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DISTRIBUTIVE MULTIPLICATIONS TO SEMIGROUP OPERATIONS

By

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Let S be a semigroup, that is, a set with an associative binary operation, which is denoted by $x+y$.

$$(x+y)+z=x+(y+z)\ .$$

We shall give S another binary operation xy such that

$$\left.\begin{array}{ll}(1) & (x+y)a=xa+ya \\ (2) & x(a+b)=xa+xb\end{array}\right\} \text{ for every } x, y, a, b \in S\,.$$

The first operation is called "addition", and the second operation is called "multiplication". S in which such an addition and multiplication are defined is called "distributive additive-multiplicative system" or "d-system". If we watch only addition, S is denoted by S_+; watching only multiplication, it is denoted by $S_\times$ which is called a multiplication system to S_+; and a d-system S is expressed as $S=(S_+, S_\times)$. Especially if the multiplication is associative, S is called a "semiring".

The purpose of this paper is to investigate a method how to find all distributive multiplications to a given semigroup operation, and to obtain all types of d-systems of order 2 and 3.

§1. Principles.

Now we regard the multiplication xa as a mapping of S_+ into S_+ which associates x of S_+ with xa in S_+ for fixed a.

Denote $$f_a(x)=xa\,.$$

Then the conditions (1) and (2) are expressed as

$$(1') \qquad f_a(x+y)=f_a(x)+f_a(y)$$

$$(2') \qquad f_{a+b}(x)=f_a(x)+f_b(x)$$

respectively. Consider the set $F=\{f_a;\ a\in S\}$.

(1′) means that any f_a is an endomorphism of S_+, i. e., a homomorphism of S_+ into S_+.

Let us introduce an addition into F as follows.

f_a+f_b means a mapping which associates x with $f_a(x)+f_b(x)$.

(2′) means that F is closed with the above addition, and hence S_+ is homomorphic onto F under the correspondence $a\to f_a$.

$E(S_+)$ denotes the set of all the endomorphisms of S_+, which is called "the endomorphism set of S_+". $E(S_+)$ is a semigroup under the operation fg defined $(fg)(x)=f(g(x))$, but $E(S_+)$ is not necessarily closed with the above addition. See, for example, the endomorphism set of Π_3. (cf. § 5) We have

Theorem 1. *In order to obtain a multiplication which is distributive to S_+, we may find a homomorphism $a\to f_a$ of S_+ into the endomorphism set $E(S_+)$ whose operation is addition. Then $S_\times$ is defined as $xa=f_a(x)$. Any $S_\times$ to a given S_+ is defined in such a way.*

If S_+ is commutative, then it is easily shown that $f_{a+b}(x+y)=f_{a+b}(x)+f_{a+b}(y)$. $E(S_+)$ is closed under the addition and hence an additive semigroup.

Further we see easily that if S_+ is commutative, $E(S_+)$ is a semiring with commutative addition.

The d-system $S=(S_+, S_\times)$ is isomorphic onto $T=(T_+, T_\times)$ if there is a one-to-one mapping f of S onto T which causes an isomorphism of S_+ onto T_+ and of $S_\times$ onto $T_\times$ at the same time.

Lemma 1. *If S_+ is isomorphic onto T_+, then, for any d-system $S=(S_+, S_\times)$, there is $T=(T_+, T_\times)$ to which S is isomorphic.*

Proof. Let f be an isomorphism of S_+ onto T_+. We may define the product $f(x)f(y)$ of elements $f(x)$ and $f(y)$ of $T_\times$ as

$$f(x)f(y)=f(xy).$$

It is easily proved that T is a d-system.

Corollary 1. *$(S_+, S_\times)$ is isomorphic onto $(S_+, S'_\times)$ if and only if $S_\times$ is isomorphic onto $S'_\times$ under an automorphism of S_+.*

In particular, if $S_+=T_+$, then f is the so-called automorphism of S_+.

Corollary 2. *If S_+ has no automorphism except the identical mapping, then $(S_+, S_\times)$ and $(S_+, T_\times)$ are isomorphic if and only if $S_\times=T_\times$, which means that the operation of $S_\times$ and $T_\times$ are equal.*

§ 2. Special cases.

In this paragraph we shall research the multiplications in case S_+ has the special types: a singular semigroup, a semigroup with a constant product, and a chain. In these cases, we shall be able to discuss $S_\times$ not by the principle in § 1, but by (1) and (2) directly.

1. Let S_+ be a right singular semigroup which is defined as $x+y=y$.

Theorem 2. *A right singular semigroup S_+ has as $S_\times$ an arbitrary multiplicative system which is defined in all the elements of S_+. $(S_+, S_\times)$ and $(S_+, S'_\times)$ are isomorphic if and only if $S_\times$ and $S'_\times$ are isomorphic.*

Proof. The before half of this theorem is easily proved by

$$(x+y)a=ya=xa+ya,\quad x(a+b)=xb=xa+xb.$$

Now it is shown that the singular semigroup S_+ has any mapping of it into itself as an automorphism. Indeed, for a mapping f of S_+ in to itself, we see

$$f(x+y)=f(y)=f(x)+f(y).$$

The latter half is immediately proved by Corollary 1.

2. Let S_+ be a semigroup defined as $x+y=0$.

Theorem 3. *$S_\times$ is a multiplicative system to S_+ if and only if $S_\times$ has 0 as two-sided zero. $(S_+, S_\times)$ and $(S_+, S'_\times)$ are isomorphic if and only if $S_\times$ is isomorphic onto $S'_\times$ under the one to one mapping fixing 0.*

Proof. Suppose that a d-system $(S_+, S'_\times)$ is obtained. Then, for all z,

$$0z=(x+y)z=xz+yz=0, \quad z0=z(x+y)=zx+zy=0,$$

whence 0 is a two-sided zero of $S_\times$. The converse is easily shown. Now it is proved that an automorphism of S_+ is nothing but any mapping of S_+ into itself which fixes 0. Indeed, if f is an automorphism of S_+, then

$$f(0)=f(x+y)=f(x)+f(y)=0.$$

Conversely if f is a one-to-one mapping of S_+ onto S_+ and $f(0)=0$, then $f(x+y)=f(0)=0$, $f(x)+f(y)=0$, whence $f(x+y)=f(x)+f(y)$. The latter half of this theorem is got by Corollary 1. q. e d.

3. Let S_+ be a chain, that is, a semilattice with linear oredr $x+y=\max(x, y)$.

Theorem 4. *$S_\times$ is a multiplicative system which is distributive to S_+, if and only if $S_\times$ satisfies*

(3) $\quad x\geqq y$ *implies* $xz\geqq yz$, $zx\geqq zy$.

$(S_+, S_\times)$ and $(S_+, T_\times)$ are isomorphic if and only if $S_\times=T_\times$.

Proof. Suppose that $(S_+, S_\times)$ is got. Since $x+y=x$,

$$xz=(x+y)z=xz+yz, \quad zx=z(x+y)=zx+zy.$$

Hence $xz\geqq yz$ and $zx\geqq zy$. Conversely it is easily seen that if (3) holds, $(S_+, S_\times)$ is a d-system. The proof of the latter half is clear by the fact that there is no automorphism except the identical mapping.

4. Let $S_{+(n)}=\{0, 1, \cdots, n-1\}$ be a cyclic group of order n. As an example of $S=(S_{+(n)}, S_\times)$, we can consider the factor ring R_n of integer ring modulo n. Denote by ab the multiplication in R_n. We shall state the theorems without proof.

Theorem 5. *$S_\times$ is a multiplicative system to the cyclic group $S_{+(n)}$ if and only if $S_\times$ is a semigroup defined as $a\times b=abk$ where k is a fixed element of $S_{+(n)}$, and abk is considered as the product of a, b, and k in R_n. Let $S(k)$ denote a d-system $S(k)=(S_{+(n)}, S_\times)$ where $S_\times$ is given by k. $S(k_1)$ and $S(k_2)$ are isomorphic if and only if the greatest common divisor of the integers n and k_1 is equal to that of n and k_2. Accordingly the number of isomorphically distinct $(S_{+(n)}, S_\times)$ is equal to the number of divisors of n.*

Next, let S_+ be an abelian group of order n. S_+ is a direct sum of cyclic groups: $S_+ = S_{+(n_1)} \oplus \cdots \oplus S_{+(n_r)}$ $n = n_1 \cdots n_r$.
Denote by $\boldsymbol{e}_i$ the base of $S_{+(n_i)}$. Then S_+ is considered as an additive group of all the forms

$$\sum_{i=1}^{r} x_i \boldsymbol{e}_i{}^{1)} \qquad (x_i \text{ being a positive integer})$$

with the addition defined by

$$\sum_i x_i \boldsymbol{e}_i + \sum_i y_i \boldsymbol{e}_i = \sum_i (x_i + y_i) \boldsymbol{e}_i .$$

Theorem 6. *$S_\times$ is a multiplicative system to a commutative group S_+ if and only if $S_\times$ is a semigroup with a multiplication*

$$\sum_i a_i \boldsymbol{e}_i \times \sum_i b_i \boldsymbol{e}_i = \sum_{i,j,k} a_i b_j u_{ijk} \boldsymbol{e}_k$$

where the positive integers u_{ijk} $(i, j, k = 1, \cdots, r)$ are chosen such that the following systems are fulfilled.

$$\sum_l u_{ijl} u_{lkm} \equiv \sum_l u_{jkl} u_{ilm} \qquad (\text{mod. } n_m) \quad (i, j, m = 1, \cdots, r) .$$

Consequently $(S_+, S_\times)$ is a commutative ring.

Thus $S_\times$ is determined by u_{ijk} $(i, j, k = 1, \cdots, r)$

Theorem 7. *A ring determined by u_{ijk} is isomorphic to a ring determined by v_{ijk} $(i, j, k = 1, \cdots, r)$ if and only if there exist $p_1, \cdots, p_r$ such that $u_{ijk} \equiv p_i p_j v_{ijk}$ (mod. n_k) $(i, j, k = 1, \cdots, r)$ where p_i and n_i are relatively prime.*

§ 3. Classification of Multiplications.

There is given a multiplicative system $S_\times$ which has a multiplication λ: $x \lambda y$ for $x, y \in S_\times$. Let a be a fixed elements of $S_\times$. We introduce another multiplication λ_a into the elements of $S_\times$ as follows.

$$x \lambda_a y = x \lambda a \lambda y .$$

We shall define an ordering among all the multiplications.

$$\lambda \leqq \mu \text{ means } \mu = \lambda_a \text{ for a suitable element } a .$$

Theorem 8. $\lambda \leqq \mu$ and $\mu \leqq \nu$ imply $\lambda \leqq \nu$.

Proof. Let $\mu = \lambda_a$ and $\nu = \mu_b$. Then we have

$$x \nu y = x \mu b \mu y = x \lambda (a \lambda b \lambda a) \lambda y = x \lambda c \lambda y$$

where $c = a \lambda b \lambda a$.

Theorem 9. *If λ is distributive to the addition $+$ and if $\lambda \leqq \mu$, then μ is also distributive to $+$.*

Proof. Let $\mu = \lambda_a$. Then

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1) It means $\underbrace{\boldsymbol{e}_i + \cdots + \boldsymbol{e}_i}_{x_i}$. If and only if $x_i \equiv y_i$ (mod. n_i) $(i = 1, \cdots, r)$, then $\sum_i x_i \boldsymbol{e}_i = \sum_i y_i \boldsymbol{e}_i$.

$$(x+y)\mu z=(x+y)\lambda a\lambda z=(x\lambda a\lambda z)+(y\lambda a\lambda z)=(x\mu z)+(y\mu z)\ .$$

Similarly we get $z\mu(x+y)=(z\mu x)+(z\mu y)$.

Theorem 10. *If λ is associative and $\lambda\leqq\mu$, then μ is also associative.*

Proof. Let $\mu=\lambda_a$. Then

$$(x\mu y)\mu z=(x\lambda a\lambda y)\lambda a\lambda z=x\lambda a\lambda(y\lambda a\lambda z)=x\mu(y\mu z)\ .$$

Especially, let us consider the set $\mathfrak{M}_+$ of all the associative multiplications to an associative addition S_+ and introduce a modified ordering into $\mathfrak{M}_+$. $\lambda\underline{\leqq}\mu$ means that either

(1) S_λ is isomorphic to S_μ under the autmorphism of S_+

or (2) $\lambda\leqq\mu$.

Then the ordering $\underline{\leqq}$ is a quasi-ordering of $\mathfrak{M}_+$.

§4. All *d*-systems of Order 2 and 3.

In this paragraph, we shall obtain all the *d*-systems of order 2 and 3 as the simplest examples.

We give all the dually-isomorphically distinct semigroups of order 2 and 3 in Table 1, where all the types lying in the first column are all the isomorphically and dually-isomorphically distinct semigroups (cf. in [1], [2]), and

1 (*abc*), 2 (*acb*), 3 (*bac*), 4 (*bca*), 5 (*cab*), 6 (*cba*)

mean the permutations

$$1\begin{pmatrix}abc\\abc\end{pmatrix},\quad 2\begin{pmatrix}abc\\acb\end{pmatrix},\quad 3\begin{pmatrix}abc\\bac\end{pmatrix},\quad 4\begin{pmatrix}abc\\bca\end{pmatrix},\quad 5\begin{pmatrix}abc\\cab\end{pmatrix},\quad 6\begin{pmatrix}abc\\cba\end{pmatrix}$$

respectively so that, for example,

$$\begin{array}{|ccc|}b&c&b\\c&b&c\\b&c&b\end{array}\ \text{is transferred from}\ \begin{array}{|ccc|}a&b&b\\b&a&a\\b&a&a\end{array}\ \text{by the permutation 4 }(bca).$$

Table 1.

Semigroups of order 2.

	1(*ab*)	2(*ba*)
1.	$\begin{array}{\|cc\|}a&b\\a&b\end{array}$	$\begin{array}{\|cc\|}a&b\\a&b\end{array}$
2.	$\begin{array}{\|cc\|}a&a\\a&a\end{array}$	$\begin{array}{\|cc\|}b&b\\b&b\end{array}$
3.	$\begin{array}{\|cc\|}a&b\\b&a\end{array}$	$\begin{array}{\|cc\|}b&a\\a&b\end{array}$
4.	$\begin{array}{\|cc\|}a&a\\a&b\end{array}$	$\begin{array}{\|cc\|}a&b\\b&b\end{array}$

Semigroups of order 3.

	1(abc)	2(acb)	3(bac)	4(bca)	5(cab)	6(cba)
1.	$a\ b\ c$ $a\ b\ c$ $a\ b\ c$	$a\ b\ c$ $a\ b\ c$ $a\ b\ c$	$a\ b\ c$ $a\ b\ c$ $a\ b\ c$	$a\ b\ c$ $a\ b\ c$ $a\ b\ c$	$a\ b\ c$ $a\ b\ c$ $a\ b\ c$	$a\ b\ c$ $a\ b\ c$ $a\ b\ c$
2.	$a\ a\ a$ $a\ a\ a$ $a\ a\ a$	$a\ a\ a$ $a\ a\ a$ $a\ a\ a$	$b\ b\ b$ $b\ b\ b$ $b\ b\ b$	$b\ b\ b$ $b\ b\ b$ $b\ b\ b$	$c\ c\ c$ $c\ c\ c$ $c\ c\ c$	$c\ c\ c$ $c\ c\ c$ $c\ c\ c$
3.	$a\ a\ a$ $a\ a\ a$ $a\ a\ b$	$a\ a\ a$ $a\ c\ a$ $a\ a\ a$	$b\ b\ b$ $b\ b\ b$ $b\ b\ a$	$c\ b\ b$ $b\ b\ b$ $b\ b\ b$	$c\ c\ c$ $c\ a\ c$ $c\ c\ c$	$b\ c\ c$ $c\ c\ c$ $c\ c\ c$
4.	$a\ b\ a$ $b\ a\ b$ $a\ b\ a$	$a\ a\ c$ $a\ a\ c$ $c\ c\ a$	$b\ a\ a$ $a\ b\ b$ $a\ b\ b$	$b\ b\ c$ $b\ b\ c$ $c\ c\ b$	$c\ a\ a$ $a\ c\ c$ $a\ c\ c$	$c\ b\ c$ $b\ c\ b$ $c\ b\ c$
5.	$a\ b\ b$ $b\ a\ a$ $b\ a\ a$	$a\ c\ c$ $c\ a\ a$ $c\ a\ a$	$b\ a\ b$ $a\ b\ a$ $b\ a\ b$	$b\ c\ b$ $c\ b\ c$ $b\ c\ b$	$c\ c\ a$ $c\ c\ a$ $a\ a\ c$	$c\ c\ b$ $c\ c\ b$ $b\ b\ c$
6.	$a\ b\ c$ $b\ c\ a$ $c\ a\ b$	$a\ b\ c$ $b\ c\ a$ $c\ a\ b$	$c\ a\ b$ $a\ b\ c$ $b\ c\ a$	$c\ a\ b$ $a\ b\ c$ $b\ c\ a$	$b\ c\ a$ $c\ a\ b$ $a\ b\ c$	$b\ c\ a$ $c\ a\ b$ $a\ b\ c$
7.	$a\ b\ a$ $a\ b\ a$ $a\ b\ a$	$a\ a\ c$ $a\ a\ c$ $a\ a\ c$	$a\ b\ b$ $a\ b\ b$ $a\ b\ b$	$b\ b\ c$ $b\ b\ c$ $b\ b\ c$	$a\ c\ c$ $a\ c\ c$ $a\ c\ c$	$c\ b\ c$ $c\ b\ c$ $c\ b\ c$
8.	$a\ a\ a$ $a\ b\ c$ $a\ b\ c$	$a\ a\ a$ $a\ b\ c$ $a\ b\ c$	$a\ b\ c$ $b\ b\ b$ $a\ b\ c$	$a\ b\ c$ $b\ b\ b$ $a\ b\ c$	$a\ b\ c$ $a\ b\ c$ $c\ c\ c$	$a\ b\ c$ $a\ b\ c$ $c\ c\ c$
9.	$a\ a\ a$ $a\ b\ b$ $a\ b\ b$	$a\ a\ a$ $a\ c\ c$ $a\ c\ c$	$a\ b\ a$ $b\ b\ b$ $a\ b\ a$	$c\ b\ c$ $b\ b\ b$ $c\ b\ c$	$a\ a\ c$ $a\ a\ c$ $c\ c\ c$	$b\ b\ c$ $b\ b\ c$ $c\ c\ c$
10.	$a\ a\ a$ $a\ b\ c$ $a\ c\ b$	$a\ a\ a$ $a\ c\ b$ $a\ b\ c$	$a\ b\ c$ $b\ b\ b$ $c\ b\ a$	$c\ b\ a$ $b\ b\ b$ $a\ b\ c$	$a\ b\ c$ $b\ a\ c$ $c\ c\ c$	$b\ a\ c$ $a\ b\ c$ $c\ c\ c$
11.	$a\ b\ a$ $a\ b\ b$ $a\ b\ c$	$a\ a\ c$ $a\ b\ c$ $a\ c\ c$	$a\ b\ a$ $a\ b\ b$ $a\ b\ c$	$a\ b\ c$ $b\ b\ c$ $c\ b\ c$	$a\ a\ c$ $a\ b\ c$ $a\ c\ c$	$a\ b\ c$ $b\ b\ c$ $c\ b\ c$
12.	$a\ b\ b$ $a\ b\ b$ $a\ b\ c$	$a\ c\ c$ $a\ b\ c$ $a\ c\ c$	$a\ b\ a$ $a\ b\ a$ $a\ b\ c$	$a\ b\ c$ $c\ b\ c$ $c\ b\ c$	$a\ a\ c$ $a\ b\ c$ $a\ a\ c$	$a\ b\ c$ $b\ b\ c$ $b\ b\ c$
13.	$a\ a\ a$ $a\ a\ a$ $a\ a\ c$	$a\ a\ a$ $a\ b\ a$ $a\ a\ a$	$b\ b\ b$ $b\ b\ b$ $b\ b\ c$	$a\ b\ b$ $b\ b\ b$ $b\ b\ b$	$c\ c\ c$ $c\ b\ c$ $c\ c\ c$	$a\ c\ c$ $c\ c\ c$ $c\ c\ c$
14.	$a\ a\ a$ $a\ a\ b$ $a\ a\ c$	$a\ a\ a$ $a\ b\ a$ $a\ c\ a$	$b\ b\ a$ $b\ b\ b$ $b\ b\ c$	$a\ b\ b$ $b\ b\ b$ $c\ b\ b$	$c\ a\ c$ $c\ b\ c$ $c\ c\ c$	$a\ c\ c$ $b\ c\ c$ $c\ c\ c$
15.	$a\ a\ a$ $a\ a\ b$ $a\ b\ c$	$a\ a\ a$ $a\ b\ c$ $a\ c\ a$	$b\ b\ a$ $b\ b\ b$ $a\ b\ c$	$a\ b\ c$ $b\ b\ b$ $c\ b\ b$	$c\ a\ c$ $a\ b\ c$ $c\ c\ c$	$a\ b\ c$ $b\ c\ c$ $c\ c\ c$
16.	$a\ b\ a$ $b\ a\ b$ $a\ b\ c$	$a\ a\ c$ $a\ b\ c$ $c\ c\ a$	$b\ a\ a$ $a\ b\ b$ $a\ b\ c$	$a\ b\ c$ $b\ b\ c$ $c\ c\ b$	$c\ a\ a$ $a\ b\ c$ $a\ c\ c$	$a\ b\ c$ $b\ c\ b$ $c\ b\ c$
17.	$a\ a\ a$ $a\ b\ a$ $a\ a\ c$	$a\ a\ a$ $a\ b\ a$ $a\ a\ c$	$a\ b\ b$ $b\ b\ b$ $b\ b\ c$	$a\ b\ b$ $b\ b\ b$ $b\ b\ c$	$a\ c\ c$ $c\ b\ c$ $c\ c\ c$	$a\ c\ c$ $c\ b\ c$ $c\ c\ c$
18.	$a\ a\ a$ $a\ b\ b$ $a\ b\ c$	$a\ a\ a$ $a\ b\ c$ $a\ c\ c$	$a\ b\ a$ $b\ b\ b$ $a\ b\ c$	$a\ b\ c$ $b\ b\ b$ $c\ b\ c$	$a\ a\ c$ $a\ b\ c$ $c\ c\ c$	$a\ b\ c$ $b\ b\ c$ $c\ c\ c$

Table 2 shows all the endomorphisms of the semigroups of order 2 and 3, where, for example,

$$5_3\ B=(abb) \text{ means an endomorphism } \begin{pmatrix} abc \\ abb \end{pmatrix} \text{ of } 5_3 .$$

We add that 3_2 is the 3rd semigroup of order 2, and 3_3 is the 3rd semigroup of order 3.

Table 2.

Endomorphisms of Semigroups of Order 2 and 3.

3_2	$A=(aa)$	$B=(ab)$					
4_2	$A=(aa)$	$B=(bb)$	$C=(ab)$				
3_3	$A=(aaa)$	$B=(aab)$	$C=(abc)$				
4_3	$A=(aaa)$	$B=(aba)$	$C=(aac)$	$D=(abc)$			
5_3	$A=(aaa)$	$B=(abb)$	$C=(abc)$				
6_3	$A=(aaa)$	$B=(abc)$	$C=(acb)$				
7_3	$A=(aaa)$	$B=(bbb)$	$C=(aba)$	$D=(bab)$	$E=(aac)$	$F=(abc)$	
8_3	$A=(aaa)$	$B=(bbb)$	$C=(ccc)$	$D=(abb)$	$E=(acc)$	$F=(abc)$	$G=(acb)$
9_3	$A=(aaa)$	$B=(bbb)$	$C=(abb)$	$D=(bbc)$	$E=(abc)$		
10_3	$A=(aaa)$	$B=(bbb)$	$C=(abb)$	$D=(abc)$			
11_3	$A=(aaa)$	$B=(bbb)$	$C=(ccc)$	$D=(aac)$	$E=(bbc)$	$F=(abc)$	$G=(bac)$
12_3	$A=(aaa)$	$B=(bbb)$	$C=(ccc)$	$D=(abb)$	$E=(baa)$	$F=(bbc)$	$G=(abc)$
13_3	$A=(aaa)$	$B=(ccc)$	$C=(aba)$	$D=(aac)$	$E=(abc)$		
14_3	$A=(aaa)$	$B=(ccc)$	$C=(aac)$	$D=(abc)$			
15_3	$A=(aaa)$	$B=(ccc)$	$C=(aac)$	$D=(abc)$			
16_3	$A=(aaa)$	$B=(ccc)$	$C=(aba)$	$D=(aac)$	$E=(abc)$		
17_3	$A=(aaa)$	$B=(bbb)$	$C=(ccc)$	$D=(aab)$	$E=(aba)$	$F=(aac)$	$G=(aca)$
	$H=(abc)$	$I=(acb)$					

In Table 3, we show addition table which is introduced into the endomorphism set (cf. Table 2) of each semigroup. (See § 1). As easily seen, $E(11_3)$ is not closed under the addition.

Table 3.

Addition of Endomorphisms.

Order 2.

3.

	A	B
A	A	B
B	B	A

4.

	A	B	C
A	A	A	A
B	A	B	C
C	A	C	C

Order 3.

3.

	A	B	C
A	A	A	A
B	A	A	A
C	A	A	B

4.

	A	B	C	D
A	A	B	A	B
B	B	A	B	A
C	A	B	A	B
D	B	A	B	A

5.

	A	B	C
A	A	B	B
B	B	A	A
C	B	A	A

6.

	A	B	C
A	A	B	C
B	B	C	A
C	C	A	B

7.

	A	B	C	D	E	F
A	A	B	C	D	A	C
B	A	B	C	D	A	C
C	A	B	C	D	A	C
D	A	B	C	D	A	C
E	A	B	C	D	A	C
F	A	B	C	D	A	C

8.

	A	B	C	D	E	F	G
A	A	A	A	A	A	A	A
B	A	B	C	D	E	F	G
C	A	B	C	D	E	F	G
D	A	D	E	D	E	F	G
E	A	D	E	D	E	F	G
F	A	D	E	D	E	F	G
G	A	D	E	D	E	F	G

9.

	A	B	C	D	E
A	A	A	A	A	A
B	A	B	C	B	C
C	A	C	C	C	C
D	A	B	C	B	C
E	A	C	C	C	C

10.

	A	B	C	D
A	A	A	A	A
B	A	B	C	D
C	A	C	C	D
D	A	D	D	C

11.

	A	B	C	D	E	F	G
A	A	B	A	A	X	X	X
B	A	B	B	X	B	X	X
C	A	B	C	D	E	F	G
D	A	B	D	D	E	F	G
E	A	B	E	D	E	F	G
F	A	B	F	D	E	F	G
G	A	B	G	D	E	F	G

12.

	A	B	C	D	E	F	G
A	A	B	B	D	E	B	D
B	A	B	B	D	E	B	D
C	A	B	C	D	E	F	G
D	A	B	B	D	E	B	D
E	A	B	B	D	E	B	D
F	A	B	F	D	E	F	G
G	A	B	F	D	E	F	G

13.

	A	B	C	D	E
A	A	A	A	A	A
B	A	B	A	D	D
C	A	A	A	A	A
D	A	D	A	D	D
E	A	D	A	D	D

14.

	A	B	C	D
A	A	A	A	A
B	A	B	C	C
C	A	C	C	C
D	A	D	C	C

15.

	A	B	C	D
A	A	A	A	A
B	A	B	C	D
C	A	C	C	C
D	A	D	C	C

16.

	A	B	C	D	E
A	A	A	C	A	C
B	A	B	C	B	E
C	C	C	A	C	A
D	A	D	C	D	E
E	C	E	A	E	D

17.

	A	B	C	D	E	F	G	H	I
A	A	A	A	A	A	A	A	A	A
B	A	B	A	D	E	A	A	E	D
C	A	A	C	A	A	F	G	F	G
D	A	D	A	D	A	A	A	A	D
E	A	E	A	A	E	A	A	E	A
F	A	A	F	A	A	F	A	F	A
G	A	A	G	A	A	A	G	A	G
H	A	E	F	A	E	F	A	H	A
I	A	D	G	D	A	A	G	A	I

In Table 4, we arrange all the multiplications which are distributive to each addition. For example, we have, for the addition 3_3, the multiplications (ABC) (AAB) (AAA) where (ABC) denotes

	a	b	c
a			
b	A	B	C
c			

i. e.

	a	b	c
a	a	a	a
b	a	a	b
c	a	b	c

in words, we rewrite the endomorphisms of 3_3

$$A=(aaa), \qquad B=(aab), \qquad C=(abc)$$

in column, and we place them in row.

In Table 4, the multiplications for the additions 1_2, 2_2, 1_3, 2_3, 18_3 are omitted, because they have been determined in § 2.

We remark that the multiplications $1\sim 14$ for the addition 7_3 are associative, but $15\sim 24$, for 7_3, are not associative.

In Table 5, we show the modified ordering of $\mathfrak{M}_+$ for each the addition S_+ of order 2 and 3. (cf. § 3). For example, in the associative multiplication for the addition 1_3,

$$17.1 \xrightarrow{c} 13.1$$

means that if λ denotes the multiplication 17.1 then the multiplication 13.1 is expressed as λ_c (cf. § 3) ; furthermore if μ denotes 1.1, then μ_x is isomorphic to μ for every x under an automorphism of the addition 1_3.

Table 4.

Multiplications to additive Semigroups.

Order 2.

3. 1(AA) 2(AB)
4. 1(AA) 2(AB) 3(AC) 4(BB) 5(CB) 6(CC)

Order 3.

3. 1(AAA) 2(AAB) 3(ABC)
4. 1(AAA) 2(AAC) 3(ABA) 4(ABC)
5. 1(AAA) 2(ABB) 3(ABC)
6. 1(AAA) 2(ABC) 3(ACB)
7. 1(AAA) 2(AAE) 3(ABA) 4(ABE) 5(ACA) 6(ACE) 7(BBB) 8(CBC)
9(CBE) 10(AFF) 11(CCF) 12(CDC) 13(CDF) 14(DCD)
15(ADA) 16(ADE) 17(BAB) 18(BCB) 19(BDB) 20(CAC) 21(CAF) 22(DAD)
23(DBD) 24(DDD)
8. 1(AAA) 2(ABB) 3(ABC) 4(ACC) 5(ADD) 6(ADE) 7(ADF) 8(AEE)
9(AFE) 10(AFF) 11(AFG) 12(AGF) 13(BBB) 14(CCC) 15(DBB) 16(DDD)
17(ECC) 18(EEE) 19(FFF)
20(ACB) 21(ADG) 22(AED) 23(AEF) 24(AEG) 25(AED) 26(AGD) 27(AGE)
28(AGG) 29(GGG)
9. 1(AAA) 2(ABB) 3(ABD) 4(ACC) 5(ACE) 6(BBB) 7(BBD) 8(CBB)
9(CBD) 10(CCC) 11(CCE)
10. 1(AAA) 2(ABB) 3(ACC) 4(ACD) 5(BBB) 6(CBB) 7(CCC)
11. 1(AAA) 2(AAC) 3(AAD) 4(ABC) 5(BBB) 6(BBC) 7(BBE) 8(CCC)
9(DDC) 10(DDD) 11(DEC) 12(DFC) 13(EEC) 14(EEE) 15(FEC) 16(FFC)
17(FFF) 18(FGC) 19(GFC)
20(BAC) 21(DGC) 22(EDC) 23(EFC) 24(EGC) 25(FDC) 26(GDC) 27(GEC)
28(GGC) 29(GGG)
12. 1(AAA) 2(ABB) 3(ABC) 4(ABF) 5(ADD) 6(ADG) 7(BBB) 8(BBC)
9(BBF) 10(CCC) 11(DBB) 12(DBC) 13(DBF) 14(DDD) 15(DDG) 16(DEE)
17(EDD) 18(EDG) 19(FFC) 20(FFF) 21(GFC) 22(GFF) 23(GGG)
24(AEE) 25(BAA) 26(BDD) 27(BDG) 28(BEE) 29(DAA) 30(EAA) 31(EBB)
32(EBC) 33(EEE) 34(FGG)
13. 1(AAA) 2(AAB) 3(AAD) 4(ACA) 5(ACB) 6(ACD) 7(BBB) 8(DDB)
9(DDD) 10(DEB) 11(DED)
14. 1(AAA) 2(AAB) 3(AAC) 4(BBB) 5(CCB) 6(CCC) 7(CDB)
15. 1(AAA) 2(AAB) 3(AAC) 4(BBB) 5(CCB) 6(CCC) 7(CDB)
16. 1(AAA) 2(AAB) 3(AAD) 4(ACA) 5(ACB) 6(ACD) 7(BBB) 8(DDB)
9(DDD) 10(DEB) 11(DED)
17. 1(AAA) 2(AAC) 3(AAD) 4(AAF) 5(AAH) 6(ABA) 7(ABC) 8(ABF)
9(ADF) 10(ADH) 11(AEA) 12(AEF) 13(AEG) 14(AGA) 15(AHA) 16(AHI)
17(AIH) 18(BBB) 19(CCC) 20(EBE) 21(EBH) 22(EEE) 23(EEH) 24(FFC)
25(FFF) 26(FHC) 27(HHH)
28(AAB) 29(AAE) 30(AAG) 31(AAI) 32(ABG) 33(ACA) 34(ACB) 35(ACD)
36(ACE) 37(ADA) 38(ADC) 39(ADE) 40(ADG) 41(AEC) 42(AED) 43(AEI)
44(AFA) 45(AFB) 46(AFD) 47(AFE) 48(AFG) 49(AFI) 50(AGB) 51(AGD)
52(AGE) 53(AGF) 54(AGH) 55(AHD) 56(AHG) 57(AIA) 58(AIF) 59(DBD)
60(DBI) 61(DDB) 62(DDD) 63(DDI) 64(DIB) 65(DID) 66(EEB) 67(EHB)
68(EHE) 69(FCF) 70(FCH) 71(FFH) 72(FHF) 73(GCG) 74(GCI) 75(GGC)
76(GGG) 77(GGI) 78(GIC) 79(GIG) 80(III)

Table 5. Distributive Associative Multiplication for Semigroups Order 2 and 3.

Order 2.

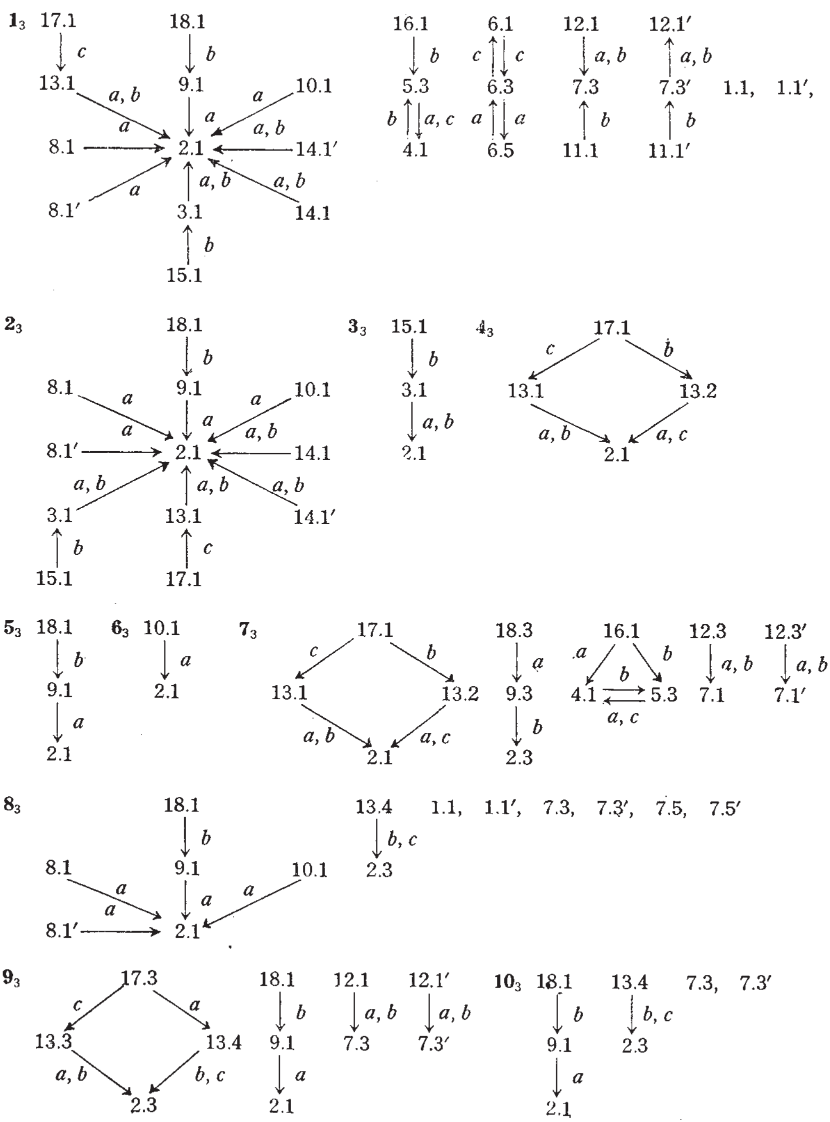
$\mathbf{1}_2$ 1.1, 1.1′, 3.1, $4.1 \xrightarrow{a} 2.1$

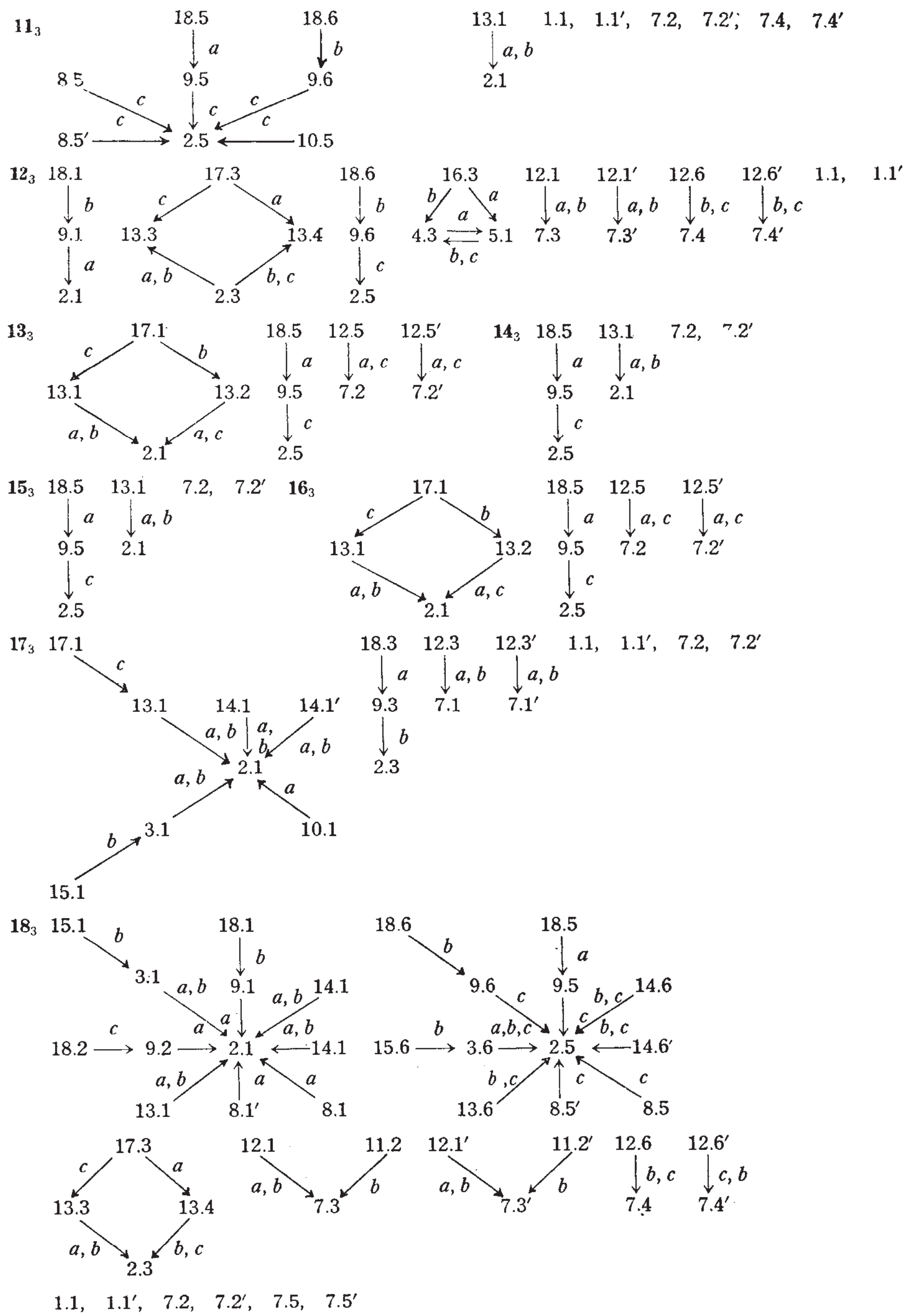
$\mathbf{2}_2$ $4.1 \xrightarrow{a} 2.1$

$\mathbf{3}_2$ $4.1 \xrightarrow{a} 2.1$

$\mathbf{4}_2$ 1.1, 1.1′, $4.1 \xrightarrow{a} 2.1$, $4.2 \xrightarrow{b} 2.2$,

Order 3.





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