S_1 = 0$. On the same way, we have a closed circle C_2 such that $C_1 \supset C_2$, $C_2 \cdot S_2 = 0$, and so on, we have a sequence of closed circle $\{C_k\}$ such that $C_1 \supset C_2 \supset C_3 \supset \cdots \supset C_k \supset \cdots$, $C_k \cdot S_k = 0$. Clearly, $\prod_1^\infty C_k \neq 0$. Then, for an arbitrary point y in $\prod_1^\infty C_k$, we have $M(y) = +\infty$, because, if $M(y) < K$, $y \in S_k$ contradicting to the fact that $y \in C_k$. Clearly, y is an inner point of D_1 and then $M(y)$ must be finite which contradicts to $M(y) = +\infty$. Thus we see that there exists at least S_k which includes a closed circle C. Then,

$$|f(x, y)| \leqq K ,$$

when (x, y) lies on (D_1, C).

Appealing to the extended theorem of Osgood, we see that $f(x, y)$ is regular in (D_1, C). D_1 is an arbitrary closed bounded domain including $\{x_n\}$ in D, and so $f(x, y)$ is regular in (D, C)[3]. D_1' is an arbitrary bounded closed domain in D', we can select D' as a closure of a neighbourhood of an arbitrary point in D_1'. Thus, we see that $f(x, y)$ is regular almost everywhere in (D, D').

Theorem 2. *If $f(x, y)$ defined in (D, D') satisfies following conditions 1) $f(x, y)$ is regular with respect to x in D for an arbitrarily fixed y in D', 2) for a sequence $\{x_n\}$ in D converging an inner point x_0 in D, $f(x_n, y)$ is regular with respect to y in D', then D' is divided into at most denumerable simply connected domains $\{E_m\}$ such that $f(x, y)$ is regular in (D, E_m) and $E_i \cdot E_j$ is a set of one dimension.*

Proof. Appealing to Theorem 1, we see that there exists a circle C in D' such that $f(x, y)$ is regular in (D, C). Extending $f(x, y)$ analytically from the domain (D, C), we have (D, E), where E is a domain in D' and $f(x, y)$ is regular in (D, E). Let K be an arbitrary simple closed Jordan curve having the length.

Put
$$F(x, y) = \frac{1}{2\pi i} \int_K \frac{f(x, \zeta)}{\zeta - y} d\zeta ,$$

where y lies in the inside of K. Let G be a domain which lies inside of K.

Then $F(x, y)$ is regular in (D, G). On the other hand,

$$F(x_n, y) = \frac{1}{2\pi i}\int_K \frac{f(x_n, \zeta)}{\zeta - y} d\zeta = f(x_n, y) ,$$

by the assumption 2). Thus we have $F(x_n, y) = f(x_n, y)$, for $n=1, 2, \cdots\cdots$, for fixed y arbitrarily in E. Then we see that $f(x, y) \equiv F(x, y)$ in (D, E) and $f(x, y)$ is extended analytically in (D, G). That is, there does not exist any singular point of $f(x, y)$ for y in E. This shoes that E is the simply connected set. Generally, such a set E are denumerable at most in D' and moreover clearly $E_i \cdot E_j$ is a set of one dimension.

References

1) Proof: Let x_n tends to x_0 in D and T be a circle such that $|x-x_0| \leqq r$ which is included in D and C be the boundary of T. Without losing generality, we may think that $\{x_n\}$ lie in T. For an arbitrary y in D', we have

$$f(x, y) = \sum_{n=0}^{\infty} f_n(y)(x-x_0)^n ,$$

where

$$f_n(y) = \frac{1}{2\pi i}\int_c \frac{f(x, y)}{(x-x_0)^{n+1}} dx , \quad \text{for} \quad n=0, 1, 2, \cdots\cdots .$$

Since $f(x, y)$ is bounded in (D, D'), there exists a positive number M such that $|f(x, y)| \leqq M$ in (D, D'). Then we have

$$|f_n(y)| \leqq \left| \frac{1}{2\pi i}\int_c \frac{f(x, y)}{(x-x_0)^{n+1}} dx \right|$$
$$\leqq \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(x, y)|}{r^{n+1}} r d\theta$$
$$\leqq \frac{M}{r^n} ,$$

where $x = x_0 + re^{i\theta}$ and $dx = ire^{i\theta}d\theta$. Let $x = x_m$,

$$f(x_m, y) = f_0(y) + \sum_{n=0}^{\infty} f_n(y)(x_m - x_0)^n .$$

Then

$$|f(x_m, y) - f_0(y)| = |\sum_{n=1}^{\infty} f_n(y)(x_m - x_0)^n|$$
$$\leqq M \sum_1^{\infty} \left(\frac{|x_m - x_0|}{r}\right)^n$$
$$= \frac{M|x_m - x_0|}{r - |x_m - x_0|} .$$

Let x_m tend to x_0, then we see that $f(x_m, y)$ converges uniformly to $f_0(y)$. Since $f(x_m, y)$ is regular with respect to y, $f_0(y)$ is also regular. By the mathematical induction we see that the regular function

$$\frac{1}{(x_m - x_0)^n}\{f(x_m, y) - f_0(y) - f_1(y)(x_m - x_0) - \cdots - f_{n-1}(y)(x_m - x_0)^{n-1}\}$$

converges uniformly to $f_n(y)$ and so $f_n(y)$ is regular. Thus we see that

$$f(x, y) = \sum_{n=0}^{\infty} f_n(y)(x - x_0)^n$$

converges uniformly in (T, D') and so $f(x, y)$ is regular in (T, D'). By the method of the analytical continuation, $f(x, y)$ becomes regular in (D, D').

2) S. Bochner and W. T. Martin: Several complex variables.

3) D_1 is extended to D by the usual way, since $f(x, y)$ is regular with respect to x in D for an arbitrary fixed y in D'.

*) I expect the advice of the general public, since I don't know much of this field.

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COMMUTATIVE NONPOTENT ARCHIMEDEAN SEMIGROUP WITH CANCELATION LOW I

By

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We established in the paper [1] that a commutative semigroup is decomposed into the class sum of unipotent or nonpotent semigroups. In the present paper we shall investigate the structure of a commutative nonpotent archimedean semigroup admitting cancelation law. We shall see that such a semigroup will be determined by the additive semigroup of non-negative integers and the indexed group.

If, for any elements a and b of a commutative semigroup S, there exist a positive integer m and an element c of S such that

$$a^m = bc\,,$$

then S is called archimedean. By "nonpotent" we mean "without idempotent".

§1. Unique Factorization.

Lemma 1. *Let S be a commutative nonpotent archimedean semigroup. Then*

$$\bigcap_{n=1}^{\infty} a^n S = \emptyset \quad \textit{for every} \quad a \in S\,.$$

Proof. Let $D = \bigcap_{n=1}^{\infty} a^n S$. Suppose that D is not empty. Then we shall prove the following (1.1), (1.2), (1.3), and (1.4) step by step.

(1.1) *D is an ideal of S.*

Since any y in D is expressed as $y = a^n t$ where $t \in S$, we get

$$yx = (a^n t)x = a^n(tx) \in a^n S \quad \text{for all } n$$

whence $yx \in D$ and so $Dx \subset D$.

(1.2) *$D \subset zS$ for any $z \in S$.*

Since S is archimedean, there are $m > 0$ and $x \in S$ such that $a^m = zx$. Then any $d \in D$ is expressed as $d = a^m y = (zx)y = z(xy) \in zS$. Therefore $D \subset zS$.

(1.3) *D is the least ideal of S.*

Let I be any ideal of S, that is, $IS \subset I$. For $y \in I$, $D \subset yS \subset IS \subset I$ by (1.2).

(1.4) *$D = dD$ for any $d \in D$.*

Using (1.1), $(dD)S = d(DS) \subset dD$, and so dD is an ideal of S. According to

(1.3), we have $D \subset dD$, while, of course, $dD \subset D$. At last we have $D = dD$.

Since D is commutative, it follows that D is a group. Consequently D contains an idempotent, contradicting with the assumption. Thus the proof of the lemma has been finished.

Denote $T_0 = S - aS$, $T_i = a^i S - a^{i+1} S$ $(i = 1, 2, \cdots)$. Then T_i $(i = 1, 2, \cdots)$ are not empty. Because, if T_i is emtpy, we get

$$a^i S = a^{i+1} S = \cdots$$

which leads to $D \neq \emptyset$, contradicting with Lemma 1.

Corollary 1. $S = \sum_{i=0}^{\infty} T_i$, $T_i \neq \emptyset$, $T_i \cap T_j = \emptyset$ $(i \neq j)$.

Lemma 2. *In a commutative archimedean semigroup S, S is nonpotent if and only if $a \neq ab$ for every $a, b \in S$.*

Proof. Suppose $a = ab$ in spite of nonpotentness of S. Then we have

$$a = ab = ab^2 = \cdots = ab^n = \cdots$$

whence $\bigcap_{n=1}^{\infty} b^n S \neq \emptyset$, contradicting with Lemma 1. Thus we see that if S is nonpotent, $a \neq ab$ for any $a, b \in S$. The converse is clear: if S has an idempotent e, then $e = ee$. q. e. d.

Hereafter a denotes a fixed element of a commutative nonpotent archimedean semigroup S with cancelation.

According to Corollary 1, for an element x of aS, a positive integer n is uniquely determined such that

$$x \in T_n \quad \text{that is,} \quad x = a^n z .$$

Further we can see that z lies in $S - aS$. Indeed, if $z = au$, then $x = a^{n+1} u \in T_{n+1}$ which conflicts with $x \in T_n$ and $T_n \cap T_{n+1} = \emptyset$. Uniqueness of z is assured by the cancelation law.

Let us introduce a symbol a°:

$$a^\circ b \quad \text{means} \quad b,$$

in words, a° is not an element, but is considered as a symbolical operation. Then, if $x \in S - aS$, x is expressed as $x = a^\circ x$. We can summarize the above description as follows.

Theorem 1. *An element x of S determines uniquely a non-negative integer n and an element z of $S - aS$ such that $x = a^n z$.*

§ 2. Homomorphism to a Group.

Now let us introduce a relation $x \sim y$ among all the elements of a commutative nonpotent archimedean semigroup with cancelation. Denote $x \sim y$ if there is a non-negative integer n such that either $x = a^n y$ or $y = a^n x$. This relation is an equivalence relation. Indeed $x \sim x$ since $x = a^\circ x$; the symmetric law is obvious. We shall prove only the transitive law in the four cases:

(2. 1) $x=a^n y$, $y=a^m z$, (2. 2) $x=a^n y$, $z=a^m y$,
(2. 3) $y=a^n x$, $y=a^m z$, (2. 4) $y=a^n x$, $z=a^m y$.

Then we have

in the case (2. 1) $x=a^{n+m}z$,

in the case (2. 2) $\begin{cases} x=a^{n-m}z & \text{if } n>m, \\ z=a^{m-n}x & \text{if } n<m, \\ x=z & \text{if } n=m, \end{cases}$

in the case (2. 3) $\begin{cases} z=a^{n-m}x & \text{if } n>m, \\ x=a^{m-n}z & \text{if } n<m, \\ x=z & \text{if } n=m, \end{cases}$

in the case (2. 4) $z=a^{n+m}x$.

Hence the transitive law holds. Further we see easily that $x\sim y$ implies $xu\sim yu$. Thus we get

Lemma 4. (2. 5) $a^n\sim a^m$. $(n, m=1, 2, \cdots)$

(2. 6) $x\sim a^n x$. $(n=1, 2, \cdots)$

(2. 7) *For any x, there is y such that $xy\sim a$.*

(2. 8) *If $x, y\in S-aS$, and $x\neq y$, then $x\not\sim y$.*

Proof. (2. 5), (2. 6), and (2. 8) are obvious by the definition of the equivalence relation; (2. 7) is led from archimedeaness as follows. For any x, there is y such that $xy=a^m\sim a$. q. e. d.

Now all the elements of S is classified by the relation $x\sim y$. S is the set union of S_α where we denote by S^* the set of all indices α. $S=\sum_{\alpha\in S^*} S_\alpha$ $S_\alpha\cap S_\beta=\emptyset$ $(\alpha\neq\beta)$.
In particular, denote by S_ε the class containing a:

$$S_\varepsilon=\{a^n;\ n=1,\ 2\ \cdots\}.$$

Since the relation is a congruence relation, $S_\alpha S_\beta\subset S_\gamma$ for some γ by which the product $\alpha\beta$ of elements α and β is defined as $\gamma=\alpha\beta$. By Lemma 4, we have

Theorem 2. *S^* is a group, and S is homomorphic onto S^*.*

§ 3. Linear Order in S_α.

We shall define an ordering between the elements of a class S_α as follows.

$$x>y \quad (x, y\in S_\alpha)$$

if and only if $x\neq y$ and there is a positive integer n such that $x=a^n y$ where a is the fixed element.

Lemma 5. (3. 1) $x\not> x$ (3. 2) *$x>y$ and $y>x$ are incompatible.* (3. 3) *$x>y$ and $y>z$ imply $x>z$.*

Proof. If $x>x$, then $x=a^n x$ for some n; if $x>y$ and $y>x$, then we have $x=a^m x$ for some m. These are impossible according to Lemma 2. Thus (3.1) and (3.2) have been proved. (3.3) is also obtained as follows:

$$x=a^n y, \text{ and } y=a^m z \text{ imply } x=a^{n+m}z.$$

Lemma 6. *Suppose that* $x>y$ *or* $x=a^n y$, $n\geqq 1$, $x, y\in S_\alpha$. *Then* $x\geqq u\geqq y$ *implies* $u=a^i y$ $(0\leqq i\leqq n)$.

Proof. $u=a^k y$, and $x=a^l u$ (for certain $k, l\geqq 0$) follow from $u\geqq y$ and $x\geqq u$ respectively; and so $x=a^{k+l}y=a^n y$. By Theorem 1, we have $k+l=n$. Hence $0\leqq k\leqq n$, q. e. d.

Consequently the interval between x and y is composed of $x_i=a^i y$ $(i=0, 1, \cdots, n)$ such that

$$x=a^n y>a^{n-1}y>\cdots>ay>y.$$

Lemma 7. S_α *satisfies the descending chain condition, that is, a sequence* $x_1>x_2>\cdots>x_n>\cdots$ *ceases at finite term.*

Proof. Suppose that there is an infinite sequence.

$$x_1>x_2>\cdots>x_n>\cdots$$

where $x_i=a^{m_i}x_{i+1}$, $(i=1, 2, \cdots, n, \cdots)$ and $m_i>0$. Letting $k_n=m_1+m_2+\cdots+m_n$, $k_1<k_2<\cdots<k_n<\cdots$ and $x_1=a^{k_1}x_2=a^{k_2}x_3=\cdots=a^{k_n}x_{n+1}=\cdots$ which arrives at $x_1\in\bigcap_{i=1}^{\infty}a^{k_i}S\neq\emptyset$ contradicting with Lemma 1. q. e. d.

According to Lemmas 6 and 7, we see that there is a minimal element in S_α. Denote $T_1=S-aS$.

Lemma 8. *A minimal element of* S_α *lies in* T_1, *and conversely an element of* T_1 *is minimal in certain* S_α.

Proof. If a minimal element z of S_α belongs to aS, then $z=au$, $u\in S$, where $au\sim u$ by Lemma 4, and hence $u\in S_\alpha$, $u<z$. This contradicts with the fact that z is minimal in S_α. Therefore $z\bar{\in} aS$. Conversely if $z\in S-aS$ and $z\in S_\alpha$; then there is no $u<z$.

By the definition of the relation $x\sim y$ and the ordering $x>y$, $T_1\cap S_\alpha$ consists of only one element denoted by x_α.

Theorem 3. *Each* S_α *is a linearly ordered set with respect to the ordering* $x>y$, *and any element* x *of* S_α *is expressed as* $x=a^n x_\alpha$ *where* $n\geqq 0$, *and* x_α *is a unique element of* T_1 *contained in* S_α.

§ 4. Construction.

Since S is homomorphic onto S^* by Theorem 2, $x_\alpha\in S_\alpha\cap T_1$ and $x_\beta\in S_\beta\cap T_1$ determine $\gamma\in S^*$ and a non-negative integer n such that $x_\alpha x_\beta=a^n x_\gamma$ where $x_\gamma\in S_\gamma\cap T_1$. This n is called the index of a pair of x_α and x_β, which is denoted by $n=I(\alpha, \beta)$. Of course $I(\alpha, \beta)=I(\beta, \alpha)\geqq 0$.

Let $(x_\alpha x_\beta)x_\gamma=x_\alpha(x_\beta x_\gamma)\in S_\pi$ and let $I(\alpha, \beta)=n$, $I(\alpha\beta, \gamma)=p$, $I(\alpha, \beta\gamma)=q$, $I(\beta, \gamma)=m$. Then $(x_\alpha x_\beta)x_\gamma=a^{n+p}x_\pi$, $x_\alpha(x_\beta x_\gamma)=a^{q+m}x_\pi$, so that we have $n+p=q+m$ by Theorem 1, or $$I(\alpha, \beta)+I(\alpha\beta, \gamma)=I(\alpha, \beta\gamma)+I(\beta, \gamma) .$$

Since the minimal element of $S_\varepsilon=\{a^i ; i=1, 2, \cdots\}$ is a, we get $I(\varepsilon, \varepsilon)=1$. Because of archimedaness, there is $m+1>1$ such that $x_\alpha^{m+1}=x_\alpha^m x_\alpha \in aS$, therefore $I(\alpha^m, \alpha)>0$ for some $m>0$. Thus a group S^* with an index is determined from S. The group S^* with an index is called "the fundamental group" of S.

Conversely, consider an abstract commutative group G and a non-negative integer-valued function $I(x, y)$ defined on all the pairs of elements of G satisfying the following conditions:

(4.1) $I(x, y)=I(y, x)$ for any $x, y \in G$.

(4.2) $I(x, y)+I(xy, z)=I(x, yz)+I(y, z)$ for any $x, y, z \in G$.

(4.3) For any $x \in G$, there is $m>0$ (depending on x) such that $I(x^m, x)>0$.

(4.4) $I(e, e)=1$ where e is an identity of G.

This I is called "index" again, and G with I is called "an indexed group"

Lemma 9. *$I(e, x)=I(e, e)=1$ for all $x \in G$.*

Proof. Setting x, y, z as e, e, x respectively in (4.2),

$$I(e, e)+I(e, x)=I(e, x)+I(e, x)$$

from which $I(e, x)=I(e, e)$ is derived.

Theorem 4. *For a commutative group G with an index I satisfying the conditions* (4.1), (4.2), (4.3), *and* (4.4), *there is a commutative nonpotent archimedean semigroup S' with cancelation law, the fundamental group of which is isomorphic to the indexed group G.*

Remark. We say that G_1 with I_1 is isomorphic to G_2 with I_2 if the isomorphism f of a group G_1 to G_2 satisfies $I_1(x, y)=I_2(f(x), f(y))$.

Proof. Consider the set S' of all ordered pairs (n, x) of non-negative integer and an element of G: $S'=\{(n, x) ; n=0, 1, 2, \cdots, x \in G\}$. Equality of elements of S' is defined as

$$(n_1, x_1)=(n_2, x_2) \text{ if and only if } n_1=n_2,\ x_1=x_2 ;$$

the product of (n, x) and (m, y) is defined as

$$(n, x)(m, y)=(k, z)$$

where $k=n+m+I(x, y)$, $z=xy$ in G.

S' is a semigroup, for

$$\{(n, x)(m, y)\}(l, z)=(n+m+I(x, y),\ xy)(l, z)$$
$$=(n+m+l+I(x, y)+I(xy, z),\ (xy)z),$$
$$(n, x)(\{m, y)(l, z)\}=(n, x)(m+l+I(y, z),\ yz)$$
$$=(n+m+l+I(x, yz)+I(y, z),\ x(yz)).$$

By the condition (4.2), we obtain

$$\{(n, x)(m, y)\}(l, z)=(n, x)\{(m, y)(l, z)\}.$$

It goes without saying that S' is commutative.

Let us prove that S' is nonpotent. Suppose that there is an idempotent (n, x), $(n, x)(n, x)=(2n+I(x, x), x^2)=(n, x)$. From $x^2=x$, we have $x=e$; from $2n+I(e, e)=n$, we have $n+I(e, e)=0$. This is impossible by (4.4). Hence S' is nonpotent.

Proof of Archimedeaness. We shall show that for (n, x) and (m, y), there are $p>0$ and (l, u) such that $(n, x)^p=(m, y)(l, u)$.

i) In the case $n\geqq 1$. Since $(n, x)=(0, e)(n-1, x)$, we may show the existence of p and (k, z) such that $(0, e)^p=(m, y)(k, z)$. Choose p such that $p-1>m+I(y, y^{-1})$ and let $k=p-1-m-I(y, y^{-1})$, and let $z=y^{-1}$. Then we get $(m, y)(k, z)=(m+k+I(y, y^{-1}), e)=(p-1, e)$, while $(0, e)^p=(I(e^{p-1}, e)+\cdots+I(e, e), e^p)=(p-1, e)$. Accordingly we have $(0, e)^p=(m, y)(k, z)$. At last $(n, x)^p=(0, e)^p(n-1, x)^p=(m, y)(k, z)(n-1, x)^p$. Hence we may adopt $(k, z)(n-1, x)^p$ as (l, u).

ii) In the case $n=0$. Due to the condition (4.3), there is $m>0$: $I(x^m, x)>0$. Choose q such that $q\geqq m$, then $(0, x)^q=(s, x^q)$, for some $s\geqq 1$. For (s, x^q), we find p and (k, z) for (m, y) such that

$$(s, x^q)^p=(m, y)(k, z)$$

and hence $(0, x)^{qp}=(m, y)(k, z)$.

Proof of Cancelation. From $(n, x)(m, y)=(n, x)(k, z)$ or $(n+m+I(x, y), xy)=(n+k+I(x, z), xz)$, we get $xy=xz$, hence $y=z$; further from $n+m+I(x, y)=n+k+I(x, y)$, we have $m=k$. Thus it has been proved that $(n, x)(m, y)=(n, x)(k, z)$ implies $(n, y)=(k, z)$.

Consider the mappping $(n, x)\to x$. From the definition of multiplication in S', it follows that S' is homomorphic onto G under the mapping. Let us consider the relation with respect to $(0, e)$, which is defined at the beginning of §2. Then there is $n\geqq 0$ such that $(k, x)=(0, e)^n(l, y)$, if and only if $k\geqq l+1$ and $x=y$. Accordingly we have $(k, x)\sim(l, y)$ if and only if $x=y$, so that S'^* corresponds to G one to one. Further,

$$T_0=S'-(0, e)\cdot S'=\{(0, x)\ ;\ x\in G\}$$

and we have $(0, x)(0, y)=(I(x, y), xy)=(0, e)^{I(x,y)}(0, xy)$ from which we see that the fundamental group S'^* is isomorphic to the given indexed group G.

The following theorem is clear.

Theorem 5. *Let S^* be the fundamental group of a commutative non potent archimedean semigroup S with cancelation. Suppose that there is given an indexed group G which is isomorphic to S^*. If we construct the semigroup S' from G by the method of Theorem 4, then S is isomorphic to S'.*

Proof. S is isomorphic to S' under the mapping $a^n x\to(n, x)$.

Remark. In the present paper, we leave the following problems unsolved.

(1) what is the relation between the fundamental group as to $a\in S$ and the fundamental group as to $b\in S$?

(2) Under what condition, is S_1' constructed from G_1 with I_1 isomorphic to S_2' from G_2 with I_2?

These problems will be discussed in the continued paper II.

References

[1] T. Tamura and N. Kimura: On decompositions of a commutative semigroup, Kōdai Math. Sem. Rep., No. 4. Dec. 1954, 109–112.

Remark

In this paper, the notation $A \subseteq B$ means that A is a proper subset of B or $A=B$,

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EINIGE ERWEITERUNG DES PÓLYASCHE IRRFAHRTPROBLEM[1)]

Von

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Es seien $z=0$, $z=1$, zwei parallele ebene ähnliche Straßennetze, die durch Elevator in jedem Kreuze—Knotenpunkte, d.h. Punkte mit ganzzahligen Koordinaten—vertikal kombiniert werden, so daß der im Straßennetze herumwandernde Punkt, welcher zur Zeit $t=0$ im Anfangspunkt des Koordinatensystems seine Irrfahrt beginnt mit der Geschwindigkeit 1, in jedem Zeit-punkt $t=1, 2, \cdots$, an gewißen Knotenpunkt erreicht und dort mit Wahrscheinlichkeit 1/5 für eine der 5 Richtungen, deren vier horizontale Kreuzen aber eine vertikales Auf- oder Abgehen sind, aufs Geratewohl sich entscheiden läßt. Zuerst behaupte ich, daß *die Wahrscheinlichkeit dafür, daß zur Zeit $t=m$ der wandernde Punkt im Knotenpunkt (x, y, z) sich befindet, folgendermaßen dargestellt werden soll* :

$$(1) \qquad P_m(x, y, z)=\frac{1}{(2\pi)^3}\iiint_0^{2\pi}\left(\frac{2\cos\varphi+2\cos\psi+e^{i\theta}}{5}\right)^m \exp\{-ix\varphi-iy\psi\}\sum_\nu \exp\{-iz_\nu\theta\}\,d\varphi d\psi d\theta\,,$$

wobei $z=0$ oder 1 ist, während jedes $z_\nu \equiv z$ (mod 2) und die Summe $\sum_\nu$ mit z beginnt und durch $z_l=m-|x|-|y|$ vollendet, jedennoch für die Fälle, daß entweder $m<|x|+|y|+z$ oder $m\not\equiv x+y+z$ (mod 2) ist, es in $\sum_\nu$ einziges Glied z allein zu einnehmen ist, und dann $P_m(x, y, z)$ auf Null reduziert. Zweitens beweise ich, daß $P_{2n}(0, 0, 0)\simeq 5/8n\pi$ für $n\to\infty$, so daß $\sum P_{2n}(0, 0, 0)$ divergiert und deshalb gegen 1 strebt die Wahrscheinlichkeit Ω_n, dafür, daß der wandernde Punkt innerhalb der Zeitspann $0<t\leq 2n$ mindestens einmal wieder in dem Anfangspunkt zurückkehrt. Ferner betrachte ich dasjenige ebene Straßennetz, wofür $\sum P_{2n}(0, 0)$ konvergiert, und also Ω_n gegen $\Omega(<1)$ strebt, und schließlich den Fall, daß außer Bewegungen längs jene zu Koordinatenachsen parallelen Richtungen noch ihre Superposition möglich kommt, sowie auch zeitweilige Pause zugelassen wird.

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1) G. Pólya, Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz, Math. Ann. **84** (1924). Y. Watanabe, Aufgaben betreffend das Irrfahrtproblem, dieses Journ. Vol. **6** (1955), S. 41, dessen § 2 aber einen Irrtum enthält, was ich jetzt beseitigen will.

§ 1.

Ich habe die Formel (1) zu nachweisen. Da aber $\exp\{-ix\varphi-iy\psi\}=\cos x\varphi\cos y\psi-\sin x\varphi\sin y\psi-i\sin x\varphi\cos y\psi-i\cos x\varphi\sin y\psi$, und wegen

$$(2)\qquad \cos^n\alpha=\frac{1}{2^{n-1}}\sum_{\kappa=0}^{[n/2]}\binom{n}{k}\cos(n-2k)\alpha\,,$$

wobei für gerades n das letzte absolute Glied noch mit $\frac{1}{2}$ multipliziert werden soll, ersichtlich $\int_0^{2\pi}\cos^n\frac{\varphi}{\psi}\sin\frac{x\varphi}{y\psi}d\frac{\varphi}{\psi}=0$ gelten, so möge (1) auch als

$$(3)\qquad P_m(x,y,z)=\frac{1}{8\pi^3}\iiint_0^{2\pi}\left(\frac{2\cos\varphi+2\cos\psi+e^{i\theta}}{5}\right)^m\cos x\varphi\cos y\psi \sum_\nu\exp\{-iz_\nu\theta\}d\varphi d\psi d\theta$$

gegeben werden.

Vorerst will ich einige Hilfssätze vorausschicken. Obgleich diese im hinblick auf Wahrscheinlichkeit fast selbstverständlich sind, soll es bewiesen werden, daß dieselbe auch für die entsprechenden Ausdrücke (1) gelten:

1° *Es lauten aus* (3)

$$P_0(x,y,0)=\frac{1}{8\pi^3}\iiint_0^{2\pi}\cos x\varphi\cos y\psi d\varphi d\psi d\theta=1\qquad bei\ x=y=0$$

aber $$=0\qquad bei\ (x,y)\neq(0,0)$$

und $$P_0(x,y,1)=\frac{1}{8\pi^3}\iiint_0^{2\pi}\cos x\varphi\cos y\psi\cdot e^{-i\theta}d\varphi d\psi d\theta=0\,.$$

Im allgemeinen bei $m>0$ schreibe ich das Integral

$$(4)\qquad \begin{aligned}&\iiint(2\cos\varphi+2\cos\psi+e^{i\theta})^m\cos x\varphi\cos y\psi\sum_\nu\exp\{-iz_\nu\theta\}d\varphi d\psi d\theta\\&=\sum_\nu\sum_{r=0}^m\binom{m}{r}\sum_{s=0}^r\binom{r}{s}\iiint 2^r\cos^s\varphi\cos^{r-s}\psi\cos|x|\varphi\cos|y|\psi\\&\qquad\exp[(m-r-z_\nu)\theta i]d\varphi d\psi d\theta\,.\end{aligned}$$

2° *Für* $0<m<|x|+|y|+z$ *ist* $P_m(x,y,z)=0$. *In diesem Falle freilich braucht es als* $z_\nu=z$ *in der Summe* $\sum_\nu$ *von* (1) *nur einziges Glied* $e^{-i\theta z}$ *genommen zu werden.*

Wieder der Formel (2) nach verschwinden alle Integrale in (4) außer demjenige mit $s-2k=|x|$, $r-s-2l=|y|$ und folglich $r=|x|+|y|+2k+2l$, wo $0\leqq k\leqq[s/2]$, $0\leqq l\leqq[(r-s)/2]$. Anderseits verschwindet das Integral für $m-r-z\neq0$, während die Identität $m=r+z=|x|+|y|+z+2k+2l$ mit der urspringlichen Annahme widerspricht, w.z.b.w.

3° *Auch für* $m\not\equiv x+y+z$ (mod 2) *wird* $P_m(x,y,z)=0$.

Z. B. seien (i) $x+y=$ gerad und $z=0$ so daß m ungerad. Dann werden alle z_ν gerad, damit für diejenige inbezugauf θ integrierte nicht verschwindende Integral in (4) es $m-r=z_\nu$ also $r=m-z_\nu$ ungerad sein soll. Daher muß im Produkt $\cos^s\varphi\cos^{r-s}\psi$ ein der s und $r-s$ gerad und das andere ungerad sein. Anderseits sind x und y beide gerad oder beide ungerad. Daraus folgt wegen (2) eben

$$\iint \cos^s\varphi\cos^{r-s}\psi\cos x\varphi\cos y\psi d\varphi d\psi=0\,, \qquad \text{w.z.b.w.}$$

Andere mögliche Fälle (ii) $x+y=$gerad, $z=1$, (iii) $x+y=$ungerad, $z=0$, (iv) $x+y=$ungerad, $z=1$ sämtlich können ganz ebenso wie in (i) bewiesen werden. Ins besondere gilt für $x+y\equiv 0 \pmod 2$

$$(5)\quad P_{2n+1}(x,\,y,\,0)=\frac{1}{8\pi^3}\iiint\left[\frac{2\cos\varphi+2\cos\psi+e^{i\theta}}{5}\right]^{2n+1}\cos x\varphi\cos y\psi d\varphi d\psi d\theta=0\,.$$

4° *Man kann die Summe* $\sum\limits_\nu$ *in* (1) *folgenderweise herstellen*: *Für* $z=0$

$$\begin{aligned}\sum_\nu &= 1+\exp(-2\theta i)+\cdots+\exp\{-(m-|x|-|y|)\theta i\}\\ &= e^{i\theta}[\exp(-\theta i)+\exp(-3\theta i)+\cdots+\exp\{-(m+1-|x|-|y|)\theta i\}]\,,\end{aligned}$$

während für $z=1$

$$\sum_\nu=\exp(-\theta i)+\exp(-3\theta i)+\cdots+\exp\{-(m-|x|-|y|)\theta i\}$$

die durch Addierung von $e^{\theta i}$, *was wegen lauter Identität*

$$\iiint (2\cos\varphi+2\cos\psi+e^{i\theta})^m\cos x\varphi\cos y\psi\cdot e^{i\theta}d\varphi d\psi d\theta=0$$

zulässig ist, folgendergestalt geschrieben werden kann:

$$e^{\theta i}[1+\exp(-2\theta i)+\cdots+\exp\{-(m+1-|x|-|y|)\theta i\}]\,.$$

Im Falle $m\leqq|x|+|y|+z$ wird $\sum\limits_\nu$ bloß $e^{-i\theta z}$; deswegen hat man bei $z=0$ für $e^0=1$ lediglich $e^{i\theta}e^{-i\theta}=1$ zu einsetzen, und bei $z=1$ das ursprüngliche $e^{-i\theta}$ mit der Summe $e^{i\theta}(1+e^{-2\theta i})$ zu ersetzen.

5° *Man darf auch zur Summe* $\sum\limits_\nu$ *ein Glied* $\exp\{-(z_l+2)\theta i\}$, d.h. $\exp\{-(m-|x|-|y|+2)\theta i\}$ *hinzufügen*; *freilich im Falle* $m\leqq|x|+|y|+z$ *reduziert das hinzufügende Glied lediglich auf* $\exp\{-(z+2)\theta i\}$.

Um dies zu bestätigen, genügt es zu zeigen, daß

$$\iiint\sum_{r=0}^{m}\binom{m}{r}\sum_{s=0}^{r}\binom{r}{s}2^r\cos^s\varphi\cos^{r-s}\psi\cdot\cos x\varphi\cos y\psi\exp\{(m-r-z_l-2)\theta i\}d\varphi d\psi d\theta=0\,.$$

Für etwaiges nicht-verschwindendes Glied soll es $m-r=z_l+2$, also $r=|x|+|y|-2$ sein. Anderseits braucht es $s-2k=|x|, r-s-2l=|y|$ mit nicht negative k and l, so daß $r=|x|+|y|+2k+2l$, was aber mit dem vorigen Schluß widerspricht. Daher sollen alle integrierte Glieder sämtlich verschwinden, w.z.b.w.

Jetzt da *personae dramatis* sich zusammengezogen haben, sind wir im stande die Formel (1) zu dartun.

Es seien $m+1 \geqq |x|+|y|+z$ und $m+1 \equiv x+y+z \pmod 2$, da sonst $P_{m+1}(x, y, z)=0$, was schon in 2° und 3° erwiesen worden war. Nach 1° besteht (1) für $m=0$. Angenommen (1) für gewißes m, hat man sein nochmaliges Bestehen für $m+1$ zu beweisen. Dazu beachtet man

$$P_{m+1}(x, y, z) = \sum_j \frac{1}{5} P_m(x'_j, y'_j, z'_j),$$

wobei die Summe über die zu (x, y, z) nächstliegenden 5 Punkte erstreckt. Diese 5 Punkte sind $(x\pm1, y, z)$, $(x, y\pm1, z)$ und $(x, y, 1-z)$, für die bei m (1) schon anwendbar sind:

$$P_{m+1}(x, y, z) = \frac{1}{5}\sum_{j=1}^{5}\frac{1}{8\pi^3}\iiint\left(\frac{2\cos\varphi+2\cos\psi+e^{i\theta}}{5}\right)^m \exp\{-ix\varphi-iy\psi\}F_j d\varphi d\psi d\theta,$$

wobei umständlich, wenn $m > |x|+|y|+z+1$ ist,

$$F_1 = e^{-ix\varphi}\sum_\nu \exp\{-z_\nu\theta i\}, \quad \text{wo } z_\nu \text{ von } z \text{ bis zu } z_l = m-|x+1|-|y|,$$

$$F_2 = e^{ix\varphi}\sum_\nu \exp\{-z_\nu\theta i\}, \quad \text{wo } z_\nu \text{ von } z \text{ bis zu } z_l = m-|x-1|-|y|,$$

$$F_3 = e^{-iy\psi}\sum_\nu \exp\{-z_\nu\theta i\}, \quad \text{wo } z_\nu \text{ von } z \text{ bis zu } z_l = m-|x|-|y+1|,$$

$$F_4 = e^{iy\psi}\sum_\nu \exp\{-z_\nu\theta i\}, \quad \text{wo } z_\nu \text{ von } z \text{ bis zu } z_l = m-|x|-|y-1|,$$

und $F_5 = e^{i\theta}\sum_\nu \exp\{-z_\nu\theta i\}$, wo z_ν von z bis zu $z_l = m+1-|x|-|y|$ wegen 4° sind, während bei $m=|x|+|y|+z+1$ oder $|x|+|y|+z-1$ auch gewiße $\sum_\nu$ aus bloßes $e^{-iz\theta}$ allein sich entstehen sollen, aber die Fälle $m=|x|+|y|+z$, $|x|+|y|+z+2, \cdots\cdots$ ausgenommen wegen ursprünglichen zweiten Annahme $m+1 \equiv x+y+z \pmod 2$.

Die erste 4 Summen können sämtlich wie selbst steht oder mit gebrauch von 5°, durch diejenige des F_5 ersetzt werden. Daher

$$P_{m+1}(x, y, z) = \frac{1}{8\pi^3}\iiint_0^{2\pi}\left(\frac{2\cos\varphi+2\cos\psi+e^{i\theta}}{5}\right)^{m+1}\exp\{-x\varphi i-y\psi i\} \sum_\nu \exp\{-iz_\nu\theta\} d\varphi d\psi d\theta,$$

wobei die Summe $\sum_\nu$ mit z beginnt und durch $m+1-|x|-|y|$ vollendet, w.z.b.w.

Wegen Periodizität des Integrandes kann das Integrationsgebiet als jeder Würfel W: $-\alpha \leqq \varphi, \psi, \theta \leqq 2\pi-\alpha$ willkürlich genommen werden.

§ 2.

Nach (1) hat man besonders

$$(6) \quad P_{2n}(0, 0, 0) = \frac{1}{8\pi^3}\iiint_{-\pi/2}^{3\pi/2}\left(\frac{2\cos\varphi+2\cos\psi+e^{i\theta}}{5}\right)^{2n}\sum_{k=0}^{n}\exp\{-2ik\theta\} d\varphi d\psi d\theta,$$

das unten abgeschätzt werden wird. Man sieht leicht, daß

$$\sum_{k=0}^{n} = \sum_{k=0}^{n} (\cos 2k\theta - i \sin 2k\theta) = \frac{\sin (n+1)\theta}{\sin\theta} [\cos n\theta - i \sin n\theta],$$

und folglish

$$|\Sigma| = \left| \frac{\sin (n+1)\,\theta}{\sin\theta} \right| \leq n+1 \,.^{2)}$$

Da aber der Faktor nach absolutem Betrage

$$R = \left| \frac{2\cos\varphi + 2\cos\psi + e^{i\theta}}{5} \right|$$

für $\varphi=\psi=\theta=0$ oder π innerhalb des Integrationsgebietes W sein Maximum 1 erreicht, so beträgt $R \leq \rho < 1$ in $W-W_0-W_\pi = U$, wobei W_0 und W_π zwei offene Würfel mit jedem Mittelpunkt (0, 0, 0), (π, π, π) von beliebig kleiner aber doch wohl bestimmter Kantenlänge $2a$ bezeichnen. Damit wird

$$|J_U| = \left| \iiint_U \right| < (n+1)\rho^{2n} = 0\,(1/n^N) \qquad \text{für} \quad n\to\infty,$$

wobei N irgendwie große aber feste positive Zahl sein mag. Anderseits ist der Beitrage aus W_0

$$J_{W_0} = \frac{1}{8\pi^3} \iiint_{-a}^{a} \left[\frac{1}{5}\left(2\cos\frac{t_1}{\sqrt{n}} + 2\cos\frac{t_2}{\sqrt{n}} + \cos\frac{t_3}{n} + i\sin\frac{t_3}{n} \right)\right]^{2n}$$

$$\frac{\sin (n+1)\, t_3/n}{\sin t_3/n} e^{-it_3} \frac{dt_1 dt_2 dt_3}{n^2} \qquad (\varphi = t_1/\sqrt{n},\ \psi = t_2/\sqrt{n},\ \theta = t_3/n)$$

$$\cong \frac{1}{8\pi^3 n^2} \int_{-\sqrt{n}a}^{\sqrt{n}a} \int_{-\sqrt{n}a}^{\sqrt{n}a} \int_{-na}^{na} \exp\left[-\frac{2}{5}(t_1^2+t_2^2) + \frac{2}{5} it_3 + 0\left(\frac{1}{n}\right)\right] \frac{\sin\left(1+\frac{1}{n}\right) t_3}{\sin t_3/n}$$

$$\exp\{-it_3\} dt_1 dt_2 dt_3$$

$$\cong \frac{1}{8\pi^3 n^2} \int_{-\infty}^{\infty} \exp\left(-\frac{2}{5}t_1^2\right) dt_1 \int_{-\infty}^{\infty} \exp\left(-\frac{2}{5}t_2^2\right) dt_2 \int_{-na}^{na} \frac{\sin\left(1+\frac{1}{n}\right)t_3}{\sin t_3/n} \exp\left\{-\frac{3}{5}it_3\right\} dt_3$$

$$\cong \frac{5}{16n\pi^2} \int_{-na}^{na} \frac{\sin t}{n \sin t/n} \left[\cos\frac{3}{5}t - i \sin\frac{3}{5}t \right] dt \qquad \text{für} \quad n\to\infty,$$

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2) Um diese Abschätzung zu prüfen, setze ich $\left\{\frac{\sin (n+1)\,\theta}{(n+1)\sin\theta}\right\}^2 = F(\theta)$. Da diese Funktion periodisch, sogar symmetrisch in bezug auf $\theta = p\pi/2$ $(p=0, 1, 2, \cdots)$ ist, so braucht nur das Intervall $0 \leq \theta \leq \pi/2$ betrachtet zu werden. Ersichtlich ist $F(0\pm)=1$ und

$$F'(\theta) = \frac{2}{(n+1)^2} \frac{\sin (n+1)\,\theta}{\sin^3\theta} [(n+1)\cos (n+1)\,\theta \sin\theta - \sin (n+1)\,\theta\cos\theta],$$

worin $\sin\theta > 0$ in $0 < \theta \leq \pi/2$. Nun gibt $F'(\theta)=0$ entweder

(i) $\sin (n+1)\,\theta = 0$ oder (ii) $\tan (n+1)\,\theta = (n+1)\tan\theta$,

aus deren letzteres $\sin^2 (n+1)\,\theta = (n+1)^2 \sin^2\theta / \{\cos^2\theta + (n+1)^2 \sin^2\theta\}$ folgt. Offenbar liefert das ersteres den Minimumwert 0, während das letzteres den Maximumwert $F(\theta) = \frac{\sin^2 (n+1)\,\theta}{(n+1)^2 \sin^2\theta}$ $= \frac{1}{1+n(n+2)\sin^2\theta} \leq 1$, w. z. b. w.

worin der imaginäre Teil wegen des ungeraden Faktors verschwindet. Da aber $0<t/n<a$ als genug klein gewählt worden war, so strebt $n \sin t/n \simeq t$, und

$$J_{W_0} \simeq \frac{5}{16n\pi^2}\int_0^{na}\left(\sin\frac{8}{5}t+\sin\frac{2}{5}t\right)\frac{dt}{t} \simeq \frac{5}{16n\pi} \qquad \text{bei} \quad n\to\infty,$$

auf daß

$$\int_0^\infty \sin ct\frac{dt}{t} = \frac{\pi}{2} \qquad \text{für} \quad c>0.$$

Ganz derselbe ist der Beitrag aus W_π. *Daher gilt*

$$P_{2n}(0, 0, 0) \simeq \frac{5}{8n\pi}, \tag{7}$$

während, wie schon in (5) *gesehen,* $P_{2n+1}(0, 0, 0)=0$ *ist.*

Obgleich die obige Ergebnisse etwas verschieden von diejenige für Pólyasches Ebenenstraßennetz: $P_{2n}(0, 0) \simeq 1/n\pi$, $P_{2n+1}(0, 0)=0$ scheint, doch divergiert gleichviel gegenwärtige Reihe $\sum P_m(0, 0, 0)$ und damit stimmt weiterer Schluß mit Pólyaschem überein. Seien nämlich Ω_n die Wahrscheinlichkeit dafür, daß der Wanderer innerhalb der Zeitspann $0<t\leq n$ nach dem Anfangspunkt zurückkehrt, und ω_m die Wahrscheinlichkeit dafür, daß der Wanderer zum ersten Male in $t=m$ auf den Anfangspunkt zurückkommt, so gilt $\Omega_n=\sum_{m=1}^{n}\omega_m$ mit $\omega_{2\nu+1}=0$. Aber ist die Wahrscheinlichkeit $P_m(0, 0, 0)$ nichts anders als der Koeffizient von z^m in die Entwickelung des Bruches $(1-\omega_2 z^2-\omega_4 z^4-\cdots)^{-1}$, und deshalb besteht wohl

$$f(z)=1-\sum_{m=1}^{\infty}\omega_m z^m = 1/\sum_{m=0}^{\infty}P_m(0, 0, 0)z^m.$$

Daraus folgt $f(1-0)=0$ und $\sum_{m=1}^{\infty}\omega_m=1$, wenn $\sum_{m=0}^{\infty}P_m(0, 0, 0)=\infty$. Also wird $\Omega=\lim_{n\to\infty}\Omega_n=1$, d.h. es ist gewiß, daß der Wanderer innerhalb endlicher Zeitspann wenigstens einmal und folglich auch unendlich oft auf den Anfangspunkt zurückkommt.

Überdies geht der Wanderer über jeden irgendwo in Straß liegenden Punkt (x, y, z), sogar unendlich oftmal. Denn, da $P_m(x, y, z)>0$ für solches $m\geq|x|+|y|+z$ und $m\equiv x+y+z \pmod 2$ ist, und für irgendgroßes N er gewiß N mal auf 0 zurückkehrt und wiederaufbricht, so wird wegen Regel der Großenzahlen er ungefähr zu $NP_m(x, y, z)$ Malen den Punkt (x, y, z) besuchen. Ferner seien (x_1, y_1, z_1), (x_2, y_2, z_2) irgend zwei Punkte auf Straße. Auf daß der Wanderer diese beide Punkte gewissermaßen besucht, ergibt sich daß es einen Verbindungsgang zwischen den zwei Punkte gibt, also *Tutte le strade conducono a Roma.*

3) Somit hier und in meinem früheren Essay würde es besser sein, anstatt von "Elevator" vielmehr mit "Ersteiger" (=risers, wie Herr J. Riordan in Mathematical Review gesagt hatte) zu ausdrücken

§ 3.

Im Falle, daß *der Elevator nur aufwärts aber nicht abwärts gehen kann*[3], soll die betreffende Wahrscheinlichkeit das folgende Gestalt annehmen:

$$(8)\ P_m(x, y, z) = \frac{1}{8\pi^3}\iiint\limits_0^{2\pi}\left(\frac{2\cos\varphi+2\cos\psi+e^{i\theta}}{5}\right)^m \exp\{-ix\varphi-iy\psi-iz\theta\}d\varphi d\psi d\theta,$$

wobei z eine nicht-negative ganzzahlige dritte Variable ist, und nochmal gilt

$$P_0(0, 0, 0) = 1 \quad \text{bzw.} \quad P_0(x, y, z) = 0 \quad \text{für} \quad (x, y, z) \neq (0, 0, 0)\,.$$

Ins besondere ist

$$P_m(0, 0, 0) = \frac{1}{4\pi^2}\left(\frac{2}{5}\right)^m \iint\limits_0^{2\pi} (\cos\varphi+\cos\psi)^m d\varphi d\psi\,,$$

demnach

$$P_{2n+1}(0, 0, 0) = 0\,,$$

während

$$P_{2n}(0, 0, 0) = \frac{1}{4\pi^2}\left(\frac{4}{25}\right)^n \sum_{k=0}^{n}\binom{2n}{2k}\iint\limits_0^{2\pi}\cos^{2k}\varphi\cos^{2n-2k}\psi d\varphi d\psi$$

$$= \frac{4}{\pi^2}\left(\frac{4}{25}\right)^n \sum_{k=0}^{n}\frac{(2n)\,!}{(2k)\,!\,(2n-2k)\,!}\int_0^{\pi/2}\cos^{2k}\varphi\, d\varphi\int_0^{\pi/2}\cos^{2n-2k}\psi\, d\psi\,.$$

Deshalb gilt

$$(9)\qquad P_{2n}(0, 0, 0) = \frac{1}{25^n}\binom{2n}{n}^2,$$

und

$$(10)\qquad P_{2n}(0, 0, 0) \simeq \frac{1}{n\pi}\left(\frac{16}{25}\right)^n \qquad \text{für} \quad n\to\infty\,.$$

Hiemit konvergiert die Reihe $\sum_{n=0}^{\infty} P_{2n}(0, 0, 0)\ (>1)$, so daß

$$\Omega = \lim_{n\to\infty}\Omega_n = \sum_{m=1}^{\infty}\omega_m = 1/\sum_{m=0}^{\infty}P_m(0, 0, 0) < 1\,.$$

Daher bei genügend großes n ist $\Omega_n^\nu \simeq \Omega^\nu$, was gegen 0 für $\nu\to\infty$ strebt, d.h. *nach ziemlich länger Zeit* $t=n\nu$ *gibt es fast kein Zurückkehrende, und die Straßenebene* $z=0$ *wird fast unbewohnte Ruine.*

Die Sache wird vielmehr einfach bei ebener Bewegung. *Der wandernde Punkt, aufbrechend zur Zeit* $t=0$ *von 0 aus, herumwandere auf xy-Ebene mit der Geschwindigkeit* 1, *und an jedem Knotenpunkte entscheide nächsten Lauf mit der Wahrscheinlichkeit* 1/3 *für eine der* 3 *Richtungen, parallel zu x-sowie y-Achse.*[4] Zwar auf jeder x-Parallele ist der Verkehr ganz frei, indessen auf jeder y-Parallele es alleiniges Einbahnstraße, d.h. nur nach +Direktion gangbar ist. Bei solches Straße läßt sich die Wahrscheinlichkeit dafür, daß der Wanderer zur Zeit $t=m$ an Knotenpunkt (x, y) sich findet, durch

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4) Der Fall mit solchen einseitigem Wege ist auch von Sherman Lehman für ebenes hexagonales Straßennetz betrachtet worden: A Problem on Random Walk, Proc. of Symposium on Statistics, Dep. Matn. California Univ., p. 263 (1951).

$$P_m(x, y) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left(\frac{2\cos\varphi+e^{i\psi}}{3}\right)^m \exp\{-ix\varphi-iy\psi\}d\varphi d\psi$$

ausdrücken. Man denke sich nur den Fall, daß nicht bloß $m \equiv x+y \pmod 2$ sondern auch $m \geqq |x|+y$ ist, weil sonst $P_m(x, y)=0$, was ebenso wie 2°, 3° in §1 beweisbar ist. Unter dieser Annahme erhält man

$$P_m(x, y) = \frac{1}{2\pi}\frac{2^{m-y}}{3^m}\binom{m}{m-y}\int_{-\pi}^{\pi}\cos^{m-y}\varphi\cos|x|\varphi d\varphi\,.$$

Da aber

$$\cos^{m-y}\varphi = \frac{1}{2^{m-y-1}}\sum_{k=0}^{[(m-y)/2]}\binom{m-y}{k}\cos(m-y-2k)\varphi\,,$$

wobei, falls $m-y$ gerad ist, das letzte absolute Glied noch mit $\frac{1}{2}$ multipliziert werden soll, so wird das nach Integration bleibende Glied

$$\frac{1}{3^m}\cdot\frac{\underline{|m}}{\underline{|y}\,\underline{|(m-y+|x|)/2}\,\underline{|(m-y-|x|)/2}},$$

was wegen Voraussetzungen, daß $m-|x|-y\equiv 0 \pmod 2$, sowie $m\geqq|x|+y$, ein positiver Bruch ist. Ins besondere ist $P_{2n+1}(0, 0)=0$ und $P_0(0, 0)=1$, während

(11) $$P_{2n}(0, 0) = \frac{1}{3^{2n}}\binom{2n}{n} = \frac{1}{9^n}\frac{(2n)!}{n!\,n!},$$

und für $n\to\infty$

(12) $$P_{2n}(0, 0) \simeq \frac{1}{\sqrt{n\pi}}\left(\frac{4}{9}\right)^n$$

gilt.[5] Demgemäß ist $\sum_{n=0}^{\infty}P_{2n}(0, 0)\ (>1)$ gewiß konvergent, und hieraus folgt

$$1-\sum_{n=1}^{\infty}\omega_{2n} = 1/\sum_{n=0}^{\infty}P_{2n}(0, 0) > 1\,,$$

also bleibt die Wahrscheinlichkeit $\Omega=\sum_{n=1}^{\infty}\omega_{2n}$ dafür, daß der Wanderer wenigstens einmal wieder nach dem Anfangspunkt zurückkehrt, gewiß ein positiver Bruch <1. Bei alledem kann man ebendenselben Schlüß ziehen, wie der oben illustrierte Fall des mit alleinig aufwärtssteigendem Ersteigern versehenden ∞-stöckigen Raumes: *Nach ziemlich längem Zeitverlaufe geben es fast keine Vorübergehenden auf Straßenbahnen* $y=0, 1, 2, \cdots$.

Es ist etwas bemerkenswert, daß beim gewöhnlichen Irrfahrtproblem im d-dimensionalen Straßennetz, $\Omega=1$ für $d=1$ oder 2 (lineare oder ebene Dimension) aber für $d\geqq 3$ (räumliche Dimension) $\Omega<1$ war, dennoch für

5) Wir erhalten ähnlichen Zahlenwert $\frac{1}{2^{2n}}\binom{2n}{n}\simeq\frac{1}{\sqrt{n\pi}}$ für dieselbe Wahrscheinlichkeit beim gewöhnlichen ebenen Straßennetz, oder doch bei vielfachen Münzenwerfenspiele; cf. W. Feller, An Introduction to Probability and its Applications, Vol. 1, p. 247 &c. (1950).

unser zweistöckigen Raum in §1 zwar $\Omega=1$ ist, während sogar $\Omega<1$ für Ebenesstraßennetz, in welchem nur alleinige Einbahnstraße längs y-Richtung gebaut ist.

Wir haben nach (9), (10) oder (11), (12), bezw. $P_{2n}(0, 0, 0)$ oder $P_{2n}(0, 0)$ berechnet:

n	P_{2n} (0, 0, 0)		P_{2n} (0, 0)		aus Pólyashen Formel	
	aus (9)	aus (10)	aus (11)	aus (12)	$P_{2n}(0,0)=\left(\frac{1}{4^n}\cdot\binom{2n}{n}\right)^2$	$\simeq\frac{1}{n\pi}$
0	1.	—	1.	—	1.	—
1	0.16	0.2037	0.2222	0.2508	0.25	0.3183
2	0.0576	0.0652	0.0741	0.0788	0.1406	0.1592
3	0.0256	0.0278	0.0274	0.0286	0.0977	0.1061
4	0.0127	0.0134	0.0107	0.0110	0.0748	0.0796
5	0.0065	0.0068	0.0043	0.0044	0.0606	0.0637
6	0.0035	0.0036	0.0017	0.0018	0.0509	0.0531
7	0.0019	0.0020	0.0007	0.0007	0.0439	0.0455
8	0.0011	0.0011	0.0003	0.0003	0.0386	0.0398
9	0.0006	0.0006	0.0001	0.0001	0.0344	0.0354
10	0.0004	0.0004	0.0001	0.0001	0.0311	0.0318
11	0.0002	0.0002	0.0000	0.0000	0.0283	0.0289
12	0.0001	0.0001	0.0000	0.0000	0.0260	0.0265
13	0.0001	0.0001	0.0000	0.0000	0.0240	0.0245
14	0.0000	0.0000	0.0000	0.0000	0.0223	0.0227
15	0.0000	0.0000	0.0000	0.0000	0.0209	0.0212
Summe $\sum$	1.2703		1.3416		∞	
$\sum^{-1}$	0.7872		0.7454		0	
$\Omega=1-\sum^{-1}$	0.2128		0.2546		1	

§ 4.

Schließlich wünsche ich die Annahme, daß der Wanderer keine Pause hat, zu verschwächen[6]. Es seien p_1, p_2 und p_0 die Wahrscheinlichkeiten dafür, daß die auf Ebene herumwandernde Molekel je um ± 1 parallel zur x–, y– Achse in Einheitzeit sich bewegt, bzw. um $\pm\sqrt{2}$ parallel zur Linie $y=x$ oder $y=-x$, was aus Superposition der Verschiebungen längs x– sowie y-Richtung resultiert—wie etwa dem Zusammenstöße zwischen Molekeln gemäß Brownsche[7] Bewegung verursacht wird—und schließlich während nächster Zeiteinheit in demselben Punkt ruft, sodaß $p_0+4p_1+4p_2=1$ gilt.

Für jeden Knotenpunkt (x, y) sind es 8 Nachbarpunkte: $(x\pm 1, y)$, $(x, y\pm 1)$, $(x\pm 1, y\pm 1)$ und $(x\pm 1, y\mp 1)$, wobei die doppelten Zeichen in Klammern je ober oder unter gleichzeitig genommen werden sollen. Die Wahrscheinlichkeit dafür, daß im Zeitpunkt $t=m$ der wandernde Punkt im Knotenpunkt (x, y) sich findet,

6) Was ich in der früheren Note loc. cit. bereits gemacht hatte, aber hier wird es etwas verfeinert werden.

7) Die Irrfahrt-Approximation zur Brownschen Bewegung ist etwa in A. Dvoretzky and P. Erdös: Some Problems on random Walk in Space, in bevor unter 4) zitierten Literatur, p. 367, und noch weiter A. Dvoretzky: Brownian motion in space and subharmonic functions, behandelt worden.

ist

$$(13)\qquad P_m(x,y)=\frac{1}{4\pi^2}\iint\limits_{-\alpha}^{2\pi-\alpha}[p_0+2p_1(\cos\varphi+\cos\psi)+2p_2\{\cos(\varphi+\psi)+\cos(\varphi-\psi)\}]^m \exp\{-ix\varphi-iy\psi\}d\varphi d\psi\,,$$

wobei α irgendeine positive Zahl $\leqq\pi$ bedeutet. Offenbar ist (13) richtig für $m=0$. Falls (13) für gewißes m gültig ist, so geht es auch für $m+1$. Denn, da

$$P_{m+1}(x,y)=p_0P_m(x,y)+p_1P_m(x\pm1,y)+p_1P_m(x,y\pm1)+ p_2P_m(x\pm1,y\pm1)+p_2P_m(x\pm1,y\mp1)\,,$$

setzt man P_m im rechtstehenden Ausdrücke durch (13) ein, so folgt

$$P_{m+1}(x,y)=\frac{1}{4\pi^2}\iint[p_0+2\{p_1\cos\varphi+p_1\cos\psi+p_2\cos(\varphi+\psi)+p_2\cos(\varphi-\psi)\}]^m e^{-ix\varphi-iy\psi}$$
$$\times[p_0+p_1(e^{\varphi i}+e^{-\varphi i})+p_1(e^{\psi i}+e^{-\psi i})+p_2(e^{(\varphi+\psi)i}-e^{-(\varphi+\psi)i})$$
$$+p_2(e^{(\varphi-\psi)i}+e^{-(\varphi-\psi)i})]d\varphi d\psi$$
$$=\frac{1}{4\pi^2}\iint[p_0+2\{p_1\cos\varphi_1+p_1\cos\psi+p_2\cos(\varphi+\psi)+p_2\cos(\varphi-\psi)\}]^{m+1}$$
$$\exp\{-ix\varphi-iy\psi\}d\varphi d\psi\,,$$

w.z.b.w. Ins besondere gilt

$$P_m(0,0)=\frac{1}{4\pi^2}\iint\limits_{-\alpha}^{2\pi-\alpha}[p_0+2\{p_1\cos\varphi+p_1\cos\psi+p_2\cos(\varphi+\psi)+p_2\cos(\varphi-\psi)\}]^m d\varphi d\psi.$$

Wir wollen dies für genug großes m abschätzen, und dazu den Integrand nach absolutem Betrag $|A|$ erwägen. Dies erreicht sein Maximum 1 für $\varphi=\psi=0$ innerhalb des ganzen Integrationsgebietes Q. Daher wird $|A|\leqq\rho^m\to0$ (m^{-N}) im abgeschlossenen Gebiet $Q-Q_0$, wo Q_0 ein kleines offenes Quadrat mit Mittelpunkt (0, 0) von Kantenlänge $2a(0<a<1)$ bedeutet. Anderseits ist der Beitrag aus Q_0

$$\iint\limits_{Q_0}=\frac{1}{4\pi^2}\iint\limits_{-a}^{a}[p_0+2\{p_1\cos\varphi+p_1\cos\psi+p_2\cos(\varphi+\psi)+p_2\cos(\varphi-\psi)\}]^m d\varphi d\psi$$

$$=\frac{1}{4\pi^2 m}\iint\limits_{-a\sqrt{m}}^{a\sqrt{m}}\exp m\log\left[p_0+2p_1\left(\cos\frac{t}{\sqrt{m}}+\cos\frac{t'}{\sqrt{m}}\right)\right.$$
$$\left.+2p_2\left(\cos\frac{t+t'}{\sqrt{m}}+\cos\frac{t-t'}{\sqrt{m}}\right)\right]dtdt'\qquad(\varphi=t/\sqrt{m},\ \psi=t'/\sqrt{m})$$

$$\simeq\frac{1}{4\pi^2 m}\iint\limits_{-a\sqrt{m}}^{a\sqrt{m}}\exp m\log\left[1-\frac{1}{m}(p_1+2p_2)(t^2+t'^2)\right]dtdt'$$
(falls m groß und a klein)

$$\simeq\frac{1}{4\pi^2 m}\left[\int_{-\infty}^{\infty}\exp\{-(p_1+2p_2)t^2\}\,dt\right]^2=\frac{1}{4\pi m(p_1+2p_2)}\,.$$

Also besteht für passend großes m

(14) $$P_m(0, 0) \simeq 1/4\pi(p_1+2p_2)m\,,$$

und $\sum P_m(0, 0)$ divergiert gegen ∞. Die Wahrscheinlichkeit Ω_n dafür, daß die Molekel innerhalb $0 < t \leqq n$ auf den Anfangspunkt zurückkommt, strebt gegen 1 für $n \to \infty$, in Übereinstimmung mit dem Pólyaschen Resultat.

Ganz ebenso kann man auch für Raum fortfahren: Zunächst wird die betreffende Wahrscheinlichkeit beim dreidimensionalen Raum folgenderweise dargestellt:

(15) $$P_m(x, y, z) = \frac{1}{8\pi^3}\iiint_{-\alpha}^{2\pi-\alpha} E^m \exp\{-ix\varphi - iy\psi - iz\}\, d\varphi d\psi d\chi\,,$$

wobei

$$E = p_0 + 2p_1(\cos\varphi + \cos\psi + \cos\chi) + 2p_2[\cos(\varphi \pm \psi) + \cos(\varphi \pm \chi) + \cos(\psi \pm \chi)]$$
$$+ 2p_3[\cos(\varphi + \psi \pm \chi) + \cos(\pm\varphi \mp \psi + \chi)]$$

und $$p_0 + 6p_1 + 12p_2 + 8p_3 = 1$$

sind. Zweitens sind es 27 Nachbarpunkte für jeden Knotenpunkt (x, y, z):

(i) 6 Punkte $(x, y, z\pm 1)$, $(x, y\pm 1, z)$, $(x\pm 1, y, z)$;

(ii) 12 Punkte $(x\pm 1, y\pm 1, z)$, $(x\pm 1, y\mp 1, z)$ und dgl,;

(iii) 8 Punkte $(x\pm 1, y\pm 1, z+1)$, $(x\pm 1, y\pm 1, z-1)$, $(x\pm 1, y\mp 1, z+1)$, $(x\pm 1, y\mp 1, z-1)$;

(iv) der nämliche Punkt (x, y, z);

von einjeder (i), (ii), (iii), (iv) aus es nach Einheitzeit mit Wahrscheinlichkeit p_1, p_2, p_3, p_0 bezw. zum Punkt (x, y, z) erreichbar sind. Dadurch aber leicht kann die vollständige Induktion der Formel (15) von m zu $m+1$ geleistet werden. Ferner nimmt der faktor E in (15) nach absolutem Betrage ersichtlich seinen Maximumwert 1 für $\varphi = \psi = \chi = 0$, und zuletzt können wir schließen, daß

(16) $$P_m(0, 0, 0) \simeq [8\pi(p_1+4p_2+4p_3)m]^{-3/2} \qquad \text{für } m \to \infty\,,$$

damit $\sum P_m(0, 0, 0)$ konvergent und $\Omega = \lim \Omega_n < 1$ ist. Also erscheint ganz anderes als der vorige ebene Fall: der Wanderer kommt nicht notwendigerweise auf Anfangsstelle zurück.

Am Ende sei es zum Vergleiche bemerkt, daß der Fall, worin jede Bewegungsrichtung *überallhin* genommen werden möge, erst von Pearson gefragt und später von Kluyver und Rayleigh schönweise folgendermaßen gelöst worden ist[8]:

Es sei nämlich ein Wanderer auf r, θ–Polarkoordinatenebene, der zur Zeit

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8) K. Pearson: Nature, **72** (1905), p. 294.
J. Kluyver: Proc. Section of Sci., K. Akad. Amsterdam, **8** (1906), p. 604.
Lord Layleigh: Phil. Mag., **37** (1919), p. 321.
G. Watson: Theory of Bessel Functions (1922), p. 419.
Y. Fusimi: Wahrscheinlichkeitsrechnung und Statistik (im Japanische).

$t=0$ von O aus mit Geschwindigkeit 1 aufbräche und zur Zeit $t=0, 1, 2, \cdots$ seine nächste Laufsrichtung überallhin aufs Geratewohl auswähle. Die Wahrscheinlichkeit dafür, daß er zur Zeit $t=n$ innerhalb des Kreises von Radius r sich findet, nach Kluyver ist

$$P_n(r)=r\int_0^\infty J_1(rt)\,[J_0(t)]^n dt\,,$$

wobei J_0, J_1 Besselfunktionen bedeuten, und woraus

$$P_n(1)=P_n=\frac{1}{n+1}$$

folgt. Bezeichnet man nun mit Ω_n und ω_m die respektive Wahrscheinlichkeit dafür, daß der Wanderer innerhalb der Zeitspann $0<t\leqq n$ bzw. erstemal in $t=m\leqq n$ auf K: $r\leqq 1$ zurückkommt, so gilt $\sum_{m=1}^{n}\omega_m=\Omega_n$. Da aber

$$P_n=\omega_n+\sum_{h+k=n}\omega_h\omega_k+\sum_{h+k+l=n}\omega_h\omega_k\omega_l+\cdots$$

ist,[9] deswegen

$$\sum_{n=0}^{\infty}P_n z^n=(1-\sum_{m=1}^{\infty}\omega_m z^m)^{-1}\,,$$

wobei beide unendliche Reihen für $|z|<1$ konvergieren, weil außer $P_0=1$ sämtliche P_n, ω_m positive genuine Brüche sind. Hiermit bei $z\to 1-0$

$$1-\sum_{m=1}^{\infty}\omega_m=1/\sum_{n=0}^{\infty}\frac{1}{n+1}=0\,,\quad \text{d.h.}\quad \lim_{n\to\infty}\Omega_n=\Omega=1\,.$$

Es ist also ganz gewiß, daß der Wanderer wenigstens einmal und folglich auch unendlich oft zu K: $r\leqq 1$ zurückkehrt.

Im Falle der räumliche Wanderung nach Lord Layleigh ist die entsprechende Wahrscheinlichkeit für großes n

$$P_n^{(3)}(r)\cong\sqrt{\frac{2}{\pi}}\int_0^{r/\sigma}t^2\,\exp\left\{-\frac{1}{2}t^2\right\}dt\,,$$

mit $\sigma^2=n/3\sigma$. Daraus erkennen wir, daß

$$P_n^{(3)}(1)\cong\sqrt{\frac{6}{\pi}}\,\frac{1}{n^{3/2}}$$

und $\sum_{n=0}^{\infty}P_n^{(3)}(1)\,(>1)$ konvergiert, so daß $0<\Omega=\lim_{n\to\infty}\Omega_n=1-1/\sum_{n=0}^{\infty}P_n^{(3)}(1)<1$. Daher ist $1-\Omega>0$, was meint, daß die Wahrscheinlichkeit dafür, daß der Wanderer ohne sogar einmalige Zurückkehrung ins Kugelgebiet K: $r\leqq 1$ lediglich ins Unendliche entgeht, eben positiv ist und also im Raum gewiß sich verbreiten die Molekeln ins Unendliche (Diffusion).

9) Hier, aus Stetigkeitsprinzip ist es angenommen, die Wahrscheinlichkeit $\doteqdot\omega_n$ ungefähr zu sein, dafür daß der zur Zeit $t=0$ von einem Punkte in K aufbrächende Wanderer in $t=n$ erstemal innerhalb des Kreises K zurückkommt.

Ganz ebenso für den Fall der Wanderung im $d(\geqq 3)$-dimensionalen Raume wird die Wahrscheinlichkeit

$$P_n^{(d)}(r) \simeq \frac{r}{\Gamma\left(\frac{1}{2}d\right)} \int_0^\infty \left(\frac{1}{2}rt\right)^{d/2-1} J_{d/2}(rt) \exp\left\{-\frac{nt^2}{2d}\right\} dt$$

$$= \frac{1}{\Gamma\left(\frac{d}{2}+1\right)} \left(\frac{r^2 d}{2n}\right)^{d/2} \cdot {}_1F_1\left(\frac{1}{2}d,\ \frac{1}{2}d+1\ ;\ -\frac{r^2 d}{2n}\right),$$

wobei ${}_1F_1(\alpha, \beta, z) = \sum_{\nu=0}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+\nu-1)}{\beta(\beta+1)\cdots(\beta+\nu-1)} \frac{z^\nu}{\nu!}$, also ${}_1F_1\left(\frac{1}{2}d,\ \frac{1}{2}d+1\ ;\ -\frac{r^2 d}{2n}\right) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} \frac{d}{d+2\nu} \left(\frac{r^2 d}{2n}\right)^\nu$ ist. Daher ist

$$P_n^{(d)}(1) = \frac{1}{\Gamma\left(\frac{d}{2}+1\right)} \left(\frac{d}{2n}\right)^{d/2} \cdot {}_1F_1\left(\frac{d}{2},\ \frac{d}{2}+1,\ -\frac{d}{2n}\right).$$

Aber, da ${}_1F_1\left(\frac{1}{2}d,\ \frac{d}{2}+1,\ -\frac{d}{2n}\right) \simeq 1$ für $n \to \infty$ gilt, so strebt

$$P_n^{(d)}(1) \simeq \left(\frac{d}{2}\right)^{d/2} \frac{1}{\Gamma(d/2+1)} \cdot \frac{1}{n^{d/2}} \quad (d \geqq 3)\ .$$

Also ist $\sum_{n=0}^{\infty} P_n^{(d)}(1)\ (>1)$ konvergent, infolgedessen $0 < \Omega = \lim \Omega_n = 1 - 1/\sum_{n=0}^{\infty} P_n^{(d)}(1) < 1$ und deswegen entgeht der Wanderunde Punkt schließlich ins Unendliche.

Nachträglich sei es noch bemerkt, daß unsre Formel (6) nach Ausführung der Integration in mehr klarer Form frei von imaginären Potenzen folgendermaßen ausgedrückt werden können:

$$P_{2n}(0,0,0) = \frac{1}{5^{2n}} \sum_{r=0}^{n} \binom{2n}{2r} \binom{2r}{r}^2, \tag{17}$$

und woraus folgende Werte berechnet werden mögen:

n	0	1	2	3	4	5
$P_{2n}(0,0,0)$	1	0.2	0.0976	0.0641	0.0480	0.386 &c.

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ON DARBOUX LINES CONTAINED IN A RIEMANNIAN SPACE

By

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§ **1.** Miss Prvanovitch[1] recently defined a Darboux line contained in an n-dimensional sub-space V_n immersed in a Riemann space V_m, obtained differential equations of this curve and found some properties.

According to her definition, a Darboux line is a curve on each point of which the vector

$$R_1 t_2^\alpha + \frac{dR_1}{ds} R_2 t_3^\alpha \text{ }^{2)}$$

is normal to a sub-space V_n immersed in V_m, where t_2^α, t_3^α are the first and the second principal normal vectors of this curve, R_1, R_2 the first and the second radius of curvature respectively.

Let $a_{\alpha\beta}dy^\alpha dy^\beta$ and $g_{ij}dx^i dx^j$ be the fundamental forms of V_m and V_n, and let H^P_{ij} be the second fundamental tensor of V_n immersed in V_m, ξ^α_P be $(m-n)$ mutually orthogonal unit vectors in V_m normal to V_n in V_m, s be arc length, then the equation of a Darboux line is given by[3]

$$(1,1) \qquad a_{\alpha\beta}\frac{\delta^3 y^\alpha}{ds^3}\xi^\beta_R = \left(\frac{\partial H^R_{ij}}{\partial x^k} + H^R_{kl}\left\{ {l \atop ij} \right\} - \sum_P \mu_{PR|k} H^P_{ij}\right)\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds} + 3H^R_{ij}\frac{d^2x^i}{ds^2}\frac{dx^j}{ds} = 0\,,$$

where ξ^β_R (R is fixed) is a considering normal vector of V_n and the quantities $\mu_{PR|k}$ are given by

$$(1,2) \qquad \mu_{PR|k} = a_{\alpha\beta}\xi^\alpha_P\xi^\beta_{R,k} + [\gamma\delta,\beta]_a\, y^\gamma_{,k}\xi^\delta_R\xi^\beta_P\,.$$

Now we can rewrite this equation such as

$$(1,3) \qquad H^R_{ij;K}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds} + 3H^R_{ij}\frac{\delta^2 x^i}{ds^2}\frac{dx^j}{ds} - \sum_P \mu_{PR|k}H^P_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0\,,$$

1) M. Prvanovitch; Lignes de Darboux dans l'espace riemannien. (Bull. Sci. Math. (2) 78 1954 p.p. 9–14)
; Hyperligne de Darboux appartenant à l'espace riemannien. (Bull. Sci. Math. (2) 78 1954 p.p. 89–97)

2) In this paragraph we shall denote by $\alpha, \beta, \gamma, \cdots$ the suffices which take the value $1, 2, \cdots, m$; by $h, i, j, k, l, \cdots$ those which take the value $\dot{1}, \dot{2}, \dot{3}, \cdots, \dot{n}$; by $P, Q, R, S, \cdots$ those which take the value $\dot{n}+\dot{1}, \dot{n}+\dot{2}, \cdots, \dot{m}$.

3) M. Prvanovitch; loc. cit.

or

$$\frac{\delta}{ds}\left(H^R_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right)+H^R_{ij}\frac{\delta^2 x^i}{ds^2}\frac{dx^j}{ds}-\sum_P \mu_{PR|K}H^P_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}=0. \tag{1, 4}$$

In the case $m=n+1$, V_n is a hypersurface of V_{n+1}, and

$$\mu_{PR|k}=0. \tag{1, 5}$$

That is to say, if we denote the normal of V_n by ξ^α and put

$$H_{(ij;k)}=\frac{1}{3}\left[H_{ij;k}+H_{jk;i}+H_{ki;j}\right]=H_{ijk},$$

then the equation of a Darboux line is

$$H_{ijk}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}+3H_{ij}\frac{\delta^2 x^i}{ds^2}\frac{dx^j}{ds}=0, \tag{1, 6}$$

or

$$3\frac{\delta}{ds}\left(H_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right)-H_{ijk}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}=0. \tag{1, 7}$$

The curves whose equations are given by

$$H_{ijk}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}=0, \tag{1, 8}$$

$$K_{ijk}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}=0. \tag{1, 9}$$

are, by Prof. Kanitani[4], called a Darboux line of the first kind and the second kind respectively, where we put

$$K_{ijk}=H_{ijk}-\frac{1}{n+2}\left[H_{ij}H_k+H_{jk}H_i+H_{ki}H_j\right],$$

$$H_i=H^{jk}H_{ijk}.$$

Also he calls the hypersurface which satisfies $K_{ijk}=0$ a hyperquadric. Taking V_n a hyperquadric, we see the equation of a Darboux line defined by Prvanovitch may be written as

$$\frac{\delta}{ds}\left(H_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right)-\frac{1}{n+2}\left(H_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right)H_k\frac{dx^k}{ds}=0. \tag{1, 10}$$

Hence, if a curve contained in a hyperquadric in V_{n+1} is an asymptotic line, it is a Darboux line defined by Prvanovitch.

As a special case, V_n being a hyperquadric which satisfies the relation $H_{ijk}=0$, we see that, as $H_k=0$, equation (1, 10) is

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4) J. Kanitani; Les equation fondamentales d'une surface plongée dans un espace à connexion projective. (Mem. of Ryojun college of Engineering. Vol. XII. No. 3 p.p. 61-88 1939)

; Sur un Espace à connexion Projective Renferment des Hyperquadriques. (Proc. of Physico- Math. Soc. of Japan. 25 p.p. 617-621 1943)

$$\frac{d}{ds}\left(H_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right)=0. \tag{1, 11}$$

Therefore from the equation (1, 11) we see when V_n is a hyperquadric satisfying $H_{ijk}=0$, a Darboux line defined by Prvanovitch is a curve whose normal curvature $H_{ij}\dfrac{dx^i}{ds}\dfrac{dx^j}{ds}$ is constant.

From now we call a Darboux line defined by Prvanovitch the Darboux line of the third kind. In this paper we mean the Darboux line of the third kind by Darboux line.

Here, we shall find the necessary and sufficient condition that every curve contained in this hyperquadric immersed in V_{n+1} be a Darboux line.

For this purpose putting

$$H_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}=\tau,$$

we obtain from (1, 10)

$$\frac{d}{ds}\log\tau=\frac{1}{n+2}H_k\frac{dx^k}{ds},$$

since this equation must be indeterminate, we see τ is a function of x^i and

$$\frac{1}{n+2}H_k=\frac{\partial\log\tau}{\partial x^k}.$$

Moreover when we put $H=(n+2)\log\tau$, we have $H_k=\dfrac{\partial H}{\partial x^k}$ and $e^{\frac{H}{n+2}}=H_{ij}\dfrac{dx^i}{ds}\dfrac{dx^j}{ds}$.

Of course this last result must be identical, then we have

$$H_{ij}=e^{\frac{H}{n+2}}g_{ij}.$$

Because of this relation we have

$$H_{ijk}=\frac{1}{3(n+2)}e^{\frac{H}{n+2}}(g_{ij}H_k+g_{jk}H_i+g_{ki}H_j)\quad\text{and}\quad H_i=H^{jk}H_{ijk}=\frac{1}{3}H_i.$$

Accordingly we have $H_i=0$, that is, H is constant.

Thus we obtain $H_{ij}=\rho g_{ij}$ (ρ is constant).

Conversly, when $H_{ij}=\rho g_{ij}$ ($\rho=$const.), we see easily $H_{ijk}=0$, $H_i=0$, that is, V_n is a hyperquadric and Darboux lines are indeterminate in it.

Consequently we obtain *the necessary and sufficient condition that every curve contained in a hyperquadric immersed in V_{n+1} be Darboux lines is this hyperquadric be properly totally umbilic hypersurface.*

§ **2.** Suppose $G_{AB}dz^Adz^B$ and $a_{\alpha\beta}dy^\alpha dy^\beta$ be the fundamental forms of V_l ($l\geqq 4$) and V_m ($l>m\geqq 3$) which is subvariety of V_l, and $B_{\alpha}^{\cdot A}=\dfrac{\partial z^A}{\partial y^\alpha}$ and ξ_σ^A be components of a tangent vector of V_m and $(l-m)$ mutually orthogonal unit vectors in V_l normal to V_m respectively then we have well known relations

(2, 1) $$a_{\alpha\beta} = G_{AB}B_{\alpha}^{\cdot A}B_{\beta}^{\cdot B}\,.$$ [4]

Moreover when we indicate by V_n a subspace V_m, by $B_i^{\cdot\alpha} = \dfrac{\partial y^\alpha}{\partial x^i}$ tangent vectors of V_n, by ξ_a^α $(m-n)$ mutually orthogonal unit vectors in V_m normal to V_n, we see the metric tensor g_{ij} of V_n is given by

(2, 2) $$g_{ij} = a_{\alpha\beta}B_i^{\cdot\alpha}B_j^{\cdot\beta}\,.$$

From the relations $B_i^{\cdot A} = B_\alpha^{\cdot A}B_i^{\cdot\alpha}$ we may put

(2, 3) $$\xi_a^A = B_\alpha^{\cdot A}\xi_a^\alpha\,,$$

and indicate (ξ_a^A, ξ_σ^A) by ξ_P^A. Moreover we indicate by $H_{\alpha\beta}^A$, H_{ij}^α and H_{ij}^A the components of normal curvature tensor of V_m in V_l, of V_n in V_m, and of V_n in V_l respectively, then we can represent

$$H_{\alpha\beta}^A = B_{\alpha;\beta}^{\cdot A}\,, \quad H_{ij}^\alpha = B_{i;j}^{\cdot\alpha}\,, \quad H_{ij}^A = B_{i;j}^{\cdot A}\,.$$

Since $H_{\alpha\beta}^A$ are orthogonal to B_γ^A, we can put

(2, 4) $$H_{\alpha\beta}^A = \sum_\sigma H_{\alpha\beta}^\sigma \xi_\sigma^A\,, \quad H_{ij}^\alpha = \sum_a H_{ij}^a \xi_a^\alpha\,, \quad H_{ij}^A = \sum_a H_{ij}^P \xi_P^A\,,$$

and obtain

$$\sum_P H_{ij}^P \xi_P^A = H_{ij}^A = (B_\alpha^{\cdot A}B_i^{\cdot\alpha})_{;j} = \sum_\sigma H_{\alpha\beta}^\sigma B_i^{\cdot\alpha}B_j^{\cdot\beta}\xi_\sigma^A + \sum_\sigma H_{ij}^a \xi_a^A\,.$$

Hence, from the definition of ξ_P^A, comparing the coefficients of vectors ξ_P^A, we have

(2, 5) $$H_{ij}^\tau = H_{\alpha\beta}^\tau B_i^{\cdot\alpha}B_j^{\cdot\beta}\,, \quad [H_{ij}^a]_{V_m} = [H_{ij}^a]_{V_n}\,.$$

When $x^i(s)$ is a Darboux line of V_n in V_m, denoting by ξ_e^α (e is fixed) the normal vector of V_n in V_m, whose components are determined by the vector

$$R_1 t_2^\alpha + R_1' R^2 t_3^\alpha$$

we see the equation of Darboux line is given by

(2, 6) $$\frac{\delta}{ds}\left(H_{ij}^e \frac{dx^i}{ds}\frac{dx^j}{ds}\right) + H_{ij}^e \frac{\delta^2 x^i}{ds^2}\frac{dx^j}{ds} - \sum_a [\mu_{ae|k}]_{V_m}\cdot H_{ij}^a \frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0\,.$$

On the other hand when we represent by $x^i(s)$ a Darboux line in V_n in V_l concerning ξ_e^A from (2, 5) the equation of this curve is given by

(2, 7) $$\frac{\delta}{ds}\left(H_{ij}^e \frac{dx^i}{ds}\frac{dx^j}{ds}\right) + H_{ij}^e \frac{\delta^2 x^i}{ds^2}\frac{dx^j}{ds} - \sum_P [\mu_{Pe|k}]_{V_l} H_{ij}^P \frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0\,,$$

where $[\mu_{Pe|k}]_{V_l}$ is given by

$$[\mu_{Pe|k}]_{V_l} = G_{AB}\xi_P^A \xi_{e,k}^B + [CD, B]_{V_l} z_{,k}^C \xi_e^D \xi_P^B\,.$$

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4) In this paragraph we shall denote by $A, B, C, \cdots, H$ the suffices which take the value $1, 2, \cdots, l$; by $\alpha, \beta, \gamma, \delta$ those which take the value $\dot{1}, \dot{2}, \dot{3}, \cdots, \dot{m}$; by h, i, j, k those which take the value $\ddot{1}, \ddot{2}, \cdots, \ddot{n}$; by a, b, c, d, e, f those which take the value $\ddot{n}+\ddot{1}, \ddot{n}+\ddot{2}, \cdots, \ddot{m}$; by μ, σ, τ those which take the value $\dot{m}+\dot{1}, \dot{m}+\dot{2}, \cdots, \dot{l}$; bu P, Q, R, S those which take the value $\dot{n}+\dot{1}, \dot{n}+\dot{2}, \cdots, \dot{l}$.

We consider, therefore, two cases, that is, $n+1 \leqq P \leqq m$ and $m < P \leqq l$. First in the case $n+1 \leqq P \leqq m$, from the relation (2, 1), (2, 3),

$$[\mu_{ae|k}]_{V_l} = G_{AB}\xi^A_a H^B_{\alpha\gamma} B^{\cdot\gamma}_k \xi^\alpha_e + G_{AB}\xi^A_a B^{\cdot B}_\alpha \xi^\alpha_{e;k},$$

and
$$G_{AB}\xi^A_a \xi^B_\sigma = G_{AB} B^{\cdot A}_\alpha \xi^B_\sigma \xi^\alpha_a = 0,$$

we have

(2, 8)
$$[\mu_{ae|k}]_{V_l} = [\mu_{ae|k}]_{V_m}.$$

In the second case we have similarly

(2, 9)
$$[\mu_{\sigma e|k}]_{V_l} = H^\sigma_{\alpha\beta}\xi^\alpha_e \xi^\beta_k.$$

Substituting this into (2, 7) we find

(2, 10)
$$\frac{\delta}{ds}\left(H^e_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right) + H^e_{ij}\frac{\delta^2 x^i}{ds^2}\frac{dx^j}{ds} - \sum_a [\mu_{ae|k}]_{V_m} \cdot H^a_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}$$
$$- \sum_\sigma H^\sigma_{\alpha\beta}\xi^\alpha_e B^{\cdot\beta}_k H^\sigma_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0,$$

and comparing with (2, 6) we obtain *the necessary and sufficient condition that a Darboux line contained in V_n immersed in V_l concerning the vector $\xi^A_e = B^{\cdot A}_\alpha \xi^\alpha_e$ of V_n be a Darboux line contained in V_n immersed in V_m concerning the vector ξ^x is that along this curve the relation.*

(2, 11)
$$\sum_\sigma H^\sigma_{\alpha\beta} H^\sigma_{ij}\xi^\alpha_e B^{\cdot\beta}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

be satisfied.

In particular, if we take $m = n+1$ and $l = n+2$, we have from (2, 8)

$$[\mu_{ae|k}]_{V_l} = [\mu_{ae|k}]_{V_m} = 0.$$

When we put $\sigma = \mathrm{I}$, $e = \mathrm{II}$ and denote by η^α components of a unit vector in V_{n+1} normal to V_n, by ξ^A_{I}, $\xi^A_{\mathrm{II}} = B^{\cdot A}_\alpha \eta^\alpha$ components of two mutually orthogonal unit vectors in V_{n+2} normal to V_n, we have

$$H^{\mathrm{I}}_{ij} = H_{\alpha\beta} B^{\cdot\alpha}_i B^{\cdot\beta}_j, \quad H^{\mathrm{II}}_{ij} = H_{ij},$$

then the equation (2, 11) is

$$H_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} \cdot \mu_{\mathrm{I}\,\mathrm{II}|k}\frac{dx^k}{ds} = 0,$$

and the quantities $\mu_{\mathrm{I}\,\mathrm{II}|k}$ are given by

$$\mu_{\mathrm{I}\,\mathrm{II}|k} = G_{AB}\xi^A_{\mathrm{I}}\xi^B_{\mathrm{II};k} = H_{\alpha\beta} B^{\cdot\beta}_k \eta^\alpha.$$

Hence we obtain *the necessary and sufficient condition that a Darboux line contained in V_n immersed in V_{n+2} concerning the normal vector $\xi^A_{\mathrm{II}} = B^{\cdot A}_\alpha \eta^\alpha$, where η^α is a normal vector to V_n in V_{n+1}, be a Darboux line contained in V_n immersed in V_{n+1} is that along this curve one of the following two conditions be satisfied*

(2, 12) $$H_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}=0,$$

(2, 13) $$H_{\alpha\beta}\eta^\alpha\frac{dy^\beta}{ds}=0.$$

§ **3.** Using the same notations in preceding paragraph we consider V_n in V_{n+1} immersed in V_{n+2}. Taking a notice that the equations of a Darboux line contained in V_{n+1} immersed in V_{n+2} is given by

$$G_{AB}\frac{\delta^3 z^A}{ds^3}\xi^B_{\mathrm{I}}=0,$$

we call V_n a Darboux variety of V_{n+1} immersed in V_{n+2} when, in V_n, for all indices i, j, k the equation

(3, 1) $$G_{AB}z^A_{;i;j;k}\xi^B_{\mathrm{I}}=0$$

are satisfied.

Since

$$z^A_{;i;j;k}=H_{\alpha\beta;\gamma}B_i^{\cdot\alpha}B_j^{\cdot\beta}B_k^{\cdot\gamma}\xi^A_{\mathrm{I}}+2H_{\alpha\beta}B_i^{\cdot\alpha}H_{jk}\eta^\beta\xi^A_{\mathrm{I}}+H_{\alpha\beta}B_i^{\cdot\alpha}B_j^{\cdot\beta}\xi^A_{\mathrm{I};k}$$
$$+H_{ij;k}\eta^\alpha B_\alpha^{\cdot A}+H_{ij}H_{\alpha\beta}B_k^{\cdot\beta}\eta^\alpha\xi^A_{\mathrm{I}}+H_{ij}\eta^\alpha_{;k}B_\alpha^{\cdot A}.$$

and $$G_{AB}\xi^A_{\mathrm{I};k}\xi^B_{\mathrm{I}}=0.$$

The equations (3, 1) can be written as

(3, 2) $$H_{\alpha\beta;\gamma}B_i^{\cdot\alpha}B_j^{\cdot\beta}B_k^{\cdot\gamma}+H_{\alpha\beta}\eta^\beta[2B_i^{\cdot\alpha}H_{jk}+B_k^{\cdot\alpha}H_{ij}]=0.$$

However, (3, 2) are written as

(3, 3) $$M_{ijk}=0,$$

where we put

(3, 4) $$M_{ijk}=H_{\alpha\beta\gamma}B_i^{\cdot\alpha}B_j^{\cdot\beta}B_k^{\cdot\gamma}+3H_{\alpha\beta}\eta^\beta B^\alpha_{[i}H_{jk]}.$$

Hence we obtain the necessary and sufficient condition that V_n immersed in V_{n+1} immersed in V_{n+2} be a Darboux variety is

$$M_{ijk}=0.$$

Especially if we take $n=1$ they evidently become equations of a Darboux line. Hence we can consider that a Darboux variety is a extension of a Darboux line contained in V_2 immersed in V_3 to n-dimensional sub-space.

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ON p-VALENT FUNCTIONS

By

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1. Introduction.

Let $f(z)$ be a function of the class of ones

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n ,$$

which are regular and schlicht in $|z|<1$. Then

$$|a_n| < en .$$

This result is well known as Littlewood's theorem. In this paper first we shall extend this result to the case of weakly p-valent functions defined by Hayman [1], which contain p-valent functions, and solve Hayman's conjecture on the coefficient of weakly p-valent functions [1].

Secondly we consider the following functions

$$w(z) = z^{-p}(1 + a_1 z + \cdots\cdots) ,$$

which are regular and p-valent in $0<|z|<1$. These functions were studied first by Prof. Kobori [2]. We shall study the values taken by $w(z)$ and the distortion theorem.

Lastly we shall remark that we can extend the following theorem of Hayman

Suppose that $w = f(z) = 1/z + a_0 + a_1 z + \cdots\cdots$ is meromorphic in $|z|<1$ and has a simple pole of residue 1 at the origin. Let D_f be the domain of all values w taken by $f(z)$ in $|z|<1$, and let E_f be the complement of D_f in the closed plane. Then

$$d(E_f) \leq 1 ,$$

where $d(E_f)$ denotes the transfinite diameter of E_f. Equality hold if and only if $f(z)$ is univalent.

and derive some analogous results to the ones derived by Hayman [1] from this theorem.

2.

According to Hayman's definition [1] we say that $f(z)$ is weakly p-valent, if for every $r>0$ the equation $f(z) = w$ either

(i) has exactly p roots in the unit circle for every value on the circle $|w|=r$ or

(ii) has less than p roots in the unit circle for some w on the circle $|w|=r$.

Of course p-valent or mean p-valent functions defined by Biernacki are weakly p-valent. We will begin with the proof of the following lemma and prove it by Mandelbrojt's method [3].

Lemma 1. *Let*

$$f(z)=z^p+\sum_{n=2}^{\infty} a_{n+p-1} z^{n+p-1},$$

be regular and weakly p-valent in $|z|<1$, then

$$\frac{1}{2\pi}\int_0^{2\pi}|f(z)|\,d\theta<\frac{r^p}{(1-r)^{2p-1}}, \qquad (|z|=r<1).$$

Proof. We put

$$f(z)=Re^{i\Theta}, \qquad z=re^{i\theta}.$$

$f(z)\neq 0$ in $|z|<1$ except $z=0$. Therefore $\log f(z)$ is a regular function of $\log z$ in $0<r_1\leq|z|\leq r_2<1$. According to Cauchy-Riemann equation we have

$$\frac{\partial R}{\partial r}=\frac{R}{r}\frac{\partial\Theta}{\partial\theta}. \tag{1}$$

Then the following relations are got by means of the above equation, where C is the image curve of the circle $|z|=r$.

$$\begin{aligned}\frac{d}{dr}\int_0^{2R}|f(re^{i\theta})|\,d\theta &= \frac{d}{dr}\int_0^{2\pi}R\,d\theta \\ &= \int_0^{2\pi}\frac{\partial R}{\partial r}d\theta=\frac{1}{r}\int_C R\,d\Theta.\end{aligned} \tag{2}$$

On the other hand $f(z)$ has neither zero points or poles except only one zero point at the origin because of weak p-valence of $f(z)$.

Hence according to the argument principle

$$\int_C d\arg f(z)=\int_C d\Theta=2\pi p. \tag{3}$$

By integrating the formula (2)

$$\int_{r_1}^{r_2}\frac{dr}{r}\int_C R\,d\Theta=\int_0^{2\pi}|f(r_2e^{i\theta})|\,d\theta-\int_0^{2\pi}|f(r_1e^{i\theta})|\,d\theta.$$

Because of $f(0)=0$ we have the next relation by tending r_1 to zero and substituting r for r_2

$$\int_0^{2\pi}|f(re^{i\theta})|\,d\theta=\int_0^r\frac{dr}{r}\int_C R\,d\Theta.$$

By Hayman's result [1]

$$|f(z)|=R\leq\frac{r^p}{(1-r)^{2p}}, \qquad (|z|=r<1).$$

Hence we have by (3)

$$\int_C R d\Theta < \int_C \frac{r^p}{(1-r)^{2p}} d\Theta = \frac{2\pi p r^p}{(1-r)^{2p}}.$$

Therefore

$$\int_0^{2\pi} |f(re^{i\theta})|\, d\theta < 2\pi p \int_0^r \frac{r^{p-1}}{(1-r)^{2p}} dr.$$

On the other hand

$$p \int_0^r \frac{r^{p-1}}{(1-r)^{2p}} dr \leq \frac{r^p}{(1-r)^{2p-1}},$$

because

$$\frac{d}{dr}\left(\frac{r^p}{(1-r)^{2p-1}}\right) - \frac{p r^{p-1}}{(1-r)^{2p}} = \frac{(p-1) r^p}{(1-r)^{2p}} \geq 0.$$

This completes the proof.

Theorem 1. *Let $f(z)$ satisfy the condition in lemma* 1. *Then*

$$|a_{n+p-1}| < e^{2p-1}\left(1+\frac{n-1}{2p-1}\right)^{2p-1} = O(n^{2p-1}).$$

Proof. By lemma 1

$$|a_{n+p-1}| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(z)|}{r^{n+p-1}} d\theta < \frac{1}{r^{n-1}(1-r)^{2p-1}}.$$

$r^{n-1}(1-r)^{2p-1}$ takes the greatest value when $r=(n-1)(n+2p-2)^{-1}$.

Hence we have theorem 1.

Remark. Hayman [1] conjectured the order of the coefficients in theorem 1.

3.

Let $w=f(z)=z^{-p}(1+a_1 z+\cdots\cdots)$ be regular and p-valent in $0<|z|<1$. Hereafter this family of functions will be denoted by F.

First we consider the case where $f(z)$ has no zero point in $0<|z|<1$. Then

$$\frac{1}{f(z)} = z^p(1-a_1 z+\cdots\cdots)$$

is regular and p-valent in $|z|<1$, and has only one zero point of order p at the origin. Hence we have the following theorem by means of Hayman's result [1] and Biernacki's one [4].

Theorem 2. *Let $f(z)\in F$ and has no zero point in $|z|<1$. Then*

(i) *The image by $f(z)$ covers the circle $|w|>4$, and*

(ii) *moreover covers $|w|>4^p$ exactly p times.*

(iii) $$r^{-p}(1-r)^{2p} \leq |f(z)| \leq r^{-p}(1+r)^{2p}, \qquad (|z|=r<1).$$

These estimates are sharp as is shown by $z^{-p}(1-z)^{2p}$ and $z^{-p}(1-z^p)^2$. The results (ii) *and* (iii) *hold still when we substitute the condition of weak p-valence for the one of p-valence.*

Secondly we consider the case where $f(z)\in F$ has zero points in $|z|<1$. It is sufficient to extend Montel-Bieberbach's theorem [5] as follows in order to have a result on the values by $f(z)$.

Lemma 2. *Let* $w(z)=z^p+b_{p+1}z^{p+1}+\cdots\cdots$ *be* p*-valent and meromorphic, then the image by* $w(z)$ *covers the circle* $|w|<\delta=\sqrt{5}-2$ *or the circle* $|w|>\frac{1}{\delta}=\sqrt{5}+2$. *This result is sharp as is shown by*

$$w_0(z)=\delta\frac{z^p}{(1-z^p)^2}\Big/\left(\frac{z^p}{(1-z^p)^2}+\delta\right)=\frac{\delta z^p}{z^p+\delta(1-z^p)^2}=z^p+\cdots\cdots.$$

Proof. Let α be one of the points on the boundary of the domain mapped by $w(z)$, which are nearest from the origin. We may suppose without loss of generality that α is positive. We put

$$\zeta(z)=\frac{\alpha w(z)}{\alpha-w(z)}=z^p+\cdots\cdots.$$

$\zeta(z)$ is regular and p-valent. Therefore we see that the image by $f(z)$ covers the circle $|\zeta|<1/4$ by means of Biernacki's theorem [4].

On the other hand

$$w(z)=\frac{\alpha\zeta}{\alpha+\zeta}.$$

Hence the image by $w(z)$ covers the exterior of the circle which has the segment $(\alpha/(1+4\alpha),\ \alpha/(1-4\alpha))$ on the real axis. And $\delta/(1-4\delta)=1/\delta$, $(\delta=\sqrt{5}-2)$. Therefore the image by $w(z)$ covers the circle $|w|>1/\delta$ when $\alpha<\delta$, or the circle $|w|<\delta$ when $\alpha\geqq\delta$. This estimate is sharp clearly, because

$$w_0(1)=\delta,\qquad w_0(e^{\pi i/p})=\frac{\delta}{1-4\delta}.$$

Here we have the following theorem directly.

Theorem 3. *Let* $f(z)\in F$ *and have zero points. Then the image by* $f(z)$ *covers the circle* $|w|<\delta$ *or the circle* $|w|>\frac{1}{\delta}$. *This estimate is best possible as is shown by* $f_0(z)=(z^p+\delta(1-z^p)^2)/\delta z^p=z^{-p}+\cdots\cdots$. $(\delta=\sqrt{5}-2)$.

Now we shall study the circle covered by $f(z)$ p times under the condition thaf $f(z)$ has only one zero point of order p.

First we will prove the following lemma.

Lemma 3. *Let* $w(z)=z+a_2z^2+\cdots\cdots$ *be meromorphic and weakly* 1*-valent. Then the image by* $w(z)$ *covers the circle* $|w|<\delta$ *or the circle* $|w|>\frac{1}{\delta}$, $(\delta=\sqrt{5}-2)$. *This estimate is sharp as is shown by*

$$w_0(z)=\frac{\delta z}{(1-z)^2}\Big/\left(\frac{z}{(1-z)^2}+\delta\right)=z+\cdots\cdots.$$

Proof. This proof is quite analogous to lemma 2, that is, we may use Hayman's one-quarter theorem of weakly 1-valent functions for Koebe's one with respect to schlicht functions.

Theorem 4. *Let* $f(z)\in F$ *and has one zero point of order* p *in* $|z|<1$. *Then the image by* $f(z)$ *covers the circle* $|w|<\delta^p$ *or the circle* $|w|>\frac{1}{\delta^p}$, p *times.*

These bounds are best possible as is shown by

$$f_0(z)=\left(\frac{\delta z}{z+\delta(1-z)^2}\right)^{-p}=z^{-p}+\cdots\cdots . \quad (\delta=\sqrt{5}-2) .$$

Proof. We can prove this theorem even when $f(z)$ is weakly p-valent.

$$\frac{1}{f(z)}=z^p+\cdots\cdots$$

is weakly p-valent and regular in $|z|<1$ except one pole of order p. We consider $(1/f(z))^{1/p}$ and if we use the slight modification of Hayman's lemma [1], we see that this function is weakly 1-valent, and therefore we can use lemma 3. This completes the proof.

Theorem 5. *Let $f(z)$ satisfy the same conditions in theorem 4. Then we have one of the following estimates for all $|z|=r$ $(r<1)$.*

$$|f(z)|\leq\frac{1}{(4\delta)^p}\times r^{-p}(1+r)^{2p},$$

or

$$|f(z)|\geq\left(\frac{\delta}{4}\right)^p\times r^{-p}(1+r)^{2p},$$

where $\delta=\sqrt{5}-2$.

Proof. Without loss of generality we can assume $z=-|z|=-r<0$ and therefore it is sufficient for this proof to evaluate $|f(-r)|$. We remark that the function $h(z)$ mapping univalently the circle $|z|<1$ to the unit circle slitted by the segment $(-1, -r)$ under the conditions $h(0)=0$ and $h'(0)>0$ is given uniquely as follows [1] or [5].

$$\frac{h(z)}{(1-h(z))^2}=q\frac{z}{(1-z)^2}, \qquad q=\frac{4r}{(1+r)^2}, \qquad h'(0)=q .$$

$$g(h(z))=z+\cdots\cdots, \quad \left(g(z)=\left(\frac{1}{f(z)}\right)^{\frac{1}{p}}\right)$$

is weakly 1-valent because of weak 1-valence of $g(z)$ and meromorphic in $|z|<1$, and therefore we can use lemma 3 for this function. On the other hand the value $g(-r)/q$ which corresponds to $z=-1$, is not taken by this function. Hence we have

$$|g(-r)|\geq q\delta=4\delta\times\frac{r}{(1+r)^2}$$

or

$$|g(-r)|\leq q\times\frac{1}{\delta}=\frac{4}{\delta}\times\frac{r}{(1+r)^2}.$$

This completes the proof.

4.

First we will extend Hayman's theorem showed in the introduction. Each of E_f, D_f and $d(E_f)$ denotes the one indicated in the introduction.

Theorem 6. *Let $f(z)=z^{-p}(1+c_1z+\cdots\cdots)$ be meromorphic in $|z|<1$, where p is a positive integer, and E_f denote the complement of D_f which is the domain of the values taken by $f(z)$. Then $d(E_f)\leqq 1$. Equality occurs only when $f(z)=g(z^p)$, where $g(z)=z^{-1}(1+a_1z+\cdots\cdots)$ is an univalent function.*

Proof. We may do the slight modification of Hayman's proof [1]. Let $G(w)=G(w, D_f)$ denote the Green function of D_f which has a pole at ∞. Then $G(w)-\log|w|\to -\log d(E_f)$ as $w\to\infty$. We put

$$u(z) = G(f(z))-\log\frac{1}{|z|^p}.$$

$u(z)$ is harmonic in $|z|<1$ except at the points where $f(z)$ has a pole other than $z=0$. And by means of the above stated property of $G(w)$ we have the following equality in the neighbourhood of the origin.

$$u(z) = \log|z^{-p}(1+c_1z+\cdots\cdots)|-\log d(E_f)-\log\frac{1}{|z|^p}+o(1),$$

From this

$$u(z) = \log d(E_f)+o(1).$$

Hence $u(z)$ is bounded and therefore harmonic at $z=0$.

On the other hand

$$\lim_{|z|\to 1} u(z) = \lim_{|z|\to 1} G(f(z))\geqq 0.$$

Therefore $u(z)$ is non-negative in $|z|<1$ and $-\log d(E_f)\geqq 0$, that is,

$$d(E_f)\leqq 1.$$

Equality occurs only when $u(z)\equiv 0$ in $|z|<1$. In this case $f(z)$ must approach the boundary of D_f as $|z|\to 1$ by the property of Green function and $f(z)$ is able to have only one pole of order p at the origin also. Here we use the following lemma of Heins-Hayman [1].

Suppose that $F(z)$ is meromorphic in a domain Δ, that the values which $F(z)$ takes in Δ lie in a domain D, and that as z tends to the boudary of Δ in any manner, $F(z)$ always approaches the boundary of D. Then $F(z)$ takes every value of D an equal finite number of times in Δ.

Hence we see that $f(z)$ takes every value exactly p times, and furthermore $w=f(z)$ must be a function in the form of $g(z^p)$, where $g(z)=z^{-1}(1+a_1z+\cdots\cdots)$ is an univalent function.

At this time the Green function is given by

$$\log\frac{1}{|g^{-1}(w)|} = \log\frac{1}{|z|^p}.$$

Theorem 7. *Let e be a bounded closed set of real numbers x whose Lesbesgue measure is at least 4. For each x in e let $C(x)$ be a closed set of points in the w-plane such that, if w_1, w_2 be any points on $C(x_1)$, $C(x_2)$ respectively, we have always*

$$|w_2-w_1|\geqq|x_2-x_1|.$$

Then with the hypotheses of theorem 6, D_f contains at least one of the set of $C(x)$, except possibly when e, E_f are intervals of length 4, and

$$f(z) = \frac{1}{z^p} + a_0 + z^p e^{i\lambda},$$

a_0 arbitrary, λ real arbitrary.

Proof. We can prove this theorem in the same method with Hayman's one [1]. We remark that $f(z)$ with respect to the exceptional case must be $g(z^p) = \frac{1}{z^p} + a_0 + \cdots\cdots$, where $g(z)$ is an univalent function, and E_f must contain two points w_1, w_2, such that $|w_2 - w_1| \geqq 4$, and therefore

$$f(z) = z^{-p} + a_0 + z^p e^{i\lambda}.$$

Lemma 4. *Let $f(z) = a_0 + a_p z^p + \cdots\cdots$ be meromorphic in $|z| < 1$ and let E be the set of all real positive r for which the circle $|w| = r$ meets E_f. Then we have*

$$|a_p| \int_{E_f} \frac{dr}{(|a_0| + r^2)} \leqq 4.$$

Proof. We may consider

$$w = \varphi(z) = \frac{a_p}{f(z) - a_0} = \frac{1}{z^p} + \cdots\cdots.$$

and use theorem 7 in the same way with Hayman's one [1].

Theorem 8. *Let $w = f(z) = a_0 + a_p z^p + \cdots\cdots$ is regular in $|z| < 1$. Then we have $|a_p| \leqq 4(|a_0| + l_f)$, where l_f denotes the Lebesgue measure of the set of all positive r, for which the circle $|w| = r$ lies entirely inside D_f, and it is assumed that l_f is finite. Equality occurs only when*

$$f(z) = \frac{a_p z^p}{(1 - e^{i\alpha} z^p)}, \qquad a_0 = 0$$

$$f(z) = a_0 + \frac{a_0 \lambda z^p e^{i\alpha}}{(1 - z^p e^{i\alpha})^2}, \qquad a_0 \neq 0.$$

Proof. Let

$$I = \int_E \frac{dr}{(|a_0| + r^2)}.$$

Then Hayman proved the following inequalities [1].

$$|a_0| + l_f \geqq \frac{1}{I}$$

Therefore by means of lemma 4 we have this theorem.

Accordingly we can derive the following theorem clearly from this.

Theorem 9. *Let $f(z) = z^p + a_{p+1} z^{p+1} + \cdots\cdots$ be regular in $|z| < 1$. D_f contains a circle $|w| = r$ with $r > \frac{1}{4}$ except when*

$$f(z) = \frac{z^p}{(1 - e^{i\alpha} z^p)^2}.$$

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SOME CONTRIBUTIONS TO ORDER STATISTICS

By

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Contents

Introduction. The classical problem to estimate the mean of a normal distribution is early and thoroughly discussed by K. Pearson[1] for large samples, while for small samples e.g. by T. Hôjô[2], and recently by H. J. Godwin, H. L. Jones and others[3]. Also H. Cramér[4] puts in his treatise the following example as instructive: Considering a small sample with size 3 drawn from

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1) Karl Pearson, On the probable errors of frequency constants (editorial), Part I, Biometrika, Vol. 2 (1903), pp. 273–281; Part II, Vol. 9 (1913), pp. 1–10; Part III, Vol. 13 (1921), pp. 113–132.

2) Tokishige Hôjô, Distribution of the median, quartiles and interquartile distance in samples from a normal population, Biometrika Vol. 23 (1931), pp. 315–360.

Also, Tokishige Hôjô, A further note on the relation between the median and quartiles in small samplings from a normal population, Biometrika, Vol. 25 (1933), pp. 79–90.

3) H. J. Godwin, Some low moments of order statistics, Ann. of Math. Statis. Vol. 20 (1949), pp. 279–285; H. L. Jones, Exact lower moments of order statistics in small samples from a normal distribution, Ann. of Math. Statis., Vol. 19 (1948), pp. 270–273.

4) Herald Cramér, Mathematical Methods of Statistics, (1946), p. 483.

a normal population $N(m, \sigma^2)$ and arranged in order of magnitude: $x_1 \leqq x_2 \leqq x_3$, the weighted mean $z = cx_1 + (1-2c)x_2 + cx_3$ would afford an unbiased estimate of the population mean with the variance $D^2(z) = \frac{\sigma^2}{3} + 3\sigma^2\left(2 - \frac{3\sqrt{3}}{\pi}\right)\left(c - \frac{1}{3}\right)^2$. This being generalized, we may inquire what would be $E(z)$ and $D^2(z)$, when $x_1 \leqq x_2 \leqq \cdots \leqq x_n$ be a sample drawn from the population $N(m, \sigma^2)$, and we form a weighted mean $z = \sum_{i=1}^{n} c_i x_i$, where $\sum c_i = 1$ with all $c_i \geqq 0$. In the present note we have solved this general problem in outline (Part I), and obtained somewhat detailed results for the particular cases: $n = 3, 4, \cdots, 7$ (Part II), at the end of which some applications are also illustrated. We have likewise schemed (as Part III) to investigate the third and fourth moments. However, their computations being too much enormous, their studies, except some fews, are deferred for future.

PART I

§1. Frequency Functions. We make preliminarily the variable standardized: $x = m + \sigma t$, and the distribution function $F(x) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x} \exp\left\{-\frac{1}{2\sigma^2}(x-m)^2\right\} \times dx$ reduces to $\Phi(t) = \int_{-\infty}^{t} \varphi(t)dt = \int_{-\infty}^{t} d\Phi(t)$, where $\varphi(t) = \frac{1}{\sqrt{2\pi}}\left\{-\frac{1}{2}t^2\right\}$, so that $\Phi(-\infty) = 0$, $\Phi(\infty) = 1$ and $\lim_{t \to \pm\infty} t^N \varphi(t) = 0$. Firstly, observing that the joint probability to obtain the sample $\{-\infty < x_1 \leqq x_2 \leqq \cdots \leqq x_n < \infty\}$, or $\{-\infty < t_1 \leqq t_2 \leqq \cdots \leqq t_n < \infty\}$ is

$$n!\, d\Phi_n d\Phi_{n-1} \cdots d\Phi_1,$$

where $d\Phi_i$ stands for $d\Phi(t_i) = \varphi(t_i)dt_i$, the total probability shall be given by

$$n! \int_{-\infty}^{\infty} d\Phi_n \int_{-\infty}^{n} d\Phi_{n-1} \cdots \int_{-\infty}^{2} d\Phi_1 = 1\,, \tag{1.1}$$

where $\int_{-\infty}^{i} d\Phi_{i-1}$ means $\int_{-\infty}^{t_i} d\Phi(t_{i-1})$. Still more abbreviating we write simply $\int^{i}$, $\int_{i}$ and $\int$ in place of $\int_{-\infty}^{i}$, $\int_{i}^{\infty}$ and $\int_{-\infty}^{\infty}$, respectively, and also e.g. for $\int_{-\infty}^{\infty} t_i^2 \varphi^3(t_i)\, \Phi^4(t_i)\, d\Phi(t_i)$, dropping the suffix i as well as defining argument, simply as $\int t^2 \varphi^3 \Phi^4 d\Phi$, when there is no fear of misunderstanding. On the other hand, if it needs to be made clear that the size of sample is n, we write $t_{i|n}$ instead of mere t_i.

To prove (1.1) we see successively

$$\int^{2} d\Phi_1 = \Phi_2 - \Phi_{-\infty} = \Phi_2\,, \qquad \int^{3} d\Phi_2 \int^{2} d\Phi_1 = \int^{3} \Phi_2 d\Phi_2 = \frac{1}{2!}\Phi_3^2\,,$$

and in general, if

$$\int^{i} d\Phi_{i-1} \cdots \int^{2} d\Phi_1 = \frac{1}{(i-1)!}\Phi_i^{i-1}$$

holds, then also

$$\int^{i+1} d\Phi_i \int^{i} d\Phi_{i-1} \cdots \int^{2} d\Phi_1 = \frac{1}{(i-1)!}\int^{i+1} \Phi_i^{i-1} d\Phi_i = \frac{1}{i!}\Phi_{i+1}^{i},$$

thus the induction is completed. Therefore

$$n!\int d\Phi_n \int^{n} d\Phi_{n-1} \cdots \int^{2} d\Phi_1 = n\int \Phi_n^{n-1} d\Phi_n = \Phi_n^n \Big|_{-\infty}^{\infty} = 1, \qquad \text{Q.E.D.}$$

Otherwise, if the order of integrations in the repeated integrals be interchanged, it becomes

$$n!\int d\Phi_1 \int_1 d\Phi_2 \cdots \int_{n-1} d\Phi_n = 1.$$

Now we can thereby find the frequency function of $t_{i|n}$. For this purpose we consider the repeated integrals of (1. 1)

$$n!\int d\Phi_n \int^{n} d\Phi_{n-1} \cdots \int^{i+2} d\Phi_{i+1} \int^{i+1} d\Phi_i \int^{i} d\Phi_{i-1} \cdots \int^{2} d\Phi_1,$$

and integrate the $i-1$ integrals beginning from Φ_1 up to Φ_{i-1}

$$n!\int d\Phi_n \int^{n} d\Phi_{n-1} \cdots \int^{i+2} d\Phi_{i+1} \int^{i+1} \frac{\Phi_i^{i-1}}{(i-1)!} d\Phi_i.$$

Further, interchanging the order of integrations

$$\frac{n!}{(i-1)!}\int \Phi_i^{i-1} d\Phi_i \int_i d\Phi_n \int_i^{n} d\Phi_{n-1} \cdots \int_i^{i+2} d\Phi_{i+1},$$

and finally integrating in regards to $\Phi_{i+1}, \cdots, \Phi_n$, successively, we get

$$\frac{n!}{(i-1)!\,(n-i)!}\int \Phi_i^{i-1}(1-\Phi_i)^{n-i} d\Phi_i = \beta_{i|n}\int \Phi^{i-1}(1-\Phi)^{n-i}\varphi dt,$$

where $\beta_{i|n}$ denotes the reciprocal of Betafunction $B(i, n-i+1) = \dfrac{\Gamma(i)\Gamma(n-i+1)}{\Gamma(n+1)}$, so that[5)]

(1. 2) $$\beta_{i|n} = \beta_{n-i+1|n} = \frac{n!}{(i-1)!\,(n-i)!} = n\binom{n-1}{i-1},$$

and the required frequency function is given by

(1. 3) $$f(t_{i|n}) = \beta_{i|n}\Phi^{i-1}(1-\Phi)^{n-i}\varphi.$$

Next, to find the joint frequency function of $t_{i|n}$ and $t_{k|n}$, where $1 \leq i < k \leq n$, we rewrite (1. 1) so as

$$n!\int d\Phi_n \int^{n} d\Phi_{n-1} \cdots \int^{k+2} d\Phi_{k+1} \int^{k+1} d\Phi_k \int^{k} d\Phi_{k-1} \cdots \int^{i+2} d\Phi_{i+1} \int^{i+1} \frac{\Phi_i^{i-1}}{(i-1)!} d\Phi_i,$$

which, on interchanging the order of integrations two by two respectively, beginning at Φ_{k+1} with Φ_k, then Φ_{k+2} with Φ_k, $\cdots$, and similarly beginning at Φ_{i+1} with Φ_i, then Φ_{i+2} with Φ_i, $\cdots$, becomes

5) Here the successive factors in denominator of $\beta_{i|n}$ are nothing but the factorials of the successive indices of factors in the integrand. This remark is also applicable to (1. 4) &c. below.

$$n!\int d\Phi_k \int_k d\Phi_n \int_k^n d\Phi_{n-1} \cdots \int_k^{k+2} d\Phi_{k+1} \int^k \frac{\Phi_i^{i-1}}{(i-1)!} d\Phi_i \int_i^k d\Phi_{k-1} \cdots \int_i^{i+2} d\Phi_{i+1} .$$

Now integrating successively leftwards beginning at Φ_{k+1} and Φ_{i+1} respectively,

$$n!\int \frac{(1-\Phi_k)^{n-k}}{(n-k)!} d\Phi_k \int^k \frac{\Phi_i^{i-1}(\Phi_k-\Phi_i)^{k-i-1}}{(i-1)!\,(k-i-1)!} d\Phi_i \ (=1) .$$

Hence, the required joint frequency function of t_i, t_k is found to be

$$(1.4) \quad f(t_{i|n},\ t_{k|n}) = \frac{n!}{(n-k)!\,(i-1)!\,(k-i-1)!}(1-\Phi_k)^{n-k}\varphi_k \cdot \Phi_i^{i-1}(\Phi_k-\Phi_i)^{k-i-1}\varphi_i ,$$
$$(1 \leq i < k \leq n) .$$

Here the numerical coefficient $\gamma_{i,k|n}$ say, is the reciprocal of the product of Beta-functions, because, on writing $\Phi_k=u$, $\Phi_i=v\Phi_k$ in the foregoing integral, it yields

$$\gamma_{i,k|n}\int_0^1 (1-u)^{n-k}u^{k-1}du \int_0^1 v^{i-1}(1-v)^{k-i-1}dv = \gamma_{i,k|n}B(n-k+1,\ k)\,B(i,\ k-i) = 1 .$$

Hence

$$(1.5) \quad \gamma_{i,k|n} = \frac{1}{B(n-k+1,\ k)B(i,\ k-i)} = \frac{n!}{(n-k)!\,(i-1)!\,(k-i-1)!} = \gamma_{n-k+1,\ n-i+1|n} .$$

Thus, in ordered statistics any two variables t_i, t_k are by no means independent of each other, while, if t_i, t_k are two individuals in unordered sample, they are independent and their joint frequency is simply $\varphi_i\varphi_k$.

Quite similarly we obtain

$$(1.6) \quad f(t_{i|n},\ t_{j|n},\ t_{k|n}) = \frac{n!}{(n-k)!\,(k-j-1)!\,(j-i-1)!\,(i-1)!}(1-\Phi_k)^{n-k}$$
$$\times (\Phi_k-\Phi_j)^{k-j-1}(\Phi_j-\Phi_i)^{j-i-1}\Phi_i^{i-1}\varphi_{ijk} \qquad (i<j<k)$$

where $\varphi_{ijk}=\varphi_j\varphi_j\varphi_k$, which, however, does not occur unless $n \geqq 3$. E.g. if $n=3$, we have $f(t_1,\ t_2,\ t_3)=6\ \varphi_{123}$ and if $n=4$,

$$(1.6.1) \quad f(t_1,\ t_2,\ t_3) = 24(1-\Phi_3)\,\varphi_{123} , \qquad f(t_1,\ t_2,\ t_4) = 24(\Phi_4-\Phi_2)\,\varphi_{124} ,$$
$$f(t_1,\ t_3,\ t_4) = 24(\Phi_3-\Phi_1)\,\varphi_{134} , \quad f(t_2,\ t_3,\ t_4) = 24\Phi_2\,\varphi_{234} .$$

Also, for $n=5$,

$$(1.6.2) \quad f(t_1,\ t_2,\ t_3) = 60(1-\Phi_3)^2\varphi_{123} , \quad f(t_1,\ t_2,\ t_4) = 120(1-\Phi_4)(\Phi_4-\Phi_2)\,\varphi_{124} ,$$
$$f(t_1,\ t_2,\ t_5) = 60(\Phi_5-\Phi_2)^2\varphi_{125} , \quad f(t_1,\ t_3,\ t_4) = 120(1-\Phi_4)(\Phi_3-\Phi_1)\,\varphi_{134} ,$$
$$f(t_1,\ t_3,\ t_5) = 120(\Phi_5-\Phi_3)(\Phi_3-\Phi_1)\,\varphi_{135} , \quad f(t_1,\ t_4,\ t_5) = 60(\Phi_4-\Phi_1)^2\varphi_{145} ,$$
$$f(t_2,\ t_3,\ t_4) = 120(1-\Phi_4)\Phi_2\,\varphi_{234} , \quad f(t_2,\ t_3,\ t_5) = 120(\Phi_5-\Phi_3)\Phi_2\,\varphi_{235} ,$$
$$f(t_2,\ t_4,\ t_5) = 120(\Phi_4-\Phi_2)\Phi_2\,\varphi_{245} , \quad f(t_3,\ t_4,\ t_5) = 60\Phi_3^2\,\varphi_{345} .$$

Furthermore

$$(1.7) \quad f(t_{i|n},\ t_{j|n},\ t_{k|n},\ t_{l|n})$$
$$= \frac{n!(1-\Phi_l)^{n-l}(\Phi_l-\Phi_k)^{l-k-1}(\Phi_k-\Phi_j)^{k-j-1}(\Phi_j-\Phi_i)^{j-i-1}\Phi^{i-1}}{(n-l)!\,(l-k-1)!\,(k-j-1)!\,(j-i-1)!\,(i-1)!}\varphi_{ijkl} \quad (i<j<k<l) .$$

In particular, for $n=5$,

(1.7.1) $f(t_1, t_2, t_3, t_4) = 24(1-\Phi_4)\,\varphi_{1234}$, $f(t_1, t_2, t_3, t_5) = 24(\Phi_5-\Phi_3)\,\varphi_{1235}$,
$f(t_1, t_2, t_4, t_5) = 24(\Phi_4-\Phi_2)\,\varphi_{1245}$, $f(t_1, t_3, t_4, t_5) = 24(\Phi_3-\Phi_1)\,\varphi_{1345}$,
$f(t_2, t_3, t_4, t_5) = 24\Phi_1\,\varphi_{2345}$, and so on.

Remark. Although we have confined ourselves to the case of normal population, all the above remain the same even for any continuous distribution with a frequency function $f(t)$, and the distribution $F(t)=\int^t f(t)dt$. Thus

$$n!\int dF_n \int^n dF_{n-1}\cdots\int^2 dF_1 = 1\,, \qquad f(t_{i|n}) = \beta_{i|n}F^{i-1}(1-F)^{n-i}f\,,^{6)}$$

$$f(t_{i|n},\, t_{k|n}) = \gamma_{i,k|n}(1-F_k)^{n-k}(F_k-F_i)^{k-i-1}F_i^{i-1}f_k f_i \quad (i<k) \quad \&c.$$

§ **2. The Expectation of $t^p_{i|n}$.** This is given after (1.3) by

$$E(t^p_{i|n}) = \beta_{i|n}\int \Phi^{i-1}(1-\Phi)^{n-i}\varphi t^p dt \quad \text{with} \quad \beta_{i|n}=\frac{n!}{(n-i)!\,(i-1)!}\,. \tag{2.1}$$

First, for $p=1$, $i=n$,

$$E(t_{n|n}) = n\int \Phi^{n-1}(-\varphi')dt\,,$$

which, integrated by parts, yields

$$-n\Phi^{n-1}\varphi\Big|_{-\infty}^{\infty} + n(n-1)\int \Phi^{n-2}\varphi^2 dt\,,$$

and since the integrated parts vanish, we have

$$E(t_{n|n}) = n(n-1)\int \Phi^{n-2}\varphi^2 dt\,. \tag{2.2}$$

For broader applications let us define

$$J^{(\alpha)}_\lambda = \int \Phi^\lambda \varphi^\alpha dt\,, \tag{2.3}$$

which shall be requisite to obtain $E(t_{n|n})=n(n-1)J^{(2)}_{n-2}$. Some of them are explicitly found in Part II.

In general, we obtain by integration by parts

$$E(t_{i|n}) = \beta_{i|n}\int \Phi^{i-1}(1-\Phi)^{n-i}(-\varphi')dt = \beta_{i|n}\int \frac{d}{d\Phi}[\Phi^{i-1}(1-\Phi)^{n-i}]\varphi^2 dt\,, \tag{2.4}$$

which becomes a sum of integrals of the form $J^{(2)}_\lambda$. Since the integrand in (2.4) is a polynomial in Φ of $n-2$ degrees, it suffices to know $J^{(2)}_\lambda$ for $\lambda=0, 1, \cdots, n-2$.

Next, for $p=2$, we have again by (2.1)

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6) In fact, if the i-th value from the top, i.e. the $(n-i+1)$-th value from the bottom be considered, we obtain by (1.3) $f(t_{n-i+1})=\beta_{i|n}F^{n-i}(1-F)^{i-1}f$, which agrees with Cramér's (28.6.1), p. 370, loc. cit.

(2. 5) $$E(t^2_{i|n}) = \beta_{i|n}\int t^2\Phi^{i-1}(1-\Phi)^{n-i}\varphi dt \qquad (t\varphi = -\varphi')$$

which integrated by parts yields

$$\beta_{i|n}\int \Phi^{i-1}(1-\Phi)^{n-i}d\Phi+\beta_{i|n}\int\frac{d}{d\Phi}\{\Phi^{i-1}(1-\Phi)^{n-1}\}\varphi^2 tdt\,.$$

The first integral reduces to 1, while, the second being

$$-\int\frac{d}{d\Phi}\{\Phi^{i-1}(1-\Phi)^{n-i}\}\varphi\varphi' dt\,,$$

again integrated by parts, it becomes

$$\int\frac{d^2}{d\Phi^2}\{\Phi^{i-1}(1-\Phi)^{n-i}\}\varphi^3 dt+\int\frac{d}{d\Phi}\{\Phi^{i-1}(1-\Phi)^{n-i}\}\varphi\varphi' dt\,,$$

and therefore is equal to

$$\frac{1}{2}\int\frac{d^2}{d\Phi^2}\{\Phi^{i-1}(1-\Phi)^{n-i}\}\varphi^3 dt\,.$$

Thus we get

(2. 6) $$E(t^2_{i|n}) = 1+\frac{1}{2}\beta_{i|n}\int\frac{d^2}{d\Phi^2}\{\Phi^{i-1}(1-\Phi)^{n-i}\}\varphi^3 dt\,.$$

Particularly for $i=n$

(2. 6. 1) $$E(t^2_{n|n}) = 1+\frac{1}{2}n(n-1)(n-2)\int\Phi^{n-3}\varphi^3 dt\,.$$

The explicit forms of $J^{(3)}_\lambda=\int\Phi^\lambda\varphi^3 dt$ for $\lambda=0, 1, 2, \dots$ have been evaluated in Part II, and whence all $E(t^2_{i|n})$ computed up to $n=7$.

Furthermore we obtain

(2. 7) $$\begin{aligned} E(t^3_{i|n}) &= \beta_{i|n}\int t^3\Phi^{i-1}(1-\Phi)^{n-i}\varphi dt \\ &= \frac{5}{2}E(t_{i|n})+\frac{1}{6}\beta_{i|n}\int\frac{d^3}{d\Phi^3}[\Phi^{i-1}(1-\Phi)^{n-i}]\varphi^4 dt\,, \end{aligned}$$

(2. 7. 1) $$E(t^3_{n|n}) = \frac{5}{2}E(t_{n|n})+\frac{1}{6}n(n-1)(n-2)(n-3)\int\Phi^{n-4}\varphi^4 dt\,.$$

and

(2. 8) $$E(t^4_{i|n}) = \frac{13}{3}E(t^2_{i|n})-\frac{4}{3}+\frac{1}{24}\beta_{i|n}\int\frac{d^4}{d\Phi^4}[\Phi^{i-1}(1-\Phi)^{n-i}]\varphi^5 dt\,,$$

(2. 8. 1) $$E(t^4_{n|n}) = \frac{13}{3}E(t^2_{n|n})-\frac{4}{3}+\frac{1}{24}n(n-1)(n-2)(n-3)(n-4)\int\Phi^{n-5}\varphi^5 dt\,, \quad \&c.$$

However their actual computations are deferred as future task.

We are very liable to write erroneous indices. To avoid this, it may be remarked that, in any successively obtained integrand, the sum of indices of Φ, $1-\Phi$ and φ should always be n; of course, if differentiated j times, j must be subtracted from the sum.

§ 3. Some Properties concerning $E(t^p_{i|n})$, $p=1, 2, \cdots$.

We have generally the following identities:

(3.1) $\sum_{i=1}^{n} E(t^p_{i|n}) = 0$, or $1, 3, 5, \cdots (p-1)n$, according as p is odd or even;

(3.2) $$E(t^p_{n-i+1|n}) = (-1)^p E(t^p_{i|n}) .$$

First, to prove (3.1) we cite (2.1) and (1.2); We see readily

$$\sum E(t^p_{i|n}) = n \sum \binom{n-1}{i-1} \int \Phi^{i-1}(1-\Phi)^{n-i} \varphi t^p\, dt = n \int (\Phi+1-\Phi)^{n-1} \varphi t^p\, dt$$
$$= n \int \varphi t^p\, dt = 0, \quad \text{or} \quad 1, 3, 5, \cdots (p-1)n .$$

Next, to prove (3.2) we take $U = \Phi - \frac{1}{2} = \int_0^t \varphi(t) dt = F(t)$, as independent variable, and then t becomes its inverse function: $t = F^{-1}(U)$. They are both odd monotonic functions, and (2.1) may be expressed as

$$E(t^p_{i|n}) = \beta_{i|n} \int_{-1/2}^{1/2} t^p \left(\frac{1}{2}+U\right)^{i-1} \left(\frac{1}{2}-U\right)^{n-i} dU .$$

The variable t_i is the i-th from the bottom, while the i-th from the top is t_{n-i+1}, for which

$$E(t^p_{n-i+1}) = \beta_{n-i+1|n} \int_0^1 t^p \Phi^{n-i}(1-\Phi)^{i-1} d\Phi = \beta_{i|n} \int_{-1/2}^{1/2} t^p \left(\frac{1}{2}+U\right)^{n-i} \left(\frac{1}{2}-U\right)^{i-1} dU,$$

and we shall show that the above two integrals are equal. In fact, two functions $y_1 = \left(\frac{1}{2}+U\right)^{i-1} \left(\frac{1}{2}-U\right)^{n-i}$ and $y_2 = \left(\frac{1}{2}+U\right)^{n-i} \left(\frac{1}{2}-U\right)^{i-1}$ are situated to each other symmetrically, having the axis of symmetry $U=0$, so that $y_2(U) = y_1(-U)$. But $t = F^{-1}(U)$ being odd, t^p is either odd or even, according as p is odd or even. So we have $t^p(U) y_2(U) = (-1)^p t^p(-U) y_1(-U)$, and therefore

$$\int_{-1/2}^{1/2} t^p(U) y_2(U) dU = (-1)^p \int_{-1/2}^{1/2} t^p(-U) y_1(-U) dU = (-1)^p \int_{-1/2}^{1/2} t^p(V) y_1(V) dV,$$

if $V = -U$, and this proves (3.2).

Specially, for $p=1$ and 2, we obtain

(3.2.1) $$E(t_{n-i+1|n}) = -E(t_{i|n}) \quad \text{and} \quad E(t^2_{n-i+1|n}) = E(t^2_{i|n}) .$$

Besides, if n be an odd integer $2\nu+1$, the middle variable $t_{\nu+1}$ becomes its median, m_i, whose mean is

(3.3) $$E(t_{\nu+1|2\nu+1}) = \frac{(2\nu+1)!}{\nu!\,\nu!} \int t \Phi^\nu (1-\Phi)^\nu d\Phi = 0, \quad \text{and} \quad E(x_{\nu+1}) = m .$$

(Incidentally we may remark that $E(t^p_{\nu+1|2\nu+1}) = 0$, if p odd.)

Also, when n is even and 2ν, the median is $m_i = \frac{1}{2}(t_\nu + t_{\nu+1})$, and by (3.2.1) still $E(m_i) = 0$, $E\left[\frac{1}{2}(x_\nu + x_{\nu+1})\right] = m$. The sample median is already an unbiased estimate of the population mean (=median).

Generally for $z=\sum c_i x_i = m+\sigma\sum c_i t_i$, we have

$$E(z)=m+\sigma\sum c_i E(t_i)=m+\sigma\sum_{i=1}^{[n/2]}(c_i-c_{n-i+1})E(t_i)\,.$$

As a matter of course $E(t_i)<E(t_k)$ for $i<k$, and all $E(t_{n-i+1})=-E(t_i)>0$ for $i=1, 2, \dots [n/2]$. Now z, as an estimate of mean, is unbiased, when and only when

$$\sum c_i E(t_i)=0\,,\quad \text{viz.}\ \sum_{i=1}^{[n/2]}(c_i-c_{n-i+1})E(t_i)=0\,.$$

For this, it is sufficient that all $c_i=c_{n-i+1}$ hold. In particular, let $c_i=c_{n-i+1}=\frac{1}{2}$ for a certain i and all other c_j's$=0$. Then, in view of (3. 2. 1),

$$z_i=\frac{1}{2}(x_i+x_{n-i+1})$$

becomes again an unbiased estimate of the population mean. Thus every mean of the i-ths ($i=1, 2, \dots, [n/2]$) from the bottom and the top, affords an unbiased estimate of the population mean, among them the sample median may also be adopted, as above mentioned. However, which z_i is more efficient, should be discussed from the values of $D^2(z_i)$, that will be investigated in §11.

§4. The Expectations of Products $t_{i|n}t_{k|n}$ for $1\leq i<k\leq n$.

We have by (1. 4)

$$E(t_{i|n}t_{k|n})=\gamma_{i,k|n}\int t_k(1-\Phi_k)^{n-k}d\Phi_k\int^k t_i\Phi_i^{i-1}(\Phi_k-\Phi_i)^{k-i-1}d\Phi_i\,. \tag{4. 1}$$

Or, dropping unnecessary suffices and adopting convenient one,

$$E(t_{i|n}t_{k|n})=\gamma_{i,k|n}\int(1-\Phi)^{n-k}\varphi' dt\int^t\Phi_1^{i-1}(\Phi-\Phi_1)^{k-i-1}\varphi_1' dt_1\,. \tag{4. 2}$$

Firstly make $k=i+1$. Since, on integrating the inner integral by parts, it yields

$$\int^t\Phi_1^{i-1}\varphi_1' dt_1=\Phi^{i-1}\varphi-(i-1)\int^t\Phi_1^{i-2}\varphi_1^2 dt_1\,,$$

so the whole integral becomes

$$\int(1-\Phi)^{n-k}\Phi^{i-1}\varphi\varphi' dt-(i-1)\int(1-\Phi)^{n-k}\varphi' dt\int^t\Phi_1^{i-2}\varphi_1^2 dt_1\,.$$

These being once more integrated by parts, respectively, we obtain

$$E(t_{i|n}t_{i+1|n})=\frac{n!}{(n-i-1)!\,(i-1)!}\Big[-\frac{1}{2}\int\frac{d}{d\Phi}[\Phi^{i-1}(1-\Phi)^{n-i-1}]\varphi^3 dt \tag{4. 3}$$
$$+(i-1)\int(1-\Phi)^{n-i-1}\Phi^{i-2}\varphi^3 dt-(i-1)(n-i-1)\int(1-\Phi)^{n-i-2}\varphi^2 dt\int^t\Phi_1^{i-2}\varphi_1^2 dt_1\Big].$$

Here both the first and second integrals are sums of $J_\lambda^{(3)}$ in (2. 3), while the third is a sum of double integrals of the form:

$$J_{\mu,\nu}^{\alpha,\beta}=\int\Phi^\mu\varphi^\alpha dt\int^t\Phi_1^\nu\varphi_1^\beta dt_1\,, \tag{4. 4}$$

and the double integrals presenting in (4.3) are those with $\alpha=\beta=2$, and $\mu+\nu=n-4$ at most[7].

However, if $i=1$ or $n-1$, the double integral disappears and simply

(4.3.1) $$E(t_{1|n}t_{2|n})=\frac{1}{2}n(n-1)(n-2)J^{(3)}_{n-3}=E(t_{n-1|n}t_{n|n}), \quad \&c.$$

Secondly, for $k=i+2$, the inner integral in (4.2) yields

$$-\int^t\frac{\partial}{\partial\Phi_1}[\Phi_1^{i-1}(\Phi-\Phi_1)]\varphi_1^2dt_1=-(i-1)\Phi\int^t\Phi_1^{i-2}\varphi_1^2dt_1+i\int^t\Phi_1^{i-1}\varphi_1^2dt_1,$$

so that by integrations by parts we get

(4.5) $$E(t_{i|n}t_{i+2|n})=\gamma_{i,i+2|n}\left\{-\int(1-\Phi)^{n-i-2}\varphi' dt\int^t\frac{\partial}{\partial\Phi_1}[\Phi_1^{i-1}(\Phi-\Phi_1)]\varphi_1^2dt_1\right\}$$
$$=\frac{n!}{(n-i-2)!\,(i-1)!}\left\{-(n-i-2)\int(1-\Phi)^{n-i-3}\varphi^2dt\int^t\frac{\partial}{\partial\Phi_1}[\Phi_1^{i-1}(\Phi-\Phi_1)]\varphi_1^2dt_1\right.$$
$$\left.+(i-1)\int(1-\Phi)^{n-i-2}\varphi^2dt\int^t\Phi_1^{i-2}\varphi_1^2dt_1-\int(1-\Phi)^{n-i-2}\Phi^{i-1}\varphi^3dt\right\}.$$

Here the first two integrals consist of $J_{\mu\nu}$ with $\mu+\nu=n-4$ at most, and the remaining integrals are all of type $J^{(3)}_\lambda$. In particular, if $i=1$,

(4.5.1) $$E(t_{1|n}t_{3|n})=n(n-1)(n-2)\left[(n-3)\int(1-\Phi)^{n-4}\varphi^2dt\int^t\varphi_1^2dt_1-J^{(3)}_{n-3}\right].$$

Thirdly, for $k\geqq i+3$, the inner integral when differentiated, becomes

$$\frac{d}{dt}\left\{-\int^t\frac{\partial}{\partial\Phi_1}[\Phi_1^{i-1}(\Phi-\Phi_1)^{k-i-1}]\varphi_1^2dt_1\right\}=-(k-i-1)\int^t\frac{\partial}{\partial\Phi_1}[\Phi_1^{i-1}(\Phi-\Phi_1)^{k-i-2}]\varphi_1^2\varphi dt_1,$$

so we obtain

(4.6) $$E(t_{i|n}t_{k|n})=\frac{n!}{(n-k)!\,(i-1)!\,(k-i-1)!}\left\{-(n-k)\int(1-\Phi)^{n-k-1}\varphi^2dt\right.$$
$$\times\int^t\frac{\partial}{\partial\Phi_1}[\Phi_1^{i-1}(\Phi-\Phi_1)^{k-i-1}]\varphi_1^2dt_1+(k-i-1)\int(1-\Phi)^{n-k}\varphi^2dt$$
$$\left.\times\int^t\frac{\partial}{\partial\Phi_1}[\Phi_1^{i-1}(\Phi-\Phi_1)^{k-i-2}]\varphi_1^2dt_1\right\}\qquad(k\geqq i+3),$$

every term of which belongs to $J_{\mu\nu}$ with $\mu+\nu=n-4$ at most. In particular, if $i=1$,

(4.6.1) $$E(t_{1|n}t_{k|n})=\frac{n!}{(n-k-1)!\,(k-3)!}\int(1-\Phi)^{n-k-1}\varphi^2dt\int^t(\Phi-\Phi_1)^{k-3}\varphi_1^2dt_1$$
$$-\frac{n!}{(n-k)!\,(k-4)!}\int(1-\Phi)^{n-k}\varphi^2dt\int^t(\Phi-\Phi_1)^{k-4}\varphi_1^2dt_1.$$

Of course, if the denominator contains a factorial of negative integer, i.e. ∞, then that term must be reckoned as zero.

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7) However, those below being almost $\alpha=\beta=2$, we write simply $J_{\mu\nu}$ for $J^{2,2}_{\mu,\nu}$

§ 5. Some Properties concerning $E(t^p_{i|n} t^q_{k|n})$ with $p, q = 1, 2, 3 \cdots$, and $i \neq k$.

Although computations of $E(t^p_i t^q_k)$ for $p, q > 1$ are now postponed for future, some general properties may here be discussed. We have indeed the following identities:

(5.1) $E(t^p_{i|n} t^q_{k|n}) = E(t^p_{i'|n} t^q_{k'|n})$, and in particular $E(t_{i|n} t_{k|n}) = E(t_{i'|n} t_{k'|n})$,

where $i' = n-i+1$, $k' = n-k+1$. This can be shown similarly as (3.2) proved: Really by (4.2), but now considering Φ, Φ_1 as two independent variables, and t, t_1 as their functions

$$E(t^p_{i|n} t^q_{k|n}) = \gamma_{i,k|n} \int_0^1 t^q (1-\Phi)^{n-k} d\Phi \int_0^\Phi t_1^p \Phi_1^{i-1} (\Phi - \Phi_1)^{k-i-1} d\Phi_1 \quad (i < k) . \tag{5.2}$$

From the relations $i' = n-i+1$, $k' = n-k+1$, $i < k$, it follows that $k' < i'$ and $n-i' = i-1$, $k'-1 = n-k$, $i'-k'-1 = k-i-1$, so that

$$E(t^p_{i'} t^q_{k'}) = \gamma_{k',i'|n} \int_0^1 t^p (1-\Phi)^{i-1} d\Phi \int_0^\Phi t_1^q \Phi_1^{n-k} (\Phi - \Phi_1)^{k-i-1} d\Phi_1 \quad (k' < i') .$$

But $\gamma_{i,k|n} = \gamma_{k',i'|n}$ by (1.5). Hence we have only to show that the above two integrals are equal. The latter integral, on interchanging the order of integrations yields

$$\int_0^1 t_1^q \Phi_1^{n-k} d\Phi_1 \int_{\Phi_1}^1 t^p (1-\Phi)^{i-1} (\Phi - \Phi_1)^{k-i-1} d\Phi ,$$

which, on changing the names of independent variables Φ, Φ_1 and their functions t, t_1, anew by $\Psi_1 = 1-\Phi$, $\Psi = 1-\Phi_1$ and T_1, T, respectively, reduces to

$$\int_0^1 T^q (1-\Psi)^{n-k} d\Psi \int_0^\Psi T_1^p \Psi_1^{i-1} (\Psi - \Psi_1)^{k-i-1} d\Psi_1 ,$$

The last integral, however, just equals that of (5.2), because the definite integral is quite immaterial to the letters of integration variables.

Also, we can compute the value of

$$E_{pq} \equiv \sum_{k=2}^{n} \sum_{i=1}^{k-1} E(t^p_{i|n} t^q_{k|n}) \qquad \text{for} \quad p, q = 1, 2, \cdots , \tag{5.3}$$

which is useful for purpose to check calculations of cross moments of order $p+q$. In fact we obtain by means of (4.1)

$$\begin{aligned} E_{pq} &= \sum_{k=2}^{n} \sum_{i=1}^{k-1} \frac{n!}{(n-k)!\,(i-1)!\,(k-i-1)!} \int t^q (1-\Phi)^{n-k} \varphi\, dt \int^t t_1^p \Phi_1^{i-1} (\Phi - \Phi_1)^{k-i-1} \varphi_1 dt_1 \\ &= n(n-1) \int \sum_{k=2}^{n} \binom{n-2}{n-k} t^q (1-\Phi)^{n-k} \varphi\, dt \int^t \sum_{i=1}^{k-1} \binom{k-2}{i-1} t_1^p \Phi_1^{i-1} (\Phi - \Phi_1)^{k-i-1} \varphi_1 dt_1 \\ &= n(n-1) \int \sum_{k=2}^{n} \binom{n-2}{k-2} t^q (1-\Phi)^{n-k} \Phi^{k-2} \varphi\, dt \int^t t_1^p \varphi_1 dt_1 \\ &= n(n-1) \int t^q \varphi\, dt \int^t t_1^p \varphi_1 dt_1 . \end{aligned}$$

To evaluate this, we transform (t_1, t) into polar co-ordinates (r, θ):

$$E_{pq}=\frac{n(n-1)}{2\pi}\int_{\pi/4}^{5\pi/4}\sin^q\theta\cos^p\theta\,d\theta\int_0^\infty \exp\left\{-\frac{1}{2}r^2\right\}r^{p+q+1}dr$$

$$=\frac{n(n-1)}{2\pi}I_{pq}\int_0^\infty (2u)^{\frac{p+q}{2}}e^{-u}du\ \ (2u=r^2)=\frac{n(n-1)}{2\pi}2^{\frac{p+q}{2}}\Gamma\left(\frac{p+q}{2}+1\right)I_{pq}$$

where $I_{pq}=\int_{\pi/4}^{5\pi/4}\sin^q\theta\cos^p\theta\,d\theta$ and their values as well as the required sum of expectations are tabulated in the following:

p	1	2	1	2	3	1
q	1	1	2	2	1	3
I_{pq}	0	$\frac{1}{3\sqrt{2}}$	$-\frac{1}{3\sqrt{2}}$	$\frac{\pi}{8}$	0	0
$\frac{E}{n(n-1)}$	0	$\frac{1}{4\sqrt{\pi}}$	$-\frac{1}{4\sqrt{\pi}}$	$\frac{1}{2}$	0	0

More generally we have the following results:

(i) When p and q both odd, $E_{pq}=0$.

(ii) When one odd and the other even, e.g. let $p=2r$ and $q=2s+1$, then

$$E_{pq}=\frac{n(n-1)}{2\pi}2^{r+s+1/2}\Gamma\left(r+s+\frac{3}{2}\right)I_{pq},$$

where

$$I_{pq}=2\int_0^{1/\sqrt{2}}u^{2r}(1-u^2)^s\,du=\sqrt{2}\sum_{\nu=0}^{s}(-1)^\nu\binom{\nu}{s}\frac{1}{(2r+2s+1)\,2^{r+s}}>0,$$

whereas the value $E_{qp}=-E_{pq}$.

(iii) When p and q both even, the formulaes become somewhat intricate. For simplicity, e.g. if we assume that $p=q=2r$, we get

$$E_{pp}=\frac{n(n-1)}{2}\left(\frac{(2r)!}{2^r r!}\right)^2.$$

Although for the present the above fragments suffice, with the purpose of later reference to higher moments, let us consider some still further general cases:

When the number of related arguments are 3, we obtain, similarly as in (5.3),

(5.4) $$\sum_{k=3}^{n}\sum_{j=2}^{k-1}\sum_{i=1}^{j=1}E(t_{i/n}^p t_{j/n}^q t_{k/n}^r)\qquad(1\leq i<j<k\leq n,\ n\geq 3)$$

$$=n!\sum\sum\sum\int\frac{(1-\Phi_k)^{n-k}}{(n-k)!}\varphi_k t_k^r dt_k\int^{t_k}\frac{(\Phi_k-\Phi_i)^{k-j-1}}{(k-j-1)!}\varphi_j t_j^q dt_j$$

$$\times\int^{t_j}\frac{(\Phi_j-\Phi_i)^{j-i-1}\Phi_i^{i-1}}{(j-i-1)!\ (i-1)!}\varphi_i t_i^p dt_i$$

$$=\frac{n(n-1)(n-2)}{\sqrt{2\pi}^3}\int t_k^r\varphi_k dt_k\int^{t_k}t_j^q\varphi_j dt_j\int^{t_j}t_i^p\varphi_i dt_i,$$

which vanishes, if all p, q, r be odd, and in particular

(5.5) $$\sum\sum\sum E(t_i t_j t_k)=0.$$

Also

$$(5.6)\quad \sum\sum\sum E(t_i^2 t_j t_k)=\sum\sum\sum E(t_i t_j t_k^2)=-\frac{1}{2}\sum\sum\sum E(t_i t_j^2 t_k)=\frac{n(n-1)(n-2)}{12\pi\sqrt{3}},$$

and consequently

$$(5.7)\qquad \sum\sum\sum \{E(t_i^2 t_j t_k)+E(t_i t_j^2 t_k)+E(t_i t_j t_k^2)\}=0\,.$$

With 4 arguments, we have

$$(5.8)\quad \sum_{l=4}^{n}\sum_{k=3}^{l-1}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1} E(t_i^p t_j^q t_k^r t_l^s)\qquad (1\leqq i<j<k<l\leqq n,\ n\geqq 4)$$

$$=n!\int\frac{(1-\Phi)^{n-l}}{(n-l)!}\varphi_l t_l^s dt_l\int^{t_l}\frac{(\Phi_l-\Phi_k)^{l-k-1}}{(l-k-1)!}\varphi_k t_k^q dt_k$$

$$\times\int^{t_k}\frac{(\Phi_k-\Phi_j)^{k-j-1}}{(k-j-1)!}\varphi_j t_j^q dt_j\int^{t_j}\frac{(\Phi_j-\Phi_i)^{j-i-1}}{(j-i-1)!}\frac{\Phi_i{}^{i-1}}{(i-1)!}\varphi_i t_i^p dt_i$$

$$=n(n-1)(n-2)(n-3)\int t_l^s\varphi_l dt_l\int^{t_l}t_k^r\varphi_k dt_k\int^{t_k}t_j^q\varphi_j dt_j\int^{t_j}t_i^2\varphi_i dt_i\,,$$

and particularly

$$(5.9)\qquad \sum\sum\sum\sum E(t_i t_j t_k t_l)=0\,.$$

No further values of $p, q, r\cdots$ are needed if we confine ourselves up to moments of order 3 and 4.

Returning to the case with 2 arguments, we have a remarkable identity

$$(5.10)\qquad \sum_{k=1}^{n}E(t_{i|n}t_{k|n})=1,\qquad (i=1, 2, \cdots, n \text{ being fixed})$$

which is very useful, as check, when all $E(t_{i|n}t_{k|n})$ are computed.

To prove (5.10), let us put

$$\sum_{k=1}^{n}=\sum_{k=1}^{i-1}+E(t_{i|n}^2)+\sum_{k=i+1}^{n}=(\mathrm{i})+(\mathrm{ii})+(\mathrm{iii})\,.$$

First by (4.2)

$$(\mathrm{i})=\sum_{k=1}^{i-1}\frac{n!}{(n-i)!(k-1)!(i-k-1)!}\int(1-\Phi)^{n-i}\varphi' dt\int^{t}\Phi_1^{k-1}(\Phi-\Phi_1)^{i-k-1}\varphi_1' dt_1$$

$$=\frac{n!}{(n-i)!(i-2)!}\int(1-\Phi)^{n-i}\varphi' dt\int^{t}\sum_{k=1}^{i-1}\frac{(i-2)!}{(k-1)!(i-k-1)!}\Phi_1^{k-1}(\Phi-\Phi_1)^{i-k-1}\varphi_1' dt_1$$

$$=\frac{n!}{(n-i)!(i-2)!}\int(1-\Phi)^{n-i}\varphi' dt\int^{t}\Phi^{i-2}\varphi_1' dt_1$$

$$=\frac{n!}{(n-i)!(i-2)!}\int(1-\Phi)^{n-i}\Phi^{i-2}\varphi\varphi' dt\,.$$

Next by (2.5)

$$(\mathrm{ii})=E(t_{i|n}^2)=1-\frac{n!}{(n-i)!(i-1)!}\int\frac{d}{d\Phi}[\Phi^{i-1}(1-\Phi)^{n-i}]\varphi\varphi' dt\,.$$

And lastly again by (4.2)

$$(\mathrm{iii})=\sum_{k=i+1}^{n}\int\frac{n!}{(n-k)!(i-1)!(k-i-1)!}\int(1-\Phi)^{n-k}\varphi' dt\int^{t}\Phi_1^{i-1}(\Phi-\Phi_1)^{k-i-1}\varphi_1' dt_1\,,$$

which, on interchanging the order of integrations, and rearranging factors, equals

$$\frac{n!}{(n-i-1)!\,(i-1)!}\int \Phi_1^{i-1}\varphi_1'dt_1\int_{t_1}\sum_{k=i+1}^{n}\frac{(n-i-1)!}{(n-k)!(k-i-1)!}(1-\Phi)^{n-k}(\Phi-\Phi_1)^{k-i-1}\varphi' dt$$

$$=\frac{n!}{(n-i-1)!\,(i-1)!}\int \Phi^{i-1}(1-\Phi)^{n-i-1}(-\varphi\varphi')dt\,.$$

Therefore

$$\sum_{k=1}^{n}E(t_{i|n}t_{k|n})=1+\int\frac{n!}{(n-i)!\,i!}\int\Big[i(i-1)\Phi^{i-2}(1-\Phi)^{n-i}-i(n-i)\Phi^{i-1}(1-\Phi)^{n-i-1}$$
$$-i\frac{d}{d\Phi}[\Phi^{i-1}(1-\Phi)^{n-i}]\Big]\varphi\varphi' dt\,,$$

in which the integrand reduces to zero, and consequently the whole expression reduces to unity, Q. E. D.

By virtue of (3.2·1) we have only to calculate $E(t_{i|n})$ and $E(t^2_{i|n})$ for each set

$$A: t_1<t_2<\cdots<t_{\left[\frac{n+1}{2}\right]},\quad \text{or}\quad B: t_n>t_{n-1},>\cdots>t_{\left[\frac{n+1}{2}\right]}.$$

Of course

$$|E(t_1)|=E(t_n)>|E(t_2)|=E(t_{n-1})>\cdots,\ \text{and}\ E(t_1^2)=E(t_n^2)>E(t_2^2)=E(t_{n-1}^2)>\cdots.$$

The mean $E(t_it_k)$ is positive if $i<k$ both belong to A only or B only, and by (5.1) it suffices to calculate about A only. On the other hand, if i and k be taken each from A and B, respectively, then $E(t_it_k)$ is negative. And indeed $E(t_it_k)$ has the larger absolute value, the farther they lie from center.

§6. The Frequency Function of $\zeta=\sum c_it_i$ &c.

So far we have discussed about $E(t_{i|n})$, $E(t^2_{i|n})$ and $E(t_{i|n}t_{k|n})$. Thereby all the following, which concern with $x_i=m+\sigma t_i$, could be computed:

$$E(x_i)=m+\sigma E(t_i)\,,\quad E(x_i^2)=m^2+2m\sigma E(t_i)+\sigma^2E(t_i^2)\,,\quad D^2(x_i)=E(x_i^2)-E(x_i)^2\,,$$
$$E(x_ix_k)=m^2+m\sigma[E(t_i)+E(t_k)]+\sigma^2E(t_it_k)\,.$$

Further, putting $z=\sum_{i=1}^{n}c_ix_i$ and $\zeta=\sum_{i=1}^{n}c_it_i$ under $\sum c_i=1$ with $c_i\geqq 0$, all the following also can be carried out:

$$z=\sum c_i(m+\sigma t_i)=m+\sigma\zeta,\ E(z)=m+\sigma E(\zeta)\,,\ E(z^2)=m^2+2m\sigma E(\zeta)+\sigma^2E(\zeta^2)\,,$$
$$E(\zeta)=\sum c_iE(t_i),\quad E(\zeta^2)=\sum c_i^2E(t_i^2)+2\sum_{j<k}c_jc_kE(t_jt_k)\,,$$

and finally

$$D^2(\zeta)=E(\zeta^2)-E(\zeta)^2=\sum c_i^2[E(t_i^2)-E(t_i)^2]+2\sum_{j<k}c_jc_k[E(t_jt_k)-E(t_j)E(t_k)]\,,$$
$$D^2(z)=\sigma^2D^2(\zeta)=\sigma^2\sum c_i^2\,\mathrm{Var}\,(t_i)+2\sigma^2\sum_{j<k}c_jc_k\,\mathrm{Cov}\,(t_j,\,t_k)\,.$$

However, before discussing $D^2(z)$, as Cramér advises[8)], it is desirable first to find the fr. f. $g(z, m)$. Let us consider his example and find the fr. f.

8) Cramér, loc. cit., p. 483, l. 8.

$g(\zeta, 0)$ of $\zeta = ct_1 + (1-2c)t_2 + ct_3$. Here the probability element is

(6.1) $$dP = 6d\Phi_3 d\Phi_2 d\Phi_1 = \frac{6}{\sqrt{2\pi^3}} \exp\left\{-\frac{1}{2}(t_1^2+t_2^2+t_3^2)\right\} dt_1 dt_2 dt_3, \quad (t_1 \leqq t_2 \leqq t_3).$$

Now transforming $\{t_1, t_2, t_3\}$ into $\{y_1, y_2, y_3\}$ orthogonally, as e.g.

$$y_1 = -\frac{1}{\sqrt{2}} t_1 + \frac{1}{\sqrt{2}} t_3,$$
$$y_2 = \frac{(1-2c)t_1 - 2ct_2 + (1-2c)t_3}{\sqrt{2}\,\gamma}, \qquad \text{with } \gamma = \sqrt{2c^2 + (1-2c)^2} = \sqrt{1-4c+6c^2} > 0$$
$$y_3 = \frac{ct_1 + (1-2c)t_2 + ct_3}{\gamma}\left(= \frac{\zeta}{\gamma}\right), \qquad \text{and } J = \frac{\partial(y_1, y_2, y_3)}{\partial(t_1, t_2, t_3)} = 1,$$

we get, on writing rows in columns,

$$t_1 = -\frac{1}{\sqrt{2}} y_1 + \frac{1-2c}{\sqrt{2}\,\gamma} y_2 + \frac{c}{\gamma} y_3,$$
$$t_2 = -\frac{2c}{\sqrt{2}\,\gamma} y_2 + \frac{1-2c}{\gamma} y_3,$$
$$t_3 = \frac{1}{\sqrt{2}} y_1 + \frac{1-2c}{\sqrt{2}\,\gamma} y_2 + \frac{c}{\gamma} y_3,$$

and, because of orthogonal transformation,

$$t_1^2 + t_2^2 + t_3^2 = y_1^2 + y_2^2 + y_3^2.$$

The order $t_1 \leqq t_2 \leqq t_3$ yields

$$\frac{\sqrt{2}(1-3c)\zeta}{\gamma} - \gamma y_1 (= A(y_1, \zeta)) \leqq y_2 \leqq \frac{\sqrt{2}(1-3c)\zeta}{\gamma} + \gamma y_1 (= B(y_1, \zeta) \text{say}),$$

and $$y_1 \geqq 0.$$

With this order we obtain

(6.2) $$dP = \frac{6}{\sqrt{2\pi^3}} \exp\left\{-\frac{1}{2}(y_1^2+y_2^2+y_3^2)\right\} dy_1 dy_2 dy_3, \qquad \left(y_3 = \frac{\zeta}{\gamma}\right),$$

and whence the required fr. f. to be

(6.3) $$g(\zeta) = \frac{3}{\pi\sqrt{2\pi}\,\gamma} \exp\left\{-\frac{\zeta^2}{2\gamma^2}\right\} \int_0^\infty \exp\left\{-\frac{1}{2} y_1^2\right\} dy_1 \int_{A(y_1,\zeta)}^{B(y_1,\zeta)} \exp\left\{-\frac{1}{2} y_2^2\right\} dy_2.$$

The last double integral yields, on transforming integration-variables (y_1, y_2) into polar co-ordinates (r, θ)

(6.4) $$\iint_D \exp\left\{-\frac{1}{2} r^2\right\} r\,dr\,d\theta$$

in which D denotes the domain bounded by the two half straight lines KL, KH (Fig. 1, 2 or 3 according as the y_2-intercept $b \gtrless 0$ or zero), whose equations are $y_2 = A(y_1, \zeta)$ and $y_2 = B(y_1, \zeta)$, namely

$$y_2 = \mp\gamma y_1 + \sqrt{2}(1-3c)\zeta/\gamma = y_1 \tan(\mp\alpha) + b,$$

where $\alpha = \mathrm{Tan}^{-1}\gamma$ denotes a positive accute angle and $b = \sqrt{2}(1-3c)\zeta/\gamma$. Or, their Hessian equations are

$$y_1 \cos\varphi + y_2 \sin\varphi = p,$$

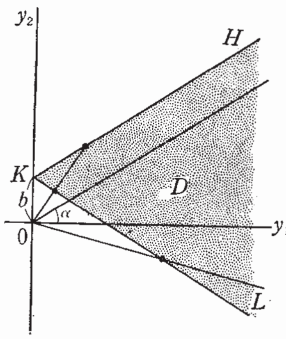

Fig. 1

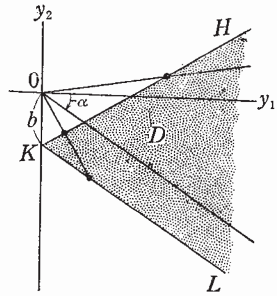

Fig. 2

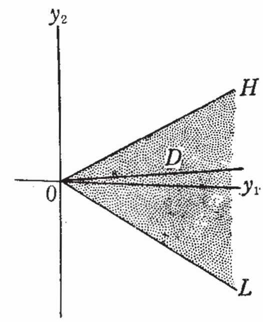

Fig. 3

where

$$p=\frac{|b|}{\sqrt{1+\gamma^2}}=\frac{|\sqrt{2}(1-3c)\zeta|}{\gamma\sqrt{1+\gamma^2}}=\text{the perpendicular distance from origin} \geqq 0\,,$$

$$\varphi=\text{angle the perpendicular makes with } +y_1\text{-axis},$$

so that, in detail, if $\beta=\frac{\pi}{2}-\alpha$ (positive accute angle), 1° in Fig. 1 ($b>0$) $\varphi=\frac{\pi}{2}-\alpha=\beta$ for KL and 2° $\varphi=\frac{\pi}{2}+\alpha=\pi-\beta$ for KH, while 3° in Fig. 2 ($b<0$), $\varphi=-\left(\frac{\pi}{2}-\alpha\right)=-\beta$ for KH, and 4° $\varphi=-\left(\frac{\pi}{2}+\alpha\right)=-(\pi-\beta)$ for KL.

We have, therefore, in all cases as their polar equations

$$R=p\sec(\theta-\varphi)\,.$$

Again, in detail, 1° $R_1=p\sec(\theta-\beta)$, 2° $R_2=-p\sec(\theta+\beta)$, 3° $R_2=p\sec(\theta+\beta)$, 4° $R_1=-p\sec(\theta-\beta)$. Hence we have

Case I. For Fig. 1 ($b=\sqrt{2}(1-3c)\zeta/\gamma>0$) (6.4) yields

$$\iint_D=\int_{-\alpha}^{\alpha}d\theta\int_{R_1}^{\infty}e^{-r^2/2}r\,dr+\int_{\alpha}^{\pi/2}d\theta\int_{R_1}^{R_2}e^{-r^2/2}r\,dr=\int_{-\alpha}^{\alpha}e^{-R_1^2/2}d\theta+\int_{\alpha}^{\pi/2}\left[e^{-R_1^2/2}-e^{-R_2^2/2}\right]d\theta$$

$$=\int_{-\alpha}^{\pi/2}e^{-R_1^2/2}d\theta-\int_{\alpha}^{\pi/2}e^{-R_2^2/2}d\theta$$

$$=\int_{-\alpha}^{\pi/2}\exp\left\{-\frac{p^2}{2}\sec^2(\theta-\beta)\right\}d\theta-\int_{\alpha}^{\pi/2}\exp\left\{-\frac{p^2}{2}\sec(\theta+\beta)\right\}d\theta=J\text{ say}\,.$$

Or, upon putting $\theta-\beta=u$ and $\theta+\beta=u+\pi$, respectively, and $p^2=\frac{2(1-3c)^2}{\gamma^2(1+\gamma^2)}\zeta^2=N\zeta^2$, we have, in view of $\alpha+\beta=\pi/2$,

$$J=\int_{-\pi/2}^{\alpha}\exp\left\{-\frac{N}{2}\zeta^2\sec^2u\right\}du-\int_{-\pi/2}^{-\alpha}\exp\left\{-\frac{N}{2}\zeta^2\sec^2u\right\}du$$

$$=\int_{-\alpha}^{\alpha}=2\int_0^{\alpha}\exp\left\{-\frac{N}{2}\zeta^2\sec^2u\right\}du\,.$$

Case II. For Fig. 2 ($b=\sqrt{2}(1-3c)\zeta/\gamma<0$)

$$\iint_D=\int_{-\pi/2}^{-\alpha}d\theta\int_{R_2}^{R_1}e^{-r^2/2}r\,dr+\int_{-\alpha}^{\alpha}d\theta\int_{R_2}^{\infty}e^{-r^2/2}r\,dr$$

$$=\int_{-\pi/2}^{-\alpha}\left[e^{-R_2^2/2}-e^{-R_1^2/2}\right]d\theta\quad+\int_{-\alpha}^{\alpha}e^{-R_2^2/2}\,d\theta=\int_{-\pi/2}^{\alpha}e^{-R_2^2/2}\,d\theta-\int_{-\pi/2}^{-\alpha}e^{-R_1^2/2}\,d\theta$$

$$= \int_{-\pi/2}^{\alpha} \exp\left\{-\frac{p^2}{2}\sec^2(\theta+\beta)\right\} d\theta - \int_{-\pi/2}^{-\alpha} \exp\left\{-\frac{p^2}{2}\sec^2(\theta-\beta)\right\} d\theta$$

$$= \int_{-\alpha}^{\pi/2} \exp\left\{-\frac{p^2}{2}\sec^2 u\right\} du - \int_{\alpha}^{\pi/2} \exp\left\{-\frac{p^2}{2}\sec^2 u\right\} du = J.$$

Case. III. For Fig. 3, $b=\sqrt{2}(1-3c)\zeta/\gamma=0$. In this case the two straight lines OL, OH reduce to $\theta=\mp\alpha$, so that

$$\iint_D = \int_{-\alpha}^{\alpha} d\theta \int_0^{\infty} e^{-r^2/2}\, r\, dr = 2\alpha .$$

But, when $c \neq \frac{1}{3}$, $p\to 0$, as $\zeta\to 0$ and $\lim_{\zeta\to 0} J=2\alpha$, so that Case III may be attached to Case I or II by deeming its open integration-interval as closed: $b\geqq 0$ or $b\leqq 0$. However, when $c=\frac{1}{3}$, we should consider this case as a special one: Really, in this trivial case, $\alpha=\tan^{-1}\gamma=\tan^{-1}\frac{1}{\sqrt{3}}$, so that the required fr. f. reduces to

$$(6.5) \qquad g(\zeta) = \frac{1}{\sqrt{2\pi}/\sqrt{3}} \exp\left\{-\frac{\zeta^2}{2/3}\right\}, \qquad \text{namely } N\left(0, \frac{1}{3}\right)$$

which is the fr. f. of the A. M.

Returning to the general case that $c \neq \frac{1}{3}$, $\zeta \gtreqless 0$, we have to contemplate

$$(6.6) \qquad g(\zeta) = \frac{3}{\pi\gamma}\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\zeta^2}{2\gamma^2}\right\} \cdot J = \frac{6}{\pi\gamma\sqrt{2\pi}} \exp\left\{-\frac{\zeta^2}{2\gamma^2}\right\}$$

$$\times \int_0^{\alpha} \exp\left\{-\frac{N}{2}\zeta^2\sec^2 u\right\} du = \frac{6}{\pi\gamma\sqrt{2\pi}} \int_0^{\alpha} \exp\left\{-\frac{\zeta^2}{2}(M+N\sec^2 u)\right\} du,$$

where
$$M=\frac{1}{\gamma^2}, \quad N=\frac{2(1-3c)^2}{\gamma^2(1+\gamma^2)}.$$

Thus $g(\zeta)$ is an even function, and accordingly

$$(6.7) \qquad E(\zeta^{2p+1}) = \int \zeta^{2p+1} g(\zeta) d\zeta = 0 .$$

Further

$$E(\zeta^2) = \frac{6}{\pi\gamma\sqrt{2\pi}} \int \zeta^2 g(\zeta) d\zeta = \frac{6}{\pi\gamma}\int_0^{\alpha}\left[M+N\sec^2 u\right]^{-\frac{3}{2}} du = \frac{6}{\pi\gamma}\int_0^{\alpha} \frac{\cos^3 u\, du}{\sqrt{M\cos^2 u+N}^3} .$$

On computing the corresponding indefinite integral, we obtain

$$\int \frac{\cos^3 u\, du}{\sqrt{M\cos^2 u+N}^3} = \frac{1}{M}\int \frac{\cos u\, du}{\sqrt{M\cos^2 u+N}} - \frac{N}{M}\int \frac{\cos u\, du}{\sqrt{M\cos^2 u+N}^3}$$

$$= \frac{1}{M}\int \frac{\cos u\, du}{\sqrt{M+N-M\sin^2 u}} - \frac{N}{M}\int \frac{\cos u\, du}{\sqrt{M+N-M\sin^2 u}^3}$$

$$= \frac{1}{M^{\frac{3}{2}}} \sin^{-1}\left(\sqrt{\frac{M}{M+N}}\sin u\right) - \frac{N}{M(M+N)} \frac{\sin u}{\sqrt{M+N-M\sin^2 u}} .$$

Substituting this result in the foregoing definite integral, we get

(6. 8) $$\mu_2 = E(\zeta^2) = \frac{6}{\pi\gamma}\left[\gamma^3 \sin^{-1}\frac{1}{2} - \frac{(3\gamma^2-1)\gamma}{4\sqrt{3}}\right] = \gamma^2 - \frac{(3\gamma^2-1)\sqrt{3}}{2\pi}$$
$$= \frac{1}{3} + \frac{3}{\pi}[2\pi - 3\sqrt{3}]\left(c-\frac{1}{3}\right)^2 \geqq \frac{1}{3}.$$

The above reduction is too much lengthy. We have already seen in § 3 that $E(\zeta)=0$, while $\mu_2 = D^2(\zeta)$ will later be readily obtained in § 11 by aid of $E(t_{i|3}t_{k'3})$. Notwithstanding we dared to deduce the latter from the fr. f. $g(\zeta)$ at the cost of a duplication, which, however, is intended to show that $\mu_4 = E(\zeta^4)$ can be likewise computed. In fact

$$\mu_4 = E(\zeta^4) = \int \zeta^4 g(\zeta) d\zeta = \frac{6}{\pi\gamma}\int_0^{\alpha} du \int \zeta^4 \exp\left\{-\frac{\zeta^2}{2}(M+N\sec^2 u)\right\}\frac{d\zeta}{\sqrt{2\pi}}$$
$$= \frac{18}{\pi\gamma}\int_0^{\alpha}\frac{\cos^5 u\, du}{\sqrt{M\cos^2 u + N}^5} = \frac{18\gamma^4}{\pi}\int_0^{\sin\alpha}\frac{(1-v^2)^2 dv}{\sqrt{L^2-v^2}^5},$$

where $v = \sin u$ and $L = \sqrt{\dfrac{M+N}{M}} = \dfrac{2\gamma}{\sqrt{1+\gamma^2}}$. The corresponding indefinite integral is

$$\int \frac{(v^4-2v^2+1)dv}{(L^2-v^2)^2\sqrt{L^2-v^2}} = \int \frac{dv}{\sqrt{L^2-v^2}} - 2(L^2-1)\int\frac{dv}{\sqrt{L^2-v^2}^3} + (L^2-1)^2\int\frac{dv}{\sqrt{L^2-v^2}^5}$$
$$= \sin^{-1}\frac{v}{L} + \frac{2(1-L^2)v}{L^2\sqrt{L^2-v^2}} + \frac{(L^2-1)^2}{L^4}\left[\frac{2v}{3\sqrt{L^2-v^2}} + \frac{L^2 v}{3\sqrt{L^2-v^2}^3}\right],$$

in which, substituted $v = \sin\alpha = \dfrac{\gamma}{\sqrt{1+\gamma^2}}$ and $L = \dfrac{2\gamma}{\sqrt{1+\gamma^2}}$, yields

(6. 9) $$\mu_4 = 3\gamma^4 + \frac{(1-3\gamma^2)(5+21\gamma^2)}{4\sqrt{3}\,\pi}.$$

We have seen that the coefficient of skewness is $\dfrac{\mu_3}{\sqrt{\mu_2}^3} = 0$, but now the coefficient of excess

(6. 10) $$\frac{\mu^4}{\mu_2^2} - 3 = \frac{3\gamma^4 + (1-3\gamma^2)(5+21\gamma^2)/4\sqrt{3}\pi}{[\gamma^2 + (1-3\gamma^2)\sqrt{3}/2\pi]^2} - 3$$
$$= \frac{(1-3\gamma^2)^2}{4\sqrt{3}\pi^2}\left(5\pi - 9\sqrt{3}\right)\Big/\left[\gamma^2 + (1-3\gamma^2)\frac{\sqrt{3}}{2\pi}\right]^2 > 0, \text{ if } \gamma^2 \neq \frac{1}{3}.$$

Therefore, $g(\zeta)$ cannot indeed be normal, unless $\gamma^2 = 1 - 4c + 6c^2 = \dfrac{1}{3}$, i.e. $c = \dfrac{1}{3}$.

In general, the fr. f. of $\zeta = \sum_{i=1}^{n} c_i t_i$ with $\sum c_i = 1$, might be found from the probability element

$$dP = n!\, d\Phi_1 d\Phi_2 \cdots d\Phi_n = \frac{n!}{\sqrt{2\pi}^n}\exp\left\{-\frac{1}{2}\sum_{i=1}^{n} t_i^2\right\} dt_1 dt_2 \cdots dt_n$$

by a similar way, as shown for the case of A.M. in Cramér's treatise[9], or certain Fisher-like geometrical considerations, but certainly not without some ingenious devices. Probably it will be expressed as a cumbrous definite multiple integral. So here we content ourselves simply with the reconsideration about the well known case of A. M.

As usual, we perform an orthogonal transformation of variables $(t_1, t_2, \cdots, t_n)$ into $(y_1, y_2, \cdots, y_n)$. To fix idea, let it be e.g.

$$y_1 = -\frac{t_1}{\sqrt{1.2}} + \frac{t_2}{\sqrt{1.2}} \quad (\geqq 0),$$

$$y_2 = -\frac{t_1}{\sqrt{2.3}} - \frac{t_2}{\sqrt{2.3}} + \frac{2}{\sqrt{2.3}} t_3,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y_i = -\frac{t_1}{\sqrt{i(i+1)}} - \frac{t_2}{\sqrt{i(i+1)}} - \frac{t_3}{\sqrt{i(i+1)}} - \cdots - \frac{t_i}{\sqrt{i(i+1)}} + \frac{i}{\sqrt{i(i+1)}} t_{i+1},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y_{n-1} = -\frac{t_1}{\sqrt{(n-1)n}} - \frac{t_2}{\sqrt{(n-1)n}} - \frac{t_3}{\sqrt{(n-1)n}} - \cdots - \frac{t_i}{\sqrt{(n-1)n}} - \cdots\cdots - \frac{t_{n-1}}{\sqrt{(n-1)n}} + \frac{n-1}{\sqrt{(n-1)n}} t_n,$$

$$y_n = \frac{t_1}{\sqrt{n}} + \frac{t_2}{\sqrt{n}} + \frac{t_3}{\sqrt{n}} + \cdots + \frac{t_i}{\sqrt{n}} + \cdots\cdots + \frac{t_{n-1}}{\sqrt{n}} + \frac{t_n}{\sqrt{n}}.$$

where Jacobian J becomes $(-1)^{n-1}$, so that $|J| = 1$[10]. On taking coefficients of every column in the foregoing substitutions as those of successive rows, we may write

$$t_1 = -\frac{y_1}{\sqrt{1.2}} - \frac{y_2}{\sqrt{2.3}} - \frac{y_3}{\sqrt{3.4}} - \cdots - \frac{y_i}{\sqrt{i(i+1)}} - \cdots\cdots - \frac{y_{n-2}}{\sqrt{(n-1)(n-2)}} - \frac{y_{n-1}}{\sqrt{n(n-1)}} + \frac{y_n}{\sqrt{n}},$$

$$t_2 = +\frac{y_1}{\sqrt{1.2}} - \frac{y_2}{\sqrt{2.3}} - \frac{y_3}{\sqrt{3.4}} - \cdots - \frac{y_i}{\sqrt{i(i+1)}} - \cdots\cdots - \frac{y_{n-2}}{\sqrt{(n-1)(n-2)}} - \frac{y_{n-1}}{\sqrt{n(n-1)}} + \frac{y_n}{\sqrt{n}},$$

$$t_3 = +\frac{2}{\sqrt{2.3}} y_2 - \frac{y_3}{\sqrt{3.4}} - \cdots - \frac{y_i}{\sqrt{i(i+1)}} - \cdots\cdots - \frac{y_{n-2}}{\sqrt{(n-1)(n-2)}} - \frac{y_{n-1}}{\sqrt{n(n-1)}} + \frac{y_n}{\sqrt{n}},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$t_{i+1} = +\frac{i}{\sqrt{i(i+1)}} y_i - \frac{y_{i+1}}{\sqrt{(i+1)(i+2)}} - \cdots - \frac{y_{n-2}}{\sqrt{(n-1)(n-2)}} - \frac{y_{n-1}}{\sqrt{n(n-1)}} + \frac{y_n}{\sqrt{n}},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$t_{n-1} = +\frac{n-2}{\sqrt{(n-1)(n-2)}} y_{n-2} - \frac{y_{n-1}}{\sqrt{n(n-1)}} + \frac{y_n}{\sqrt{n}},$$

$$t_n = +\frac{n-1}{\sqrt{n(n-1)}} y_{n-1} + \frac{y_n}{\sqrt{n}}.$$

But, as $t_1 \leqq t_2 \leqq \cdots \leqq t_n$,

$$t_{i+1} - t_i = \sqrt{\frac{i+1}{i}}\, y_i - \sqrt{\frac{i-1}{i}}\, y_{i-1} \geqq 0,$$

---

9) Cramér, loc. cit., p. 383.

10) Consequently our transformation is properly or improperly orthogonal, according as n is odd or even.

therefore $$y_i \geqq \sqrt{\frac{i-1}{i+1}}\, y_{i-1}\,, \qquad (i=1, 2, \cdots, n-1)\,.$$

By these $n-1$ inequalities every $y_1, y_2, \cdots, y_{n-1}$ is positive and bounded downwards, however $y_n=\sqrt{n}\,\zeta \gtrless 0$, and unbounded. Also, as consequence of orthogonal transformation, it follows that

$$\sum_{i=1}^{n} t_i^2 = \sum_{i=1}^{n} y_i^2\,,$$

and the probability element now becomes

$$dP = \frac{n!}{\sqrt{2\pi}^n} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} t_i^2\right\} dt_1 dt_2 \cdots dt_n$$
$$= \frac{n!}{\sqrt{2\pi}^n} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} y_i^2\right\} \cdot dy_1 dy_2 \cdots dy_n\,, \qquad (y_n=\sqrt{n}\,\zeta)\,.$$

Therefore the fr. f. $g(\zeta)$ is given by

(6.11) $$g(\zeta)d\zeta = \frac{n!}{\sqrt{2\pi}^n} \exp\left\{-\frac{n}{2}\zeta^2\right\} \sqrt{n}\, d\zeta \int_0 \exp\{-y_1^2/2\}\, dy_1 \times$$
$$\times \int_{\sqrt{\frac{1}{3}}y_1} \exp\{-y_2^2/2\}\, dy_2 \cdots \int_{\sqrt{\frac{i-1}{i+1}}y_{i-1}} \exp\{-y_i^2/2\}\, dy_i \cdots \int_{\sqrt{\frac{n-2}{n}}y_{n-2}} \exp\{-y_{n-1}^2/2\}\, dy_{n-1}\,.$$

Here the $(n-1)$-ple integral being independent of all y's, it reduces to some constant C_n, and we have

(6.12) $$g(\zeta)d\zeta = \frac{n!}{\sqrt{2\pi}^n} C_n \exp\left\{-\frac{1}{2}n\zeta^2\right\} \cdot \sqrt{n}\, d\zeta\,.$$

But

(6.13) $$1 = \int g(\zeta)d\zeta = \frac{n!}{\sqrt{2\pi}^{n-1}} C_n\,.$$

Hence[11)]

(6.14) $$C_n = \frac{\sqrt{2\pi}^{n-1}}{n!}$$

and we obtain finally

(6.15) $$g(\zeta) = \frac{1}{\sqrt{2\pi\Big/\frac{1}{n}}} \exp\left\{-\zeta^2\Big/\frac{2}{n}\right\}, \text{ so } g(z) = \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left\{-\frac{1}{2}(z-m)^2\Big/\frac{\sigma^2}{n}\right\};$$

i.e. the theorem: The A.M. $z=\frac{1}{n}\sum x_i$ distributes normally with mean m and variance σ^2/n, which was the case when $x_1, x_2, \cdots, x_n$ are unordered and independent of each other.

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11) Really $C_2=\int_0 e^{-y_1^2/2}\, dy_1=\frac{\sqrt{2\pi}}{2}$, $C_3=\int_0 e^{-y_1^2/2}\int_{\sqrt{y_1/3}} e^{-y_2^2/2}\, dy_2=\int_{\pi/6}^{\pi/2} d\theta \int_0^{\infty} e^{-r^2/2}\, r dr=\frac{2\pi}{3!}$, and so on. Since we have obtained (6.14) as a logical consequence from (6.11) (6.12) (6.13), we need not prove (6.14) expressly, e.g. by mathematical induction, though it might be considered as a rather difficult but superfluous exercise.

PART II

§7. Computation of $J_\lambda^{(\alpha)}$. Rather more conveniently we shall compute

(7.1) $K_\lambda^{(\alpha)}=\int U^\lambda \varphi^\alpha dt$, where $U=\int_0^t \varphi dt$, $\varphi=\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$, $(\alpha=2, 3, \cdots)$.

Since $U(t)$ is an odd function, it yields always

(7.2) $$K_{2p+1}^{(\alpha)}=0.$$

Hence, we have

(7.3) $J_\lambda^{(\alpha)}=\int \Phi^\lambda \varphi^\alpha dt=\int\left(\frac{1}{2}+U\right)^\lambda \varphi^\alpha dt=\sum_{\nu=1}^{p}\binom{\lambda}{2\nu}\left(\frac{1}{2}\right)^{\lambda-2\nu}K_{2\nu}^{(\alpha)}$, where $p=\left[\frac{\lambda}{2}\right]$.

Thus, to obtain $J_\lambda^{(\alpha)}$ it suffices to know $K_{2\nu}^{(\alpha)}$ for $\nu=0, 1, 2, \cdots, p=\left[\frac{\lambda}{2}\right]$. We shall compute $K_\lambda^{(\alpha)}$ and $J_\lambda^{(\alpha)}$ successively.

1° For $\lambda=0$ evidently, if $\alpha>0$,

(7.4) $$K_0^{(\alpha)}=J_0^{(\alpha)}=\int \varphi^\alpha dt=\frac{1}{\sqrt{2\pi^\alpha}}\int e^{-\frac{\alpha}{2}t^2}dt=\frac{1}{\sqrt{\alpha}\sqrt{2\pi^{\alpha-1}}}=c_\alpha, \text{ say.}$$

In particular

(7.4.1) $$c_2=K_0^{(2)}=J_0^{(2)}=\frac{1}{2\sqrt{\pi}} \quad\text{and}\quad c_3=K_0^{(3)}=J_0^{(3)}=\frac{1}{2\pi\sqrt{3}}.$$

2° For $\lambda=1$, $K_1^{(\alpha)}=0$. But

(7.5) $$J_1^{(\alpha)}=\int \Phi\varphi^\alpha dt=\int\left(U+\frac{1}{2}\right)\varphi^\alpha dt=\frac{1}{2}\int\varphi^\alpha dt=\frac{1}{2}c_\alpha.$$

In particular

(7.5.1) $$J_1^{(2)}=\frac{1}{4\sqrt{\pi}} \quad\text{and}\quad J_1^{(3)}=\frac{1}{4\pi\sqrt{3}}.$$

3° For $\lambda=2$, we observe, by means of polar co-ordinates, that

(7.6) $$U(t)^2=\left[\int_0^t\varphi dt\right]^2=\frac{1}{2\pi}\int_0^t\int_0^t\exp\left\{-\frac{1}{2}(x^2+y^2)\right\}dxdy$$
$$=\frac{1}{\pi}\int_0^{\pi/4}d\theta\int_0^{t\sec\theta}\exp\left\{-\frac{1}{2}r^2\right\}rdr=\frac{1}{\pi}\int_0^{\pi/4}\left[1-\exp\left\{-\frac{1}{2}t^2\sec^2\theta\right\}\right]d\theta$$
$$=\frac{1}{4}-\frac{1}{\pi}\int_0^{\pi/4}\exp\left\{-\frac{1}{2}t^2\sec^2\theta\right\}d\theta.$$

Consequently

$$K_2^{(\alpha)}=\int U^2\varphi^\alpha dt=c_\alpha\left[\frac{1}{4}-\frac{\sqrt{\alpha}}{\pi}\int_0^{\pi/4}\frac{d\theta}{\sqrt{\alpha+\sec^2\theta}}\right].$$

Here, owing to the absolute convergence of the concerned integral, the order of integrations was interchanged. The alike would be tacitly and frequently applied below. Now the above last integration performed, we have

(7.7) $$\int_0^{\pi/4}\frac{d\theta}{\sqrt{\alpha+\sec^2\theta}}=\frac{1}{\sqrt{\alpha}}\sin^{-1}\left\{\sqrt{\frac{\alpha}{1+\alpha}}\sin\theta\right\}\Big|_0^{\pi/4}=\frac{1}{\sqrt{\alpha}}\sin^{-1}\sqrt{\frac{\alpha}{2(1+\alpha)}}.$$

Since the function plays very important roles in the subsequent, we shall expressively denote it by

$$(7.8)\qquad S(\alpha)=\frac{\sqrt{\alpha}}{\pi}\int_0^{\pi/4}\frac{d\theta}{\sqrt{\alpha+\sec^2\theta}}=\frac{1}{\pi}\sin^{-1}\sqrt{\frac{\alpha}{2(1+\alpha)}}\,.$$

E. g.

$$(7.8.1)\qquad \begin{cases} \pi S(2)=\sin^{-1}\dfrac{1}{\sqrt{3}}=\cos^{-1}\sqrt{\dfrac{2}{3}}=\tan^{-1}\dfrac{1}{\sqrt{2}}\,, \\ \pi S(3)=\sin^{-1}\sqrt{\dfrac{3}{8}}=\cos^{-1}\sqrt{\dfrac{5}{8}}=\tan^{-1}\sqrt{\dfrac{3}{5}}\,. \end{cases}$$

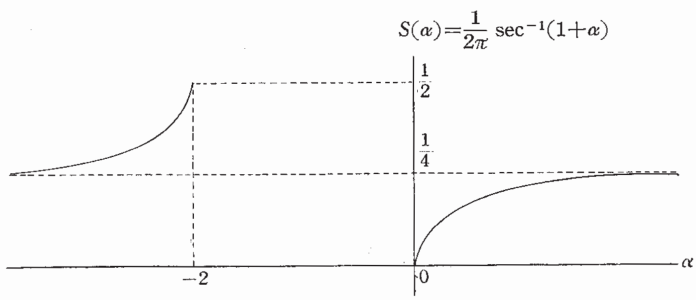
If $\xi=\sqrt{\dfrac{\alpha}{2(1+\alpha)}}$, $\alpha=\dfrac{2\xi^2}{1-2\xi^2}$ and $S(\alpha)=\dfrac{1}{\pi}\sin^{-1}\xi$, so that, e.g.

ξ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\xi>1$ or imag.	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{2}\sqrt{\frac{3}{2}}$ &c.
α	0	1	$\pm\infty$	-3	-2	$-2<\alpha<0$	$\frac{1}{2}$	2	3 &c.
$S(\alpha)$	0	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	imaginary	0.1339	0.1959	0.2098 &c.

In fact, our $S(\alpha)$ is nothing but

$$(7.8.2)\qquad S(\alpha)=\frac{1}{2\pi}\cos^{-1}\frac{1}{1+\alpha}=\frac{1}{2\pi}\sec^{-1}(1+\alpha)$$

and its graph is as shown in Fig. 4.



Eig. 4.

Returning to our text, we have

$$(7.9)\qquad K_2^{(\alpha)}=c_\alpha\left[\frac{1}{4}-\frac{1}{\pi}\sin^{-1}\sqrt{\frac{\alpha}{2(\alpha+1)}}\right]=c_\alpha\left[\frac{1}{4}-S(\alpha)\right],$$

$$(7.9.1)\qquad K_2^{(2)}=\frac{1}{2\sqrt{\pi}}\left[\frac{1}{4}-S(2)\right],\quad K_2^{(3)}=\frac{1}{2\pi\sqrt{3}}\left[\frac{1}{4}-S(3)\right],\quad \&c.$$

Consequently

$$(7.10)\qquad J_2^{(\alpha)}=\int\left(U^2+\frac{1}{4}\right)\varphi^\alpha dt=\frac{c_\alpha}{2}\left[1-2S(\alpha)\right].$$

$$(7.10.1)\qquad J_2^{(2)}=\frac{1}{4\sqrt{\pi}}\left[1-2S(2)\right],\quad J_2^{(3)}=\frac{1}{4\pi\sqrt{3}}\left[1-2S(3)\right],\quad \&c.$$

4° For $\lambda=3$, also $K_3^{(\alpha)}=0$, but

(7.11) $$J_3^{(\alpha)}=\int\left[\frac{1}{8}+\frac{3}{2}U^2\right]\varphi^\alpha dt=\frac{1}{8}K_0^{(\alpha)}+\frac{3}{2}K_2^{(\alpha)}=\frac{1}{2}c_\alpha[1-3S(\alpha)].$$

(7.11.1) $$J_3^{(2)}=\frac{1}{4\sqrt{\pi}}[1-3S(2)],\quad J_3^{(3)}=\frac{1}{4\pi\sqrt{3}}[1-3S(3)],\quad \&c.$$

5° However, for $\lambda=4$, we have

$$K_4^{(\alpha)}=\int U^4\varphi^\alpha dt,$$

where

$$U^4=\frac{1}{4\pi^2}\iiiint_0^t \exp\left[-\frac{1}{2}(\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2)\right]d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$

Transformed these 4-dimensional rectangular co-ordinates into general polar co-ordinates[12], we have

$$U^4=\frac{1}{\pi^2}\int_0^{\pi/4}d\theta_1\int_0^{\tau_1}\cos\theta_2 d\theta_2\int_0^{\tau_2}\cos^2\theta_3 d\theta_3\int_0^{\tau_3}\exp\left\{-\frac{1}{2}\rho^2\right\}\rho^3 d\rho,$$

where $\tau_1=\mathrm{Tan}^{-1}\cos\theta_1$, $\tau_2=\tan^{-1}(\cos\theta_1\cos\theta_2)$, $\tau_3=t\sec\theta_1\sec\theta_2\sec\theta_3$. However, the above repeated integrals are too complicate to be actually performed. Rather we prefer to avail the before obtained integral of (7.6):

$$\begin{aligned}K_4^{(\alpha)}&=\int U^4\varphi^\alpha dt=\int\left[\frac{1}{4}-\frac{1}{\pi}\int_0^{\pi/4}\exp\left\{-\frac{1}{2}t^2\sec\theta\right\}d\theta\right]^2\varphi^\alpha dt\\&=\frac{1}{16}\int\varphi^\alpha dt-\frac{1}{2\pi}\int\varphi^\alpha dt\int_0^{\pi/4}\exp\left\{-\frac{t^2}{2}\sec^2\theta\right\}d\theta\\&\quad+\frac{1}{\pi^2}\int\varphi^\alpha dt\int_0^{\pi/4}\int_0^{\pi/4}\exp\left\{-\frac{t^2}{2}(\sec^2\theta+\sec^2\theta')\right\}d\theta d\theta'\\&=c_\alpha\left[\frac{1}{16}-\frac{1}{2}S(\alpha)+\frac{1}{\pi^2}\sqrt{\frac{\alpha}{2\pi}}\int_0^{\pi/4}d\theta\int_0^{\pi/4}d\theta'\int\exp\left\{-\frac{t^2}{2}(\alpha+\sec^2\theta+\sec^2\theta')\right\}dt\right].\end{aligned}$$

---

12) Generally, if $P(\xi_1,\cdots,\xi_n)$ be transformed into general polar coordinates:

$$\begin{aligned}\xi_1&=\rho\cos\theta_{n-1}\cos\theta_{n-2}\cdots\cdots\cos\theta_3\cos\theta_2\cos\theta_1\\\xi_2&=\rho\cos\theta_{n-1}\cos\theta_{n-2}\cdots\cdots\cos\theta_3\cos\theta_2\sin\theta_1\\\xi_3&=\rho\cos\theta_{n-1}\cos\theta_{n-2}\cdots\cdots\cos\theta_3\sin\theta_2\\&\cdots\cdots\cdots\cdots\cdots\cdots\\\xi_{n-1}&=\rho\cos\theta_{n-1}\sin\theta_{n-2}\\\xi_n&=\rho\sin\theta_{n-1}\end{aligned}$$

where $\rho=OP$ and θ_i denotes the angle which the normal projection of the straight line OP on the $(i+1)$-dimensional subspace formed by ξ_1-, ξ_2-, ……, ξ_{i+1}-axes, makes with the i-dimensional subspace formed by ξ_1-, ξ_2-, ……, ξ_i-axes. Then, the integral

$$U^n=\frac{1}{\sqrt{2\pi}^n}\int_0^t\int_0^t\cdots\cdots\int_0^t\exp\left\{-\frac{1}{2}\sum_{i=1}^n\xi_i^2\right\}d\xi_1 d\xi_2\cdots\cdots d\xi_n$$

is transformed into

$$U^n=\frac{n}{\sqrt{2\pi}^n}\int_0^{\pi/4}d\theta_1\int_0^{\tau_1}d\theta_2\int_0^{\tau_2}d\theta_3\cdots\cdots\int_0^{\tau_{n-2}}\int_0^{\tau_{n-1}}\exp\left\{-\frac{1}{2}\rho^2\right\}|J|d\rho,$$

where $\tau_1=\mathrm{Tan}^{-1}(\cos\theta_1)$, $\tau_2=\mathrm{Tan}^{-1}(\cos\theta_1\cos\theta_2)$, ……, $\tau_{n-2}=\mathrm{Tan}^{-1}(\cos\theta_1\cos\theta_2\cdots\cdots\cos\theta_{n-2})$, but the last $\tau_{n-1}=t\sec\theta_1\sec\theta_2\cdots\cdots\sec\theta_{n-1}$, and $|J|$ denotes the Jacobian of transformation $\dfrac{\partial(\xi_1,\xi_2,\cdots,\xi_n)}{\partial(\rho,\theta_1,\cdots,\theta_{n-1})}$; thus for $n=4$ really $|J|=\rho^3\cos\theta_2\cos^2\theta_3$.

The last triple integral ($I_4^{(\alpha)}$ say), again (7.8) applied, reduces to

$$I_4^{(\alpha)}=\frac{\sqrt{\alpha}}{\pi^2}\int_0^{\pi/4}d\theta\int_0^{\pi/4}\frac{d\theta'}{\sqrt{\alpha+\sec^2\theta+\sec^2\theta'}}$$
$$=\frac{\sqrt{\alpha}}{\pi^2}\int_0^{\pi/4}\frac{d\theta}{\sqrt{\alpha+\sec^2\theta}}\sin^{-1}\sqrt{\frac{\alpha+\sec^2\theta}{2[1+\alpha+\sec^2\theta]}},$$

which, if the integration variable be changed into $u=\sin\theta$, becomes

$$I_4^{(\alpha)}=\frac{1}{\pi^2}\int_0^{1/\sqrt{2}}\frac{\sqrt{\alpha}\,du}{\sqrt{\alpha+1-\alpha u^2}}\sin^{-1}\sqrt{\frac{\alpha+1-\alpha u^2}{2[\alpha+2-(\alpha+1)u^2]}}$$
$$=\frac{1}{2\pi^2}\int_0^{1/\sqrt{2}}\frac{\sqrt{\alpha}\,du}{\sqrt{\alpha+1-\alpha u^2}}\sec^{-1}\left(\alpha+1+\frac{1}{1-u^2}\right)^{13)}.$$

Or, upon putting $\alpha\sin^2\theta=(\alpha+1)\sin^2\psi$, we have by (7.7)

$$\frac{d\psi}{d\theta}=\frac{\sqrt{\alpha}}{\sqrt{\alpha+\sec^2\theta}}$$

and the integral $I_4^{(\alpha)}$ becomes

$$I_4^{(\alpha)}=\frac{1}{\pi^2}\int_0^{\pi S(\alpha)}\sin^{-1}\sqrt{\frac{\alpha(\alpha+1)/2}{\alpha(\alpha+2)-\tan^2\psi}}\,d\psi. \tag{7.12}$$

Hence, we obtain

$$K_4^{(\alpha)}=\frac{c_\alpha}{2}\left[\frac{1}{8}-S(\alpha)+2I_4^{(\alpha)}\right] \tag{7.13}$$

and accordingly

$$J_4^{(\alpha)}=\frac{c_\alpha}{2}\left[1-4S(\alpha)+2I_4^{(\alpha)}\right]. \tag{7.14}$$

In particular

$$K_4^{(2)}=\frac{1}{4\sqrt{\pi}}\left[\frac{1}{8}-S(2)+2I_4^{(2)}\right],\quad K_4^{(3)}=\frac{1}{4\pi\sqrt{3}}\left[\frac{1}{8}-S(3)+2I_4^{(3)}\right], \tag{7.13.1}$$

$$J_4^{(2)}=\frac{1}{4\sqrt{\pi}}[1-4S(2)+2I_4^{(2)}],\quad J_4^{(3)}=\frac{1}{4\pi\sqrt{3}}[1-4S(3)+2I_4^{(3)}], \tag{7.14.1}$$

where[13]

$$I_4^{(2)}=\frac{1}{\pi^2}\int_0^{\pi S(2)}\sin^{-1}\sqrt{\frac{3}{8-\tan^2\psi}}\,d\psi=\frac{1}{2\pi}\int_0^{\pi S(2)}\frac{1}{\pi}\sec^{-1}\left(1+\frac{6}{2-\tan^2\psi}\right)d\psi,$$
$$I_4^{(3)}=\frac{1}{\pi^2}\int_0^{\pi S(3)}\sin^{-1}\sqrt{\frac{6}{15-\tan^2\psi}}\,d\psi=\frac{1}{2\pi}\int_0^{\pi S(3)}\frac{1}{\pi}\sec^{-1}\left(1+\frac{12}{3-\tan^2\psi}\right)d\psi.$$

6° For $\lambda=5$, $K_5^{(\alpha)}=0$,

$$J_5^{(\alpha)}=\frac{1}{32}K_0^{(\alpha)}+\frac{5}{4}K_2^{(\alpha)}+\frac{5}{2}K_4^{(\alpha)}=\frac{c_\alpha}{2}[1-5S(\alpha)+5I_4^{(\alpha)}], \tag{7.15}$$

13) By numerical computations the second forms with inverse secants are convenient, because they are free from radical signs.

and

(7.15.1) $J_5^{(2)} = \frac{1}{4\sqrt{\pi}}[1-5S(2)+5I_4^{(2)}]$, $J_5^{(3)} = \frac{1}{4\pi\sqrt{3}}[1-5S(3)+5I_4^{(3)}]$.

However, if we carry out similarly with $K_6^{(\alpha)} = \int U^6\varphi^\alpha dt$, we must treat a furthermore complicated integral

$$\int_0^{\pi/4} \frac{d\theta'}{\sqrt{\alpha'}} \int_0^{\pi S(\alpha')} \sin^{-1}\sqrt{\frac{\alpha'(\alpha'+1)/2}{\alpha'(\alpha'+2)-\tan^2\psi}}\, d\psi \quad (\alpha' = \alpha+\sec^2\theta),$$

so we give up to continue any more.

§8. Computations of $J_{\mu,\nu}^{\alpha,\beta}$ for $\alpha, \beta=1, 2, 3, \cdots; \mu, \nu=0, 1, 2, 3, \cdots$ [14].

We wish to find

(8.1) $$J_{\mu,\nu}^{\alpha,\beta} = \int \Phi^\mu\varphi^\alpha dt \int^t \Phi_1^\nu\varphi_1^\beta dt_1,$$

which are obtainable from

(8.2) $$K_{\mu,\nu}^{\alpha,\beta} = \int U^\mu\varphi^\alpha dt \int^t U_1^\nu\varphi_1^\beta dt_1,$$

because

(8.3) $$J_{\mu,\nu}^{\alpha,\beta} = \int\left(\frac{1}{2}+U\right)^\mu \varphi^\alpha dt \int^t\left(\frac{1}{2}+U_1\right)^\nu \varphi_1^\beta dt_1 = \sum_{j=1}^{\mu}\sum_{k=1}^{\nu}\binom{\mu}{j}\binom{\nu}{k}\left(\frac{1}{2}\right)^{\mu+\nu-j-k} K_{j\,k}^{\alpha,\beta}.$$

Also, in general,

$$K_{\nu,\mu}^{\beta,\alpha} = \int U^\nu\varphi^\beta dt \int^t U_1^\mu\varphi_1^\alpha dt_1 = \int U_1^\mu\varphi_1^\alpha dt_1 \int_{t_1} U^\nu\varphi^\beta dt$$
$$= \int U_1^\mu\varphi_1^\alpha dt_1\left[\int U^\nu\varphi^\beta dt - \int^{t_1} U^\nu\varphi^\beta dt\right] = K_\mu^{(\alpha)}K_\nu^{(\beta)} - K_{\mu,\nu}^{\alpha,\beta}.$$

Therefore

(8.4) $$K_{\mu,\nu}^{\alpha,\beta} + K_{\nu,\mu}^{\beta,\alpha} = K_\mu^{(\alpha)}K_\nu^{(\beta)}.$$

In particular, if at least one of μ, ν be odd, it yields, in view of (7.2),

(8.4.1) $K_{\mu,\nu}^{\alpha,\beta} + K_{\nu,\mu}^{\beta,\alpha} = 0$, i.e. $K_{\nu,\mu}^{\beta,\alpha} = -K_{\mu,\nu}^{\alpha,\beta}$ (one of μ, ν odd).

Quite similarly

(8.5) $$J_{\mu,\nu}^{\alpha,\beta} + J_{\nu,\mu}^{\beta,\alpha} = J_\mu^{(\alpha)}J_\nu^{(\beta)}.$$

Now we shall compute $K_{\mu,\nu}^{\alpha,\beta}$ and $J_{\mu,\nu}^{\alpha,\beta}$ successively.

1° $\mu=\nu=0$.

$$K_{00}^{\alpha,\beta} = J_{00}^{\alpha,\beta} = \int\varphi^\alpha dt \int^t \varphi_1^\beta dt_1 = \frac{1}{\sqrt{2\pi}^{\,\alpha+\beta}} \iint_D \exp\left\{-\frac{1}{2}(\alpha t^2+\beta t_1^2)\right\} dt dt_1.$$

where D denotes the domain $t \geq t_1$ in the tt_1-rectangular co-ordinates plane. So that if we transform them into polar co-ordinates (r, θ), D consists of

14) Although, for calculations of the second moments, it is enough to consider the case $\alpha=\beta=2$ only, we have endeavoured to obtain rather more general formulas, because they are required for calculations of higher ordered moments.

$\frac{\pi}{4} \leqq \theta \leqq \frac{5\pi}{4}$, $0 \leqq r < \infty$, and therefore

$$\iint_D = \int_{\pi/4}^{5\pi/4} d\theta \int_0^{\infty} \exp\left\{-\frac{1}{2} r^2(\alpha \sin^2\theta + \beta\cos^2\theta)\right\} r dr = \int_{\pi/4}^{5\pi/4} \frac{\sec^2\theta d\theta}{\alpha\tan^2\theta + \beta}$$

$$= \frac{1}{\sqrt{\alpha\beta}} \tan^{-1}\left\{\sqrt{\frac{\alpha}{\beta}}\tan\theta\right\}\Bigg|_{\pi/4}^{5\pi/4} = \frac{1}{\sqrt{\alpha\beta}}\left[\cdots\Big|_{\pi/4}^{\pi/2} + \cdots\Big|_{\pi/2+0}^{\pi-0} + \cdots\Big|_{\pi}^{5\pi/4}\right] = \frac{\pi}{\sqrt{\alpha\beta}}.$$

Thus

(8.6) $$K_{00}^{\alpha,\beta} = J_{00}^{\alpha,\beta} = \frac{1}{2\sqrt{\alpha\beta}\sqrt{2\pi}^{\alpha+\beta-2}} = \frac{1}{2} c_\alpha c_\beta .$$

(8.6.1)

α	1	1	2	1	3	2
β	1	2	1	3	1	2
$K_{00}^{\alpha,\beta}$	$\frac{1}{2}$	$\frac{1}{4\sqrt{\pi}}$	$\frac{1}{4\sqrt{\pi}}$	$\frac{1}{4\pi\sqrt{3}}$	$\frac{1}{4\pi\sqrt{3}}$	$\frac{1}{8\pi}$ &c.

2° $\mu=1$, $\nu=0$. Since

$$\int^t \varphi_1^\beta dt_1 = \int^0 + \int_0^t = \frac{1}{2\sqrt{\beta}\sqrt{2\pi}^{\beta-1}} + \int_0^t \varphi_1^\beta dt_1 ,$$

and U is odd, it yields that

$$K_{10}^{\alpha,\beta} = \int U\varphi^\alpha dt \int^t \varphi_1^\beta dt_1 = \int \varphi^\alpha dt \int_0^t \varphi_2 dt_2 \int_0^t \varphi_1^\beta dt_1 = \int \varphi^\alpha I(t)\, dt ,$$

of which the inner integral $I(t)$ is to be calculated below: Transforming again into polar co-ordinates (r, ψ), as in (7.6), but now observing that here the contributions from domains $0 \leqq \psi \leqq \frac{\pi}{4}$ and $\frac{\pi}{4} \leqq \psi \leqq \frac{\pi}{2}$ are different, we get

$$\sqrt{2\pi}^{\beta+1} I(t) = \int_0^t\int_0^t \exp\left\{-\frac{1}{2}(t_2^2 + \beta t_1^2)\right\} dt_1 dt_2$$
$$= \int_0^{\pi/4} d\psi \int_0^{t\sec\psi} \exp\left\{-\frac{r^2}{2}(\sin^2\psi + \beta\cos^2\psi)\right\} r dr$$
$$+ \int_{\pi/4}^{\pi/2} d\psi \int_0^{t\,\mathrm{cosec}\,\psi} \exp\left\{-\frac{r^2}{2}(\sin^2\psi + \beta\cos^2\psi)\right\} r dr .$$

After integrating about r and writing $\tan\psi = u$, we obtain

(8.7)
$$I(t) = \frac{1}{\sqrt{2\pi}^{\beta+1}} \int_0^1 \frac{1-\exp\left\{-\frac{t^2}{2}(u^2+\beta)\right\}}{u^2+\beta} du + \frac{1}{\sqrt{2\pi}^{\beta+1}} \int_1^{\infty} \frac{1-\exp\left\{-\frac{t^2}{2}\left(1+\frac{\beta}{u^2}\right)\right\}}{u^2+\beta} du$$

Here we have assumed to be $t>0$; however it remains the same with $t<0$ because $I(t)$ is even. On substituting (8.7) in $K_{10}^{\alpha,\beta}$, we have

$$\sqrt{2\pi}^{\alpha+\beta} K_{10}^{\alpha,\beta} = \frac{1}{\sqrt{2\pi}} \int_0^1 \frac{du}{u^2+\beta} \int \left[\exp\left\{-\frac{\alpha}{2}t^2\right\} - \exp\left\{-\frac{t^2}{2}(u^2+\alpha+\beta)\right\}\right] dt$$
$$+ \frac{1}{\sqrt{2\pi}} \int_1^{\infty} \frac{du}{u^2+\beta} \int \left[\exp\left\{-\frac{\alpha}{2}t^2\right\} - \exp\left\{-\frac{t^2}{2}\left(\alpha+1+\frac{\beta}{u^2}\right)\right\}\right] dt$$

$$=\int_0^1 \frac{du}{u^2+\beta}\left[\frac{1}{\sqrt{\alpha}}-\frac{1}{\sqrt{u^2+\alpha+\beta}}\right]+\int_1^\infty \frac{du}{u^2+\beta}\left[\frac{1}{\sqrt{\alpha}}-\frac{u}{\sqrt{(\alpha+1)u^2+\beta}}\right]$$

$$=\frac{1}{\sqrt{\alpha}}\int_0^\infty \frac{du}{u^2+\beta}-\int_0^1 \frac{du}{(u^2+\beta)\sqrt{u^2+\alpha+\beta}}-\int_1^\infty \frac{udu}{(u^2+\beta)\sqrt{(\alpha+1)u^2+\beta}}.$$

On performing integrations we get

$$(8.8)\qquad K_{10}^{\alpha,\beta}=\frac{c_\alpha c_\beta}{2\pi}\left[\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}-\sin^{-1}\sqrt{\frac{\alpha}{(\alpha+\beta)(\beta+1)}}\right]=-K_{01}^{\beta,\alpha}.$$

In particular, if $\beta=\alpha$,

$$(8.8.1)\qquad K_{10}^{\alpha,\alpha}=\frac{c_\alpha^2}{2\pi}\left[\sin^{-1}\frac{\sqrt{2\alpha+1}}{\alpha+1}-\sin^{-1}\frac{1}{\sqrt{2(\alpha+1)}}\right]$$

$$=\frac{c_\alpha^2}{2\pi}\sin^{-1}\frac{1}{\sqrt{2(\alpha+1)}}=\frac{1}{2}c_\alpha^2 S\left(\frac{1}{\alpha}\right)=-K_{01}^{\alpha,\alpha};$$

and more particularly e. g.

$$(8.8.1.1)\qquad K_{10}^{2,2}=\frac{1}{8\pi}S\left(\frac{1}{2}\right)=\frac{1}{8\pi^2}\sin^{-1}\frac{1}{\sqrt{6}}=-K_{01}^{2,2}.$$

Accordingly

$$(8.9)\qquad J_{10}^{\alpha,\beta}=\int\left(\frac{1}{2}+U\right)\varphi^\alpha dt\int^t\varphi_1^\beta dt_1=\frac{1}{2}K_{00}^{\alpha,\beta}+K_{10}^{\alpha,\beta}$$

$$=\frac{1}{2}c_\alpha c_\beta\left[\frac{1}{2}+\frac{1}{\pi}\left(\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}-\sin^{-1}\sqrt{\frac{\alpha}{(\alpha+\beta)(\beta+1)}}\right)\right],$$

$$(8.9.1)\qquad J_{10}^{\alpha,\alpha}=\frac{1}{2}c_\alpha^2\left[\frac{1}{2}+S\left(\frac{1}{\alpha}\right)\right],\quad J_{10}^{2,2}=\frac{1}{8\pi}\left[\frac{1}{2}+S\left(\frac{1}{2}\right)\right].$$

Also

$$(8.10)\qquad J_{01}^{\alpha,\beta}=J_0^{(\alpha)}J_1^{(\beta)}-J_{10}^{\beta,\alpha}=\frac{1}{2}c_\alpha c_\beta-J_{10}^{\beta,\alpha}$$

$$=\frac{1}{2}c_\alpha c_\beta\left[\frac{1}{2}-\frac{1}{\pi}\left(\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}-\sin^{-1}\sqrt{\frac{\beta}{(\alpha+\beta)(\alpha+1)}}\right)\right].$$

$$(8.10.1)\qquad J_{01}^{\alpha,\alpha}=\frac{1}{2}c_\alpha^2\left[\frac{1}{2}-S\left(\frac{1}{\alpha}\right)\right],\quad J_{01}^{2,2}=\frac{1}{8\pi}\left[\frac{1}{2}-S\left(\frac{1}{2}\right)\right].$$

3° $\mu=\nu=1$.

$$K_{11}^{\alpha,\beta}=\int U\varphi^\alpha dt\int^t U_1\varphi_1^\beta dt_1.$$

Here evidently the inner integral becomes even and the whole integrand odd, so that

$$(8.11)\qquad K_{11}^{\alpha,\beta}=0.$$

However

$$(8.12)\qquad J_{11}^{\alpha,\beta}=\int\left(\frac{1}{2}+U\right)\varphi^\alpha dt\int^t\left(\frac{1}{2}+U_1\right)\varphi_1^\beta dt_1=\frac{1}{4}K_{00}^{\alpha,\beta}=\frac{1}{8}c_\alpha c_\beta.$$

4° $\mu=2,\ \nu=0$. Observing that $\int_0^t \varphi_1^\beta dt$ is odd, we get readily

(8.13) $K_{20}^{\alpha,\beta}=\int U^2\varphi^\alpha dt\left[\int^0 \varphi_1^\beta dt_1+\int_0^t \varphi_1^\beta dt_1\right]=\frac{c_\beta}{2}K_2^{(\alpha)}=\frac{1}{2}c_\alpha c_\beta\left[\frac{1}{4}-S(\alpha)\right].$

(8.14) $J_{20}^{\alpha,\beta}=\frac{1}{4}K_{00}^{\alpha,\beta}+K_{10}^{\alpha,\beta}+K_{20}^{\alpha,\beta}$

$$=\frac{1}{2}c_\alpha c_\beta\left[\frac{1}{2}-S(\alpha)+\frac{1}{\pi}\left(\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}-\sin^{-1}\sqrt{\frac{\alpha}{(\alpha+\beta)(\beta+1)}}\right)\right].$$

(8.14.1) $J_{20}^{\alpha,\alpha}=\frac{1}{2}c_\alpha^2\left[\frac{1}{2}-S(\alpha)+S\left(\frac{1}{\alpha}\right)\right],\quad J_{20}^{2,2}=\frac{1}{8\pi}\left[\frac{1}{2}-S(2)+S\left(\frac{1}{2}\right)\right].$

Also for $\mu=0,\ \nu=2$:

(8.15) $K_{02}^{\alpha,\beta}=K_0^{(\alpha)}K_2^{(\beta)}-K_{20}^{\beta,\alpha}=\frac{1}{2}c_\alpha c_\beta\left[\frac{1}{4}-S(\beta)\right].$

(8.16) $J_{02}^{\alpha,\beta}=J_0^{(\alpha)}J_2^{(\beta)}-J_{20}^{\beta,\alpha}$

$$=\frac{1}{2}c_\alpha c_\beta\left[\frac{1}{2}-S(\beta)-\frac{1}{\pi}\left(\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}-\sin^{-1}\sqrt{\frac{\beta}{(\alpha+\beta)(\alpha+1)}}\right)\right],$$

(8.16.1) $J_{02}^{\alpha,\alpha}=\frac{1}{2}c_\alpha^2\left\{\frac{1}{2}-S(\alpha)-S\left(\frac{1}{\alpha}\right)\right\},\quad J_{02}^{2,2}=\frac{1}{8\pi}\left\{\frac{1}{2}-S(2)-S\left(\frac{1}{2}\right)\right\}.$

5° $\mu=2,\ \nu=1$.

$$K_{21}^{\alpha,\beta}=\int U^2\varphi^\alpha dt\int^t U_1\varphi_1^\beta dt_1,\quad \text{where}\quad U_1=\int_0^{t_1}\varphi(\tau)\,d\tau\,.$$

The inner integral is

$$\int^t U_1\varphi_1^\beta dt_1=\int^0+\int_0^t=\text{(i)}+\text{(ii)}\,.$$

Here

$$\sqrt{2\pi}^{\,\beta+1}\ \text{(i)}=\int^0 dt_1\int_0^{t_1}\exp\left\{-\frac{1}{2}(\beta t_1^2+\tau^2)\right\}d\tau$$

$$=-\int_{5\pi/4}^{3\pi/2}d\psi\int_0^\infty \exp\left\{-\frac{r^2}{2}(\beta\sin^2\psi+\cos^2\psi)\right\}r\,dr$$

$$=-\int_1^\infty\frac{du}{1+\beta u^2}\ (u=\tan\psi)=-\left[\frac{1}{\sqrt{\beta}}\tan\sqrt{\beta}u\right]_1^\infty=-\frac{1}{\sqrt{\beta}}\tan\frac{1}{\sqrt{\beta}},$$

and accordingly we have

$$\text{(i)}=-\frac{1}{\sqrt{\beta}\sqrt{2\pi}^{\,\beta+1}}\tan^{-1}\frac{1}{\sqrt{\beta}}\,.$$

Hence the corresponding whole integral becomes

$$\text{(I)}=-\frac{c_\beta}{2\pi}\tan^{-1}\frac{1}{\sqrt{\beta}}\cdot K_2^{(\alpha)}=-\frac{c_\alpha c_\beta}{2\pi}\tan^{-1}\frac{1}{\sqrt{\beta}}\left[\frac{1}{4}-S(\alpha)\right].$$

On the other hand

$$\sqrt{2\pi}^{\beta+1}\ \text{(ii)}=\int_0^t \exp\left\{-\frac{\beta}{2}t_1^2\right\}dt_1\int_0^{t_1}\exp\left\{-\frac{1}{2}\tau^2\right\}d\tau$$
$$=\int_{\pi/4}^{\pi/2}d\psi\int_0^{t\,\mathrm{cosec}\,\psi}\exp\left\{-\frac{r^2}{2}(\beta\sin^2\psi+\cos^2\psi)\right\}rdr$$
$$=\int_1^\infty\frac{dv}{1+\beta v^2}\left[1-\exp\left\{-\frac{t^2}{2}\left(\beta+\frac{1}{v^2}\right)\right\}\right]\quad (v=\tan\psi)\,.$$

Here we have assumed $t>0$. However for $t<0$ it results the same. Hence the contribution from (ii) is, making use of the above and (7.6),

$$\text{(II)}=\int U^2\varphi^\alpha dt\int_0^t U_1\varphi_1^\beta dt_1$$
$$=\frac{1}{\sqrt{2\pi}^{\alpha+\beta+1}}\int\exp\left\{-\frac{1}{2}\alpha t^2\right\}\left[\frac{1}{4}-\frac{1}{\pi}\int_0^{\pi/4}\exp\left\{-\frac{t^2}{2}\sec^2\theta\right\}d\theta\right]dt$$
$$\times\int_1^\infty\frac{dv}{1+\beta v^2}\left[1-\exp\left\{-\frac{t^2}{2}\left(\beta+\frac{1}{v^2}\right)\right\}\right]$$
$$=\frac{1}{\sqrt{2\pi}^{\alpha+\beta+1}}\int_1^\infty\frac{dv}{1+\beta v^2}\int\left[\exp\left\{-\frac{\alpha}{2}t^2\right\}-\exp\left\{-\frac{t^2}{2}\left(\alpha+\beta+\frac{1}{v^2}\right)\right\}\right]$$
$$\times\left[\frac{1}{4}-\frac{1}{\pi}\int_0^{\pi/4}\exp\left\{-\frac{t^2}{2}\sec^2\theta\right\}d\theta\right]dt$$
$$=\text{(II)}'-\text{(II)}''\,.$$

where (II)′ and (II)″ correspond to two terms in the latest square brackets.

Integrating (II)′ in regards to t,

$$\text{(II)}'=\frac{1}{4\sqrt{2\pi}^{\alpha+\beta}}\left[\frac{1}{\sqrt{\alpha}}\int_1^\infty\frac{dv}{1+\beta v^2}-\int_1^\infty\frac{vdv}{(1+\beta\gamma^2)\sqrt{1+(\alpha+\beta)v^2}}\right]$$
$$=\frac{c_\alpha c_\beta}{8\pi}\left[\sin^{-1}\sqrt{\frac{\beta(1+\alpha+\beta)}{(\alpha+\beta)(1+\beta)}}-\tan^{-1}\sqrt{\beta}\right].$$

Also (7.7) being applied,

$$\text{(II)}''=\frac{1}{\pi\sqrt{2\pi}^{\alpha+\beta+1}}\int_1^\infty\frac{dv}{1+\beta v^2}\int_0^{\pi/4}d\theta\int\left[\exp\left\{-\frac{t^2}{2}(\alpha+\sec^2\theta)\right\}\right.$$
$$\left.-\exp\left\{-\frac{t^2}{2}\left(\alpha+\beta+\frac{1}{v^2}+\sec^2\theta\right)\right\}\right]dt$$
$$=\frac{1}{\pi\sqrt{2\pi}^{\alpha+\beta}}\int_1^\infty\frac{dv}{1+\beta v^2}\int_0^{\pi/4}\left[\frac{1}{\sqrt{\alpha+\sec^2\theta}}-\frac{1}{\sqrt{\alpha+\beta+\frac{1}{v^2}+\sec^2\theta}}\right]d\theta$$
$$=\frac{1}{2}c_\alpha c_\beta\left[\frac{1}{\pi}\tan^{-1}\frac{1}{\sqrt{\beta}}\cdot S(\alpha)\right.$$
$$\left.-\frac{\sqrt{\alpha\beta}}{\pi^2}\int_1^\infty\frac{vdv}{(1+\beta v^2)\sqrt{1+(\alpha+\beta)v^2}}\sin^{-1}\sqrt{\frac{1+(\alpha+\beta)v^2}{2[1+(1+\alpha+\beta)v^2]}}\right].$$

Therefore

$$\text{(8.17)}\quad K_{21}^{\alpha,\beta}=\text{(I)}+\text{(II)}'-\text{(II)}''=\frac{1}{2}c_\alpha c_\beta\left[-\frac{1}{4\pi}\sin^{-1}\sqrt{\frac{\alpha}{(\alpha+\beta)(1+\beta)}}+I_{21}^{\alpha,\beta}\right],$$

where

(8.18) $$I_{21}^{\alpha,\beta}=\frac{\sqrt{\alpha\beta}}{\pi^2}\int_1^\infty\frac{vdv}{(1+\beta v^2)\sqrt{1+(\alpha+\beta)v^2}}\sin^{-1}\sqrt{\frac{1+(\alpha+\beta)v^2}{2[1+(1+\alpha+\beta)v^2]}}$$
$$=\frac{\sqrt{\alpha\beta}}{\pi^2}\int_0^1\frac{du}{(u^2+\beta)\sqrt{u^2+\alpha+\beta}}\sin^{-1}\sqrt{\frac{u^2+\alpha+\beta}{2(u^2+\alpha+\beta+1)}}.$$

(8.18.1) $$I_{21}^{\alpha,\alpha}=\frac{\alpha}{\pi^2}\int_0^1\frac{du}{(u^2+\alpha)\sqrt{u^2+2\alpha}}\sin^{-1}\sqrt{\frac{u^2+2\alpha}{2(u^2+2\alpha+1)}}$$
$$=\frac{\alpha}{2\pi^2}\int_0^1\frac{\sec^{-1}(u^2+2\alpha+1)}{(u^2+\alpha)\sqrt{u^2+2\alpha}}du.$$

(8.17.1) $$K_{21}^{\alpha,\alpha}=\frac{1}{2}c_\alpha^2\left[-\frac{1}{4}S\left(\frac{1}{\alpha}\right)+I_{21}^{\alpha,\alpha}\right],$$

(8.17.1.1) $$K_{21}^{2,2}=\frac{1}{8\pi}\left[-\frac{1}{4}S\left(\frac{1}{2}\right)+I_{21}^{2,2}\right].$$

And

(8.19) $$J_{21}^{\alpha,\beta}=\int\left(\frac{1}{2}+U\right)^2\varphi^\alpha dt\int^t\left(\frac{1}{2}+U_1\right)\varphi_1^\beta dt_1$$
$$=\frac{1}{8}K_{00}^{\alpha,\beta}+\frac{1}{2}K_{10}^{\alpha,\beta}+\frac{1}{4}K_{01}^{\alpha,\beta}+\frac{1}{2}K_{20}^{\alpha,\beta}+K_{21}^{\alpha,\beta}$$
$$=\frac{c_\alpha c_\beta}{2}\left[\frac{1}{4}-\frac{1}{2}S(\alpha)+\frac{1}{4\pi}\left(\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}\right.\right.$$
$$\left.\left.-3\sin^{-1}\sqrt{\frac{\alpha}{(\alpha+\beta)(\beta+1)}}+\sin^{-1}\sqrt{\frac{\beta}{(\alpha+\beta)(\alpha+1)}}\right)+I_{21}^{\alpha,\beta}\right].$$

(8.19.1) $$J_{21}^{\alpha,\alpha}=\frac{1}{2}c_\alpha^2\left[\frac{1}{4}-\frac{1}{2}S(\alpha)+I_{21}^{\alpha,\alpha}\right],$$

here the relation $\sin^{-1}\frac{\sqrt{1+2\alpha}}{1+\alpha}=2\sin^{-1}\frac{1}{\sqrt{2(1+\alpha)}}$ has been used. And

(8.19.1.1) $$J_{21}^{2,2}=\frac{1}{8\pi}\left[\frac{1}{4}-\frac{1}{2}S(2)+I_{21}^{2,2}\right]\quad\text{with}$$

(8.19.1.2) $$I_{21}^{2,2}=\frac{1}{\pi^2}\int_0^1\frac{\sec^{-1}(u^2+5)}{(u^2+2)\sqrt{u^2+4}}du.$$

Also for $\mu=1$, $\nu=2$,

$$K_{12}^{\alpha,\beta}=-K_{21}^{\beta,\alpha}\quad\text{and}$$

(8.20) $$J_{12}^{\alpha,\beta}=J_1^{(\alpha)}J_2^{(\beta)}-J_{21}^{\beta,\alpha}$$
$$=\frac{c_\alpha c_\beta}{2}\left[\frac{1}{4}-\frac{1}{2}S(\beta)-\frac{1}{4\pi}\left(\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}\right.\right.$$
$$\left.\left.-3\sin^{-1}\sqrt{\frac{\beta}{(\alpha+\beta)(\alpha+1)}}+\sin^{-1}\sqrt{\frac{\alpha}{(\alpha+\beta)(\beta+1)}}\right)-I_{21}^{\beta,\alpha}\right].$$

(8.20.1) $$J_{12}^{\alpha,\alpha}=\frac{1}{2}c_\alpha^2\left[\frac{1}{4}-\frac{1}{2}S(\alpha)-I_{21}^{\alpha,\alpha}\right],$$

(8.20.1.1) $$J_{12}^{2,2}=\frac{1}{8\pi}\left[\frac{1}{4}-\frac{1}{2}S(2)-I_{21}^{2,2}\right].$$

6° $\mu=3$, $\nu=0$. $U^3(t)$ being odd,

$$K_{30}^{\alpha,\beta}=\int U^3\varphi^\alpha dt\int^t\varphi_1^\beta dt_1=\int U^3\varphi^\alpha dt\int_0^t\varphi_1^\beta dt_1=\int U^2\varphi^\alpha dt\int_0^t\varphi_2 dt_2\int_0^t\varphi_1^\beta dt_1$$
$$=\int\left[\frac{1}{4}-\frac{1}{\pi}\int_0^{\pi/4}\exp\left\{-\frac{1}{2}t^2\sec^2\theta\right\}d\theta\right]\varphi^\alpha\cdot I(t)\,dt\quad\text{(availed (7.6))}$$
$$=(\mathrm{I})-(\mathrm{II}),$$

where (I) and (II) correspond to the two terms in the square brackets, and $I(t)$ is the same as in (8.7). Hence by (8.8)

$$(\mathrm{I})=\frac{1}{4}K_{10}^{\alpha,\beta}=\frac{c_\alpha c_\beta}{8\pi}\left[\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}-\sin^{-1}\sqrt{\frac{\alpha}{(\alpha+\beta)(\beta+1)}}\right].$$

As to (II), still availing (8.7) and integrating about t, we obtain

$$(\mathrm{II})=\frac{1}{\pi\sqrt{2\pi}^{\,\alpha+\beta}}\int_0^{\pi/4}d\theta\left[\int_0^\infty\frac{du}{(u^2+\beta)\sqrt{\alpha+\sec^2\theta}}-\int_0^1\frac{du}{(u^2+\beta)\sqrt{u^2+\alpha+\beta+\sec^2\theta}}\right.$$
$$\left.-\int_1^\infty\frac{du}{(u^2+\beta)\sqrt{1+\alpha+\sec^2\theta+\dfrac{\beta}{u^2}}}\right],$$

which, on interchanging the order of integrations and applying (7.8), yields

(8.21) $$(\mathrm{II})=\frac{1}{2}c_\alpha c_\beta\left[\frac{1}{2}S(\alpha)-\frac{\sqrt{\alpha\beta}}{\pi^2}\left\{\int_0^1\frac{du}{(u^2+\beta)\sqrt{u^2+\alpha+\beta}}\sin^{-1}\sqrt{\frac{u^2+\alpha+\beta}{2(u^2+\alpha+\beta+1)}}\right.\right.$$
$$\left.\left.+\int_1^\infty\frac{udu}{(u^2+\beta)\sqrt{(1+\alpha)u^2+\beta}}\sin^{-1}\sqrt{\frac{(1+\alpha)u^2+\beta}{2[(2+\alpha)u^2+\beta]}}\right\}\right].$$

Hence we obtain

(8.22) $$K_{30}^{\alpha,\beta}=\frac{c_\alpha c_\beta}{2}\left[\frac{1}{4\pi}\left\{\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}-\sin^{-1}\sqrt{\frac{\alpha}{(\alpha+\beta)(\beta+1)}}\right\}\right.$$
$$\left.-\frac{1}{2}S(\alpha)+I_{30}^{\alpha,\beta}\right],$$

where, according to (8.21) and (8.18),

(8.23) $$I_{30}^{\alpha,\beta}=I_{21}^{\alpha,\beta}+\frac{\sqrt{\alpha\beta}}{\pi^2}\int_1^\infty\frac{udu}{(u^2+\beta)\sqrt{(1+\alpha)u^2+\beta}}\sin^{-1}\sqrt{\frac{(1+\alpha)u^2+\beta}{2[(2+\alpha)u^2+\beta]}}$$
$$=I_{21}^{\alpha,\beta}+\frac{\sqrt{\alpha\beta}}{\pi^2}\int_0^1\frac{dv}{(1+\beta v^2)\sqrt{1+\alpha+\beta v^2}}\sin^{-1}\sqrt{\frac{1+\alpha+\beta v^2}{2[2+\alpha+\beta v^2]}}.$$

In particular

(8. 23. 1) $$K_{30}^{\alpha,\alpha}=\frac{c_\alpha^2}{8}\left\{S\left(\frac{1}{\alpha}\right)-2S(\alpha)+4I_{30}^{\alpha,\alpha}\right\},$$

(8. 23. 1. 1) $$K_{30}^{2,2}=\frac{1}{8\pi}\left\{\frac{1}{4}S\left(\frac{1}{2}\right)-\frac{1}{2}S(2)+I_{30}^{2,2}\right\}=-K_{03}^{2,2}.$$

Consequently

(8. 24) $$J_{30}^{\alpha,\beta}=\int\left(\frac{1}{2}+U\right)^3\varphi^\alpha dt\int^t\varphi_1^\beta dt_1=\frac{1}{8}K_{00}^{\alpha,\beta}+\frac{3}{4}K_{10}^{\alpha,\beta}+\frac{3}{2}K_{20}^{\alpha,\beta}+K_{30}^{\alpha,\beta}$$

$$=\frac{1}{2}c_\alpha c_\beta\left[\frac{1}{2}-2S(\alpha)+\frac{1}{\pi}\left\{\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}\right.\right.$$

$$\left.\left.-\sin^{-1}\sqrt{\frac{\alpha}{(\alpha+1)(\beta+1)}}\right\}+I_{30}^{\alpha,\beta}\right],$$

(8. 24. 1) $$J_{30}^{\alpha,\alpha}=\frac{c_\alpha^2}{2}\left[\frac{1}{2}-2S(\alpha)+S\left(\frac{1}{\alpha}\right)+I_{30}^{\alpha,\alpha}\right],$$

(8. 24. 1. 1) $$J_{30}^{2,2}=\frac{1}{8\pi}\left[\frac{1}{2}-2S(2)+S\left(\frac{1}{2}\right)+I_{30}^{2,2}\right]\quad\text{with}$$

(8. 24. 1. 2) $$I_{3,0}^{2,2}=I_{2,1}^{2,2}+\frac{2}{\pi^2}\int_0^1\frac{dv}{(1+2v^2)\sqrt{3+2v^2}}\sin^{-1}\sqrt{\frac{3+2v^2}{8+4v^2}}$$

$$=I_{2,1}^{2,2}+\frac{1}{\pi^2}\int_0^1\frac{\sec^{-1}(4+2v^2)}{(1+2v^2)\sqrt{3+2v^2}}dv.$$

Lastly $K_{03}^{\alpha,\beta}=-K_{30}^{\beta,\alpha}$, and

(8. 25) $$J_{03}^{\alpha,\beta}=J_0^{(\alpha)}J_3^{(\beta)}-J_{30}^{\beta,\alpha}=\frac{c_\alpha c_\beta}{2}\left[\frac{1}{2}-S(\beta)-\frac{1}{\pi}\left\{\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}\right.\right.$$

$$\left.\left.-\sin^{-1}\sqrt{\frac{\beta}{(\alpha+\beta)(\alpha+1)}}\right\}-I_{3,0}^{\beta,\alpha}\right].$$

(8. 25. 1) $$J_{03}^{\alpha,\alpha}=\frac{1}{2}c_\alpha^2\left[\frac{1}{2}-S(\alpha)-S\left(\frac{1}{\alpha}\right)-I_{30}^{\alpha,\alpha}\right]$$

(8. 25. 1. 1) $$J_{0,3}^{2,2}=\frac{1}{8\pi}\left[\frac{1}{2}-S(2)-S\left(\frac{1}{2}\right)-I_{30}^{2,2}\right].$$

Although it could be computed similarly for

$$J_{\lambda,\mu,\nu}^{\alpha,\beta,\gamma}=\int\Phi^\lambda\varphi^\alpha dt\int^t\Phi_1^\mu\varphi_1^\beta dt_1\int^{t_1}\Phi_2^\nu\varphi_2^\gamma dt_2\quad(t\geqq t_1\geqq t_2)\quad\&\text{c.},$$

which are requisite for calculations of $E(t_{i|n}^p t_{j|n}^q t_{k|n}^r)$ &c., it is postponed for future.

For purpose of later references we have tabulated the values of $K_\lambda^{(\alpha)}$, $J_\lambda^{(\alpha)}$, for $\alpha=2, 3$ (Table I) and those of $K_{\mu\nu}$, $J_{\mu\nu}$ (Table II)[15]:

15) For calculations of the 2nd moments, $K_{\mu,\nu}^{2,2}$ and $J_{\mu,\nu}^{2,2}$ being only of use, these are below simply expressed as $K_{\mu\nu}$, $J_{\mu\nu}$.

Table I

λ	$K_\lambda^{(2)}$	$K_\lambda^{(3)}$	$J_\lambda^{(2)}$	$J_\lambda^{(3)}$
0	$\frac{1}{2\sqrt{\pi}}=c_2$	$\frac{1}{2\pi\sqrt{3}}=c_3$	$\frac{1}{2\sqrt{\pi}}=c_2$	$\frac{1}{2\pi\sqrt{3}}=c_3$
1	0	0	$\frac{1}{4\sqrt{\pi}}=\frac{1}{2}c_2$	$\frac{1}{4\pi\sqrt{3}}=\frac{1}{2}c_3$
2	$\frac{1}{2\sqrt{\pi}}\left(\frac{1}{4}-S(2)\right)$	$\frac{1}{2\pi\sqrt{3}}\left(\frac{1}{4}-S(3)\right)$	$\frac{1}{4\sqrt{\pi}}[1-2S(2)]$	$\frac{1}{4\pi\sqrt{3}}[1-2S(3)]$
3	0	0	$\frac{1}{4\sqrt{\pi}}[1-3S(2)]$	$\frac{1}{4\pi\sqrt{3}}[1-3S(3)]$
4	$\frac{1}{4\sqrt{\pi}}\left[\frac{1}{8}-S(2)+2I_4^{(2)}\right]$	$\frac{1}{4\pi\sqrt{3}}\left[\frac{1}{8}-S(3)+2I_4^{(3)}\right]$	$\frac{1}{4\sqrt{\pi}}[1-4S(2)+2I_4^{(2)}]$	$\frac{1}{4\pi\sqrt{3}}[1-4S(3)+2I_4^{(3)}]$
5	0	0	$\frac{1}{4\sqrt{\pi}}[1-5S(2)+5I_4^{(2)}]$	$\frac{1}{4\pi\sqrt{3}}[1-5S(3)+5I_4^{(3)}]$

Table II

μ	ν	$K_{\mu\nu}$	$J_{\mu\nu}$
0	0	$\frac{1}{8\pi}$	$\frac{1}{8\pi}$
1	0	$\frac{1}{8\pi}S\left(\frac{1}{2}\right)$	$\frac{1}{8\pi}\left[\frac{1}{2}+S\left(\frac{1}{2}\right)\right]$
0	1	$-K_{10}$	$\frac{1}{8\pi}\left[\frac{1}{2}-S\left(\frac{1}{2}\right)\right]$
1	1	0	$\frac{1}{32\pi}$
2	0	$\frac{1}{8\pi}\left[\frac{1}{4}-S(2)\right]$	$\frac{1}{8\pi}\left[\frac{1}{2}-S(2)+S\left(\frac{1}{2}\right)\right]$
0	2	K_{20}	$\frac{1}{8\pi}\left[\frac{1}{2}-S(2)-S\left(\frac{1}{2}\right)\right]$
2	1	$\frac{1}{8\pi}\left[-\frac{1}{4}S\left(\frac{1}{2}\right)+I_{2,1}\right]$	$\frac{1}{8\pi}\left[\frac{1}{4}-\frac{1}{2}S(2)+I_{2,1}\right]$
1	2	$-K_{21}$	$\frac{1}{8\pi}\left[\frac{1}{4}-\frac{1}{2}S(2)-I_{2,1}\right]$
3	0	$\frac{1}{8\pi}\left[\frac{1}{4}S\left(\frac{1}{2}\right)-\frac{1}{2}S(2)+I_{3,0}\right]$	$\frac{1}{8\pi}\left[\frac{1}{2}-2S(2)+S\left(\frac{1}{2}\right)+I_{3,0}\right]$
0	3	$-K_{30}$	$\frac{1}{8\pi}\left[\frac{1}{2}-S(2)-S\left(\frac{1}{2}\right)-I_{3,0}\right]$

§ 9. Calculations of $E(t_{i|n})$, $E(t_{i|n}^2)$ and $E(t_{i|n}t_{k|n})$.

By means of foumulas in Part I, and the above Table I, II, we have calculated $E(t_{i|n})$, $E(t_{i|n}^2)$ as well as $E(t_{i|n}t_{k|n})$ up to $n=7$, the results of which are tabulated below in Table III and IV. (As to numerical approximations, cf. § 10).

Table III (Expectations of $t_{i|n}$ and $t^2_{i|n}$)[16]

n	i	$n-i+1$	$E(t_{i\|n})=-E(t_{n-i+1\|n})$	$E(t^2_{i\|n})=E(t^2_{n-i+1})$
2	2	1	$\frac{1}{\sqrt{\pi}}=0.5641896$	1
3	3	1	$\frac{3}{2\sqrt{\pi}}=0.8462844$	$1+\frac{\sqrt{3}}{2\pi}=1.2756644$
	2		0	$1-\frac{\sqrt{3}}{\pi}=0.4486711$
4	4	1	$\frac{3}{\sqrt{\pi}}(1-2S(2))=1.0293754$	$1+\frac{\sqrt{3}}{\pi}=1.5513289$
	3	2	$\frac{3}{\sqrt{\pi}}(-1+6S(2))=0.2970114$	$1-\frac{\sqrt{3}}{\pi}=0.4486711$
5	5	1	$\frac{5}{\sqrt{\pi}}(1-3S(2))=1.1629645$	$1+\frac{5\sqrt{3}}{2\pi}[1-2S(3)]=1.8000204$
	4	2	$\frac{5}{\sqrt{\pi}}(-1+6S(2))=0.4950190$	$1+\frac{5\sqrt{3}}{\pi}[-1+4S(3)]=0.5565627$
	3		0	$1+\frac{5\sqrt{3}}{\pi}[1-6S(3)]=0.2868337$
6	6	1	$\frac{15}{2\sqrt{\pi}}[1-4S(2)+2I_4^{(2)}]=1.2672064$	$1+\frac{5\sqrt{3}}{\pi}[1-3S(3)]=2.0217391$
	5	2	$\frac{15}{2\sqrt{\pi}}[-1+8S(2)-10I_4^{(2)}]=0.6417550$	$1+\frac{5\sqrt{3}}{\pi}[-2+9S(3)]=0.6914273$
	4	3	$\frac{30}{\sqrt{\pi}}[0-S(2)+5I_4^{(2)}]=0.2015468$	$1+\frac{5\sqrt{3}}{\pi}[1-6S(3)]=0.2868337$
7	7	1	$\frac{21}{2\sqrt{\pi}}[1-5S(2)+5I_4^{(2)}]=1.3521784$	$1+\frac{35\sqrt{3}}{\pi}\left[\frac{1}{4}-S(3)+\frac{1}{2}I_4^{(3)}\right]=2.2203043$
	6	2	$\frac{21}{2\sqrt{\pi}}[-1+10S(2)-20I_4^{(2)}]=0.7573743$	$1+\frac{35\sqrt{3}}{\pi}\left[-\frac{1}{2}+3S(3)-3I_4^{(3)}\right]=0.8303490$
	5	3	$\frac{105}{2\sqrt{\pi}}[0-S(2)+5I_4^{(2)}]=0.3527069$	$1+\frac{35\sqrt{3}}{\pi}\left[\frac{1}{4}-3S(3)+\frac{15}{2}I_4^{(3)}\right]=0.3441234$
	4		0	$1+\frac{35\sqrt{3}}{\pi}\left[0+2S(3)-10I_4^{(3)}\right]=0.2104481$

Where $S(\alpha)=\frac{1}{\pi}\sin^{-1}\sqrt{\frac{\alpha}{2(1+\alpha)}}$, so that $S(2)=\frac{1}{\pi}\sin^{-1}\frac{1}{\sqrt{3}}=0.1959132760$, $S(3)=\frac{1}{\pi}\sin^{-1}\sqrt{\frac{3}{8}}=0.2097846884$, and $S\left(\frac{1}{2}\right)=\frac{1}{\pi}\sin^{-1}\frac{1}{\sqrt{6}}=0.1338602364$. Also $I_4^{(\alpha)}=\frac{1}{\pi^2}\int_0^{\pi S(\alpha)}\sin^{-1}\sqrt{\frac{\alpha(\alpha+1)/2}{\alpha(\alpha+2)-\tan^2\psi}}\,d\psi$, so that $I_4^{(2)}=0.04156\ 42048$, and $I_4^{(3)}=0.04604\ 86206$.

16) Although some of our results apparently differ in form from those of Godwin, loc. cit. p. 284, all of them do really coincide, as e.g. $E(t_{3|4})=\frac{3}{\sqrt{\pi}}\left[\frac{6}{\pi}\sin^{-1}\frac{1}{\sqrt{3}}-1\right]=\frac{3}{\sqrt{\pi}}\left[\frac{1}{2}-\frac{3}{\pi}\sin^{-1}\frac{1}{3}\right]$, because $2\sin^{-1}\frac{1}{\sqrt{3}}+\sin^{-1}\frac{1}{3}=\frac{\pi}{2}$, &c. The same can be said about Table IV below.

Table IV (Expectations of Products $t_{i|n}t_{k|n}$ for $i \neq k$)

n	$t_i t_k$	$t_{n-k+1} t_{n-i+1}$	$E(t_{i\|n} t_{k\|n}) = E(t_{n-k+1\|n} t_{n-i+1\|n})$
2	$t_1 t_2$		0
3	$t_1 t_2$	$t_2 t_3$	$\frac{\sqrt{3}}{2\pi}=0.2756644$
	$t_1 t_3$		$-\frac{\sqrt{3}}{\pi}=-0.5513289$
4	$t_1 t_2$	$t_3 t_4$	$\frac{\sqrt{3}}{\pi}=0.5513289$
	$t_1 t_3$	$t_2 t_4$	$\frac{3-2\sqrt{3}}{\pi}=0.1477281$
	$t_1 t_4$		$-\frac{3}{\pi}=-0.9549297$
	$t_2 t_3$		$\frac{2\sqrt{3}-3}{\pi}=0.1477281$
5	$t_1 t_2$	$t_4 t_5$	$\frac{5\sqrt{3}}{\pi}\left[\frac{1}{2}-S(3)\right]=0.8000204$
	$t_1 t_3$	$t_3 t_5$	$\frac{15}{\pi}\left[\frac{1}{2}-S\left(\frac{1}{2}\right)\right]+\frac{5\sqrt{3}}{\pi}[-1+2S(3)]=0.1481477$
	$t_1 t_4$	$t_2 t_5$	$\frac{15}{\pi}\left[-\frac{1}{2}+3S\left(\frac{1}{2}\right)\right]=-0.4699175$
	$t_1 t_5$		$\frac{15}{\pi}\left[0-2S\left(\frac{1}{2}\right)\right]=-1.2782711$
	$t_2 t_3$	$t_3 t_4$	$\frac{15}{\pi}\left[-\frac{1}{2}+S\left(\frac{1}{2}\right)\right]+\frac{5\sqrt{3}}{\pi}\left[\frac{1}{2}+S(3)\right]=0.2084354$
	$t_2 t_4$		$\frac{15}{\pi}\left[1-4S\left(\frac{1}{2}\right)\right]+\frac{5\sqrt{3}}{\pi}[0-4S(3)]=-0.0951011$
6	$t_1 t_2$	$t_5 t_6$	$\frac{5\sqrt{3}}{\pi}[1-3S(3)]=1.0217391$
	$t_1 t_3$	$t_4 t_6$	$\frac{45}{\pi}\left[\frac{1}{2}-S(2)-S\left(\frac{1}{2}\right)\right]+\frac{5\sqrt{3}}{\pi}[-2+6S(3)]=0.3948367$
	$t_1 t_4$	$t_3 t_6$	$\frac{45}{\pi}\left[-1+3S(2)+3S\left(\frac{1}{2}\right)\right]=-0.1529720$
	$t_1 t_5$	$t_2 t_6$	$\frac{45}{\pi}\left[1-4S(2)-2S\left(\frac{1}{2}\right)\right]=-0.7358723$
	$t_1 t_6$		$\frac{45}{\pi}\left[-\frac{1}{2}+2S(2)\right]=-1.5494705$
	$t_2 t_3$	$t_4 t_5$	$\frac{45}{\pi}\left[-\frac{1}{2}+S(2)+S\left(\frac{1}{2}\right)\right]+\frac{5\sqrt{3}}{\pi}[1+0]=0.3183300$
	$t_2 t_4$	$t_3 t_5$	$\frac{45}{\pi}\left[\frac{3}{2}-3S(2)-5S\left(\frac{1}{2}\right)\right]+\frac{5\sqrt{3}}{\pi}[0-6S(3)]=0.0103204$
	$t_2 t_5$		$\frac{45}{\pi}\left[-2+6S(2)+6S\left(\frac{1}{2}\right)\right]=-0.3059441$
	$t_3 t_4$		$\frac{45}{\pi}\left[-\frac{1}{2}+0+2S\left(\frac{1}{2}\right)\right]+\frac{5\sqrt{3}}{\pi}[0+6S(3)]=0.1426517$

Table IV (Continued)

n	$t_i t_k$	$t_{n-k+1} t_{n-i+1}$	$E(t_{i\mid n} t_{k\mid n}) = E(t_{n-k+1\mid n} t_{n-i+1\mid n})$
7	$t_1 t_k$	$t_6 t_7$	$\frac{35\sqrt{3}}{\pi}\left[\frac{1}{4}-S(3)+\frac{1}{2}I_4^{(3)}\right]=1.2203041$
	$t_1 t_3$	$t_5 t_7$	$\frac{105}{\pi}\left[\frac{1}{2}-S(2)-S\left(\frac{1}{2}\right)-I_{3,0}+0\right]+\frac{35\sqrt{3}}{\pi}\left[-\frac{1}{2}+2S(3)-I_4^{(3)}\right]=0.6090384$
	$t_1 t_4$	$t_4 t_7$	$\frac{105}{\pi}\left[-\frac{5}{4}+\frac{5}{2}S(2)+4S\left(\frac{1}{2}\right)+4I_{3,0}-3I_{2,1}\right]=0.0984870$
	$t_1 t_5$	$t_3 t_7$	$\frac{105}{\pi}\left[\frac{3}{2}-3S(2)-6S\left(\frac{1}{2}\right)-6I_{3,0}+12I_{2,1}\right]=-0.4003630$
	$t_1 t_6$	$t_2 t_7$	$\frac{105}{\pi}\left[-\frac{3}{4}+\frac{1}{2}S(2)+5S\left(\frac{1}{2}\right)+5I_{3,0}-15I_{2,1}\right]=-0.9641864$
	$t_1 t_7$		$\frac{105}{\pi}\left[0+S(2)-2S\left(\frac{1}{2}\right)-2I_{3,0}+6I_{2,1}\right]=-1.7835842$
	$t_2 t_3$	$t_5 t_6$	$\frac{105}{\pi}\left[-\frac{1}{2}+S(2)+S\left(\frac{1}{2}\right)+I_{3,0}+0\right]+\frac{35\sqrt{3}}{\pi}\left[\frac{1}{4}+0-\frac{3}{2}I_4^{(3)}\right]=0.4416147$
	$t_2 t_4$	$t_4 t_6$	$\frac{105}{\pi}\left[2-4S(2)-7S\left(\frac{1}{2}\right)-4I_{3,0}+6I_{2,1}\right]+\frac{35\sqrt{3}}{\pi}\left[0-2S(3)+4I_4^{(3)}\right]=0.1307293$
	$t_2 t_5$	$t_3 t_6$	$\frac{105}{\pi}\left[-\frac{15}{4}+\frac{21}{2}S(2)+15S\left(\frac{1}{2}\right)+6I_{3,0}-27I_{2,1}\right]=-0.1651763$
	$t_2 t_6$		$\frac{105}{\pi}\left[3-8S(2)-14S\left(\frac{1}{2}\right)-8I_{3,0}+36I_{2,1}\right]=-0.4936345$
	$t_3 t_4$	$t_4 t_5$	$\frac{105}{\pi}\left[-\frac{3}{4}+\frac{3}{2}S(2)+3S\left(\frac{1}{2}\right)+0-3I_{2,1}\right]+\frac{35\sqrt{3}}{\pi}\left[0+S(3)+I_4^{(3)}\right]=0.1655597$
	$t_3 t_5$		$\frac{105}{\pi}\left[3-9S(2)-12S\left(\frac{1}{2}\right)+0+18I_{2,1}\right]+\frac{35\sqrt{3}}{\pi}\left[0+0-6I_4^{(3)}\right]=0.0052035$

Where
$$I_{2,1}=\frac{1}{\pi^2}\int_0^1\frac{\sec^{-1}(u^2+5)}{(u^2+2)\sqrt{u^2+4}}du=0.02940\ 08395,$$
$$I_{3,0}=I_{2,1}+\frac{1}{\pi^2}\int_0^1\frac{\sec^{-1}(2v^2+4)}{(2v^2+1)\sqrt{2v^2+3}}dv=0.07898\ 12767.$$

§10. **Checks by Numerical Integrations.** We have so far obtained explicit forms of all $E(t_{i\mid n})$, $E(t_{i\mid n}^2)$ and $E(t_{i\mid n}t_{k\mid n})$ up to $n=7$. They are expressed by the combinations of constants $S(2)$, $S(3)$, $S\left(\frac{1}{2}\right)$ and integrals $I_4^{(\alpha)}$, $I_{2,1}$, $I_{3,0}$. But, to check our results e.g. by comparing with Godwin's, their numerical values are requisite.

First to compute

$$S(\alpha)=\frac{1}{\pi}\sin^{-1}\sqrt{\frac{\alpha}{2(1+\alpha)}}=\frac{1}{2\pi}\sec^{-1}(1+\alpha)\,, \tag{10. 1}$$

we need to know inverse-secants, which would be found e.g. from Chamber's seven-figures mathematical tables. In fact, by making use of the law of P. P., we found

$$S(2)=0.19591\ 32677,\quad S(3)=0.20978\ 46759,\quad S\left(\frac{1}{2}\right)=0.13386\ 02324.$$

To ascertain the precision of these figures, we have alternatively evaluated them by expanding $\sec^{-1}X$ into series:

$$(10.2)\quad \sec^{-1}X=\frac{\pi}{2}-\sum_{\nu=0}^{\infty}\frac{1.3.\cdots(2\nu-1)}{2.4.\cdots 2\nu}\frac{1}{2\nu+1}\frac{1}{X^{2\nu+1}}=\frac{\pi}{2}-\frac{1}{X}\sum_{\nu=0}^{\infty}\frac{\lfloor 2\nu}{(2^{\nu}\lfloor\nu)^2}\frac{X^{-2\nu}}{2\nu+1}$$
$$=\frac{\pi}{2}-\left[\frac{1}{X}+\frac{1}{6X^3}+\frac{3}{40}\frac{1}{X^5}+\frac{5}{112}\frac{1}{X^7}+\frac{35}{1152}\frac{1}{X^9}+\frac{63}{2816}\frac{1}{X^{11}}+\cdots\right]$$

and obtained, by taking a sufficiently many number of terms,

$$(10.3)\quad S(2)=0.19591\ 32760,\ S(3)=0.20978\ 46884,\ S\left(\frac{1}{2}\right)=0.13386\ 02364\,.$$

Thus even those before obtained from simple P. P. already agree with true values up to the seventh decimal place, so that by the following numerical integrations the values of inverse-secants were frequently evaluated simply by aid of the law of P. P. from Chamber's tables, the results of which are however reliable to seven efficient figures.

Next to evaluate

$$(10.4)\quad I_4^{(\alpha)}=\frac{1}{\pi^2}\int_0^{\pi S(\alpha)}\sin^{-1}\sqrt{\frac{\alpha(\alpha+1)/2}{\alpha(\alpha+2)-\tan^2\psi}}\,d\psi=\frac{1}{2\pi^2}\int_0^{\pi S(\alpha)}\sec^{-1}\left\{1+\frac{\alpha(\alpha+1)}{\alpha-\tan^2\psi}\right\}d\psi$$
$$(\alpha=2,\ 3)\,.$$

After Gauss, putting $\psi=\frac{1}{2}(1+t)\pi S(\alpha)$ and $\sec^{-1}\left\{1+\frac{\alpha(\alpha+1)}{\alpha-\tan^2\psi}\right\}=f(t)$, we have

$$I_4^{(\alpha)}=\frac{1}{2\pi}S(\alpha)\sum_{\nu=1}^{n}R_\nu f(t_\nu)\,,\qquad (\alpha=2,\ 3)\,.$$

Since $\tan S(2)=0.198\cdots\ll\sqrt{2}$ and $\tan S(3)=0.212\cdots\ll\sqrt{3}$, the integrand of (10.4) is surely regular in $|t|\leqq 1$, so that its Maclaurin's expansion $\sum_{\nu=0}^{\infty}c_\nu t^\nu$ converges absolutely and uniformly in $|t|\leqq 1$. Therefore, if m be taken appropriately large, the integrand may be approximated by $\sum_{\nu=0}^{m}c_\nu t^\nu$, a polynomial of degree m. Hence, Gauss' method of numerical integrations by n selected ordinates would certainly give a good approximation for the integral, if $m\leqq 2n-1$. We have taken as $n=5$, $m=9$, and obtained

$$(10.5)\qquad I_4^{(2)}=0.04156\ 420,\quad I_4^{(3)}=0.04604\ 862\,.$$

To secure how many figures are correct, we have after Gauss to find the upper bound of errors

$$F_n=c_{2n}\Omega_n\,,\qquad \text{where } c_{2n}=\frac{1}{\lfloor 2n}f^{(2n)}(0)\,.$$

However, as it is utterly cumbersome to get the $2n$-th (here tenth) derivative, so we proceed to compute Gauss' approximations at any rate, and check them for some known results. Really, we calculated $I_4^{(2)}$, $I_4^{(3)}$ from our Table III, using some explicit expressions of $E(t_{i|n})$, $E(t_{i|n}^2)$, say $E(t_{1|7})$, $E(t_{4|7}^2)$, whose numerical values are given in Godwin's paper, and found

$$I_4^{(2)} = 0.04156\ 420, \quad I_4^{(3)} = 0.04604\ 862.$$

These coincide precisely with our results (10. 5).

Lastly to evaluate

$$\text{(10. 6)} \qquad \pi^2 I_{2,1} = \int_0^1 \frac{\sec^{-1}(5+x^2)}{(2+x^2)\sqrt{4+x^2}}\,dx = \int_{-1}^0 = \frac{1}{2}\int_{-1}^1 f(x)dx.$$

Here the integrand $f(x)$ is also regular in $(-1-\varepsilon, 1+\varepsilon)$ and expansible in a Maclaurin's series, which is uniformly and absolutely convergent in $(-1, 1)$ Hence, on taking the partial sum $\sum_{\nu=0}^{m} c_\nu x^\nu$ adequately, again Gauss' method is applicable. However, now that

$$\left(1+\frac{1}{2}x^2\right)^{-1} = 1-0.5x_2+0.25x^4-0.125x^6+0.0625x^8-0.03125x^{10}- \cdots \&c.,$$

the convergency is rather slow in the vicinity of $x=1$, and if we make $n=5$, $m=9$, the error shall possibly take place at the third decimal place or thereabouts. Also, in the integral

$$\text{(10. 7)} \qquad \pi^2(I_{3,0}-I_{2,1}) = \int_0^1 \frac{\sec^{-1}(4+2x^2)}{(1+2x^2)\sqrt{3+2x^2}}\,dx = \frac{1}{2}\int_{-1}^1 f(x)dx$$

the factor $(1+2x^2)^{-1}$ being already not regular on $|x|=1/\sqrt{2}$ in the complex x-plane, the relating Maclaurin's series cannot be uniformly convergent in $|x|\leqq 1$, so that the applicability of Gauss' method becomes even doubtful. In fact, on applying this method, as $n=7$, we obtained

$$I_{2,1} = 0.02943\ 033, \quad I_{3,0} = 0.07902\ 063,$$

which are inacculate, since they make

$$\mathrm{Cov}(t_{1|7}, t_{4|7}) = E(t_{1|7}t_{4|7}) = \frac{105}{\pi}\left[-\frac{5}{4}+\frac{5}{2}S(2)+4S\left(\frac{1}{2}\right)+4I_{3,0}-3I_{2,1}\right] = 0.10079\cdots,$$

while Godwin's result informs to be 0.09849. Of course, if we put $x=\frac{1}{2}(1+t)$ and $\dfrac{\sec^{-1}\left\{5+\frac{1}{2}(1+t)^2\right\}}{[8+(1+t)^2]\sqrt{(16+(1+t)^2}} = g(t)$, we get $I_{2,1} = \frac{8}{\pi^2}\times\frac{1}{2}\int_{-1}^1 g(t)dt = \frac{8}{\pi^2}\sum_{\nu=1}^{n} R_\nu g(t_\nu)$. Whence we found, as $n=5$, $I_{2,1}=0.02940\ 083$ and similarly $I_{3,0}=0.07901\ 226$ but $\mathrm{Cov}(t_{1|7}, t_{4|7})=0.10263$. This is worse than before obtained one, in which however we took, as $n=7$.

Rather, if computed after Simpson's rule with twenty subdivisions, we obtain

$$I_{2,1} = 0.02940\ 082, \quad I_{3,0} = 0.07898\ 295 \quad \text{and} \quad \mathrm{Cov}\,(t_{1|7}, t_{4|7}) = 0.09870,$$

which is still aberrant from Godwin's result though, yet somewhat nearer than those obtained before by Gauss' method[17]. Therefore we have contrived another method of approximation as follows:

On applying (10. 2) the inverse-secant in the integrand of (10. 6) is

$$\sec^{-1}(x^2+5) = \frac{\pi}{2} - \sum_{\nu=0}^{\infty} \frac{\lfloor 2\nu}{(2^\nu \lfloor \nu)^2 (2\nu+1)(x^2+5)^{2\nu+1}},$$

in which, already for $\nu=5$, the corresponding term becomes $<\frac{1}{10^9}$ and also the total remainder inclusive this term $R_5 < \frac{1}{10^9}\left[1+\frac{1}{5^2}+\frac{1}{5^4}+\cdots\right] < \frac{2}{10^9}$. Hence, when R_5 be substituted in (10. 6) and integrated, it would be at most amount $< \frac{1}{4\pi^2} \times \frac{2}{10^9} < \frac{1}{10^{10}}$. Therefore, to obtain an approximate value of $I_{2,1}$ up to the tenth decimal place, we may neglect R_5 and take only five terms in summation. Thus it suffices to compute

$$(10.8) \qquad I_\nu = \int_0^1 \frac{dx}{(x^2+2)\sqrt{x^2+4}\,(x^2+5)^{2\nu-1}} \quad \text{for} \quad \nu = \frac{1}{2}, 1, \cdots, 5.$$

Or, putting $x=2\tan\theta$, we obtain

$$(10.9) \qquad I_\nu = \frac{1}{2}\int_0^{\tan^{-1}1/2} \frac{\cos^{4\nu-1}\theta\, d\theta}{(1+\sin^2\theta)(5-\sin^2\theta)^{2\nu-1}} = \frac{1}{2}\int_0^{1/\sqrt{5}} \frac{(1-t^2)^{2\nu-1}\, dt}{(1+t^2)(5-t^2)^{2\nu-1}} \qquad (t=\sin\theta).$$

---

17) We have also tried Markov-Berger's method of numerical integrations; Really we have from formulas in Tables II and III

$$K_{3,0} = \int U^3\varphi^2\, dt \int^t \varphi_1^2 dt_1 = \frac{1}{8\pi}\left[I_{3,0} - \frac{1}{2}S(2) + \frac{1}{4}S\left(\frac{1}{2}\right)\right].$$

On the other hand after Markov and Berger

$$K_{3,0} = \frac{1}{4\pi}\int \frac{e^{-t^2}}{\sqrt{\pi}} U^3\, dt \int^t \frac{e^{-t_1^2}}{\sqrt{\pi}}\, dt_1 = \int \frac{e^{-t^2}}{\sqrt{\pi}} f(t)\, dt = \sum_{i=1}^{n} A_i f(t_i),$$

where

$$f(t) = \frac{1}{4\pi}\left\{\int_0^t e^{-t^2/2}\frac{dt}{\sqrt{2\pi}}\right\}^3 \int_{-\infty}^t e^{-t^2}\frac{dt}{\sqrt{\pi}}$$

can be found from the tables of Probability Integrals. Likewise

$$K_{2,1} = \frac{1}{8\pi}\left[I_{2,1} - \frac{1}{4}S\left(\frac{1}{2}\right)\right]$$

and

$$K_{2,1} = \int U^2\varphi^2\, dt \int^t U_1\varphi_1^2\, dt_1 = \int \frac{e^{-t^2}}{\sqrt{\pi}} f(t)\, dt = \sum_{i=1}^{n} A_i f(t_i),$$

if $f(t) = U^2(t)V(t)$ and $V(t) = \frac{1}{2\sqrt{\pi}}\int^t U_1\varphi_1^2\, dt_1 = \int^0 + \int_0^t = \text{(i)} + \text{(ii)}$,

where

$$\text{(i)} = \frac{1}{2\sqrt{\pi}}\int^0 U_1\varphi_1^2\, dt_1 = \frac{1}{2\sqrt{\pi}}\int_0 e^{-t_1^2}\frac{dt}{2\pi}\int_0^{t_1} e^{-\tau^2/2}\frac{d\tau}{\sqrt{2\pi}}$$

$$= \frac{1}{4\pi^2\sqrt{2}}\int_{\pi/4}^{\pi/2} d\theta \int_0 \exp\left\{-\frac{r^2}{2}(\cos^2\theta + 2\sin^2\theta)\right\} r\, dr = \frac{1}{8\pi^2}\tan^{-1}\frac{1}{\sqrt{2}}$$

and

$$\text{(ii)} = \frac{1}{4\pi^2\sqrt{2}}\int_{\pi/4}^{\pi/2} d\theta \int_0^{t\,\mathrm{cosec}\,\theta} \exp\left\{-\frac{r^2}{2}(\cos^2\theta + 2\sin^2\theta)\right\} r\, dr$$

$$= \frac{1}{4\pi^2\sqrt{2}}\int_0^1 \frac{1 - \exp\{-\frac{1}{2}(1+u^2)t^2\}}{2+u^2}\, du \qquad (u = \cot\theta)$$

which for every $t=t_i$ can be computed by Gauss. Thus when $K_{3,0}$ and $K_{2,1}$ be found, we can thereby calculate $I_{3,0}$ and $I_{2,1}$. However, the results obtained as $n=7$ were very unpleasing.

These were exactly integrated, and the required integral is nearly

(10.10) $$I_{2,1}=\frac{1}{\pi^2}\left[\frac{\pi}{2}I_{\frac{1}{2}}-\left(I_1+\frac{1}{6}I_2+\frac{3}{40}I_3+\frac{5}{112}I_4+\frac{35}{1152}I_5\right)\right]=0.02940\ 08395.$$

Similarly with (10.7), we have

(10.11) $$I_{3,0}=I_{2,1}+\frac{1}{\pi^2}\left[\frac{\pi}{2}J_{\frac{1}{2}}-\left(J_1+\frac{1}{6}J_2+\frac{3}{40}J_3+\frac{5}{112}J_4+\frac{35}{1152}J_5\right)\right],$$

where

(10.12) $$\begin{aligned} J_\nu &= \int_0^1 \frac{dx}{(1+2x^2)\sqrt{3+2x^2}\,(4+2x^2)^{2\nu-1}} \\ &= \frac{1}{\sqrt{2}}\int_0^{\tan^{-1}\sqrt{2/3}} \frac{\cos^{4\nu-1}\theta\, d\theta}{(1+2\sin^2\theta)(4-\sin^2\theta)^{2\nu-1}} \quad \left(x=\sqrt{\frac{3}{2}}\tan\theta\right) \\ &= \frac{1}{\sqrt{2}}\int_0^{\sqrt{0\cdot 4}} \frac{(1-t^2)^{2\nu-1}\,dt}{(1+2t^2)(4-t^2)^{2\nu-1}} \left(t=\sin\theta,\ \nu=\frac{1}{2},1,2,\cdots,5\right). \end{aligned}$$

Whence its numerical value was obtained as

(10.13) $$I_{3,0}=0.0789812767,\quad \text{approximately.}$$

These values being substituted, now yields

$$\text{Cov }(t_{1|7},\,t_{4|7})=0.0984869917$$

which coincides with Godwin's result.

By making use of (10.3) (10.5) (10.10) and (10.13) numerical values in Tables III and IV were obtained up to the seventh decimal place, all of which agree with those in Godwin's paper.

§ **11. Variance $\mathbf{D}^2(\zeta)$.** We have seen in §6 that, if $z=\sum c_i x_i$, $\sum c_i=1$, $x_i=m+\sigma t_i$, $\zeta=\sum c_i t_i$, so $E(z)=m+\sigma E(\zeta)$, $D^2(z)=\sigma^2 D^2(\zeta)$ with $E(\zeta)=\sum c_i E(t_i)$ and

(11.1) $$\begin{aligned} D^2(\zeta) &= \sum_{i=1}^n c_i^2 \text{ Var }(t_{i|n})+\sum_{k=1}^n\sum_{k\neq i} c_i c_k \text{ Cov }(t_{i|n},\,t_{k|n}) \\ &= \sum_{i=1}^n c_i^2[E(t_i^2)-E(t_i)^2]+\sum_{i=1}^n\sum_{k\neq i} c_i c_k[E(t_i t_k)-E(t_i)E(t_k)]=u \text{ say.} \end{aligned}$$

Let us consider its relative minimum under condition $\sum_{i=1}^n c_i=1$. For this we have to find the absolute minimum of the function.

(11.2) $$w=u-2\lambda v \qquad \text{with} \qquad v=\sum c_i-1,$$

so that

(11.3) $$\frac{1}{2}\frac{\partial w}{\partial c_j}=c_j[E(t_j^2)-E(t_j)^2]+\sum_{k\neq j} c_k[E(t_j t_k)-E(t_j)E(t_k)]-\lambda=0,$$

viz.

$$\sum_{k=1}^n c_k[E(t_j t_k)-E(t_j)E(t_k)]=\lambda \qquad (j=1,2,\cdots,n).$$

Hence we obtain by G. Cramer's formula

(11.4) $$c_k = \Delta_k/\Delta \qquad (k = 1, 2, \cdots, n),$$

where

$$\Delta = \begin{vmatrix} E(t_1t_1)-E(t_1)E(t_1) & \cdots & E(t_1t_k)-E(t_1)E(t_k) & \cdots & E(t_1t_n)-E(t_1)E(t_n) \\ \cdots\cdots & & \cdots\cdots & & \cdots\cdots \\ E(t_it_1)-E(t_i)E(t_1) & \cdots & E(t_it_k)-E(t_i)E(t_k) & \cdots & E(t_it_n)-E(t_i)E(t_n) \\ \cdots\cdots & & \cdots\cdots & & \cdots\cdots \\ E(t_nt_1)-E(t_n)E(t_1) & \cdots & E(t_nt_k)-E(t_n)E(t_k) & \cdots & E(t_nt_n)-E(t_n)E(t_n) \end{vmatrix}$$

$$= \begin{vmatrix} E(t_1t_1)-E(t_1)E(t_1) & \cdots & \sum_{k=1}^{n} E(t_1t_k)-E(t_1)\sum_{k=1}^{n} E(t_k) & \cdots & E(t_1t_n)-E(t_1)E(t_n) \\ \cdots\cdots & & \cdots\cdots & & \cdots\cdots \\ E(t_it_1)-E(t_i)E(t_1) & \cdots & \sum_{k=1}^{n} E(t_it_k)-E(t_i)\sum_{k=1}^{n} E(t_k) & \cdots & E(t_it_n)-E(t_i)E(t_n) \\ \cdots\cdots & & \cdots\cdots & & \cdots\cdots \\ E(t_nt_1)-E(t_n)E(t_1) & \cdots & \sum_{k=1}^{n} E(t_nt_k)-E(t_n)\sum_{k=1}^{n} E(t_k) & \cdots & E(t_nt_n)-E(t_n)E(t_n) \end{vmatrix}.$$

But $\sum_{k=1}^{n} E(t_it_k)-E(t_i)\sum_{k=1}^{n} E(t_k)=1$ $(i=1, 2, \cdots, n)$, in view of (5.10) and (3.1), so

$$\Delta_k = \lambda\Delta \quad \text{and consequently} \quad c_k = \lambda \quad (k = 1, 2, \cdots, n).$$

Hence $c_1=c_2=\cdots=c_n=\frac{1}{n}$ and $E(\zeta)=\frac{1}{n}\sum E(t_k)=0$, $E(z)=\frac{1}{n}\sum E(x_i)=\frac{1}{n}\sum [m+\sigma E(t_i)]=m$. Thus, the theorem that the A. M. $\bar{x}=\frac{1}{n}\sum x_i$ is the efficient estimate of population mean, which is the case for any unordered sample with independent individuals, still remains true for any ordered sample with non-independent individuals also. In fact, $E(\zeta)=0$, $E(\bar{x})=m$, so $\bar{x}$ is unbiased, and besides, in virtue of (5.3) and (3.1)

$$D^2(\bar{t}) = \frac{1}{n^2}\left[\sum_{i=1}^{n} E(t_i^2)+\sum_{i=1}^{n-1}\sum_{k>i}^{n} E(t_it_k)-\left\{\sum_{i=1}^{n} E(t_i)\right\}^2\right] = \frac{1}{n^2}[n+0+0] = \frac{1}{n}.$$

Hence

(11.5) $$D^2(\bar{x}) = \sigma^2/n.$$

More specially, if we consider the case that $c_{n-i+1}=c_i$ for $i=1, 2, \cdots, [n/2]$ with $\sum_{i=1}^{n} c_i=1$, and if $n=2p+1$, making $c_{p+1}=1-2\sum_{i=1}^{p} c_i$, we see that by the above theorem that every $D^2(\zeta)$ cannot be smaller than $1/n$. This can be directly shown by formation of actual expressions, on substituting those values $E(t_{i|n})$, $E(t_{i|n}^2)$ and $E(t_{i|n}t_{k|n})$ of Tables III and IV in (11.1):

1° $n=3$ (Cramér's example): $c_1=c_3=c$, $c_2=1-2c$. We get readily

(11.6) $$D^2(\zeta) = \frac{1}{3}+\frac{3}{\pi}(2\pi-3\sqrt{3})\left(c-\frac{1}{3}\right)^2, \quad D^2(z) = \sigma^2 D^2(\zeta) \geqq \frac{\sigma^2}{3}.$$

2° $n=4$: $c_1=c_4=c$, $c_2=c_3=\frac{1}{2}-c$.

(11.7) $$D^2(\zeta) = \frac{1}{4} + \frac{1}{\pi}(2\pi + 4\sqrt{3} - 3)\left(c - \frac{1}{4}\right)^2, \quad D^2(z) \geqq \frac{\sigma^2}{4}.$$

3° $n=5$: $c_1 = c_5 = c$, $c_2 = c_4 = c'$, $c_3 = 1 - 2c - 2c'$. We assume $D^2(\zeta)$ to be of the form:

(11.8) $$D^2(\zeta) = \frac{1}{5} + A\left(c + c' - \frac{2}{5}\right)^2 + B\left(c - \frac{1}{5}\right)^2 + C\left(c' - \frac{1}{5}\right)^2.$$

To find A, we ask the coefficient of $2cc'$ in the obtained expression of $D^2(\zeta)$. This is really

$$A = 4 + \frac{5\sqrt{3}}{\pi}[7 - 38S(3)] + \frac{15}{\pi}\left[6S\left(\frac{1}{2}\right) - 1\right] > 0.$$

On subtractidg $\frac{1}{5} + A\left(c + c' - \frac{2}{5}\right)^2$ from $D^2(\zeta)$, the remainder decomposes into a sum of squares whose coefficients are

$$B = 2 + \frac{10\sqrt{3}}{\pi}[3 - 2S(3)] - \frac{15}{\pi}\left[3 + 2S\left(\frac{1}{2}\right)\right] > 0,$$

$$C = 2 - \frac{15\sqrt{3}}{\pi}[3 - 2S(3)] + \frac{15}{\pi}\left[7 - 11S\left(\frac{1}{2}\right)\right] > 0.$$

4° $n=6$: $c_1 = c_6 = c$, $c_2 = c_5 = c'$, $c_3 = c_4 = \frac{1}{2} - c - c$. Assuming again as before

(11.9) $$D^2(\zeta) = \frac{1}{6} + A\left(c + c' - \frac{1}{3}\right)^2 + B\left(c - \frac{1}{6}\right)^2 + C\left(c' - \frac{1}{6}\right)^2,$$

and asking the coefficient of $2cc'$, we find

$$A = 2 + \frac{30\sqrt{3}}{\pi}[1 - S(3)] + \frac{180}{\pi}\left[-2S(2) + S\left(\frac{1}{2}\right)\right] > 0.$$

Subtracting $\frac{1}{6} + A\left(c + c' - \frac{1}{3}\right)^2$ from the expression of $D^2(\zeta)$, the remainder decomposes into a sum of squares whose coefficients are

$$B = 2 + \frac{30\sqrt{3}}{\pi}[1 - 4S(3)] + \frac{180}{\pi}\left[S(2) - 2S\left(\frac{1}{2}\right)\right] > 0,$$

$$C = 2 + \frac{60\sqrt{3}}{\pi}[4S(3) - 1] + \frac{45}{\pi}\left[-9 + 28S(2) + 28S\left(\frac{1}{2}\right)\right] > 0.$$

5° $n=7$: $c_1 = c_7 = c$, $c_2 = c_6 = c'$, $c_3 = c_5 = c''$, $c_4 = 1 - 2c - 2c' - 2c''$. Assuming

(11.10) $$D^2(\zeta) = \frac{1}{7} + A\left(c' + c'' - \frac{2}{7}\right)^2 + B\left(c + c'' - \frac{2}{7}\right)^2 + C\left(c + c' - \frac{2}{7}\right)^2 + H\left(c - \frac{1}{7}\right)^2 + K\left(c' - \frac{1}{7}\right)^2 + L\left(c'' - \frac{1}{7}\right)^2,$$

and asking the coefficients of $2c'c''$, $2cc''$ and $2cc'$, we find

$$A = 4 + \frac{35\sqrt{3}}{\pi}\left[\frac{1}{2} + 12S(3) - 63I_4^{(3)}\right] + \frac{105}{\pi}\left[-\frac{27}{2} + 33S(2) + 48S\left(\frac{1}{2}\right) - 66I_{2,1} + 30I_{3,0}\right] > 0,$$

$$B=4+\frac{35\sqrt{3}}{\pi}\left[-1+8S(3)-46I_4^{(3)}\right]+\frac{105}{\pi}\left[12-24S(2)-42S\left(\frac{1}{2}\right)\right.$$
$$\left.+48I_{2,1}-30I_{3,0}\right]>0,$$

$$C=4+\frac{35\sqrt{3}}{\pi}\left[\frac{1}{2}+14S(3)-55I_4^{(3)}\right]+\frac{105}{\pi}\left[-\frac{9}{2}+7S(2)+22S\left(\frac{1}{2}\right)\right.$$
$$\left.-42I_{2,1}+10I_{3,0}\right]>0.$$

Subtracting $\frac{1}{7}+A\left(c'+c''-\frac{2}{7}\right)^2+B\left(c+c''-\frac{2}{7}\right)^2+C\left(c+c'-\frac{2}{7}\right)^2$ from the whole expression of $D^2(\zeta)$, we see that the remainder decomposes into a sum of three squares whose coefficients are

$$H=-2+\frac{35\sqrt{3}}{\pi}[1-16S(3)+62I_4^{(3)}]+\frac{105}{\pi}\left[\frac{5}{2}-S(2)-16S\left(\frac{1}{2}\right)\right.$$
$$\left.+30I_{2,1}-16I_{3,0}\right]>0,$$

$$K=-2+\frac{35\sqrt{3}}{\pi}[-2+4S(3)+62I_4^{(3)}]+\frac{105}{\pi}\left[8-24S(2)-42S\left(\frac{1}{2}\right)\right.$$
$$\left.+132I_{2,1}-24I_{3,0}\right]>0,$$

$$L=-2+\frac{35\sqrt{3}}{\pi}[1-26S(3)+64I_4^{(3)}]+\frac{105}{\pi}\left[\frac{27}{2}-39S(2)-54S\left(\frac{1}{2}\right)+78I_{2,1}\right]>0.$$

If we take a single $x_{i|n}$ as z, we get $E(z)=m+\sigma E(t_{i|n})$. Thus there gives rise to a biase $\sigma E(t_{i|n})$, which is $\neq 0$, except the case of median, and $D^2(z)=\sigma^2$ Var $(t_{i|n})$. All values of $E(t_{i|n})$, Var $(t_{i|n})$ as well as Cov $(t_{i|n}, t_{k|n})$ for $n=2, 3, \cdots, 10$ are given to five decimal places in Godwin's paper loc. cit., pp. 281-2. Only when $n=2p+1$, $i=p+1$, we have $E(t_{p+1|2p+1})=0$ and $x_{p+1|2p+1}$, as single observation, renders an unbiased estimate of the population mean with efficiency

(11. 11) $$\text{eff.}=\frac{1}{n}:\ \text{Var}\ (t_{p+1|2p+1})=1/nE(t^2_{p+1|2p+1})\ .$$

Really, using Godwin's Table, we get these efficiencies as those starred in Table V. Lastly, every $z=\frac{1}{2}(x_{i|n}+x_{n-i+1|n})$ or $\zeta=\frac{1}{2}(t_{i|n}+t_{n-i+1|n})$ gives an unbiased estimate, its variance being

$$D^2(\zeta)=E(\zeta)^2-E(\zeta)^2=\frac{1}{4}\{E(t_i^2+t_{n-i+1}^2+2t_it_{n-i+1})-[E(t_i)+E(t_{n-i+1})]^2\}$$
$$=\frac{2}{4}\{E(t_i^2)-E(t_i)^2+E(t_it_{n-i+1})-E(t_i)E(t_{n-i+1})\}=\frac{1}{2}[\text{Var}(t_i)+\text{Cov}(t_i,\ t_{n-i+1})].$$

Hence its efficiency is

(11. 12) $$\text{eff.}=2/n[\text{Va}(t_{i|n})+\text{Cov}(t_{i|n}t_{n-i+1|n})]\ .$$

These are also calculated from Godwin's Table and tabulated in Table V, below.

Table V. (Efficiencies of estiamates $\frac{1}{2}(t_{i|n}+t_{n-i+1|n})$)

n	i	1	2	3	4	5
2	2	1				
3	2 3	0.92038	0.74294*			
4	3 4	0.83836	0.83836			
5	3 4 5	0.76665	0.86681	0.69728*		
6	4 5 6	0.70581	0.86470	0.77613		
7	4 5 6 7	0.65423	0.84855	0.81792	0.67882*	
8	5 6 7 8	0.61014	0.82604	0.83727	0.74323	
9	5 6 7 8 9	0.57213	0.80106	0.84300	0.78460	0.66894*
10	6 7 8 9 10	0.53895	0.77555	0.84027	0.81008	0.72294

§ 12. **Truncated Samples.** If a random variable ξ distribute logarithmico-normally, viz. its fr. f. be

$$\frac{1}{\sqrt{2\pi}\,\sigma(\xi-a)}\exp\left\{-\frac{1}{2\sigma^2}[\log(\xi-a)-m]^2\right\}, \qquad (\xi>a)$$

the variable $x=\log(\xi-a)$ distributes normally, viz. its fr. f. becomes

$$\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left\{-\frac{1}{2\sigma^2}(x-m)^2\right\}, \qquad (-\infty<x<\infty).$$

Hence, putting again $x=m+\sigma t$, the problem reduces to our case.

We are now interested in the so-called truncated sample[18]: namely, when only $k(<n)$ values $\xi_1\leqq\xi_2\leqq\cdots\leqq\xi_k$ are observed, but there experiment being stopped, the remaining values $\xi_{k+1}\leqq\cdots\leqq\xi_n$ left unmeasured (missed), required is how to estimate the mean and variance of the population? To determine

18) Cf. e.g. A. C. Cohen, Estimating the mean and variance of normal populations from simply truncated and doubly truncated sample, Math. Statis., Vol. 21 (1950), pp. 557–569.

them we are used to avail the so-called likelihood function, which is obtainable as follows.

The probability element to obtain $\{\xi_1, \xi_2, \cdots, \xi_k\}$ or $\{t_1, t_2, \cdots, t_k\}$ is

$$n!\, d\Phi_1 d\Phi_2 \cdots d\Phi_k \int_k d\Phi_{k+1} \cdots \int_{n-1} d\Phi_n$$
$$= \frac{n!}{(n-k)!} (1-\Phi_k)^{n-k} d\Phi_1 d\Phi_2 \cdots d\Phi_k, \qquad (1<k<n)^{19)}$$

where

$$d\Phi_i = \varphi_i dt_i = \frac{1}{\sqrt{2\pi}\sigma} e^{-t_i^2/2} dt_i, \; t_i = \frac{x_i - m}{\sigma} \quad \text{and} \quad 1-\Phi_k = \frac{1}{\sqrt{2\pi}} \int_{t_k} e^{-t^2/2} dt.$$

Hence the required likelihood function is

$$L(x_1, \cdots, x_k;\, m, \sigma) = \frac{n!}{(n-k)!} \frac{1}{(\sqrt{2\pi}\,\sigma)^k} \exp\left\{-\frac{1}{2\sigma^2} \sum_{\nu=1}^{k} (x_\nu - m)^2\right\} \cdot (1-\Phi_k)^{n-k},$$

and consequently

$$\log L = -\frac{1}{2\sigma^2} \sum_{\nu=1}^{k} (x_\nu - m)^2 - k \log \sigma + (n-k) \log(1-\Phi_k) + \log \frac{n!}{(n-k)!\,\sqrt{2\pi}^k}.$$

According to the Principle of Maximum Likelihood,

$$(12.1) \qquad \frac{\partial \log L}{\partial m} = \frac{1}{\sigma^2} \sum_{\nu=1}^{k} (x_\nu - m) + \frac{n-k}{(1-\Phi_k)\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x_k - m)^2}{2\sigma^2}\right\} = 0,$$

$$(12.2) \qquad \frac{\partial \log L}{\partial \sigma} = \frac{1}{\sigma^3} \sum_{\nu=1}^{k} (x_\nu - m)^2 - \frac{k}{\sigma} + \frac{(n-k)(x_k - m)}{(1-\Phi_k)\sqrt{2\pi}\sigma^2} \exp\left\{-\frac{(x_k - m)^2}{2\sigma^2}\right\} = 0.$$

On multiplying the first equation by $m - x_k$, the second by σ, respectively, and adding them, we obtain,

$$\sum_{\nu=1}^{k} (x_\nu - m)^2 - (m - x_k) \sum_{\nu=1}^{k} (m - x_\nu) = k\sigma^2.$$

Therefore

$$(12.3) \qquad \sigma^2 = \frac{1}{k} \sum_{\nu=1}^{k} (m - x_\nu)(x_k - x_\nu).$$

Also (12.1) being rewritten,

$$(12.4) \qquad \frac{n-k}{\sqrt{2\pi}} \exp\left\{-\frac{(m - x_k)^2}{2\sigma^2}\right\} = (1-\Phi_k) \sum_{\nu=1}^{k} \frac{m - x_\nu}{\sigma}.$$

If the value of σ^2, (12.3) be substituted in (12.4), we obtain an equation containing m only. Now, if a first approximation for m be m_1, we may put $m = m_1 + \varepsilon$, and seek the correction ε. We assume that m_1 being already fairly fitting, ε is so small in magnitude, that all its powers of higher degree than 1 are negligible. So, we get from (12.3)

$$\sigma^2 = \frac{1}{k}\left[\sum_{\nu=1}^{k} (m_1 - x_\nu)(x_k - x_\nu) + \varepsilon \sum_{\nu=1}^{k} (x_k - x_\nu) + 0(\varepsilon^2)\right].$$

---

19) Evidently it is impossible to determine two unknown m and σ from only one observation. Also, if $k=n$, the sample becomes complete. Hence we assume to be $1<k<n$.

For the sake of abbreviations, we put[20]

$$(12.5)\quad \begin{cases} \bar{x}=\sum_{\nu=1}^{k} x_\nu/k, \quad S_1=\sum_{\nu=1}^{k}(x_k-x_\nu)=k(x_k-\bar{x}), \quad S_2=\sum_{\nu=1}^{k} x_\nu(x_k-x_\nu), \\ \sum_0=\sum_{\nu=1}^{k}(m_1-x_\nu)(x_k-x_\nu), \quad \sigma_0^2=\sum_0/k=m_1S_1-S_2, \\ S=\sum_{\nu=1}^{k}(m_1-x_\nu)=k(m_1-\bar{x}), \quad S_1/\sum_0=(x_k-\bar{x})/\sigma_0^2. \end{cases}$$

And thus

$$(12.6)\qquad \frac{1}{\sigma^2}=\frac{1}{\sigma_0^2}\left(1-\frac{S_1}{\sum_0}\varepsilon\right), \quad \frac{1}{\sigma}=\frac{1}{\sigma_0}\left[1-\frac{S_1}{2\sum_0}\varepsilon\right].$$

So that

$$(12.7)\qquad t_\nu=\frac{x_\nu-m}{\sigma}=\frac{1}{\sigma_0}\left[x_\nu-m_1-\left(1+\frac{x_\nu-m_1}{2}\frac{S_1}{\sum_0}\right)\varepsilon\right].$$

And in particular

$$(12.8)\qquad t_k=\frac{x_k-m}{\sigma}=\frac{1}{\sigma_0}\left[x_k-m_1-\left(1+\frac{x_k-m_1}{2}\frac{S_1}{\sum_0}\right)\varepsilon\right]=A-B\varepsilon,$$

where

$$(12.9)\qquad A=\frac{x_k-m_1}{\sigma_0}, \quad B=\frac{1}{\sigma_0}\left[1+\frac{x_k-m_1}{2}\frac{S_1}{\sum_0}\right].$$

Also from (12.7) and (12.5) yields

$$(12.10)\qquad \sum_{\nu=1}^{k}\frac{m-x_\nu}{\sigma}=\frac{1}{\sigma_0}\left[S+\left(k-\frac{SS_1}{2\sum_0}\right)\varepsilon\right].$$

As to Φ_k we have by (12.8)

$$(12.11)\qquad \Phi_k=\int^{t_k}\varphi dt=\int^{A-B\varepsilon}=\int^{A}+\int_{A}^{A-B\varepsilon}=\Phi(A)-B\varphi(A)\varepsilon, \quad \text{nearly.}$$

All these substituted in (12.4) and after neglection of higher power of ε, solved for ε, we attain finally

$$(12.12)\qquad \varepsilon=\varepsilon_1=\frac{(n-k)\sigma_0\varphi(A)-S[1-\varphi(A)]}{BS\varphi(A)+[k-ABS-SS_1/2\sum_0][1-\Phi(A)]}, \quad \text{approximately.}$$

Thus $\varepsilon=\varepsilon_1$ being found, we recompute σ_0, S, A, B with the corrected $m_1+\varepsilon_1=m_2$ (the second approximation), and again calculate a new correction ε_2; and over again using $m_2+\varepsilon_2=m_3$ recompute $\sum_0$ &c., find third correction ε_3, and so on (successive approximations).

However, if we could observe every value x_ν several times (each N times, say) we may utilize Gauss' Method of Least Squares. Really from $x_\nu=m+\sigma t_\nu$, we have

$$E(x_\nu)=m+\sigma E(t_\nu), \quad \nu=1, 2, \cdots, k.$$

---

20) Here Σ_0 shall be positive, because, assumed that $\sigma_0>0$, such m_1 as makes $\Sigma_0\leqq 0$ is previously to be rejected.

Especially, if N truncated observations be repeated, and N be a pretty large, $\bar{x}_\nu = \sum_{j=1}^{N} x_{\nu j}/N$ would be nearly $E(x_\nu)$, and we have

$$m+\sigma E(t_\nu) = \bar{x}_\nu \ (\nu = 1, 2, \cdots, k) . \tag{12.13}$$

Or, even when $N=1$, we may roughly consider every singly observed value x_ν as $\bar{x}_\nu$; but yet if $k>2$, we have a number of equations, more than the number of unknowns. Thus equations (12.13) give the so-called observation equations:

$$a_\nu m + b_\nu \sigma = c_\nu \quad (\nu = 1, 2, \cdots, k>2) , \tag{12.14}$$

where $a_\nu = 1$, $b_\nu = E(t_\nu)$ and $c_\nu = \bar{x}_\nu$ are all known, theoretically or experimentally. We form Gaussian brackets (sums of products): i.e.

$$\begin{cases} [aa] = k , \quad [ab] = \sum_{\nu=1}^{k} E(t_{\nu|n}) , \quad [ac] = \sum_{\nu=1}^{k} \bar{x}_\nu , \\ [bb] = \sum_{\nu=1}^{k} E(t_{\nu|n})^2, \quad [bc] = \sum_{\nu=1}^{k} E(t_{\nu|n})\bar{x}_\nu , \end{cases} \tag{12.15}$$

and write the normal equations

$$[aa]m + [ab]\sigma = [ac] ,$$
$$[ab]m + [bb]\sigma = [bc] .$$

Whence

$$m = \frac{[bb][ac]-[ab][bc]}{[aa][bb]-[ab]^2} , \quad \sigma = \frac{[aa][bc]-[ab][ac]}{[aa][bb]-[ab]^2} . \tag{12.16}$$

These would probably afford better estimates than those obtained by method of maximum likelihood, if N large.

Furthermore, if beginning i values $x_1 \leqq x_2 \leqq \cdots \leqq x_i$ were ignored, besides missed measurements $x_{i+k+1} \leqq \cdots \leqq x_n$, the actually known values are only the intermediate values: $x_{i+1} \leqq x_{i+2} \leqq \cdots \leqq x_{i+k}$. In this doubly truncated sample we may also use the above mentioned Gauss' method of least squares, especially if repeatedly observed. Here we have, as before, obseravation-equations:

$$m + \sigma E(t_{i+\nu|n}) = \bar{x}_{i+\nu} \quad (\nu = 1, 2, \cdots, k>2) ,$$

normal equations of which determine the most probable values of m and σ. However, if the experiments are not repeated or few, we should again proceed by method of maximum likelihood. Now the probability element being

$$\frac{n!}{i!\,(n-i-k)!} \Phi_{i+1}^{i} (1-\Phi_{i+k})^{n-i-k} d\Phi_{i+1} d\Phi_{i+2} \cdots d\Phi_{i+k} ,$$

where

$$\Phi_{i+1} = \frac{1}{\sqrt{2\pi}} \int^{t_{i+1}} e^{-t^2/2} dt \quad \text{and} \quad 1-\Phi_{i+k} = \frac{1}{\sqrt{2\pi}} \int_{t_{i+k}} e^{-t^2/2} dt \quad \text{with} \quad t_j = \frac{x_j - m}{\sigma} ,$$

the likelihood function is given by

$$L=\frac{n!}{i!\,(n-i-k)!}\cdot\frac{1}{(\sqrt{2\pi}\sigma)^k}\exp\left\{-\frac{1}{2\sigma^2}\sum_{\nu=1}^{k}(x_{i+\nu}-m)^2\right\}\Phi_{i+1}^{i}(1-\Phi_{i+k})^{n-i-k},$$

so that

$$\log L=-\frac{1}{2\sigma^2}\sum_{\nu=1}^{k}(x_{i+\nu}-m)^2-k\log\sigma+i\log\Phi_{i+1}+(n-i-k)\log(1-\Phi_{i+k})$$
$$+\log\frac{n!}{i!\,(n-i-k)!\,\sqrt{2\pi}^k}.$$

Hence the likelihood equations are

(12.17) $$\frac{\partial\log L}{\partial m}=\frac{1}{\sigma^2}\sum_{\nu=1}^{k}(x_{i+\nu}-m)-\frac{i}{\sqrt{2\pi}\,\sigma\Phi_{i+1}}\exp\left\{-\frac{(x_{i+1}-m)^2}{2\sigma^2}\right\}$$
$$+\frac{n-i-k}{\sqrt{2\pi}\,\sigma(1-\Phi_{i+k})}\exp\left\{-\frac{(x_{i+k}-m)^2}{2\sigma^2}\right\}=0,$$

(12.18) $$\frac{\partial\log L}{\partial\sigma}=\frac{1}{\sigma^3}\sum_{\nu=1}^{k}(x_{i+\nu}-m)^2-\frac{k}{\sigma}-\frac{i(x_{i+1}-m)}{\sqrt{2\pi}\,\sigma^2\Phi_{i+1}}\exp\left\{-\frac{(x_{i+1}-m)^2}{2\sigma^2}\right\}$$
$$+\frac{(n-i-k)(x_{i+k}-m)}{\sqrt{2\pi}\,\sigma^2(1-\Phi_{i+k})}\exp\left\{-\frac{(x_{i+k}-m)^2}{2\sigma^2}\right\}=0.$$

From these two equations, firstly eliminating Φ_{i+k} and secondly Φ_{i+1}, we obtain

$$\frac{i(x_{i+k}-x_{i+1})}{\sqrt{2\pi}\,\sigma}\exp\left\{-\frac{(x_{i+1}-m)^2}{2\sigma^2}\right\}=\left[k-\frac{1}{\sigma^2}\sum_{\nu=1}^{k}(m-x_{i+\nu})(x_{i+k}-x_{i+\nu})\right]\Phi_{i+1},$$

$$\frac{(n-i-k)(x_{i+k}-x_{i+1})}{\sqrt{2\pi}\,\sigma}\exp\left\{-\frac{(x_{i+k}-m)^2}{2\sigma^2}\right\}=\left[k-\frac{1}{\sigma^2}\sum_{\nu=1}^{k}(m-x_{i+\nu})(x_{i+1}-x_{i+\nu})\right][1-\Phi_{i+k}].$$

These two equations can together be denoted by

(12.19) $$\frac{R}{\sqrt{2\pi}\,\sigma}\exp\left\{-\frac{(x_{i+j}-m)^2}{2\sigma^2}\right\}=\left[k-\frac{1}{\sigma^2}\sum_{\nu=1}^{k}(m-x_{i+\nu})(x_{i+j'}-x_{i+\nu})\right]\Psi_{jj'}$$

where

(12.20) $$\begin{cases}R=x_{i+k}-x_{i+1}\quad\text{(range) and}\quad(j,j')=(1,k)\ \text{or}\ (k,1)\quad\text{with}\\ \Psi_{1k}=\dfrac{\Phi_{i+1}}{i}=\dfrac{1}{i}\displaystyle\int^{i+1}\frac{e^{-t^2/2}}{\sqrt{2\pi}}dt,\qquad\Psi_{k1}=\dfrac{1-\Phi_{i+k}}{n-i-k}.\end{cases}$$

The first approximation being m_1 and σ_1, let us find their corrections ε and η i.e. such quantities as

$$m=m_1+\varepsilon,\qquad\sigma=\sigma_1+\eta.$$

Assuming m_1, σ_1 to be already fair approximations, ε, η will be so small that their powers with exponents greater than 1 may be neglected. Consequently

$$t_{i+j}=\frac{x_{i+j}-m}{\sigma}=\frac{x_{i+j}-m_1}{\sigma_1}-\frac{\varepsilon}{\sigma_1}-\frac{x_{i+j}-m}{\sigma_1^2}\eta\quad\text{(nearly)}$$
$$=\alpha_j-\beta\varepsilon-\alpha_j\beta\eta,\qquad(j=1\ \text{or}\ k)$$

where

(12.21) $$\alpha_j=\frac{x_{i+j}-m_1}{\sigma_1},\qquad\beta=\frac{1}{\sigma_1}.$$

Accordingly we have also approximately

$$\exp\left\{-\frac{1}{2}t_{i+j}^2\right\}=\exp\left\{-\frac{(x_{i+j}-m)^2}{2\sigma^2}\right\}=\exp\left\{-\frac{1}{2}\alpha_j^2\right\}[1+\alpha_j\beta\varepsilon+\alpha_j^2\beta\eta],$$

i. e.

$$\varphi(t_{i+j})=\varphi(\alpha_j)[1+\alpha_j\beta\varepsilon+\alpha_j^2\beta\eta]$$

as well as

$$\Phi_{i+j}=\int^{t_{i+j}}=\int^{\alpha_j}+\int_{\alpha_j}^{t_{i+j}}=\Phi(\alpha_j)-(\beta\varepsilon+\beta\alpha_j\eta)\,\varphi(\alpha_j)\,.$$

As to the summation in the right handed side of (12.19), we have

$$\sum_{\nu=1}^{k}(m-x_{i+\nu})(x_{i+j'}-x_{i+\nu})=\sum_{\nu=1}^{k}(m_1-x_{i+\nu})(x_{i+j'}-x_{i+\nu})+\varepsilon\sum_{\nu=1}^{k}(x_{i+j'}-x_{i+\nu})\,.$$

Hence, upon putting

$$\sum_{\nu=1}^{k}(m_1-x_{i+\nu})(x_{i+j'}-x_{i+\nu})=\sum\nolimits_{j'}\quad\text{and}\quad\sum_{\nu=1}^{k}x_{i+\nu}=k\bar{x}\,,\tag{12.22}$$

we obtain

$$\frac{1}{\sigma^2}\sum_{\nu=1}^{k}(m-x_{i+\nu})(x_{i+j'}-x_{i+\nu})=\beta^2[\sum\nolimits_{j'}+k(x_{i+j'}-\bar{x})\,\varepsilon-2\beta\eta\sum\nolimits_{j'}]\,.$$

With all these approximations, (12.19) yields, when each of j, j' denotes one and the other of 1, k, respectively,

$$R\beta\varphi(\alpha_j)[1+\alpha_j\beta\varepsilon+(\alpha_j^2-1)\,\beta\eta]=[k-\beta^2\{\sum\nolimits_{j'}+k(x_{i+j'}-\bar{x})\,\varepsilon-2\beta\eta\sum\nolimits_{j'}\}]\,\Psi_{jj'}$$

where $\Psi_{jj'}$ denotes either one of

$$\Psi_{1k}=\frac{1}{i}[\Phi(\alpha_1)-(\beta\varepsilon+\beta\alpha_1\eta)\,\varphi(\alpha_1)]\,,\quad\Psi_{k1}=\frac{1}{n-i-k}[1-\Phi(\alpha_k)+(\beta\varepsilon+\beta\alpha_k\eta)\,\varphi(\alpha_k)]$$

approximately.

We rewrite these equations in detail, according as (j, j') is $(1, k)$ or $(k, 1)$: For $j=1$, $j'=k$

$$\begin{aligned}&[(k-\beta^2\sum\nolimits_k+iR\alpha_1\,\beta)\,\beta\varphi(\alpha_1)+k\beta^2(x_{i+k}-\bar{x})\,\Phi(\alpha_1)]\,\varepsilon\\&\quad+[\{(k-\beta^2\sum\nolimits_k)\,\alpha_1+iR(\alpha_1^2-1)\,\beta\varphi(\alpha_1)-2\beta^3\Phi(\alpha_1)]\,\eta\\&\quad=(k-\beta^2\sum\nolimits_k)\,\Phi(\alpha_1)-iR\beta\varphi(\alpha_1)\,,\end{aligned}\tag{12.23}$$

and for $j=k$, $j'=1$

$$\begin{aligned}&[\{(n-i-k)\,R\alpha_k\beta-(k-\beta^2\sum\nolimits_1)\}\,\beta\varphi(\alpha_k)+k\beta^2(x_{i+1}-\bar{x})\{1-\Phi(\alpha_k)\}]\,\varepsilon\\&\quad+[\{(n-i-k)\,R(\alpha_k^2-1)\,\beta-(k-\beta^2\sum\nolimits_1)\,\alpha_k\}\,\beta\varphi(\alpha_k)-2\beta^3\sum\nolimits_1\{1-\Phi(\alpha_k)\}]\,\eta\\&\quad=(k-\beta^2\sum\nolimits_1)[1-\Phi(\alpha_k)]-(n-i-k)\,R\beta\varphi(\alpha_k)\,.\end{aligned}\tag{12.24}$$

We should solve[21] (12.23) and (12.24) simultaneously for ε and η, and

21) Here we have mainly aimed only to show the principle; For practical calculations more convenient procedures are devised, see Cohen, loc. cit.

find their roots $\varepsilon=\varepsilon_1$ and $\eta=\eta_1$. Now taking $m_2=m_1+\varepsilon_1$, $\sigma_2=\sigma_1+\eta_1$ as the second approximation, recompute all of

(12.25) $$\beta=\frac{1}{\sigma_2},\quad \alpha_j=\frac{x_{i+j}-m_2}{\sigma_2},\quad \sum_j=\sum_{\nu=1}^{k}(m_2-x_{i+\nu})(x_{i+j}-x_{i+\nu})$$
for $j=1$ or k;

or, in detail,

$$\sum_1=m_2S_1-S_2,\quad \text{where}\quad S_1=\sum_{\nu=2}^{k}(x_{i+1}-x_{i+\nu}),\quad S_2=\sum_{\nu=2}^{k}x_{i+\nu}(x_{i+1}-x_{i+\nu}),$$

$$\sum_k=m_2S_1'-S_2',\ \text{where}\ S_1'=\sum_{\nu=1}^{k-1}(x_{i+k}-x_{i+\nu}),\quad S_2'=\sum_{\nu=1}^{k-1}x_{i+\nu}(x_{i+k}-x_{i+\nu}),$$

and solve thus obtained new simultaneous equations (12.23) and (12.24). If ε_2, η_2 be the new roots, $m_3=m_2+\varepsilon_2$ and $\sigma_3=\sigma_2+\eta_2$ will give the third approximations and so on.

Example 1. We get the following 3 samples, each of size 10, by drawing at random from the table of random samples from $N(1, 0)$:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
1°	−1.38	−1.13	−1.07	−0.92	0.21	—	—	—	—	—
2°	−2.48	−1.58	−1.06	−0.58	0.33	—	—	—	—	—
3°	−1.33	−1.03	−0.87	−0.03	0.24	—	—	—	—	—
sum	−5.19	−3.07	−3.00	−1.47	−0.78					

Assuming that the first five in each sample were observed, but the remaining five unmeasured, it is required to estimate population mean m and S. D. σ.

I. Solved by method of least squares,: We have here 5 observation equations

$$3(m-1.53875\,\sigma)=-5.19$$
$$3(m-1.00136\,\sigma)=-3.74$$
$$3(m-0.65606\,\sigma)=-3.00$$
$$3(m-0.37577\,\sigma)=-1.47$$
$$3(m-0.12267\,\sigma)=\ \ 0.78$$

(typically: $am+b\sigma=c$ ······ (12.14)).

Hence, Gaussian brackets are

$$[aa]=45,\quad [ab]=-33.2514,\quad [ac]=-37.86,$$
$$[bb]=\ \ 35.6143,\quad [bc]=\ \ 42.4682.$$

Solving the normal equations, we find

$$m^*=\frac{-1348.36+1412.13}{1602.64-1105.66}=0.128,\qquad \sigma^*=\frac{1911.07-1258.90}{1602.64-1105.66}=1.31.$$

II. On the other hand, if we solve the maximum likelihood estimating equations by successive approximations, we find

	m^*	σ^*
1°	0.127	1.17
2°	0.288	1.67
3°	0.235	1.03
mean	0.215	1.29

Example 2. Again, by use of the table of random samples from $N(0, 1)$, we obtained

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
—	—	−0.72	−0.52	−0.37	−0.24	0.28	0.67	—	—

I. Analysed by method of least squares:

$$\begin{aligned} m-0.65606\,\sigma &= -0.72 \\ m-0.37577\,\sigma &= -0.52 \\ m-0.12267\,\sigma &= -0.37 \\ m+0.12267\,\sigma &= -0.24 \\ m+0.37577\,\sigma &= 0.28 \\ m+0.65606\,\sigma &= 0.67 \end{aligned}$$

$$[aa]=6, \quad [ab]=0, \quad [ac]=-0.9,$$
$$[bb]=1.1733, \quad [bc]=1.2285,$$

whence

$$m^* = -0.15, \quad \sigma^* = 1.05.$$

II. Analysed by method of maximum likelihood:

$$m^* = -0.15, \qquad \sigma^* = 0.86.$$

The calculations by least squares are far easier.

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DISTRIBUTIVE MULTIPLICATIONS TO SEMIGROUP OPERATIONS

By

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Let S be a semigroup, that is, a set with an associative binary operation, which is denoted by $x+y$.

$$(x+y)+z=x+(y+z)\ .$$

We shall give S another binary operation xy such that

$$\left.\begin{array}{ll}(1) & (x+y)a=xa+ya \\ (2) & x(a+b)=xa+xb\end{array}\right\}\ \text{for every } x,\ y,\ a,\ b\in S\ .$$

The first operation is called "addition", and the second operation is called "multiplication". S in which such an addition and multiplication are defined is called "distributive additive-multiplicative system" or "d-system". If we watch only addition, S is denoted by S_+; watching only multiplication, it is denoted by $S_\times$ which is called a multiplication system to S_+; and a d-system S is expressed as $S=(S_+,\ S_\times)$. Especially if the multiplication is associative, S is called a "semiring".

The purpose of this paper is to investigate a method how to find all distributive multiplications to a given semigroup operation, and to obtain all types of d-systems of order 2 and 3.

§1. Principles.

Now we regard the multiplication xa as a mapping of S_+ into S_+ which associates x of S_+ with xa in S_+ for fixed a.

Denote $$f_a(x)=xa\ .$$

Then the conditions (1) and (2) are expressed as

$$(1')\qquad f_a(x+y)=f_a(x)+f_a(y)$$

$$(2')\qquad f_{a+b}(x)=f_a(x)+f_b(x)$$

respectively. Consider the set $F=\{f_a\ ;\ a\in S\}$.
(1′) means that any f_a is an endomorphism of S_+, i. e., a homomorphism of S_+ into S_+.

Let us introduce an addition into F as follows.

f_a+f_b means a mapping which associates x with $f_a(x)+f_b(x)$.

(2′) means that F is closed with the above addition, and hence S_+ is homomorphic onto F under the correspondence $a \to f_a$.

$E(S_+)$ denotes the set of all the endomorphisms of S_+, which is called "the endomorphism set of S_+". $E(S_+)$ is a semigroup under the operation fg defined $(fg)(x) = f(g(x))$, but $E(S_+)$ is not necessarily closed with the above addition. See, for example, the endomorphism set of II_3. (cf. § 5) We have

Theorem 1. *In order to obtain a multiplication which is distributive to S_+, we may find a homomorphism $a \to f_a$ of S_+ into the endomorphism set $E(S_+)$ whose operation is addition. Then $S_\times$ is defined as $xa = f_a(x)$. Any $S_\times$ to a given S_+ is defined in such a way.*

If S_+ is commutative, then it is easily shown that $f_{a+b}(x+y) = f_{a+b}(x) + f_{a+b}(y)$. $E(S_+)$ is closed under the addition and hence an additive semigroup.

Further we see easily that if S_+ is commutative, $E(S_+)$ is a semiring with commutative addition.

The d-system $S = (S_+, S_\times)$ is isomorphic onto $T = (T_+, T_\times)$ if there is a one-to-one mapping f of S onto T which causes an isomorphism of S_+ onto T_+ and of $S_\times$ onto $T_\times$ at the same time.

Lemma 1. *If S_+ is isomorphic onto T_+, then, for any d-system $S = (S_+, S_\times)$, there is $T = (T_+, T_\times)$ to which S is isomorphic.*

Proof. Let f be an isomorphism of S_+ onto T_+. We may define the product $f(x)f(y)$ of elements $f(x)$ and $f(y)$ of $T_\times$ as

$$f(x)f(y) = f(xy) .$$

It is easily proved that T is a d-system.

Corollary 1. *$(S_+, S_\times)$ is isomorphic onto $(S_+, S'_\times)$ if and only if $S_\times$ is isomorphic onto $S'_\times$ under an automorphism of S_+.*

In particular, if $S_+ = T_+$, then f is the so-called automorphism of S_+.

Corollary 2. *If S_+ has no automorphism except the identical mapping, then $(S_+, S_\times)$ and $(S_+, T_\times)$ are isomorphic if and only if $S_\times = T_\times$, which means that the operation of $S_\times$ and $T_\times$ are equal.*

§ 2. Special cases.

In this paragraph we shall research the multiplications in case S_+ has the special types: a singular semigroup, a semigroup with a constant product, and a chain. In these cases, we shall be able to discuss $S_\times$ not by the principle in § 1, but by (1) and (2) directly.

1. Let S_+ be a right singular semigroup which is defined as $x+y=y$.

Theorem 2. *A right singular semigroup S_+ has as $S_\times$ an arbitrary multiplicative system which is defined in all the elements of S_+. $(S_+, S_\times)$ and $(S_+, S'_\times)$ are isomorphic if and only if $S_\times$ and $S'_\times$ are isomorphic.*

Proof. The before half of this theorem is easily proved by

$$(x+y)a = ya = xa + ya, \quad x(a+b) = xb = xa + xb .$$

Now it is shown that the singular semigroup S_+ has any mapping of it into itself as an automorphism. Indeed, for a mapping f of S_+ in to itself, we see

$$f(x+y) = f(y) = f(x)+f(y) .$$

The latter half is immediately proved by Corollary 1.

2. Let S_+ be a semigroup defined as $x+y=0$.

Theorem 3. *$S_\times$ is a multiplicative system to S_+ if and only if $S_\times$ has 0 as two-sided zero. $(S_+, S_\times)$ and $(S_+, S'_\times)$ are isomorphic if and only if $S_\times$ is isomorphic onto $S'_\times$ under the one to one mapping fixing 0.*

Proof. Suppose that a d–system $(S_+, S'_\times)$ is obtained. Then, for all z,

$$0z = (x+y)z = xz+yz = 0 , \quad z0 = z(x+y) = zx+zy = 0 ,$$

whence 0 is a two-sided zero of $S_\times$. The converse is easily shown. Now it is proved that an automorphism of S_+ is nothing but any mapping of S_+ into itself which fixes 0. Indeed, if f is an automorphism of S_+, then

$$f(0) = f(x+y) = f(x)+f(y) = 0 .$$

Conversely if f is a one-to-one mapping of S_+ onto S_+ and $f(0)=0$, then $f(x+y)=f(0)=0$, $f(x)+f(y)=0$, whence $f(x+y)=f(x)+f(y)$. The latter half of this theorem is got by Corollary 1. q. e d.

3. Let S_+ be a chain, that is, a semilattice with linear oredr $x+y =\max(x, y)$.

Theorem 4. *$S_\times$ is a multiplicative system which is distributive to S_+, if and only if $S_\times$ satisfies*

(3) *$x\geqq y$ implies $xz\geqq yz$, $zx\geqq zy$.*

$(S_+, S_\times)$ and $(S_+, T_\times)$ are isomorphic if and only if $S_\times = T_\times$.

Proof. Suppose that $(S_+, S_\times)$ is got. Since $x+y=x$,

$$xz = (x+y)z = xz+yz , \quad zx = z(x+y) = zx+zy .$$

Hence $xz\geqq yz$ and $zx\geqq zy$. Conversely it is easily seen that if (3) holds, $(S_+, S_\times)$ is a d–system. The proof of the latter half is clear by the fact that there is no automorphism except the identical mapping.

4. Let $S_{+(n)}=\{0, 1, \cdots, n-1\}$ be a cyclic group of order n. As an example of $S=(S_{+(n)}, S_\times)$, we can consider the factor ring R_n of integer ring modulo n. Denote by ab the multiplication in R_n. We shall state the theorems without proof.

Theorem 5. *$S_\times$ is a multiplicative system to the cyclic group $S_{+(n)}$ if and only if $S_\times$ is a semigroup defined as $a\times b=abk$ where k is a fixed element of $S_{+(n)}$, and abk is considered as the product of a, b, and k in R_n. Let $S(k)$ denote a d–system $S(k)=(S_{+(n)}, S_\times)$ where $S_\times$ is given by k. $S(k_1)$ and $S(k_2)$ are isomorphic if and only if the greatest common divisor of the integers n and k_1 is equal to that of n and k_2. Accordingly the number of isomorphically distinct $(S_{+(n)}, S_\times)$ is equal to the number of divisors of n.*

Next, let S_+ be an abelian group of order n. S_+ is a direct sum of cyclic groups: $S_+ = S_{+(n_1)} \oplus \cdots \oplus S_{+(n_r)}$ $n = n_1 \cdots n_r$.
Denote by $\boldsymbol{e}_i$ the base of $S_{+(n_i)}$. Then S_+ is considered as an additive group of all the forms

$$\sum_{i=1}^{r} x_i \boldsymbol{e}_i{}^{1)} \qquad (x_i \text{ being a positive integer})$$

with the addition defined by

$$\sum_i x_i \boldsymbol{e}_i + \sum_i y_i \boldsymbol{e}_i = \sum_i (x_i + y_i) \boldsymbol{e}_i .$$

Theorem 6. *$S_\times$ is a multiplicative system to a commutative group S_+ if and only if $S_\times$ is a semigroup with a multiplication*

$$\sum_i a_i \boldsymbol{e}_i \times \sum_i b_i \boldsymbol{e}_i = \sum_{i,j,k} a_i b_j u_{ijk} \boldsymbol{e}_k$$

where the positive integers u_{ijk} $(i, j, k=1, \cdots, r)$ are chosen such that the following systems are fulfilled.

$$\sum_l u_{ijl} u_{lkm} \equiv \sum_l u_{jkl} u_{ilm} \qquad (\text{mod. } n_m) \quad (i, j, m = 1, \cdots, r) .$$

Consequently $(S_+, S_\times)$ is a commutative ring.

Thus $S_\times$ is determined by u_{ijk} $(i, j, k=1, \cdots, r)$

Theorem 7. *A ring determined by u_{ijk} is isomorphic to a ring determined by v_{ijk} $(i, j, k=1, \cdots, r)$ if and only if there exist $p_1, \cdots, p_r$ such that $u_{ijk} \equiv p_i p_j v_{ijk}$ (mod. n_k) $(i, j, k=1, \cdots, r)$ where p_i and n_i are relatively prime.*

§ 3. Classification of Multiplications.

There is given a multiplicative system $S_\times$ which has a multiplication λ: $x\lambda y$ for $x, y \in S_\times$. Let a be a fixed elements of $S_\times$. We introduce another multiplication λ_a into the elements of $S_\times$ as follows.

$$x \lambda_a y = x \lambda a \lambda y .$$

We shall define an ordering among all the multiplications.

$$\lambda \leqq \mu \text{ means } \mu = \lambda_a \text{ for a suitable element } a .$$

Theorem 8. $\lambda \leqq \mu$ and $\mu \leqq \nu$ imply $\lambda \leqq \nu$.

Proof. Let $\mu = \lambda_a$ and $\nu = \mu_b$. Then we have

$$x \nu y = x \mu b \mu y = x \lambda (a \lambda b \lambda a) \lambda y = x \lambda c \lambda y$$

where $c = a \lambda b \lambda a$.

Theorem 9. *If λ is distributive to the addition $+$ and if $\lambda \leqq \mu$, then μ is also distributive to $+$.*

Proof. Let $\mu = \lambda_a$. Then

1) It means $\underbrace{\boldsymbol{e}_i + \cdots + \boldsymbol{e}_i}_{x_i}$. If and only if $x_i \equiv y_i$ (mod. n_i) $(i=1, \cdots, r)$, then $\sum_i x_i \boldsymbol{e}_i = \sum_i y_i \boldsymbol{e}_i$.

$$(x+y)\mu z=(x+y)\lambda a\lambda z=(x\lambda a\lambda z)+(y\lambda a\lambda z)=(x\mu z)+(y\mu z)\ .$$

Similarly we get $z\mu(x+y)=(z\mu x)+(z\mu y)$.

Theorem 10. *If λ is associative and $\lambda\leq\mu$, then μ is also associative.*

Proof. Let $\mu=\lambda_a$. Then

$$(x\mu y)\mu z=(x\lambda a\lambda y)\lambda a\lambda z=x\lambda a\lambda(y\lambda a\lambda z)=x\mu(y\mu z)\ .$$

Especially, let us consider the set $\mathfrak{M}_+$ of all the associative multiplications to an associative addition S_+ and introduce a modified ordering into $\mathfrak{M}_+$. $\lambda\lesssim\mu$ means that either

(1) S_λ is isomorphic to S_μ under the autmorphism of S_+

or (2) $\lambda\leq\mu$.

Then the ordering $\lesssim$ is a quasi-ordering of $\mathfrak{M}_+$.

§ 4. All *d*-systems of Order 2 and 3.

In this paragraph, we shall obtain all the *d*-systems of order 2 and 3 as the simplest examples.

We give all the dually-isomorphically distinct semigroups of order 2 and 3 in Table 1, where all the types lying in the first column are all the isomorphically and dually-isomorphically distinct semigroups (cf. in [1], [2]), and

1 (*abc*), 2 (*acb*), 3 (*bac*), 4 (*bca*), 5 (*cab*), 6 (*cba*)

mean the permutations

$$1\begin{pmatrix}abc\\abc\end{pmatrix},\quad 2\begin{pmatrix}abc\\acb\end{pmatrix},\quad 3\begin{pmatrix}abc\\bac\end{pmatrix},\quad 4\begin{pmatrix}abc\\bca\end{pmatrix},\quad 5\begin{pmatrix}abc\\cab\end{pmatrix},\quad 6\begin{pmatrix}abc\\cba\end{pmatrix}$$

respectively so that, for example,

$$\begin{vmatrix}b&c&b\\c&b&c\\b&c&b\end{vmatrix}\ \text{is transferred from}\ \begin{vmatrix}a&b&b\\b&a&a\\b&a&a\end{vmatrix}\ \text{by the permutation 4 }(bca).$$

Table 1.

Semigroups of order 2.

	1(*ab*)	2(*ba*)
1.	$\begin{vmatrix}a&b\\a&b\end{vmatrix}$	$\begin{vmatrix}a&b\\a&b\end{vmatrix}$
2.	$\begin{vmatrix}a&a\\a&a\end{vmatrix}$	$\begin{vmatrix}b&b\\b&b\end{vmatrix}$
3.	$\begin{vmatrix}a&b\\b&a\end{vmatrix}$	$\begin{vmatrix}b&a\\a&b\end{vmatrix}$
4.	$\begin{vmatrix}a&a\\a&b\end{vmatrix}$	$\begin{vmatrix}a&b\\b&b\end{vmatrix}$

Semigroups of order 3.

	1(*abc*)	2(*acb*)	3(*bac*)	4(*bca*)	5(*cab*)	6(*cba*)
1.	a b c a b c a b c	a b c a b c a b c	a b c a b c a b c	a b c a b c a b c	a b c a b c a b c	a b c a b c a b c
2.	a a a a a a a a a	a a a a a a a a a	b b b b b b b b b	b b b b b b b b b	c c c c c c c c c	c c c c c c c c c
3.	a a a a a a a a b	a a a a c a a a a	b b b b b b b b a	c b b b b b b b b	c c c c a c c c c	b c c c c c c c c
4.	a b a b a b a b a	a a c a a c c c a	b a a a b b a b b	b b c b b c c c b	c a a a c c a c c	c b c b c b c b c
5.	a b b b a a b a a	a c c c a a c a a	b a b a b a b a b	b c b c b c b c b	c c a c c a a a c	c c b c c b b b c
6.	a b c b c a c a b	a b c b c a c a b	c a b a b c b c a	c a b a b c b c a	b c a c a b a b c	b c a c a b a b c
7.	a b a a b a a b a	a a c a a c a a c	a b b a b b a b b	b b c b b c b b c	a c c a c c a c c	c b c c b c c b c
8.	a a a a b c a b c	a a a a b c a b c	a b c b b b a b c	a b c b b b a b c	a b c a b c c c c	a b c a b c c c c
9.	a a a a b b a b b	a a a a c c a c c	a b a b b b a b a	c b c b b b c b c	a a c a a c c c c	b b c b b c c c c
10.	a a a a b c a c b	a a a a c b a b c	a b c b b b c b a	c b a b b b a b c	a b c b a c c c c	b a c a b c c c c
11.	a b a a b b a b c	a a c a b c a c c	a b a a b b a b c	a b c b b c c b c	a a c a b c a c c	a b c b b c c b c
12.	a b b a b b a b c	a c c a b c a c c	a b a a b a a b c	a b c c b c c b c	a a c a b c a a c	a b c b b c b b c
13.	a a a a a a a a c	a a a a b a a a a	b b b b b b b b c	a b b b b b b b b	c c c c b c c c c	a c c c c c c c c
14.	a a a a a b a a c	a a a a b a a c a	b b a b b b b b c	a b b b b b c b b	c a c c b c c c c	a c c b c c c c c
15.	a a a a a b a b c	a a a a b c a c a	b b a b b b a b c	a b c b b b c b b	c a c a b c c c c	a b c b c c c c c
16.	a b a b a b a b c	a a c a b c c c a	b a a a b b a b c	a b c b b c c c b	c a a a b c a c c	a b c b c b c b c
17.	a a a a b a a a c	a a a a b a a a c	a b b b b b b b c	a b b b b b b b c	a c c c b c c c c	a c c c b c c c c
18.	a a a a b b a b c	a a a a b c a c c	a b a b b b a b c	a b c b b b c b c	a a c a b c c c c	a b c b b c c c c

Table 2 shows all the endomorphisms of the semigroups of order 2 and 3, where, for example,

$$5_3\ B=(abb) \text{ means an endomorphism } \begin{pmatrix} abc \\ abb \end{pmatrix} \text{ of } 5_3 .$$

We add that 3_2 is the 3rd semigroup of order 2, and 3_3 is the 3rd semigroup of order 3.

Table 2.

Endomorphisms of Semigroups of Order 2 and 3.

3_2	$A=(aa)$	$B=(ab)$					
4_2	$A=(aa)$	$B=(bb)$	$C=(ab)$				
3_3	$A=(aaa)$	$B=(aab)$	$C=(abc)$				
4_3	$A=(aaa)$	$B=(aba)$	$C=(aac)$	$D=(abc)$			
5_3	$A=(aaa)$	$B=(abb)$	$C=(abc)$				
6_3	$A=(aaa)$	$B=(abc)$	$C=(acb)$				
7_3	$A=(aaa)$	$B=(bbb)$	$C=(aba)$	$D=(bab)$	$E=(aac)$	$F=(abc)$	
8_3	$A=(aaa)$	$B=(bbb)$	$C=(ccc)$	$D=(abb)$	$E=(acc)$	$F=(abc)$	$G=(acb)$
9_3	$A=(aaa)$	$B=(bbb)$	$C=(abb)$	$D=(bbc)$	$E=(abc)$		
10_3	$A=(aaa)$	$B=(bbb)$	$C=(abb)$	$D=(abc)$			
11_3	$A=(aaa)$	$B=(bbb)$	$C=(ccc)$	$D=(aac)$	$E=(bbc)$	$F=(abc)$	$G=(bac)$
12_3	$A=(aaa)$	$B=(bbb)$	$C=(ccc)$	$D=(abb)$	$E=(baa)$	$F=(bbc)$	$G=(abc)$
13_3	$A=(aaa)$	$B=(ccc)$	$C=(aba)$	$D=(aac)$	$E=(abc)$		
14_3	$A=(aaa)$	$B=(ccc)$	$C=(aac)$	$D=(abc)$			
15_3	$A=(aaa)$	$B=(ccc)$	$C=(aac)$	$D=(abc)$			
16_3	$A=(aaa)$	$B=(ccc)$	$C=(aba)$	$D=(aac)$	$E=(abc)$		
17_3	$A=(aaa)$	$B=(bbb)$	$C=(ccc)$	$D=(aab)$	$E=(aba)$	$F=(aac)$	$G=(aca)$
	$H=(abc)$	$I=(acb)$					

In Table 3, we show addition table which is introduced into the endomorphism set (cf. Table 2) of each semigroup. (See §1). As easily seen, $E(11_3)$ is not closed under the addition.

Table 3.

Addition of Endomorphisms.

Order 2.

3.

	A	B
A	A	B
B	B	A

4.

	A	B	C
A	A	A	A
B	A	B	C
C	A	C	C

Order 3.

3.

	A	B	C
A	A	A	A
B	A	A	A
C	A	A	B

4.

	A	B	C	D
A	A	B	A	B
B	B	A	B	A
C	A	B	A	B
D	B	A	B	A

5.

	A	B	C
A	A	B	B
B	B	A	A
C	B	A	A

6.

	A	B	C
A	A	B	C
B	B	C	A
C	C	A	B

7.

	A	B	C	D	E	F
A	A	B	C	D	A	C
B	A	B	C	D	A	C
C	A	B	C	D	A	C
D	A	B	C	D	A	C
E	A	B	C	D	A	C
F	A	B	C	D	A	C

8.

	A	B	C	D	E	F	G
A	A	A	A	A	A	A	A
B	A	B	C	D	E	F	G
C	A	B	C	D	E	F	G
D	A	D	E	D	E	F	G
E	A	D	E	D	E	F	G
F	A	D	E	D	E	F	G
G	A	D	E	D	E	F	G

9.

	A	B	C	D	E
A	A	A	A	A	A
B	A	B	C	B	C
C	A	C	C	C	C
D	A	B	C	B	C
E	A	C	C	C	C

10.

	A	B	C	D
A	A	A	A	A
B	A	B	C	D
C	A	C	C	D
D	A	D	D	C

11.

	A	B	C	D	E	F	G
A	A	B	A	A	X	X	X
B	A	B	B	X	B	X	X
C	A	B	C	D	E	F	G
D	A	B	D	D	E	F	G
E	A	B	E	D	E	F	G
F	A	B	F	D	E	F	G
G	A	B	G	D	E	F	G

12.

	A	B	C	D	E	F	G
A	A	B	B	D	E	B	D
B	A	B	B	D	E	B	D
C	A	B	C	D	E	F	G
D	A	B	B	D	E	B	D
E	A	B	B	D	E	B	D
F	A	B	F	D	E	F	G
G	A	B	F	D	E	F	G

13.

	A	B	C	D	E
A	A	A	A	A	A
B	A	B	A	D	D
C	A	A	A	A	A
D	A	D	A	D	D
E	A	D	A	D	D

14.

	A	B	C	D
A	A	A	A	A
B	A	B	C	C
C	A	C	C	C
D	A	D	C	C

15.

	A	B	C	D
A	A	A	A	A
B	A	B	C	D
C	A	C	C	C
D	A	D	C	C

16.

	A	B	C	D	E
A	A	A	C	A	C
B	A	B	C	B	E
C	C	C	A	C	A
D	A	D	C	D	E
E	C	E	A	E	D

17.

	A	B	C	D	E	F	G	H	I
A	A	A	A	A	A	A	A	A	A
B	A	B	A	D	E	A	A	E	D
C	A	A	C	A	A	F	G	F	G
D	A	D	A	D	A	A	A	A	D
E	A	E	A	A	E	A	A	E	A
F	A	A	F	A	A	F	A	F	A
G	A	A	G	A	A	A	G	A	G
H	A	E	F	A	E	F	A	H	A
I	A	D	G	D	A	A	G	A	I

In Table 4, we arrange all the multiplications which are distributive to each addition. For example, we have, for the addition 3_3, the multiplications (ABC) (AAB) (AAA) where (ABC) denotes

	a	b	c
a			
b	A	B	C
c			

i. e.

	a	b	c
a	a	a	a
b	a	a	b
c	a	b	c

in words, we rewrite the endomorphisms of 3_3

$$A = (aaa), \qquad B = (aab), \qquad C = (abc)$$

in column, and we place them in row.

In Table 4, the multiplications for the additions 1_2, 2_2, 1_3, 2_3, 18_3 are omitted, because they have been determined in § 2.

We remark that the multiplications $1 \sim 14$ for the addition 7_3 are associative, but $15 \sim 24$, for 7_3, are not associative.

In Table 5, we show the modified ordering of $\mathfrak{M}_+$ for each the addition S_+ of order 2 and 3. (cf. § 3). For example, in the associative multiplication for the addition 1_3,

$$17.1 \xrightarrow{c} 13.1$$

means that if λ denotes the multiplication 17.1 then the multiplication 13.1 is expressed as λ_c (cf. § 3) ; furthermore if μ denotes 1.1, then μ_x is isomorphic to μ for every x under an automorphism of the addition 1_3.

Table 4.

Multiplications to additive Semigroups.

Order 2.

3. 1(AA) 2(AB)
4. 1(AA) 2(AB) 3(AC) 4(BB) 5(CB) 6(CC)

Order 3.

3. 1(AAA) 2(AAB) 3(ABC)
4. 1(AAA) 2(AAC) 3(ABA) 4(ABC)
5. 1(AAA) 2(ABB) 3(ABC)
6. 1(AAA) 2(ABC) 3(ACB)
7. 1(AAA) 2(AAE) 3(ABA) 4(ABE) 5(ACA) 6(ACE) 7(BBB) 8(CBC)
9(CBE) 10(AFF) 11(CCF) 12(CDC) 13(CDF) 14(DCD)
15(ADA) 16(ADE) 17(BAB) 18(BCB) 19(BDB) 20(CAC) 21(CAF) 22(DAD)
23(DBD) 24(DDD)
8. 1(AAA) 2(ABB) 3(ABC) 4(ACC) 5(ADD) 6(ADE) 7(ADF) 8(AEE)
9(AFE) 10(AFF) 11(AFG) 12(AGF) 13(BBB) 14(CCC) 15(DBB) 16(DDD)
17(ECC) 18(EEE) 19(FFF)
20(ACB) 21(ADG) 22(AED) 23(AEF) 24(AEG) 25(AED) 26(AGD) 27(AGE)
28(AGG) 29(GGG)
9. 1(AAA) 2(ABB) 3(ABD) 4(ACC) 5(ACE) 6(BBB) 7(BBD) 8(CBB)
9(CBD) 10(CCC) 11(CCE)
10. 1(AAA) 2(ABB) 3(ACC) 4(ACD) 5(BBB) 6(CBB) 7(CCC)
11. 1(AAA) 2(AAC) 3(AAD) 4(ABC) 5(BBB) 6(BBC) 7(BBE) 8(CCC)
9(DDC) 10(DDD) 11(DEC) 12(DFC) 13(EEC) 14(EEE) 15(FEC) 16(FFC)
17(FFF) 18(FGC) 19(GFC)
20(BAC) 21(DGC) 22(EDC) 23(EFC) 24(EGC) 25(FDC) 26(GDC) 27(GEC)
28(GGC) 29(GGG)
12. 1(AAA) 2(ABB) 3(ABC) 4(ABF) 5(ADD) 6(ADG) 7(BBB) 8(BBC)
9(BBF) 10(CCC) 11(DBB) 12(DBC) 13(DBF) 14(DDD) 15(DDG) 16(DEE)
17(EDD) 18(EDG) 19(FFC) 20(FFF) 21(GFC) 22(GFF) 23(GGG)
24(AEE) 25(BAA) 26(BDD) 27(BDG) 28(BEE) 29(DAA) 30(EAA) 31(EBB)
32(EBC) 33(EEE) 34(FGG)
13. 1(AAA) 2(AAB) 3(AAD) 4(ACA) 5(ACB) 6(ACD) 7(BBB) 8(DDB)
9(DDD) 10(DEB) 11(DED)
14. 1(AAA) 2(AAB) 3(AAC) 4(BBB) 5(CCB) 6(CCC) 7(CDB)
15. 1(AAA) 2(AAB) 3(AAC) 4(BBB) 5(CCB) 6(CCC) 7(CDB)
16. 1(AAA) 2(AAB) 3(AAD) 4(ACA) 5(ACB) 6(ACD) 7(BBB) 8(DDB)
9(DDD) 10(DEB) 11(DED)
17. 1(AAA) 2(AAC) 3(AAD) 4(AAF) 5(AAH) 6(ABA) 7(ABC) 8(ABF)
9(ADF) 10(ADH) 11(AEA) 12(AEF) 13(AEG) 14(AGA) 15(AHA) 16(AHI)
17(AIH) 18(BBB) 19(CCC) 20(EBE) 21(EBH) 22(EEE) 23(EEH) 24(FFC)
25(FFF) 26(FHC) 27(HHH)
28(AAB) 29(AAE) 30(AAG) 31(AAI) 32(ABG) 33(ACA) 34(ACB) 35(ACD)
36(ACE) 37(ADA) 38(ADC) 39(ADE) 40(ADG) 41(AEC) 42(AED) 43(AEI)
44(AFA) 45(AFB) 46(AFD) 47(AFE) 48(AFG) 49(AFI) 50(AGB) 51(AGD)
52(AGE) 53(AGF) 54(AGH) 55(AHD) 56(AHG) 57(AIA) 58(AIF) 59(DBD)
60(DBI) 61(DDB) 62(DDD) 63(DDI) 64(DIB) 65(DID) 66(EEB) 67(EHB)
68(EHE) 69(FCF) 70(FCH) 71(FFH) 72(FHF) 73(GCG) 74(GCI) 75(GGC)
76(GGG) 77(GGI) 78(GIC) 79(GIG) 80(III)

Table 5. Distributive Associative Multiplication for Semigroups Order 2 and 3.

Order 2.

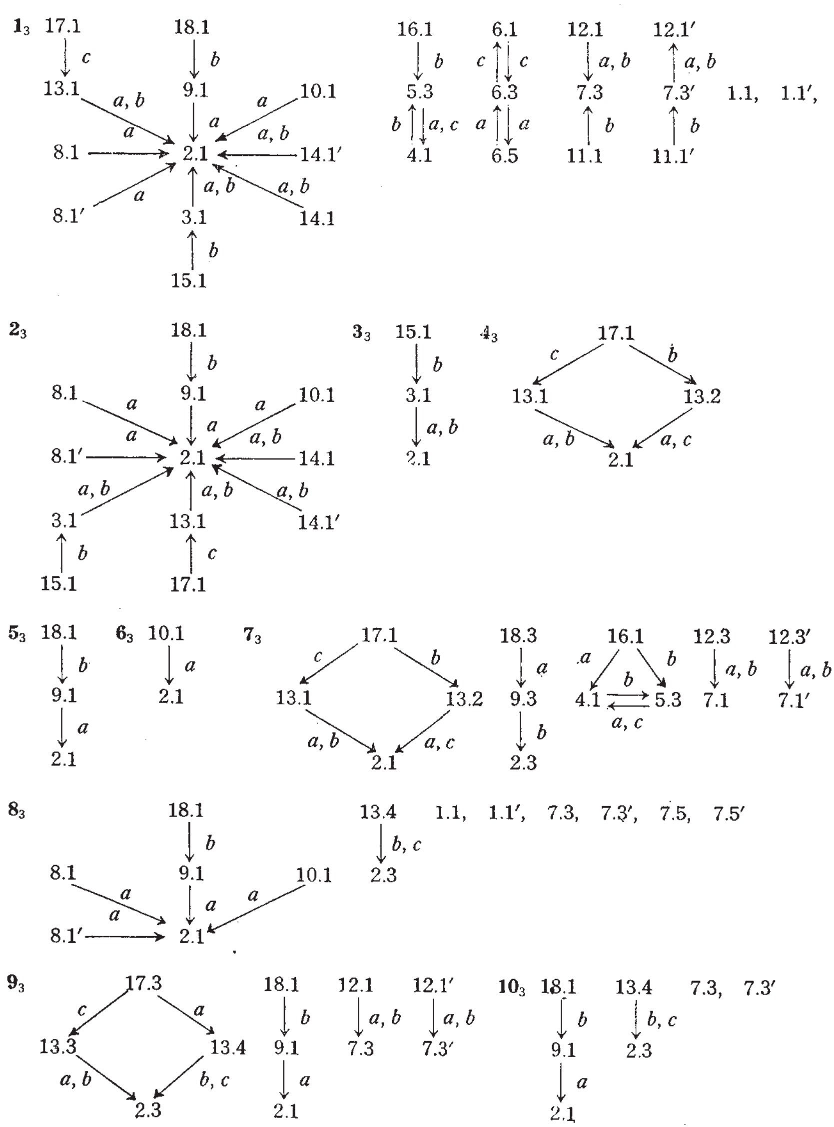
$\mathbf{1}_2$ 1.1, 1.1′, 3.1, $4.1 \xrightarrow{a} 2.1$

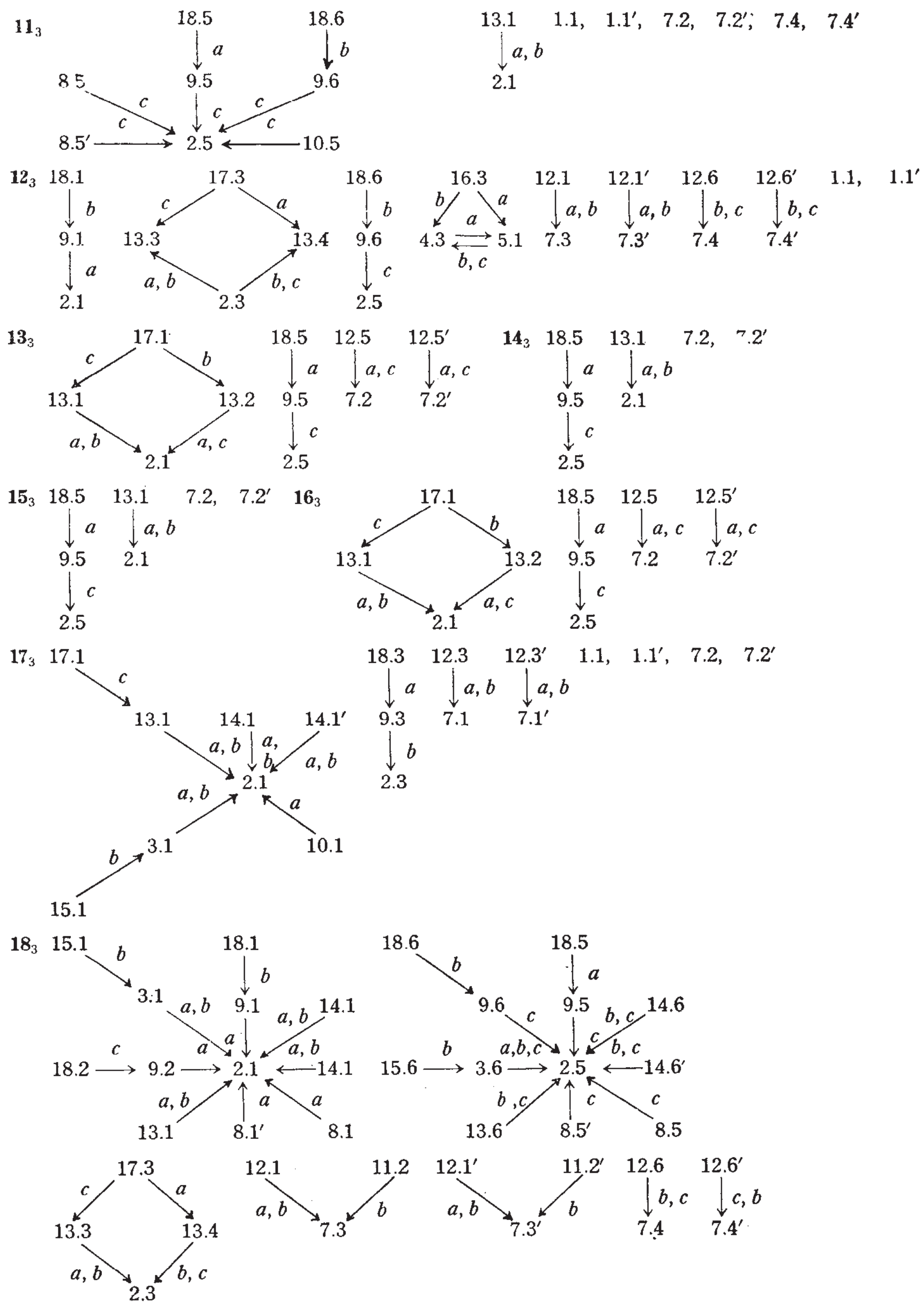
$\mathbf{2}_2$ $4.1 \xrightarrow{a} 2.1$

$\mathbf{3}_2$ $4.1 \xrightarrow{a} 2.1$

$\mathbf{4}_2$ 1.1, 1.1′, $4.1 \xrightarrow{a} 2.1$, $4.2 \xrightarrow{b} 2.2$,

Order 3.





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