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SOME CONTRIBUTIONS TO ORDER STATISTICS

By

Yosikatsu WATANABE, Makoto ISIDA, Seiichi TAGA, Yosihiro ICHIJÔ
Takaichi KAWASE, Gôsuke NISIDE, Yosiharu TAKEDA,
Akira HORISUZI and Isamu KURIYAMA

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Contents

Introduction. The classical problem to estimate the mean of a normal distribution is early and thoroughly discussed by K. Pearson[1] for large samples, while for small samples e.g. by T. Hôjô[2], and recently by H. J. Godwin, H. L. Jones and others[3]. Also H. Cramér[4] puts in his treatise the following example as instructive: Considering a small sample with size 3 drawn from

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1) Karl Pearson, On the probable errors of frequency constants (editorial), Part I, Biometrika, Vol. 2 (1903), pp. 273–281; Part II, Vol. 9 (1913), pp. 1–10; Part III, Vol. 13 (1921), pp. 113–132.

2) Tokishige Hôjô, Distribution of the median, quartiles and interquartile distance in samples from a normal population, Biometrika Vol. 23 (1931), pp. 315–360.

Also, Tokishige Hôjô, A further note on the relation between the median and quartiles in small samplings from a normal population, Biometrika, Vol. 25 (1933), pp. 79–90.

3) H. J. Godwin, Some low moments of order statistics, Ann. of Math. Statis. Vol. 20 (1949), pp. 279–285; H. L. Jones, Exact lower moments of order statistics in small samples from a normal distribution, Ann. of Math. Statis., Vol. 19 (1948), pp. 270–273.

4) Herald Cramér, Mathematical Methods of Statistics, (1946), p. 483.

a normal population $N(m, \sigma^2)$ and arranged in order of magnitude: $x_1 \leqq x_2 \leqq x_3$, the weighted mean $z = cx_1 + (1-2c)x_2 + cx_3$ would afford an unbiased estimate of the population mean with the variance $D^2(z) = \frac{\sigma^2}{3} + 3\sigma^2\left(2 - \frac{3\sqrt{3}}{\pi}\right)\left(c - \frac{1}{3}\right)^2$. This being generalized, we may inquire what would be $E(z)$ and $D^2(z)$, when $x_1 \leqq x_2 \leqq \cdots \leqq x_n$ be a sample drawn from the population $N(m, \sigma^2)$, and we form a weighted mean $z = \sum_{i=1}^{n} c_i x_i$, where $\sum c_i = 1$ with all $c_i \geqq 0$. In the present note we have solved this general problem in outline (Part I), and obtained somewhat detailed results for the particular cases: $n = 3, 4, \cdots, 7$ (Part II), at the end of which some applications are also illustrated. We have likewise schemed (as Part III) to investigate the third and fourth moments. However, their computations being too much enormous, their studies, except some fews, are deferred for future.

PART I

§ 1. **Frequency Functions.** We make preliminarily the variable standardized: $x = m + \sigma t$, and the distribution function $F(x) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x} \exp\left\{-\frac{1}{2\sigma^2}(x-m)^2\right\} \times dx$ reduces to $\Phi(t) = \int_{-\infty}^{t} \varphi(t)dt = \int_{-\infty}^{t} d\Phi(t)$, where $\varphi(t) = \frac{1}{\sqrt{2\pi}}\left\{-\frac{1}{2}t^2\right\}$, so that $\Phi(-\infty) = 0$, $\Phi(\infty) = 1$ and $\lim_{t \to \pm\infty} t^N \varphi(t) = 0$. Firstly, observing that the joint probability to obtain the sample $\{-\infty < x_1 \leqq x_2 \leqq \cdots \leqq x_n < \infty\}$, or $\{-\infty < t_1 \leqq t_2 \leqq \cdots \leqq t_n < \infty\}$ is

$$n!\, d\Phi_n d\Phi_{n-1} \cdots d\Phi_1,$$

where $d\Phi_i$ stands for $d\Phi(t_i) = \varphi(t_i)dt_i$, the total probability shall be given by

$$(1.1) \qquad n! \int_{-\infty}^{\infty} d\Phi_n \int_{-\infty}^{n} d\Phi_{n-1} \cdots \int_{-\infty}^{2} d\Phi_1 = 1,$$

where $\int_{-\infty}^{i} d\Phi_{i-1}$ means $\int_{-\infty}^{t_i} d\Phi(t_{i-1})$. Still more abbreviating we write simply $\int^i$, $\int_i$ and $\int$ in place of $\int_{-\infty}^{i}$, $\int_i^{\infty}$ and $\int_{-\infty}^{\infty}$, respectively, and also e.g. for $\int_{-\infty}^{\infty} t_i^2 \varphi^3(t_i)\, \Phi^4(t_i)\, d\Phi(t_i)$, dropping the suffix i as well as defining argument, simply as $\int t^2 \varphi^3 \Phi^4 d\Phi$, when there is no fear of misunderstanding. On the other hand, if it needs to be made clear that the size of sample is n, we write $t_{i|n}$ instead of mere t_i.

To prove (1.1) we see successively

$$\int^2 d\Phi_1 = \Phi_2 - \Phi_{-\infty} = \Phi_2, \qquad \int^3 d\Phi_2 \int^2 d\Phi_1 = \int^3 \Phi_2 d\Phi_2 = \frac{1}{2!}\Phi_3^2,$$

and in general, if

$$\int^i d\Phi_{i-1} \cdots \int^2 d\Phi_1 = \frac{1}{(i-1)!}\Phi_i^{i-1}$$

holds, then also

$$\int^{i+1} d\Phi_i \int^i d\Phi_{i-1} \cdots \int^2 d\Phi_1 = \frac{1}{(i-1)\,!} \int^{i+1} \Phi_i^{i-1} d\Phi_i = \frac{1}{i\,!} \Phi_{i+1}^i ,$$

thus the induction is completed. Therefore

$$n! \int d\Phi_n \int^n d\Phi_{n-1} \cdots \int^2 d\Phi_1 = n \int \Phi_n^{n-1} d\Phi_n = \Phi_n^n \Big|_{-\infty}^{\infty} = 1 , \qquad \text{Q.E.D.}$$

Otherwise, if the order of integrations in the repeated integrals be interchanged, it becomes

$$n! \int d\Phi_1 \int_1 d\Phi_2 \cdots \int_{n-1} d\Phi_n = 1 .$$

Now we can thereby find the frequency function of $t_{i|n}$. For this purpose we consider the repeated integrals of (1. 1)

$$n! \int d\Phi_n \int^n d\Phi_{n-1} \cdots \int^{i+2} d\Phi_{i+1} \int^{i+1} d\Phi_i \int^i d\Phi_{i-1} \cdots \int^2 d\Phi_1 ,$$

and integrate the $i-1$ integrals beginning from Φ_1 up to Φ_{i-1}

$$n! \int d\Phi_n \int^n d\Phi_{n-1} \cdots \int^{i+2} d\Phi_{i+1} \int^{i+1} \frac{\Phi_i^{i-1}}{(i-1)\,!} d\Phi_i .$$

Further, interchanging the order of integrations

$$\frac{n!}{(i-1)\,!} \int \Phi_i^{i-1} d\Phi_i \int_i d\Phi_n \int_i^n d\Phi_{n-1} \cdots \int_i^{i+2} d\Phi_{i+1} ,$$

and finally integrating in regards to $\Phi_{i+1}, \cdots, \Phi_n$, successively, we get

$$\frac{n!}{(i-1)!\,(n-i)!} \int \Phi_i^{i-1} (1-\Phi_i)^{n-i} d\Phi_i = \beta_{i|n} \int \Phi^{i-1} (1-\Phi)^{n-i} \varphi dt ,$$

where $\beta_{i|n}$ denotes the reciprocal of Betafunction $B(i, n-i+1) = \dfrac{\Gamma(i)\Gamma(n-i+1)}{\Gamma(n+1)}$, so that[5]

(1. 2) $$\beta_{i|n} = \beta_{n-i+1|n} = \frac{n!}{(i-1)!\,(n-i)!} = n \binom{n-1}{i-1} ,$$

and the required frequency function is given by

(1. 3) $$f(t_{i|n}) = \beta_{i|n} \Phi^{i-1} (1-\Phi)^{n-i} \varphi .$$

Next, to find the joint frequency function of $t_{i|n}$ and $t_{k|n}$, where $1 \leqq i < k \leqq n$, we rewrite (1. 1) so as

$$n! \int d\Phi_n \int^n d\Phi_{n-1} \cdots \int^{k+2} d\Phi_{k+1} \int^{k+1} d\Phi_k \int^k d\Phi_{k-1} \cdots \int^{i+2} d\Phi_{i+1} \int^{i+1} \frac{\Phi_i^{i-1}}{(i-1)\,!} d\Phi_i ,$$

which, on interchanging the order of integrations two by two respectively, beginning at Φ_{k+1} with Φ_k, then Φ_{k+2} with Φ_k, $\cdots$, and similarly beginning at Φ_{i+1} with Φ_i, then Φ_{i+2} with Φ_i, $\cdots$, becomes

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5) Here the successive factors in denominator of $\beta_{i|n}$ are nothing but the factorials of the successive indices of factors in the integrand. This remark is also applicable to (1. 4) &c. below.

$$n!\int d\Phi_k \int_k d\Phi_n \int_k^n d\Phi_{n-1} \cdots \int_k^{k+2} d\Phi_{k+1} \int^k \frac{\Phi_i^{i-1}}{(i-1)!} d\Phi_i \int_i^k d\Phi_{k-1} \cdots \int_i^{i+2} d\Phi_{i+1} .$$

Now integrating successively leftwards beginning at Φ_{k+1} and Φ_{i+1} respectively,

$$n!\int \frac{(1-\Phi_k)^{n-k}}{(n-k)!} d\Phi_k \int^k \frac{\Phi_i^{i-1}(\Phi_k-\Phi_i)^{k-i-1}}{(i-1)!\,(k-i-1)!} d\Phi_i \;(=1) .$$

Hence, the required joint frequency function of t_i, t_k is found to be

(1.4) $$f(t_{i|n}, t_{k|n}) = \frac{n!}{(n-k)!\,(i-1)!\,(k-i-1)!}(1-\Phi_k)^{n-k}\varphi_k \cdot \Phi_i^{i-1}(\Phi_k-\Phi_i)^{k-i-1}\varphi_i , \quad (1 \leq i < k \leq n) .$$

Here the numerical coefficient $\gamma_{i,k|n}$ say, is the reciprocal of the product of Beta-functions, because, on writing $\Phi_k = u$, $\Phi_i = v\Phi_k$ in the foregoing integral, it yields

$$\gamma_{i,k|n}\int_0^1 (1-u)^{n-k}u^{k-1}du \int_0^1 v^{i-1}(1-v)^{k-i-1}dv = \gamma_{i,k|n}B(n-k+1,\, k)\,B(i,\, k-i) = 1 .$$

Hence

(1.5) $$\gamma_{i,k|n} = \frac{1}{B(n-k+1,\, k)B(i,\, k-i)} = \frac{n!}{(n-k)!\,(i-1)!\,(k-i-1)!} = \gamma_{n-k+1,\, n-i+1|n} .$$

Thus, in ordered statistics any two variables t_i, t_k are by no means independent of each other, while, if t_i, t_k are two individuals in unordered sample, they are independent and their joint frequency is simply $\varphi_i\varphi_k$.

Quite similarly we obtain

(1.6) $$f(t_{i|n}, t_{j|n}, t_{k|n}) = \frac{n!}{(n-k)!\,(k-j-1)!\,(j-i-1)!\,(i-1)!}(1-\Phi_k)^{n-k} \times (\Phi_k-\Phi_j)^{k-j-1}(\Phi_j-\Phi_i)^{j-i-1}\Phi_i^{i-1}\varphi_{ijk} \qquad (i<j<k)$$

where $\varphi_{ijk} = \varphi_j\varphi_j\varphi_k$, which, however, does not occur unless $n \geqq 3$. E.g. if $n=3$, we have $f(t_1, t_2, t_3) = 6\ \varphi_{123}$ and if $n=4$,

(1.6.1) $$f(t_1, t_2, t_3) = 24(1-\Phi_3)\,\varphi_{123} , \qquad f(t_1, t_2, t_4) = 24(\Phi_4-\Phi_2)\,\varphi_{124} ,$$
$$f(t_1, t_3, t_4) = 24(\Phi_3-\Phi_1)\,\varphi_{134} , \quad f(t_2, t_3, t_4) = 24\Phi_2\,\varphi_{234} .$$

Also, for $n=5$,

(1.6.2) $$f(t_1, t_2, t_3) = 60(1-\Phi_3)^2\varphi_{123} , \quad f(t_1, t_2, t_4) = 120(1-\Phi_4)(\Phi_4-\Phi_2)\,\varphi_{124} ,$$
$$f(t_1, t_2, t_5) = 60(\Phi_5-\Phi_2)^2\varphi_{125} , \quad f(t_1, t_3, t_4) = 120(1-\Phi_4)(\Phi_3-\Phi_1)\,\varphi_{134} ,$$
$$f(t_1, t_3, t_5) = 120(\Phi_5-\Phi_3)(\Phi_3-\Phi_1)\,\varphi_{135} , \quad f(t_1, t_4, t_5) = 60(\Phi_4-\Phi_1)^2\varphi_{145} ,$$
$$f(t_2, t_3, t_4) = 120(1-\Phi_4)\Phi_2\,\varphi_{234} , \quad f(t_2, t_3, t_5) = 120(\Phi_5-\Phi_3)\Phi_2\,\varphi_{235} ,$$
$$f(t_2, t_4, t_5) = 120(\Phi_4-\Phi_2)\Phi_2\,\varphi_{245} , \quad f(t_3, t_4, t_5) = 60\Phi_3^2\,\varphi_{345} .$$

Furthermore

(1.7) $$f(t_{i|n}, t_{j|n}, t_{k|n}, t_{l|n}) = \frac{n!(1-\Phi_l)^{n-l}(\Phi_l-\Phi_k)^{l-k-1}(\Phi_k-\Phi_j)^{k-j-1}(\Phi_j-\Phi_i)^{j-i-1}\Phi^{i-1}}{(n-l)!\,(l-k-1)!\,(k-j-1)!\,(j-i-1)!\,(i-1)!}\varphi_{ijkl} \quad (i<j<k<l) .$$

In particular, for $n=5$,

(1.7.1) $$f(t_1, t_2, t_3, t_4) = 24(1-\Phi_4)\,\varphi_{1234}\,, \quad f(t_1, t_2, t_3, t_5) = 24(\Phi_5-\Phi_3)\,\varphi_{1235}\,,$$
$$f(t_1, t_2, t_4, t_5) = 24(\Phi_4-\Phi_2)\,\varphi_{1245}\,, \quad f(t_1, t_3, t_4, t_5) = 24(\Phi_3-\Phi_1)\,\varphi_{1345}\,,$$
$$f(t_2, t_3, t_4, t_5) = 24\Phi_1\,\varphi_{2345}\,, \quad \text{and so on.}$$

Remark. Although we have confined ourselves to the case of normal population, all the above remain the same even for any continuous distribution with a frequency function $f(t)$, and the distribution $F(t)=\int^t f(t)dt$. Thus

$$n!\int dF_n\int^n dF_{n-1}\cdots\int^2 dF_1=1\,, \qquad f(t_{i|n})=\beta_{i|n}F^{i-1}(1-F)^{n-i}f\,,^{6)}$$
$$f(t_{i|n}, t_{k|n})=\gamma_{i,k|n}(1-F_k)^{n-k}(F_k-F_i)^{k-i-1}F_i^{i-1}f_kf_i \quad (i<k) \quad \&c.$$

§ **2. The Expectation of $t^p_{i|n}$.** This is given after (1.3) by

(2.1) $$E(t^p_{i|n})=\beta_{i|n}\int\Phi^{i-1}(1-\Phi)^{n-i}\varphi t^p dt \quad \text{with} \quad \beta_{i|n}=\frac{n!}{(n-i)!\,(i-1)!}\,.$$

First, for $p=1$, $i=n$,

$$E(t_{n|n})=n\int\Phi^{n-1}(-\varphi')dt\,,$$

which, integrated by parts, yields

$$-n\Phi^{n-1}\varphi\Big|_{-\infty}^{\infty}+n(n-1)\int\Phi^{n-2}\varphi^2 dt\,,$$

and since the integrated parts vanish, we have

(2.2) $$E(t_{n|n})=n(n-1)\int\Phi^{n-2}\varphi^2 dt\,.$$

For broader applications let us define

(2.3) $$J_\lambda^{(\alpha)}=\int\Phi^\lambda\varphi^\alpha dt\,,$$

which shall be requisite to obtain $E(t_{n|n})=n(n-1)J^{(2)}_{n-2}$. Some of them are explicitly found in Part II.

In general, we obtain by integration by parts

(2.4) $$E(t_{i|n})=\beta_{i|n}\int\Phi^{i-1}(1-\Phi)^{n-i}(-\varphi')dt=\beta_{i|n}\int\frac{d}{d\Phi}[\Phi^{i-1}(1-\Phi)^{n-i}]\varphi^2 dt\,,$$

which becomes a sum of integrals of the form $J^{(2)}_\lambda$. Since the integrand in (2.4) is a polynomial in Φ of $n-2$ degrees, it suffices to know $J^{(2)}_\lambda$ for $\lambda=0, 1, \cdots, n-2$.

Next, for $p=2$, we have again by (2.1)

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6) In fact, if the i-th value from the top, i.e. the $(n-i+1)$-th value from the bottom be considered, we obtain by (1.3) $f(t_{n-i+1})=\beta_{i|n}F^{n-i}(1-F)^{i-1}f$, which agrees with Cramér's (28.6.1), p. 370, loc. cit.

(2.5) $$E(t^2_{i|n}) = \beta_{i|n}\int t^2\Phi^{i-1}(1-\Phi)^{n-i}\varphi dt \qquad (t\varphi=-\varphi')$$

which integrated by parts yields

$$\beta_{i|n}\int \Phi^{i-1}(1-\Phi)^{n-i}d\Phi+\beta_{i|n}\int\frac{d}{d\Phi}\{\Phi^{i-1}(1-\Phi)^{n-1}\}\varphi^2 tdt\,.$$

The first integral reduces to 1, while, the second being

$$-\int\frac{d}{d\Phi}\{\Phi^{i-1}(1-\Phi)^{n-i}\}\varphi\varphi' dt\,,$$

again integrated by parts, it becomes

$$\int\frac{d^2}{d\Phi^2}\{\Phi^{i-1}(1-\Phi)^{n-i}\}\varphi^3 dt+\int\frac{d}{d\Phi}\{\Phi^{i-1}(1-\Phi)^{n-i}\}\varphi\varphi' dt\,,$$

and therefore is equal to

$$\frac{1}{2}\int\frac{d^2}{d\Phi^2}\{\Phi^{i-1}(1-\Phi)^{n-i}\}\varphi^3 dt\,.$$

Thus we get

(2.6) $$E(t^2_{i|n})=1+\frac{1}{2}\beta_{i|n}\int\frac{d^2}{d\Phi^2}\{\Phi^{i-1}(1-\Phi)^{n-i}\}\varphi^3 dt\,.$$

Particularly for $i=n$

(2.6.1) $$E(t^2_{n|n})=1+\frac{1}{2}n(n-1)(n-2)\int\Phi^{n-3}\varphi^3 dt\,.$$

The explicit forms of $J^{(3)}_\lambda=\int\Phi^\lambda\varphi^3 dt$ for $\lambda=0, 1, 2, \ldots$ have been evaluated in Part II, and whence all $E(t^2_{i|n})$ computed up to $n=7$.

Furthermore we obtain

(2.7) $$E(t^3_{i|n})=\beta_{i|n}\int t^3\Phi^{i-1}(1-\Phi)^{n-i}\varphi dt$$
$$=\frac{5}{2}E(t_{i|n})+\frac{1}{6}\beta_{i|n}\int\frac{d^3}{d\Phi^3}[\Phi^{i-1}(1-\Phi)^{n-i}]\varphi^4 dt\,,$$

(2.7.1) $$E(t^3_{n|n})=\frac{5}{2}E(t_{n|n})+\frac{1}{6}n(n-1)(n-2)(n-3)\int\Phi^{n-4}\varphi^4 dt\,.$$

and

(2.8) $$E(t^4_{i|n})=\frac{13}{3}E(t^2_{i|n})-\frac{4}{3}+\frac{1}{24}\beta_{i|n}\int\frac{d^4}{d\Phi^4}[\Phi^{i-1}(1-\Phi)^{n-i}]\varphi^5 dt\,,$$

(2.8.1) $$E(t^4_{n|n})=\frac{13}{3}E(t^2_{n|n})-\frac{4}{3}+\frac{1}{24}n(n-1)(n-2)(n-3)(n-4)\int\Phi^{n-5}\varphi^5 dt\,, \quad \&c.$$

However their actual computations are deferred as future task.

We are very liable to write erroneous indices. To avoid this, it may be remarked that, in any successively obtained integrand, the sum of indices of Φ, $1-\Phi$ and φ should always be n; of course, if differentiated j times, j must be subtracted from the sum.

§ 3. Some Properties concerning $E(t^p_{i|n})$, $p=1, 2, \cdots$.

We have generally the following identities:

(3.1) $\sum_{i=1}^{n} E(t^p_{i|n}) = 0$, or $1, 3, 5, \cdots (p-1)n$, according as p is odd or even;

(3.2) $$E(t^p_{n-i+1|n}) = (-1)^p E(t^p_{i|n}) .$$

First, to prove (3.1) we cite (2.1) and (1.2); We see readily

$$\sum E(t^p_{i|n}) = n \sum \binom{n-1}{i-1} \int \Phi^{i-1}(1-\Phi)^{n-i} \varphi t^p \, dt = n \int (\Phi + 1 - \Phi)^{n-1} \varphi t^p \, dt$$
$$= n \int \varphi t^p \, dt = 0, \quad \text{or} \quad 1, 3, 5, \cdots (p-1) n .$$

Next, to prove (3.2) we take $U = \Phi - \frac{1}{2} = \int_0^t \varphi(t) dt = F(t)$, as independent variable, and then t becomes its inverse function: $t = F^{-1}(U)$. They are both odd monotonic functions, and (2.1) may be expressed as

$$E(t^p_{i|n}) = \beta_{i|n} \int_{-1/2}^{1/2} t^p \left(\frac{1}{2} + U\right)^{i-1} \left(\frac{1}{2} - U\right)^{n-i} dU .$$

The variable t_i is the i-th from the bottom, while the i-th from the top is t_{n-i+1}, for which

$$E(t^p_{n-i+1}) = \beta_{n-i+1|n} \int_0^1 t^p \Phi^{n-i}(1-\Phi)^{i-1} d\Phi = \beta_{i|n} \int_{-1/2}^{1/2} t^p \left(\frac{1}{2} + U\right)^{n-i} \left(\frac{1}{2} - U\right)^{i-1} dU ,$$

and we shall show that the above two integrals are equal. In fact, two functions $y_1 = \left(\frac{1}{2} + U\right)^{i-1} \left(\frac{1}{2} - U\right)^{n-i}$ and $y_2 = \left(\frac{1}{2} + U\right)^{n-i} \left(\frac{1}{2} - U\right)^{i-1}$ are situated to each other symmetrically, having the axis of symmetry $U=0$, so that $y_2(U) = y_1(-U)$. But $t = F^{-1}(U)$ being odd, t^p is either odd or even, according as p is odd or even. So we have $t^p(U) y_2(U) = (-1)^p t^p(-U) y_1(-U)$, and therefore

$$\int_{-1/2}^{1/2} t^p(U) y_2(U) dU = (-1)^p \int_{-1/2}^{1/2} t^p(-U) y_1(-U) dU = (-1)^p \int_{-1/2}^{1/2} t^p(V) y_1(V) dV ,$$

if $V = -U$, and this proves (3.2).

Specially, for $p=1$ and 2, we obtain

(3.2.1) $$E(t_{n-i+1|n}) = -E(t_{i|n}) \quad \text{and} \quad E(t^2_{n-i+1|n}) = E(t^2_{i|n}) .$$

Besides, if n be an odd integer $2\nu+1$, the middle variable $t_{\nu+1}$ becomes its median, m_i, whose mean is

(3.3) $$E(t_{\nu+1|2\nu+1}) = \frac{(2\nu+1)!}{\nu!\,\nu!} \int t \Phi^{\nu}(1-\Phi)^{\nu} d\Phi = 0, \quad \text{and} \quad E(x_{\nu+1}) = m .$$

(Incidentally we may remark that $E(t^p_{\nu+1|2\nu+1}) = 0$, if p odd.)

Also, when n is even and 2ν, the median is $m_i = \frac{1}{2}(t_\nu + t_{\nu+1})$, and by (3.2.1) still $E(m_i) = 0$, $E\left[\frac{1}{2}(x_\nu + x_{\nu+1})\right] = m$. The sample median is already an unbiased estimate of the population mean (=median).

Generally for $z=\sum c_i x_i = m+\sigma\sum c_i t_i$, we have

$$E(z) = m+\sigma\sum c_i E(t_i) = m+\sigma\sum_{i=1}^{[n/2]}(c_i - c_{n-i+1})E(t_i)\ .$$

As a matter of course $E(t_i) < E(t_k)$ for $i<k$, and all $E(t_{n-i+1}) = -E(t_i) > 0$ for $i=1, 2, \dots [n/2]$. Now z, as an estimate of mean, is unbiased, when and only when

$$\sum c_i E(t_i) = 0\ , \quad \text{viz.}\ \sum_{i=1}^{[n/2]}(c_i - c_{n-i+1})E(t_i) = 0\ .$$

For this, it is sufficient that all $c_i = c_{n-i+1}$ hold. In particular, let $c_i = c_{n-i+1} = \frac{1}{2}$ for a certain i and all other c_j's$=0$. Then, in view of (3. 2. 1),

$$z_i = \frac{1}{2}(x_i + x_{n-i+1})$$

becomes again an unbiased estimate of the population mean. Thus every mean of the i-ths ($i=1, 2, \dots, [n/2]$) from the bottom and the top, affords an unbiased estimate of the population mean, among them the sample median may also be adopted, as above mentioned. However, which z_i is more efficient, should be discussed from the values of $D^2(z_i)$, that will be investigated in §11.

§4. The Expectations of Products $t_{i|n}t_{k|n}$ for $1\leq i<k\leq n$.

We have by (1. 4)

$$(4.1)\qquad E(t_{i|n}t_{k|n}) = \gamma_{i,k|n}\int t_k(1-\Phi_k)^{n-k}d\Phi_k\int^k t_i\Phi_i^{i-1}(\Phi_k-\Phi_i)^{k-i-1}d\Phi_i\ .$$

Or, dropping unnecessary suffices and adopting convenient one,

$$(4.2)\qquad E(t_{i|n}t_{k|n}) = \gamma_{i,k|n}\int(1-\Phi)^{n-k}\varphi' dt\int^t\Phi_1^{i-1}(\Phi-\Phi_1)^{k-i-1}\varphi_1' dt_1\ .$$

Firstly make $k=i+1$. Since, on integrating the inner integral by parts, it yields

$$\int^t\Phi_1^{i-1}\varphi_1' dt_1 = \Phi^{i-1}\varphi-(i-1)\int^t\Phi_1^{i-2}\varphi_1^2 dt_1\ ,$$

so the whole integral becomes

$$\int(1-\Phi)^{n-k}\Phi^{i-1}\varphi\varphi' dt-(i-1)\int(1-\Phi)^{n-k}\varphi' dt\int^t\Phi_1^{i-2}\varphi_1^2 dt_1\ .$$

These being once more integrated by parts, respectively, we obtain

$$(4.3)\quad E(t_{i|n}t_{i+1|n}) = \frac{n!}{(n-i-1)!\,(i-1)!}\Big[-\frac{1}{2}\int\frac{d}{d\Phi}[\Phi^{i-1}(1-\Phi)^{n-i-1}]\varphi^3 dt$$
$$+(i-1)\int(1-\Phi)^{n-i-1}\Phi^{i-2}\varphi^3 dt-(i-1)(n-i-1)\int(1-\Phi)^{n-i-2}\varphi^2 dt\int^t\Phi_1^{i-2}\varphi_1^2 dt_1\Big].$$

Here both the first and second integrals are sums of $J_\lambda^{(3)}$ in (2. 3), while the third is a sum of double integrals of the form:

$$(4.4)\qquad J_{\mu,\nu}^{\alpha,\beta} = \int\Phi^\mu\varphi^\alpha dt\int^t\Phi_1^\nu\varphi_1^\beta dt_1\ ,$$

and the double integrals presenting in (4. 3) are those with $\alpha=\beta=2$, and $\mu+\nu=n-4$ at most[7].

However, if $i=1$ or $n-1$, the double integral disappears and simply

(4. 3. 1) $$E(t_{1|n}t_{2|n})=\frac{1}{2}n(n-1)(n-2)J^{(3)}_{n-3}=E(t_{n-1|n}t_{n|n}), \quad \&c.$$

Secondly, for $k=i+2$, the inner integral in (4. 2) yields

$$-\int^{t}\frac{\partial}{\partial\Phi_1}[\Phi_1^{i-1}(\Phi-\Phi_1)]\varphi_1^2dt_1=-(i-1)\Phi\int^{t}\Phi_1^{i-2}\varphi_1^2dt_1+i\int^{t}\Phi_1^{i-1}\varphi_1^2dt_1,$$

so that by integrations by parts we get

(4. 5) $$E(t_{i|n}t_{i+2|n})=\gamma_{i,i+2|n}\left\{-\int(1-\Phi)^{n-i-2}\varphi'dt\int^{t}\frac{\partial}{\partial\Phi_1}[\Phi_1^{i-1}(\Phi-\Phi_1)]\varphi_1^2dt_1\right\}$$
$$=\frac{n!}{(n-i-2)!\,(i-1)!}\left\{-(n-i-2)\int(1-\Phi)^{n-i-3}\varphi^2dt\int^{t}\frac{\partial}{\partial\Phi_1}[\Phi_1^{i-1}(\Phi-\Phi_1)]\varphi_1^2dt_1\right.$$
$$\left.+(i-1)\int(1-\Phi)^{n-i-2}\varphi^2dt\int^{t}\Phi_1^{i-2}\varphi_1^2dt_1-\int(1-\Phi)^{n-i-2}\Phi^{i-1}\varphi^3dt\right\}.$$

Here the first two integrals consist of $J_{\mu\nu}$ with $\mu+\nu=n-4$ at most, and the remaining integrals are all of type $J^{(3)}_{\lambda}$. In particular, if $i=1$,

(4. 5. 1) $$E(t_{1|n}t_{3|n})=n(n-1)(n-2)\left[(n-3)\int(1-\Phi)^{n-4}\varphi^2dt\int^{t}\varphi_1^2dt_1-J^{(3)}_{n-3}\right].$$

Thirdly, for $k\geqq i+3$, the inner integral when differentiated, becomes

$$\frac{d}{dt}\left\{-\int^{t}\frac{\partial}{\partial\Phi_1}[\Phi_1^{i-1}(\Phi-\Phi_1)^{k-i-1}]\varphi_1^2dt_1\right\}=-(k-i-1)\int^{t}\frac{\partial}{\partial\Phi_1}[\Phi_1^{i-1}(\Phi-\Phi_1)^{k-i-2}]\varphi_1^2\varphi dt_1,$$

so we obtain

(4. 6) $$E(t_{i|n}t_{k|n})=\frac{n!}{(n-k)!\,(i-1)!\,(k-i-1)!}\left\{-(n-k)\int(1-\Phi)^{n-k-1}\varphi^2dt\right.$$
$$\times\int^{t}\frac{\partial}{\partial\Phi_1}[\Phi_1^{i-1}(\Phi-\Phi_1)^{k-i-1}]\varphi_1^2dt_1+(k-i-1)\int(1-\Phi)^{n-k}\varphi^2dt$$
$$\left.\times\int^{t}\frac{\partial}{\partial\Phi_1}[\Phi_1^{i-1}(\Phi-\Phi_1)^{k-i-2}]\varphi_1^2dt_1\right\}\qquad(k\geqq i+3),$$

every term of which belongs to $J_{\mu\nu}$ with $\mu+\nu=n-4$ at most. In particular, if $i=1$,

(4. 6. 1) $$E(t_{1|n}t_{k|n})=\frac{n!}{(n-k-1)!\,(k-3)!}\int(1-\Phi)^{n-k-1}\varphi^2dt\int^{t}(\Phi-\Phi_1)^{k-3}\varphi_1^2dt_1$$
$$-\frac{n!}{(n-k)!\,(k-4)!}\int(1-\Phi)^{n-k}\varphi^2dt\int^{t}(\Phi-\Phi_1)^{k-4}\varphi_1^2dt_1.$$

Of course, if the denominator contains a factorial of negative integer, i.e. ∞, then that term must be reckoned as zero.

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7) However, those below being almost $\alpha=\beta=2$, we write simply $J_{\mu\nu}$ for $J^{2,2}_{\mu,\nu}$

§ 5. Some Properties concerning $E(t^p_{i|n} t^q_{k|n})$ with $p, q = 1, 2, 3 \cdots$, and $i \neq k$.

Although computations of $E(t^p_i t^q_k)$ for $p, q > 1$ are now postponed for future, some general properties may here be discussed. We have indeed the following identities:

(5. 1) $E(t^p_{i|n} t^q_{k|n}) = E(t^p_{i'|n} t^q_{k'|n})$, and in particular $E(t_{i|n} t_{k|n}) = E(t_{i'|n} t_{k'|n})$,

where $i' = n-i+1$, $k' = n-k+1$. This can be shown similarly as (3. 2) proved: Really by (4. 2), but now considering Φ, Φ_1 as two independent variables, and t, t_1 as their functions

$$E(t^p_{i|n} t^q_{k|n}) = \gamma_{i,k|n} \int_0^1 t^q (1-\Phi)^{n-k} d\Phi \int_0^{\Phi} t_1^p \Phi_1^{i-1} (\Phi - \Phi_1)^{k-i-1} d\Phi_1 \quad (i < k) . \tag{5. 2}$$

From the relations $i' = n-i+1$, $k' = n-k+1$, $i < k$, it follows that $k' < i'$ and $n-i' = i-1$, $k'-1 = n-k$, $i'-k'-1 = k-i-1$, so that

$$E(t^p_{i'} t^q_{k'}) = \gamma_{k',i'|n} \int_0^1 t^p (1-\Phi)^{i-1} d\Phi \int_0^{\Phi} t_1^q \Phi_1^{n-k} (\Phi - \Phi_1)^{k-i-1} d\Phi_1 \quad (k' < i') .$$

But $\gamma_{i,k|n} = \gamma_{k',i'|n}$ by (1. 5). Hence we have only to show that the above two integrals are equal. The latter integral, on interchanging the order of integrations yields

$$\int_0^1 t_1^q \Phi_1^{n-k} d\Phi_1 \int_{\Phi_1}^1 t^p (1-\Phi)^{i-1} (\Phi - \Phi_1)^{k-i-1} d\Phi ,$$

which, on changing the names of independent variables Φ, Φ_1 and their functions t, t_1, anew by $\Psi_1 = 1-\Phi$, $\Psi = 1-\Phi_1$ and T_1, T, respectively, reduces to

$$\int_0^1 T^q (1-\Psi)^{n-k} d\Psi \int_0^{\Psi} T_1^p \Psi_1^{i-1} (\Psi - \Psi_1)^{k-i-1} d\Psi_1 ,$$

The last integral, however, just equals that of (5. 2), because the definite integral is quite immaterial to the letters of integration variables.

Also, we can compute the value of

$$E_{pq} \equiv \sum_{k=2}^{n} \sum_{i=1}^{k-1} E(t^p_{i|n} t^q_{k|n}) \qquad \text{for} \quad p, q = 1, 2, \cdots , \tag{5. 3}$$

which is useful for purpose to check calculations of cross moments of order $p+q$. In fact we obtain by means of (4. 1)

$$\begin{aligned} E_{pq} &= \sum_{k=2}^{n} \sum_{i=1}^{k-1} \frac{n!}{(n-k)!\,(i-1)!\,(k-i-1)!} \int t^q (1-\Phi)^{n-k} \varphi\, dt \int^t t_1^p \Phi_1^{i-1} (\Phi - \Phi_1)^{k-i-1} \varphi_1\, dt_1 \\ &= n(n-1) \int \sum_{k=2}^{n} \binom{n-2}{n-k} t^q (1-\Phi)^{n-k} \varphi\, dt \int^t \sum_{i=1}^{k-1} \binom{k-2}{i-1} t_1^p \Phi_1^{i-1} (\Phi - \Phi_1)^{k-i-1} \varphi_1\, dt_1 \\ &= n(n-1) \int \sum_{k=2}^{n} \binom{n-2}{k-2} t^q (1-\Phi)^{n-k} \Phi^{k-2} \varphi\, dt \int^t t_1^p \varphi_1\, dt_1 \\ &= n(n-1) \int t^q \varphi\, dt \int^t t_1^p \varphi_1\, dt_1 . \end{aligned}$$

To evaluate this, we transform (t_1, t) into polar co-ordinates (r, θ):

$$E_{pq}=\frac{n(n-1)}{2\pi}\int_{\pi/4}^{5\pi/4}\sin^q\theta\cos^p\theta\,d\theta\int_0^\infty \exp\left\{-\frac{1}{2}r^2\right\}r^{p+q+1}dr$$
$$=\frac{n(n-1)}{2\pi}I_{pq}\int_0^\infty (2u)^{\frac{p+q}{2}}e^{-u}du\quad(2u=r^2)=\frac{n(n-1)}{2\pi}2^{\frac{p+q}{2}}\Gamma\left(\frac{p+q}{2}+1\right)I_{pq}$$

where $I_{pq}=\int_{\pi/4}^{5\pi/4}\sin^q\theta\cos^p\theta\,d\theta$ and their values as well as the required sum of expectations are tabulated in the following:

p	1	2	1	2	3	1
q	1	1	2	2	1	3
I_{pq}	0	$\frac{1}{3\sqrt{2}}$	$-\frac{1}{3\sqrt{2}}$	$\frac{\pi}{8}$	0	0
$\frac{E}{n(n-1)}$	0	$\frac{1}{4\sqrt{\pi}}$	$-\frac{1}{4\sqrt{\pi}}$	$\frac{1}{2}$	0	0

More generally we have the following results:

(i) When p and q both odd, $E_{pq}=0$.

(ii) When one odd and the other even, e.g. let $p=2r$ and $q=2s+1$, then

$$E_{pq}=\frac{n(n-1)}{2\pi}2^{r+s+1/2}\Gamma\left(r+s+\frac{3}{2}\right)I_{pq},$$

where

$$I_{pq}=2\int_0^{1/\sqrt{2}}u^{2r}(1-u^2)^s\,du=\sqrt{2}\sum_{\nu=0}^{s}(-1)^\nu\binom{\nu}{s}\frac{1}{(2r+2s+1)\,2^{r+s}}>0,$$

whereas the value $E_{qp}=-E_{pq}$.

(iii) When p and q both even, the formulaes become somewhat intricate. For simplicity, e.g. if we assume that $p=q=2r$, we get

$$E_{pp}=\frac{n(n-1)}{2}\left(\frac{(2r)!}{2^r r!}\right)^2.$$

Although for the present the above fragments suffice, with the purpose of later reference to higher moments, let us consider some still further general cases:

When the number of related arguments are 3, we obtain, similarly as in (5.3),

$$\sum_{k=3}^{n}\sum_{j=2}^{k-1}\sum_{i=1}^{j=1}E(t^p_{i/n}t^q_{j/n}t^r_{k/n})\qquad(1\leq i<j<k\leq n,\ n\geq 3)\tag{5.4}$$
$$=n!\sum\sum\sum\int\frac{(1-\Phi_k)^{n-k}}{(n-k)!}\varphi_k t_k^r dt_k\int^{t_k}\frac{(\Phi_k-\Phi_i)^{k-j-1}}{(k-j-1)!}\varphi_j t_j^q dt_j$$
$$\times\int^{t_j}\frac{(\Phi_j-\Phi_i)^{j-i-1}\Phi_i^{i-1}}{(j-i-1)!\,(i-1)!}\varphi_i t_i^p dt_i$$
$$=\frac{n(n-1)(n-2)}{\sqrt{2\pi}^3}\int t_k^r\varphi_k dt_k\int^{t_k}t_j^q\varphi_j dt_j\int^{t_j}t_i^p\varphi_i dt_i,$$

which vanishes, if all p, q, r be odd, and in particular

$$\sum\sum\sum E(t_it_jt_k)=0.\tag{5.5}$$

Also

$$(5.6)\quad \sum\sum\sum E(t_i^2t_jt_k)=\sum\sum\sum E(t_it_jt_k^2)=-\frac{1}{2}\sum\sum\sum E(t_it_j^2t_k)=\frac{n(n-1)(n-2)}{12\pi\sqrt{3}},$$

and consequently

$$(5.7)\quad \sum\sum\sum \{E(t_i^2t_jt_k)+E(t_it_j^2t_k)+E(t_it_jt_k^2)\}=0.$$

With 4 arguments, we have

$$(5.8)\quad \sum_{l=4}^{n}\sum_{k=3}^{l-1}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}E(t_i^pt_j^qt_k^rt_l^s)\qquad (1\leqq i<j<k<l\leqq n,\ n\geqq 4)$$

$$=n!\int\frac{(1-\Phi)^{n-l}}{(n-l)!}\varphi_lt_l^sdt_l\int^{t_l}\frac{(\Phi_l-\Phi_k)^{l-k-1}}{(l-k-1)!}\varphi_kt_k^qdt_k$$

$$\times\int^{t_k}\frac{(\Phi_k-\Phi_j)^{k-j-1}}{(k-j-1)!}\varphi_jt_j^qdt_j\int^{t_j}\frac{(\Phi_j-\Phi_i)^{j-i-1}}{(j-i-1)!}\frac{\Phi_i^{i-1}}{(i-1)!}\varphi_it_i^pdt_i$$

$$=n(n-1)(n-2)(n-3)\int t_l^s\varphi_ldt_l\int^{t_l}t_k^r\varphi_kdt_k\int^{t_k}t_j^q\varphi_jdt_j\int^{t_j}t_i^2\varphi_idt_i,$$

and particularly

$$(5.9)\quad \sum\sum\sum\sum E(t_it_jt_kt_l)=0.$$

No further values of $p, q, r\cdots$ are needed if we confine ourselves up to moments of order 3 and 4.

Returning to the case with 2 arguments, we have a remarkable identity

$$(5.10)\quad \sum_{k=1}^{n}E(t_{i|n}t_{k|n})=1,\qquad (i=1, 2, \cdots, n \text{ being fixed})$$

which is very useful, as check, when all $E(t_{i|n}t_{k|n})$ are computed.

To prove (5.10), let us put

$$\sum_{k=1}^{n}=\sum_{k=1}^{i-1}+E(t_{i|n}^2)+\sum_{k=i+1}^{n}=(\text{i})+(\text{ii})+(\text{iii}).$$

First by (4.2)

$$(\text{i})=\sum_{k=1}^{i-1}\frac{n!}{(n-i)!(k-1)!(i-k-1)!}\int(1-\Phi)^{n-i}\varphi'dt\int^{t}\Phi_1^{k-1}(\Phi-\Phi_1)^{i-k-1}\varphi_1'dt_1$$

$$=\frac{n!}{(n-i)!(i-2)!}\int(1-\Phi)^{n-i}\varphi'dt\int^{t}\sum_{k=1}^{i-1}\frac{(i-2)!}{(k-1)!(i-k-1)!}\Phi_1^{k-1}(\Phi-\Phi_1)^{i-k-1}\varphi_1'dt_1$$

$$=\frac{n!}{(n-i)!(i-2)!}\int(1-\Phi)^{n-i}\varphi'dt\int^{t}\Phi^{i-2}\varphi_1'dt_1$$

$$=\frac{n!}{(n-i)!(i-2)!}\int(1-\Phi)^{n-i}\Phi^{i-2}\varphi\varphi'dt.$$

Next by (2.5)

$$(\text{ii})=E(t_{i|n}^2)=1-\frac{n!}{(n-i)!(i-1)!}\int\frac{d}{d\Phi}[\Phi^{i-1}(1-\Phi)^{n-i}]\varphi\varphi'dt.$$

And lastly again by (4.2)

$$(\text{iii})=\sum_{k=i+1}^{n}\int\frac{n!}{(n-k)!(i-1)!(k-i-1)!}\int(1-\Phi)^{n-k}\varphi'dt\int^{t}\Phi_1^{i-1}(\Phi-\Phi_1)^{k-i-1}\varphi_1'dt_1,$$

which, on interchanging the order of integrations, and rearranging factors, equals

$$\frac{n!}{(n-i-1)!\,(i-1)!}\int \Phi_1^{i-1}\varphi_1' dt_1 \int_{t_1} \sum_{k=i+1}^{n} \frac{(n-i-1)!}{(n-k)!(k-i-1)!}(1-\Phi)^{n-k}(\Phi-\Phi_1)^{k-i-1}\varphi' dt$$
$$=\frac{n!}{(n-i-1)!\,(i-1)!}\int \Phi^{i-1}(1-\Phi)^{n-i-1}(-\varphi\varphi')dt\,.$$

Therefore

$$\sum_{k=1}^{n} E(t_{i|n}t_{k|n}) = 1+\int \frac{n!}{(n-i)!\,i!}\int\Big[i(i-1)\Phi^{i-2}(1-\Phi)^{n-i}-i(n-i)\Phi^{i-1}(1-\Phi)^{n-i-1}$$
$$-i\frac{d}{d\Phi}\left[\Phi^{i-1}(1-\Phi)^{n-i}\right]\Big]\varphi\varphi' dt\,,$$

in which the integrand reduces to zero, and consequently the whole expression reduces to unity, Q. E. D.

By virtue of (3. 2·1) we have only to calculate $E(t_{i|n})$ and $E(t_{i|n}^2)$ for each set

$$A: t_1 < t_2 < \cdots < t_{\left[\frac{n+1}{2}\right]}, \quad \text{or} \quad B: t_n > t_{n-1}, > \cdots > t_{\left[\frac{n+1}{2}\right]}.$$

Of course

$$|E(t_1)| = E(t_n) > |E(t_2)| = E(t_{n-1}) > \cdots, \text{ and } E(t_1^2) = E(t_n^2) > E(t_2^2) = E(t_{n-1}^2) > \cdots.$$

The mean $E(t_i t_k)$ is positive if $i<k$ both belong to A only or B only, and by (5. 1) it suffices to calculate about A only. On the other hand, if i and k be taken each from A and B, respectively, then $E(t_i t_k)$ is negative. And indeed $E(t_i t_k)$ has the larger absolute value, the farther they lie from center.

§6. The Frequency Function of $\zeta = \sum c_i t_i$ &c.

So far we have discussed about $E(t_{i|n})$, $E(t_{i|n}^2)$ and $E(t_{i|n}t_{k|n})$. Thereby all the following, which concern with $x_i = m + \sigma t_i$, could be computed:

$$E(x_i) = m+\sigma E(t_i)\,,\quad E(x_i^2) = m^2+2m\sigma E(t_i)+\sigma^2 E(t_i^2)\,,\quad D^2(x_i) = E(x_i^2)-E(x_i)^2\,,$$
$$E(x_i x_k) = m^2 + m\sigma\left[E(t_i)+E(t_k)\right]+\sigma^2 E(t_i t_k)\,.$$

Further, putting $z=\sum_{i=1}^{n} c_i x_i$ and $\zeta=\sum_{i=1}^{n} c_i t_i$ under $\sum c_i = 1$ with $c_i \geqq 0$, all the following also can be carried out:

$$z=\sum c_i(m+\sigma t_i) = m+\sigma\zeta\,,\ E(z) = m+\sigma E(\zeta)\,,\ E(z^2) = m^2+2m\sigma E(\zeta)+\sigma^2 E(\zeta^2)\,,$$
$$E(\zeta) = \sum c_i E(t_i)\,,\quad E(\zeta^2) = \sum c_i^2 E(t_i^2) + 2\sum_{j<k} c_j c_k E(t_j t_k)\,,$$

and finally

$$D^2(\zeta) = E(\zeta^2)-E(\zeta)^2 = \sum c_i^2\left[E(t_i^2)-E(t_i)^2\right]+2\sum_{j<k} c_j c_k\left[E(t_j t_k)-E(t_j)E(t_k)\right],$$
$$D^2(z) = \sigma^2 D^2(\zeta) = \sigma^2\sum c_i^2 \operatorname{Var}(t_i) + 2\sigma^2\sum_{j<k} c_j c_k \operatorname{Cov}(t_j,\,t_k)\,.$$

However, before discussing $D^2(z)$, as Cramér advises[8], it is desirable first to find the fr. f. $g(z, m)$. Let us consider his example and find the fr. f.

8) Cramér, loc. cit., p. 483, l. 8.

$g(\zeta, 0)$ of $\zeta = ct_1 + (1-2c)t_2 + ct_3$. Here the probability element is

(6.1) $$dP = 6d\Phi_3 d\Phi_2 d\Phi_1 = \frac{6}{\sqrt{2\pi^3}} \exp\left\{-\frac{1}{2}(t_1^2+t_2^2+t_3^2)\right\} dt_1 dt_2 dt_3, \quad (t_1 \leqq t_2 \leqq t_3).$$

Now transforming $\{t_1, t_2, t_3\}$ into $\{y_1, y_2, y_3\}$ orthogonally, as e.g.

$$y_1 = -\frac{1}{\sqrt{2}} t_1 + \frac{1}{\sqrt{2}} t_3,$$
$$y_2 = \frac{(1-2c)t_1 - 2ct_2 + (1-2c)t_3}{\sqrt{2}\,\gamma},$$
$$y_3 = \frac{ct_1 + (1-2c)t_2 + ct_3}{\gamma} \left(= \frac{\zeta}{\gamma}\right),$$

with $\gamma = \sqrt{2c^2 + (1-2c)^2} = \sqrt{1-4c+6c^2} > 0$

and $J = \dfrac{\partial(y_1, y_2, y_3)}{\partial(t_1, t_2, t_3)} = 1$,

we get, on writing rows in columns,

$$t_1 = -\frac{1}{\sqrt{2}} y_1 + \frac{1-2c}{\sqrt{2}\,\gamma} y_2 + \frac{c}{\gamma} y_3,$$
$$t_2 = -\frac{2c}{\sqrt{2}\,\gamma} y_2 + \frac{1-2c}{\gamma} y_3,$$
$$t_3 = \frac{1}{\sqrt{2}} y_1 + \frac{1-2c}{\sqrt{2}\,\gamma} y_2 + \frac{c}{\gamma} y_3,$$

and, because of orthogonal transformation,

$$t_1^2 + t_2^2 + t_3^2 = y_1^2 + y_2^2 + y_3^2.$$

The order $t_1 \leqq t_2 \leqq t_3$ yields

$$\frac{\sqrt{2}(1-3c)\zeta}{\gamma} - \gamma y_1 (= A(y_1, \zeta)) \leqq y_2 \leqq \frac{\sqrt{2}(1-3c)\zeta}{\gamma} + \gamma y_1 (= B(y_1, \zeta) \text{say}),$$

and $$y_1 \geqq 0.$$

With this order we obtain

(6.2) $$dP = \frac{6}{\sqrt{2\pi^3}} \exp\left\{-\frac{1}{2}(y_1^2+y_2^2+y_3^2)\right\} dy_1 dy_2 dy_3, \quad \left(y_3 = \frac{\zeta}{\gamma}\right),$$

and whence the required fr. f. to be

(6.3) $$g(\zeta) = \frac{3}{\pi\sqrt{2\pi}\,\gamma} \exp\left\{-\frac{\zeta^2}{2\gamma^2}\right\} \int_0^\infty \exp\left\{-\frac{1}{2}y_1^2\right\} dy_1 \int_{A(y_1, \zeta)}^{B(y_1, \zeta)} \exp\left\{-\frac{1}{2}y_2^2\right\} dy_2.$$

The last double integral yields, on transforming integration-variables (y_1, y_2) into polar co-ordinates (r, θ)

(6.4) $$\iint_D \exp\left\{-\frac{1}{2}r^2\right\} r\,dr\,d\theta$$

in which D denotes the domain bounded by the two half straight lines KL, KH (Fig. 1, 2 or 3 according as the y_2-intercept $b \gtrless 0$ or zero), whose equations are $y_2 = A(y_1, \zeta)$ and $y_2 = B(y_1, \zeta)$, namely

$$y_2 = \mp\gamma y_1 + \sqrt{2}(1-3c)\zeta/\gamma = y_1 \tan(\mp\alpha) + b,$$

where $\alpha = \text{Tan}^{-1}\gamma$ denotes a positive accute angle and $b = \sqrt{2}(1-3c)\zeta/\gamma$. Or, their Hessian equations are

$$y_1 \cos\varphi + y_2 \sin\varphi = p,$$

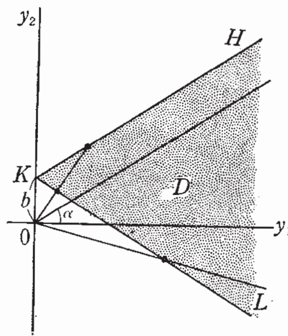


Fig. 1

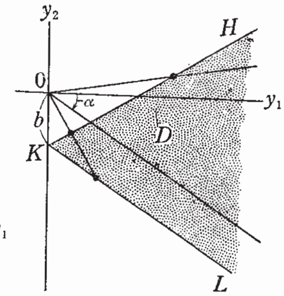


Fig. 2

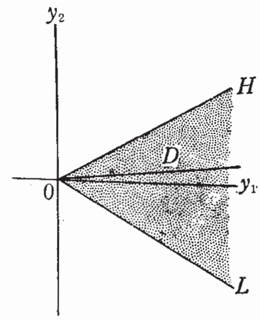


Fig. 3

where

$$p=\frac{|b|}{\sqrt{1+\gamma^2}}=\frac{|\sqrt{2}(1-3c)\zeta|}{\gamma\sqrt{1+\gamma^2}}=\text{the perpendicular distance from origin} \geqq 0\,,$$

$$\varphi=\text{angle the perpendicular makes with } +y_1\text{-axis,}$$

so that, in detail, if $\beta=\frac{\pi}{2}-\alpha$ (positive accute angle), 1° in Fig. 1 $(b>0)$ $\varphi=\frac{\pi}{2}-\alpha=\beta$ for KL and 2° $\varphi=\frac{\pi}{2}+\alpha=\pi-\beta$ for KH, while 3° in Fig. 2 $(b<0)$, $\varphi=-\left(\frac{\pi}{2}-\alpha\right)=-\beta$ for KH, and 4° $\varphi=-\left(\frac{\pi}{2}+\alpha\right)=-(\pi-\beta)$ for KL.

We have, therefore, in all cases as their polar equations

$$R=p\sec(\theta-\varphi)\,.$$

Again, in detail, 1° $R_1=p\sec(\theta-\beta)$, 2° $R_2=-p\sec(\theta+\beta)$, 3° $R_2=p\sec(\theta+\beta)$, 4° $R_1=-p\sec(\theta-\beta)$. Hence we have

Case I. For Fig. 1 $(b=\sqrt{2}(1-3c)\zeta/\gamma>0)$ (6.4) yields

$$\iint_D=\int_{-\alpha}^{\alpha}d\theta\int_{R_1}^{\infty}e^{-r^2/2}r\,dr+\int_{\alpha}^{\pi/2}d\theta\int_{R_1}^{R_2}e^{-r^2/2}r\,dr=\int_{-\alpha}^{\alpha}e^{-R_1^2/2}d\theta+\int_{\alpha}^{\pi/2}\left[e^{-R_1^2/2}-e^{-R_2^2/2}\right]d\theta$$

$$=\int_{-\alpha}^{\pi/2}e^{-R_1^2/2}d\theta-\int_{\alpha}^{\pi/2}e^{-R_2^2/2}d\theta$$

$$=\int_{-\alpha}^{\pi/2}\exp\left\{-\frac{p^2}{2}\sec^2(\theta-\beta)\right\}d\theta-\int_{\alpha}^{\pi/2}\exp\left\{-\frac{p^2}{2}\sec(\theta+\beta)\right\}d\theta=J\text{ say}\,.$$

Or, upon putting $\theta-\beta=u$ and $\theta+\beta=u+\pi$, respectively, and $p^2=\frac{2(1-3c)^2}{\gamma^2(1+\gamma^2)}\zeta^2=N\zeta^2$, we have, in view of $\alpha+\beta=\pi/2$,

$$J=\int_{-\pi/2}^{\alpha}\exp\left\{-\frac{N}{2}\zeta^2\sec^2 u\right\}du-\int_{-\pi/2}^{-\alpha}\exp\left\{-\frac{N}{2}\zeta^2\sec^2 u\right\}du$$

$$=\int_{-\alpha}^{\alpha}=2\int_0^{\alpha}\exp\left\{-\frac{N}{2}\zeta^2\sec^2 u\right\}du\,.$$

Case II. For Fig. 2 $(b=\sqrt{2}(1-3c)\zeta/\gamma<0)$

$$\iint_D=\int_{-\pi/2}^{-\alpha}d\theta\int_{R_2}^{R_1}e^{-r^2/2}r\,dr+\int_{-\alpha}^{\alpha}d\theta\int_{R_2}^{\infty}e^{-r^2/2}r\,dr$$

$$=\int_{-\pi/2}^{-\alpha}\left[e^{-R_2^2/2}-e^{-R_1^2/2}\right]d\theta\quad+\int_{-\alpha}^{\alpha}e^{-R_2^2/2}d\theta=\int_{-\pi/2}^{\alpha}e^{-R_2^2/2}d\theta-\int_{-\pi/2}^{-\alpha}e^{-R_1^2/2}d\theta$$

$$= \int_{-\pi/2}^{\alpha} \exp\left\{-\frac{p^2}{2}\sec^2(\theta+\beta)\right\} d\theta - \int_{-\pi/2}^{-\alpha} \exp\left\{-\frac{p^2}{2}\sec^2(\theta-\beta)\right\} d\theta$$

$$= \int_{-\alpha}^{\pi/2} \exp\left\{-\frac{p^2}{2}\sec^2 u\right\} du - \int_{\alpha}^{\pi/2} \exp\left\{-\frac{p^2}{2}\sec^2 u\right\} du = J.$$

Case. III. For Fig. 3, $b=\sqrt{2}(1-3c)\zeta/\gamma=0$. In this case the two straight lines OL, OH reduce to $\theta=\mp\alpha$, so that

$$\iint_D = \int_{-\alpha}^{\alpha} d\theta \int_0^{\infty} e^{-r^2/2}\, r\, dr = 2\alpha.$$

But, when $c \neq \frac{1}{3}$, $p\to 0$, as $\zeta\to 0$ and $\lim_{\zeta\to 0} J=2\alpha$, so that Case III may be attached to Case I or II by deeming its open integration-interval as closed: $b\geqq 0$ or $b\leqq 0$. However, when $c=\frac{1}{3}$, we should consider this case as a special one: Really, in this trivial case, $\alpha=\tan^{-1}\gamma=\tan^{-1}\frac{1}{\sqrt{3}}$, so that the required fr. f. reduces to

$$\text{(6.5)} \qquad g(\zeta)=\frac{1}{\sqrt{2\pi/\sqrt{3}}}\exp\left\{-\frac{\zeta^2}{2/3}\right\}, \qquad \text{namely } N\left(0, \frac{1}{3}\right)$$

which is the fr. f. of the A. M.

Returning to the general case that $c\neq\frac{1}{3}$, $\zeta \gtreqless 0$, we have to contemplate

$$\text{(6.6)} \qquad g(\zeta)=\frac{3}{\pi\gamma}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{\zeta^2}{2\gamma^2}\right\}\cdot J=\frac{6}{\pi\gamma\sqrt{2\pi}}\exp\left\{-\frac{\zeta^2}{2\gamma^2}\right\}$$

$$\times\int_0^{\alpha}\exp\left\{-\frac{N}{2}\zeta^2\sec^2 u\right\}du=\frac{6}{\pi\gamma\sqrt{2\pi}}\int_0^{\alpha}\exp\left\{-\frac{\zeta^2}{2}(M+N\sec^2 u)\right\}du,$$

where

$$M=\frac{1}{\gamma^2}, \quad N=\frac{2(1-3c)^2}{\gamma^2(1+\gamma^2)}.$$

Thus $g(\zeta)$ is an even function, and accordingly

$$\text{(6.7)} \qquad E(\zeta^{2p+1})=\int\zeta^{2p+1}g(\zeta)d\zeta=0.$$

Further

$$E(\zeta^2)=\frac{6}{\pi\gamma\sqrt{2\pi}}\int\zeta^2 g(\zeta)d\zeta=\frac{6}{\pi\gamma}\int_0^{\alpha}\Big[M+N\sec^2 u\Big]^{-\frac{3}{2}}du=\frac{6}{\pi\gamma}\int_0^{\alpha}\frac{\cos^3 u\,du}{\sqrt{M\cos^2 u+N}^3}.$$

On computing the corresponding indefinite integral, we obtain

$$\int\frac{\cos^3 u\,du}{\sqrt{M\cos^2 u+N}^3}=\frac{1}{M}\int\frac{\cos u\,du}{\sqrt{M\cos^2 u+N}}-\frac{N}{M}\int\frac{\cos u\,du}{\sqrt{M\cos^2 u+N}^3}$$

$$=\frac{1}{M}\int\frac{\cos u\,du}{\sqrt{M+N-M\sin^2 u}}-\frac{N}{M}\int\frac{\cos u\,du}{\sqrt{M+N-M\sin^2 u}^3}$$

$$=\frac{1}{M^{\frac{3}{2}}}\sin^{-1}\left(\sqrt{\frac{M}{M+N}}\sin u\right)-\frac{N}{M(M+N)}\frac{\sin u}{\sqrt{M+N-M\sin^2 u}}.$$

Substituting this result in the foregoing definite integral, we get

$$(6.8)\qquad \mu_2 = E(\zeta^2) = \frac{6}{\pi\gamma}\left[\gamma^3 \sin^{-1}\frac{1}{2} - \frac{(3\gamma^2-1)\gamma}{4\sqrt{3}}\right] = \gamma^2 - \frac{(3\gamma^2-1)\sqrt{3}}{2\pi}$$
$$= \frac{1}{3} + \frac{3}{\pi}[2\pi - 3\sqrt{3}]\left(c - \frac{1}{3}\right)^2 \geqq \frac{1}{3}.$$

The above reduction is too much lengthy. We have already seen in § 3 that $E(\zeta)=0$, while $\mu_2 = D^2(\zeta)$ will later be readily obtained in § 11 by aid of $E(t_{i|3}t_{k'3})$. Notwithstanding we dared to deduce the latter from the fr. f. $g(\zeta)$ at the cost of a duplication, which, however, is intended to show that $\mu_4 = E(\zeta^4)$ can be likewise computed. In fact

$$\mu_4 = E(\zeta^4) = \int \zeta^4 g(\zeta)d\zeta = \frac{6}{\pi\gamma}\int_0^\alpha du \int \zeta^4 \exp\left\{-\frac{\zeta^2}{2}(M+N\sec^2 u)\right\}\frac{d\zeta}{\sqrt{2\pi}}$$
$$= \frac{18}{\pi\gamma}\int_0^\alpha \frac{\cos^5 u\,du}{\sqrt{M\cos^2 u + N}^5} = \frac{18\gamma^4}{\pi}\int_0^{\sin\alpha}\frac{(1-v^2)^2 dv}{\sqrt{L^2-v^2}^5},$$

where $v=\sin u$ and $L=\sqrt{\dfrac{M+N}{M}} = \dfrac{2\gamma}{\sqrt{1+\gamma^2}}$. The corresponding indefinite integral is

$$\int \frac{(v^4-2v^2+1)dv}{(L^2-v^2)^2\sqrt{L^2-v^2}} = \int\frac{dv}{\sqrt{L^2-v^2}} - 2(L^2-1)\int\frac{dv}{\sqrt{L^2-v^2}^3} + (L^2-1)^2\int\frac{dv}{\sqrt{L^2-v^2}^5}$$
$$= \sin^{-1}\frac{v}{L} + \frac{2(1-L^2)v}{L^2\sqrt{L^2-v^2}} + \frac{(L^2-1)^2}{L^4}\left[\frac{2v}{3\sqrt{L^2-v^2}} + \frac{L^2 v}{3\sqrt{L^2-v^2}^3}\right],$$

in which, substituted $v = \sin\alpha = \dfrac{\gamma}{\sqrt{1+\gamma^2}}$ and $L = \dfrac{2\gamma}{\sqrt{1+\gamma^2}}$, yields

$$(6.9)\qquad \mu_4 = 3\gamma^4 + \frac{(1-3\gamma^2)(5+21\gamma^2)}{4\sqrt{3}\,\pi}.$$

We have seen that the coefficient of skewness is $\dfrac{\mu_3}{\sqrt{\mu_2}^3}=0$, but now the coefficient of excess

$$(6.10)\qquad \frac{\mu^4}{\mu_2^2} - 3 = \frac{3\gamma^4 + (1-3\gamma^2)(5+21\gamma^2)/4\sqrt{3}\,\pi}{[\gamma^2+(1-3\gamma^2)\sqrt{3}/2\pi]^2} - 3$$
$$= \frac{(1-3\gamma^2)^2}{4\sqrt{3}\,\pi^2}\left(5\pi - 9\sqrt{3}\right)\Big/\left[\gamma^2 + (1-3\gamma^2)\frac{\sqrt{3}}{2\pi}\right]^2 > 0, \text{ if } \gamma^2 \neq \frac{1}{3}.$$

Therefore, $g(\zeta)$ cannot indeed be normal, unless $\gamma^2 = 1-4c+6c^2 = \dfrac{1}{3}$, i.e. $c = \dfrac{1}{3}$.

In general, the fr. f. of $\zeta = \sum_{i=1}^n c_i t_i$ with $\sum c_i = 1$, might be found from the probability element

$$dP = n!\,d\Phi_1 d\Phi_2 \cdots d\Phi_n = \frac{n!}{\sqrt{2\pi}^n}\exp\left\{-\frac{1}{2}\sum_{i=1}^n t_i^2\right\}dt_1 dt_2 \cdots dt_n$$

by a similar way, as shown for the case of A.M. in Cramér's treatise[9], or certain Fisher-like geometrical considerations, but certainly not without some ingenious devices. Probably it will be expressed as a cumbrous definite multiple integral. So here we content ourselves simply with the reconsideration about the well known case of A. M.

As usual, we perform an orthogonal transformation of variables $(t_1, t_2, \cdots, t_n)$ into $(y_1, y_2, \cdots, y_n)$. To fix idea, let it be e.g.

$$y_1 = -\frac{t_1}{\sqrt{1.2}} + \frac{t_2}{\sqrt{1.2}} \quad (\geqq 0),$$

$$y_2 = -\frac{t_1}{\sqrt{2.3}} - \frac{t_2}{\sqrt{2.3}} + \frac{2}{\sqrt{2.3}} t_3,$$

....

$$y_i = -\frac{t_1}{\sqrt{i(i+1)}} - \frac{t_2}{\sqrt{i(i+1)}} - \frac{t_3}{\sqrt{i(i+1)}} - \cdots - \frac{t_i}{\sqrt{i(i+1)}} + \frac{i}{\sqrt{i(i+1)}} t_{i+1},$$

....

$$y_{n-1} = -\frac{t_1}{\sqrt{(n-1)n}} - \frac{t_2}{\sqrt{(n-1)n}} - \frac{t_3}{\sqrt{(n-1)n}} - \cdots - \frac{t_i}{\sqrt{(n-1)n}} - \cdots\cdots - \frac{t_{n-1}}{\sqrt{(n-1)n}} + \frac{n-1}{\sqrt{(n-1)n}} t_n,$$

$$y_n = \frac{t_1}{\sqrt{n}} + \frac{t_2}{\sqrt{n}} + \frac{t_3}{\sqrt{n}} + \cdots + \frac{t_i}{\sqrt{n}} + \cdots\cdots + \frac{t_{n-1}}{\sqrt{n}} + \frac{t_n}{\sqrt{n}}.$$

where Jacobian J becomes $(-1)^{n-1}$, so that $|J|=1$[10]. On taking coefficients of every column in the foregoing substitutions as those of successive rows, we may write

$$t_1 = -\frac{y_1}{\sqrt{1.2}} - \frac{y_2}{\sqrt{2.3}} - \frac{y_3}{\sqrt{3.4}} - \cdots - \frac{y_i}{\sqrt{i(i+1)}} - \cdots\cdots - \frac{y_{n-2}}{\sqrt{(n-1)(n-2)}} - \frac{y_{n-1}}{\sqrt{n(n-1)}} + \frac{y_n}{\sqrt{n}},$$

$$t_2 = +\frac{y_1}{\sqrt{1.2}} - \frac{y_2}{\sqrt{2.3}} - \frac{y_3}{\sqrt{3.4}} - \cdots - \frac{y_i}{\sqrt{i(i+1)}} - \cdots\cdots - \frac{y_{n-2}}{\sqrt{(n-1)(n-2)}} - \frac{y_{n-1}}{\sqrt{n(n-1)}} + \frac{y_n}{\sqrt{n}},$$

$$t_3 = +\frac{2}{\sqrt{2.3}} y_2 - \frac{y_3}{\sqrt{3.4}} - \cdots - \frac{y_i}{\sqrt{i(i+1)}} - \cdots\cdots - \frac{y_{n-2}}{\sqrt{(n-1)(n-2)}} - \frac{y_{n-1}}{\sqrt{n(n-1)}} + \frac{y_n}{\sqrt{n}},$$

....

$$t_{i+1} = +\frac{i}{\sqrt{i(i+1)}} y_i - \frac{y_{i+1}}{\sqrt{(i+1)(i+2)}} - \cdots - \frac{y_{n-2}}{\sqrt{(n-1)(n-2)}} - \frac{y_{n-1}}{\sqrt{n(n-1)}} + \frac{y_n}{\sqrt{n}},$$

....

$$t_{n-1} = +\frac{n-2}{\sqrt{(n-1)(n-2)}} y_{n-2} - \frac{y_{n-1}}{\sqrt{n(n-1)}} + \frac{y_n}{\sqrt{n}},$$

$$t_n = +\frac{n-1}{\sqrt{n(n-1)}} y_{n-1} + \frac{y_n}{\sqrt{n}}.$$

But, as $t_1 \leqq t_2 \leqq \cdots \leqq t_n$,

$$t_{i+1} - t_i = \sqrt{\frac{i+1}{i}}\, y_i - \sqrt{\frac{i-1}{i}}\, y_{i-1} \geqq 0,$$

9) Cramér, loc. cit., p. 383.

10) Consequently our transformation is properly or improperly orthogonal, according as n is odd or even.

therefore $$y_i \geqq \sqrt{\frac{i-1}{i+1}}\, y_{i-1}\,, \qquad (i=1, 2, \cdots, n-1)\,.$$

By these $n-1$ inequalities every $y_1, y_2, \cdots, y_{n-1}$ is positive and bounded downwards, however $y_n = \sqrt{n}\,\zeta \gtreqless 0$, and unbounded. Also, as consequence of orthogonal transformation, it follows that

$$\sum_{i=1}^{n} t_i^2 = \sum_{i=1}^{n} y_i^2\,,$$

and the probability element now becomes

$$dP = \frac{n!}{\sqrt{2\pi}^{\,n}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} t_i^2\right\} dt_1 dt_2 \cdots dt_n$$
$$= \frac{n!}{\sqrt{2\pi}^{\,n}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} y_i^2\right\}\cdot dy_1 dy_2 \cdots dy_n\,, \qquad (y_n = \sqrt{n}\,\zeta)\,.$$

Therefore the fr. f. $g(\zeta)$ is given by

(6.11) $$g(\zeta)d\zeta = \frac{n!}{\sqrt{2\pi}^{\,n}} \exp\left\{-\frac{n}{2}\zeta^2\right\}\sqrt{n}\,d\zeta \int_0 \exp\{-y_1^2/2\}dy_1 \times$$
$$\times \int_{\sqrt{\frac{1}{3}}y_1} \exp\{-y_2^2/2\}dy_2 \cdots \int_{\sqrt{\frac{i-1}{i+1}}y_{i-1}} \exp\{-y_i^2/2\}dy_i \cdots \int_{\sqrt{\frac{n-2}{n}}y_{n-2}} \exp\{-y_{n-1}^2/2\}dy_{n-1}\,.$$

Here the $(n-1)$-ple integral being independent of all y's, it reduces to some constant C_n, and we have

(6.12) $$g(\zeta)d\zeta = \frac{n!}{\sqrt{2\pi}^{\,n}} C_n \exp\left\{-\frac{1}{2}n\zeta^2\right\}\cdot\sqrt{n}\,d\zeta\,.$$

But

(6.13) $$1 = \int g(\zeta)d\zeta = \frac{n!}{\sqrt{2\pi}^{\,n-1}} C_n\,.$$

Hence[11]

(6.14) $$C_n = \frac{\sqrt{2\pi}^{\,n-1}}{n!}$$

and we obtain finally

(6.15) $$g(\zeta) = \frac{1}{\sqrt{2\pi\Big/\dfrac{1}{n}}} \exp\left\{-\zeta^2\Big/\frac{2}{n}\right\}, \text{ so } g(z) = \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left\{-\frac{1}{2}(z-m)^2\Big/\frac{\sigma^2}{n}\right\};$$

i.e. the theorem: The A.M. $z = \frac{1}{n}\sum x_i$ distributes normally with mean m and variance σ^2/n, which was the case when $x_1, x_2, \cdots, x_n$ are unordered and independent of each other.

---

11) Really $C_2 = \int_0 e^{-y_1^2/2}dy_1 = \frac{\sqrt{2\pi}}{2}$, $C_3 = \int_0 e^{-y_1^2/2}\int_{\sqrt{y_1/3}} e^{-y_2^2/2}dy_2 = \int_{\pi/6}^{\pi/2} d\theta \int_0^{\infty} e^{-r^2/2}\,r dr = \frac{2\pi}{3!}$, and so on. Since we have obtained (6.14) as a logical consequence from (6.11) (6.12) (6.13), we need not prove (6.14) expressly, e.g. by mathematical induction, though it might be considered as a rather difficult but superfluous exercise.

PART II

§7. Computation of $J_\lambda^{(\alpha)}$. Rather more conveniently we shall compute

$$(7.1)\quad K_\lambda^{(\alpha)}=\int U^\lambda \varphi^\alpha dt, \quad \text{where} \quad U=\int_0^t \varphi dt\,,\ \varphi=\frac{1}{\sqrt{2\pi}}e^{-t^2/2}, \quad (\alpha=2, 3, \cdots).$$

Since $U(t)$ is an odd function, it yields always

$$(7.2)\qquad K_{2p+1}^{(\alpha)}=0\,.$$

Hence, we have

$$(7.3)\quad J_\lambda^{(\alpha)}=\int \Phi^\lambda \varphi^\alpha dt=\int\left(\frac{1}{2}+U\right)^\lambda \varphi^\alpha dt=\sum_{\nu=1}^{p}\binom{\lambda}{2\nu}\left(\frac{1}{2}\right)^{\lambda-2\nu}K_{2\nu}^{(\alpha)}, \text{ where } p=\left[\frac{\lambda}{2}\right].$$

Thus, to obtain $J_\lambda^{(\alpha)}$ it suffices to know $K_{2\nu}^{(\alpha)}$ for $\nu=0, 1, 2, \cdots, p=\left[\frac{\lambda}{2}\right]$. We shall compute $K_\lambda^{(\alpha)}$ and $J_\lambda^{(\alpha)}$ successively.

1° For $\lambda=0$ evidently, if $\alpha>0$,

$$(7.4)\quad K_0^{(\alpha)}=J_0^{(\alpha)}=\int \varphi^\alpha dt=\frac{1}{\sqrt{2\pi^\alpha}}\int e^{-\frac{\alpha}{2}t^2}dt=\frac{1}{\sqrt{\alpha}\sqrt{2\pi^{\alpha-1}}}=c_\alpha, \quad \text{say.}$$

In particular

$$(7.4.1)\qquad c_2=K_0^{(2)}=J_0^{(2)}=\frac{1}{2\sqrt{\pi}} \quad \text{and} \quad c_3=K_0^{(3)}=J_0^{(3)}=\frac{1}{2\pi\sqrt{3}}.$$

2° For $\lambda=1$, $K_1^{(\alpha)}=0$. But

$$(7.5)\qquad J_1^{(\alpha)}=\int \Phi\varphi^\alpha dt=\int\left(U+\frac{1}{2}\right)\varphi^\alpha dt=\frac{1}{2}\int \varphi^\alpha dt=\frac{1}{2}c_\alpha\,.$$

In particular

$$(7.5.1)\qquad J_1^{(2)}=\frac{1}{4\sqrt{\pi}} \quad \text{and} \quad J_1^{(3)}=\frac{1}{4\pi\sqrt{3}}.$$

3° For $\lambda=2$, we observe, by means of polar co-ordinates, that

$$(7.6)\quad U(t)^2=\left[\int_0^t \varphi dt\right]^2=\frac{1}{2\pi}\int_0^t\int_0^t \exp\left\{-\frac{1}{2}(x^2+y^2)\right\}dxdy$$
$$=\frac{1}{\pi}\int_0^{\pi/4}d\theta\int_0^{t\sec\theta}\exp\left\{-\frac{1}{2}r^2\right\}rdr=\frac{1}{\pi}\int_0^{\pi/4}\left[1-\exp\left\{-\frac{1}{2}t^2\sec^2\theta\right\}\right]d\theta$$
$$=\frac{1}{4}-\frac{1}{\pi}\int_0^{\pi/4}\exp\left\{-\frac{1}{2}t^2\sec^2\theta\right\}d\theta\,.$$

Consequently

$$K_2^{(\alpha)}=\int U^2\varphi^\alpha dt=c_\alpha\left[\frac{1}{4}-\frac{\sqrt{\alpha}}{\pi}\int_0^{\pi/4}\frac{d\theta}{\sqrt{\alpha+\sec^2\theta}}\right].$$

Here, owing to the absolute convergence of the concerned integral, the order of integrations was interchanged. The alike would be tacitly and frequently applied below. Now the above last integration performed, we have

$$(7.7)\quad \int_0^{\pi/4}\frac{d\theta}{\sqrt{\alpha+\sec^2\theta}}=\frac{1}{\sqrt{\alpha}}\sin^{-1}\left\{\sqrt{\frac{\alpha}{1+\alpha}}\sin\theta\right\}\Big|_0^{\pi/4}=\frac{1}{\sqrt{\alpha}}\sin^{-1}\sqrt{\frac{\alpha}{2(1+\alpha)}}\,.$$

Since the function plays very important roles in the subsequent, we shall expressively denote it by

(7.8) $$S(\alpha)=\frac{\sqrt{\alpha}}{\pi}\int_0^{\pi/4}\frac{d\theta}{\sqrt{\alpha+\sec^2\theta}}=\frac{1}{\pi}\sin^{-1}\sqrt{\frac{\alpha}{2(1+\alpha)}}\,.$$

E. g.

(7.8.1) $$\begin{cases}\pi S(2)=\sin^{-1}\dfrac{1}{\sqrt{3}}=\cos^{-1}\sqrt{\dfrac{2}{3}}=\tan^{-1}\dfrac{1}{\sqrt{2}}\,,\\[2ex] \pi S(3)=\sin^{-1}\sqrt{\dfrac{3}{8}}=\cos^{-1}\sqrt{\dfrac{5}{8}}=\tan^{-1}\sqrt{\dfrac{3}{5}}\,.\end{cases}$$

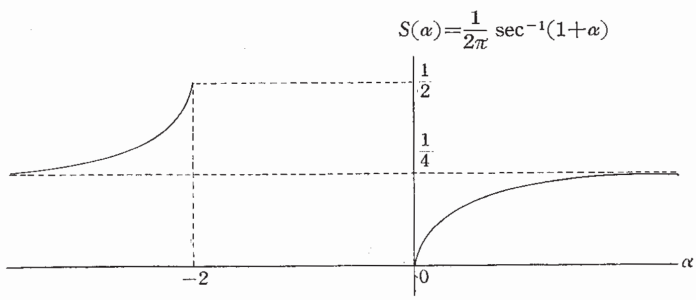
If $\xi=\sqrt{\dfrac{\alpha}{2(1+\alpha)}}$, $\alpha=\dfrac{2\xi^2}{1-2\xi^2}$ and $S(\alpha)=\dfrac{1}{\pi}\sin^{-1}\xi$, so that, e.g.

ξ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\xi>1$ or imag.	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{2}\sqrt{\frac{3}{2}}$ &c.
α	0	1	$\pm\infty$	-3	-2	$-2<\alpha<0$	$\frac{1}{2}$	2	3 &c.
$S(\alpha)$	0	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	imaginary	0.1339	0.1959	0.2098 &c.

In fact, our $S(\alpha)$ is nothing but

(7.8.2) $$S(\alpha)=\frac{1}{2\pi}\cos^{-1}\frac{1}{1+\alpha}=\frac{1}{2\pi}\sec^{-1}(1+\alpha)$$

and its graph is as shown in Fig. 4.



Eig. 4.

Returning to our text, we have

(7.9) $$K_2^{(\alpha)}=c_\alpha\left[\frac{1}{4}-\frac{1}{\pi}\sin^{-1}\sqrt{\frac{\alpha}{2(\alpha+1)}}\right]=c_\alpha\left[\frac{1}{4}-S(\alpha)\right],$$

(7.9.1) $$K_2^{(2)}=\frac{1}{2\sqrt{\pi}}\left[\frac{1}{4}-S(2)\right],\quad K_2^{(3)}=\frac{1}{2\pi\sqrt{3}}\left[\frac{1}{4}-S(3)\right],\quad \&c.$$

Consequently

(7.10) $$J_2^{(\alpha)}=\int\left(U^2+\frac{1}{4}\right)\varphi^\alpha dt=\frac{c_\alpha}{2}\left[1-2S(\alpha)\right].$$

(7.10.1) $$J_2^{(2)}=\frac{1}{4\sqrt{\pi}}\left[1-2S(2)\right],\quad J_2^{(3)}=\frac{1}{4\pi\sqrt{3}}\left[1-2S(3)\right],\quad \&c.$$

4° For $\lambda=3$, also $K_3^{(\alpha)}=0$, but

$$J_3^{(\alpha)}=\int\left[\frac{1}{8}+\frac{3}{2}U^2\right]\varphi^\alpha dt=\frac{1}{8}K_0^{(\alpha)}+\frac{3}{2}K_2^{(\alpha)}=\frac{1}{2}c_\alpha[1-3S(\alpha)]. \tag{7.11}$$

$$J_3^{(2)}=\frac{1}{4\sqrt{\pi}}[1-3S(2)],\quad J_3^{(3)}=\frac{1}{4\pi\sqrt{3}}[1-3S(3)],\quad \&c. \tag{7.11.1}$$

5° However, for $\lambda=4$, we have

$$K_4^{(\alpha)}=\int U^4\varphi^\alpha dt,$$

where

$$U^4=\frac{1}{4\pi^2}\iiiint_0^t \exp\left[-\frac{1}{2}(\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2)\right]d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$

Transformed these 4-dimensional rectangular co-ordinates into general polar co-ordinates[12], we have

$$U^4=\frac{1}{\pi^2}\int_0^{\pi/4}d\theta_1\int_0^{\tau_1}\cos\theta_2 d\theta_2\int_0^{\tau_2}\cos^2\theta_3 d\theta_3\int_0^{\tau_3}\exp\left\{-\frac{1}{2}\rho^2\right\}\rho^3 d\rho,$$

where $\tau_1=\mathrm{Tan}^{-1}\cos\theta_1$, $\tau_2=\tan^{-1}(\cos\theta_1\cos\theta_2)$, $\tau_3=t\sec\theta_1\sec\theta_2\sec\theta_3$. However, the above repeated integrals are too complicate to be actually performed. Rather we prefer to avail the before obtained integral of (7.6):

$$\begin{aligned}K_4^{(\alpha)}&=\int U^4\varphi^\alpha dt=\int\left[\frac{1}{4}-\frac{1}{\pi}\int_0^{\pi/4}\exp\left\{-\frac{1}{2}t^2\sec\theta\right\}d\theta\right]^2\varphi^\alpha dt\\&=\frac{1}{16}\int\varphi^\alpha dt-\frac{1}{2\pi}\int\varphi^\alpha dt\int_0^{\pi/4}\exp\left\{-\frac{t^2}{2}\sec^2\theta\right\}d\theta\\&\quad+\frac{1}{\pi^2}\int\varphi^\alpha dt\int_0^{\pi/4}\int_0^{\pi/4}\exp\left\{-\frac{t^2}{2}(\sec^2\theta+\sec^2\theta')\right\}d\theta d\theta'\\&=c_\alpha\left[\frac{1}{16}-\frac{1}{2}S(\alpha)+\frac{1}{\pi^2}\sqrt{\frac{\alpha}{2\pi}}\int_0^{\pi/4}d\theta\int_0^{\pi/4}d\theta'\int\exp\left\{-\frac{t^2}{2}(\alpha+\sec^2\theta+\sec^2\theta')\right\}dt\right].\end{aligned}$$

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12) Generally, if $P(\xi_1,\cdots,\xi_n)$ be transformed into general polar coordinates:

$$\begin{aligned}\xi_1&=\rho\cos\theta_{n-1}\cos\theta_{n-2}\cdots\cdots\cos\theta_3\cos\theta_2\cos\theta_1\\\xi_2&=\rho\cos\theta_{n-1}\cos\theta_{n-2}\cdots\cdots\cos\theta_3\cos\theta_2\sin\theta_1\\\xi_3&=\rho\cos\theta_{n-1}\cos\theta_{n-2}\cdots\cdots\cos\theta_3\sin\theta_2\\&\cdots\cdots\cdots\cdots\cdots\cdots\\\xi_{n-1}&=\rho\cos\theta_{n-1}\sin\theta_{n-2}\\\xi_n&=\rho\sin\theta_{n-1}\end{aligned}$$

where $\rho=OP$ and θ_i denotes the angle which the normal projection of the straight line OP on the $(i+1)$-dimensional subspace formed by ξ_1-, ξ_2-, ……, ξ_{i+1}-axes, makes with the i-dimensional subspace formed by ξ_1-, ξ_2-, ……, ξ_i-axes. Then, the integral

$$U^n=\frac{1}{\sqrt{2\pi}^n}\int_0^t\int_0^t\cdots\cdots\int_0^t\exp\left\{-\frac{1}{2}\sum_{i=1}^n\xi_i^2\right\}d\xi_1 d\xi_2\cdots\cdots d\xi_n$$

is transformed into

$$U^n=\frac{n}{\sqrt{2\pi}^n}\int_0^{\pi/4}d\theta_1\int_0^{\tau_1}d\theta_2\int_0^{\tau_2}d\theta_3\cdots\cdots\int_0^{\tau_{n-2}}\int_0^{\tau_{n-1}}\exp\left\{-\frac{1}{2}\rho^2\right\}|J|d\rho,$$

where $\tau_1=\mathrm{Tan}^{-1}(\cos\theta_1)$, $\tau_2=\mathrm{Tan}^{-1}(\cos\theta_1\cos\theta_2)$, ……, $\tau_{n-2}=\mathrm{Tan}^{-1}(\cos\theta_1\cos\theta_2\cdots\cdots\cos\theta_{n-2})$, but the last $\tau_{n-1}=t\sec\theta_1\sec\theta_2\cdots\cdots\sec\theta_{n-1}$, and $|J|$ denotes the Jacobian of transformation $\dfrac{\partial(\xi_1,\xi_2,\cdots,\xi_n)}{\partial(\rho,\theta_1,\cdots,\theta_{n-1})}$; thus for $n=4$ really $|J|=\rho^3\cos\theta_2\cos^2\theta_3$.

The last triple integral ($I_4^{(\alpha)}$ say), again (7.8) applied, reduces to

$$I_4^{(\alpha)}=\frac{\sqrt{\alpha}}{\pi^2}\int_0^{\pi/4}d\theta\int_0^{\pi/4}\frac{d\theta'}{\sqrt{\alpha+\sec^2\theta+\sec^2\theta'}}$$
$$=\frac{\sqrt{\alpha}}{\pi^2}\int_0^{\pi/4}\frac{d\theta}{\sqrt{\alpha+\sec^2\theta}}\sin^{-1}\sqrt{\frac{\alpha+\sec^2\theta}{2[1+\alpha+\sec^2\theta]}},$$

which, if the integration variable be changed into $u=\sin\theta$, becomes

$$I_4^{(\alpha)}=\frac{1}{\pi^2}\int_0^{1/\sqrt{2}}\frac{\sqrt{\alpha}\,du}{\sqrt{\alpha+1-\alpha u^2}}\sin^{-1}\sqrt{\frac{\alpha+1-\alpha u^2}{2[\alpha+2-(\alpha+1)u^2]}}$$
$$=\frac{1}{2\pi^2}\int_0^{1/\sqrt{2}}\frac{\sqrt{\alpha}\,du}{\sqrt{\alpha+1-\alpha u^2}}\sec^{-1}\left(\alpha+1+\frac{1}{1-u^2}\right)^{13)}.$$

Or, upon putting $\alpha\sin^2\theta=(\alpha+1)\sin^2\psi$, we have by (7.7)

$$\frac{d\psi}{d\theta}=\frac{\sqrt{\alpha}}{\sqrt{\alpha+\sec^2\theta}}$$

and the integral $I_4^{(\alpha)}$ becomes

(7.12) $$I_4^{(\alpha)}=\frac{1}{\pi^2}\int_0^{\pi S(\alpha)}\sin^{-1}\sqrt{\frac{\alpha(\alpha+1)/2}{\alpha(\alpha+2)-\tan^2\psi}}\,d\psi.$$

Hence, we obtain

(7.13) $$K_4^{(\alpha)}=\frac{c_\alpha}{2}\left[\frac{1}{8}-S(\alpha)+2I_4^{(\alpha)}\right]$$

and accordingly

(7.14) $$J_4^{(\alpha)}=\frac{c_\alpha}{2}\left[1-4S(\alpha)+2I_4^{(\alpha)}\right].$$

In particular

(7.13.1) $$K_4^{(2)}=\frac{1}{4\sqrt{\pi}}\left[\frac{1}{8}-S(2)+2I_4^{(2)}\right],\quad K_4^{(3)}=\frac{1}{4\pi\sqrt{3}}\left[\frac{1}{8}-S(3)+2I_4^{(3)}\right],$$

(7.14.1) $$J_4^{(2)}=\frac{1}{4\sqrt{\pi}}[1-4S(2)+2I_4^{(2)}],\quad J_4^{(3)}=\frac{1}{4\pi\sqrt{3}}[1-4S(3)+2I_4^{(3)}],$$

where[13)]

$$I_4^{(2)}=\frac{1}{\pi^2}\int_0^{\pi S(2)}\sin^{-1}\sqrt{\frac{3}{8-\tan^2\psi}}\,d\psi=\frac{1}{2\pi}\int_0^{\pi S(2)}\frac{1}{\pi}\sec^{-1}\left(1+\frac{6}{2-\tan^2\psi}\right)d\psi,$$
$$I_4^{(3)}=\frac{1}{\pi^2}\int_0^{\pi S(3)}\sin^{-1}\sqrt{\frac{6}{15-\tan^2\psi}}\,d\psi=\frac{1}{2\pi}\int_0^{\pi S(3)}\frac{1}{\pi}\sec^{-1}\left(1+\frac{12}{3-\tan^2\psi}\right)d\psi.$$

6° For $\lambda=5$, $K_5^{(\alpha)}=0$,

(7.15) $$J_5^{(\alpha)}=\frac{1}{32}K_0^{(\alpha)}+\frac{5}{4}K_2^{(\alpha)}+\frac{5}{2}K_4^{(\alpha)}=\frac{c_\alpha}{2}[1-5S(\alpha)+5I_4^{(\alpha)}],$$

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13) By numerical computations the second forms with inverse secants are convenient, because they are free from radical signs.

and

(7.15.1) $J_5^{(2)}=\frac{1}{4\sqrt{\pi}}[1-5S(2)+5I_4^{(2)}]$, $J_5^{(3)}=\frac{1}{4\pi\sqrt{3}}[1-5S(3)+5I_4^{(3)}]$.

However, if we carry out similarly with $K_6^{(\alpha)}=\int U^6\varphi^\alpha dt$, we must treat a furthermore complicated integral

$$\int_0^{\pi/4}\frac{d\theta'}{\sqrt{\alpha'}}\int_0^{\pi S(\alpha')}\sin^{-1}\sqrt{\frac{\alpha'(\alpha'+1)/2}{\alpha'(\alpha'+2)-\tan^2\psi}}\,d\psi \quad (\alpha'=\alpha+\sec^2\theta),$$

so we give up to continue any more.

§8. Computations of $J_{\mu,\nu}^{\alpha,\beta}$ for $\alpha,\beta=1,2,3,\cdots;\ \mu,\nu=0,1,2,3,\cdots$[14].

We wish to find

(8.1) $$J_{\mu,\nu}^{\alpha,\beta}=\int\Phi^\mu\varphi^\alpha dt\int^t\Phi_1^\nu\varphi_1^\beta dt_1,$$

which are obtainable from

(8.2) $$K_{\mu,\nu}^{\alpha,\beta}=\int U^\mu\varphi^\alpha dt\int^t U_1^\nu\varphi_1^\beta dt_1,$$

because

(8.3) $$J_{\mu,\nu}^{\alpha,\beta}=\int\left(\frac{1}{2}+U\right)^\mu\varphi^\alpha dt\int^t\left(\frac{1}{2}+U_1\right)^\nu\varphi_1^\beta dt_1=\sum_{j=1}^{\mu}\sum_{k=1}^{\nu}\binom{\mu}{j}\binom{\nu}{k}\left(\frac{1}{2}\right)^{\mu+\nu-j-k}K_{j\ k}^{\alpha,\beta}.$$

Also, in general,

$$K_{\nu,\mu}^{\beta,\alpha}=\int U^\nu\varphi^\beta dt\int^t U_1^\mu\varphi_1^\alpha dt_1=\int U_1^\mu\varphi_1^\alpha dt_1\int_{t_1}U^\nu\varphi^\beta dt$$
$$=\int U_1^\mu\varphi_1^\alpha dt_1\left[\int U^\nu\varphi^\beta dt-\int^{t_1}U^\nu\varphi^\beta dt\right]=K_\mu^{(\alpha)}K_\nu^{(\beta)}-K_{\mu,\nu}^{\alpha,\beta}.$$

Therefore

(8.4) $$K_{\mu,\nu}^{\alpha,\beta}+K_{\nu,\mu}^{\beta,\alpha}=K_\mu^{(\alpha)}K_\nu^{(\beta)}.$$

In particular, if at least one of μ,ν be odd, it yields, in view of (7.2),

(8.4.1) $K_{\mu,\nu}^{\alpha,\beta}+K_{\nu,\mu}^{\beta,\alpha}=0$, i.e. $K_{\nu,\mu}^{\beta,\alpha}=-K_{\mu,\nu}^{\alpha,\beta}$ (one of μ,ν odd).

Quite similarly

(8.5) $$J_{\mu,\nu}^{\alpha,\beta}+J_{\nu,\mu}^{\beta,\alpha}=J_\mu^{(\alpha)}J_\nu^{(\beta)}.$$

Now we shall compute $K_{\mu,\nu}^{\alpha,\beta}$ and $J_{\mu,\nu}^{\alpha,\beta}$ successively.

1° $\mu=\nu=0$.

$$K_{00}^{\alpha,\beta}=J_{00}^{\alpha,\beta}=\int\varphi^\alpha dt\int^t\varphi_1^\beta dt_1=\frac{1}{\sqrt{2\pi}^{\,\alpha+\beta}}\iint_D\exp\left\{-\frac{1}{2}(\alpha t^2+\beta t_1^2)\right\}dt\,dt_1.$$

where D denotes the domain $t\geq t_1$ in the tt_1-rectangular co-ordinates plane. So that if we transform them into polar co-ordinates (r,θ), D consists of

14) Although, for calculations of the second moments, it is enough to consider the case $\alpha=\beta=2$ only, we have endeavoured to obtain rather more general formulas, because they are required for calculations of higher ordered moments.

$\frac{\pi}{4} \leq \theta \leq \frac{5\pi}{4}$, $0 \leq r < \infty$, and therefore

$$\iint_D = \int_{\pi/4}^{5\pi/4} d\theta \int_0^\infty \exp\left\{-\frac{1}{2} r^2(\alpha \sin^2\theta + \beta\cos^2\theta)\right\} r\,dr = \int_{\pi/4}^{5\pi/4} \frac{\sec^2\theta\, d\theta}{\alpha\tan^2\theta + \beta}$$

$$= \frac{1}{\sqrt{\alpha\beta}} \tan^{-1}\left\{\sqrt{\frac{\alpha}{\beta}}\tan\theta\right\}\Bigg|_{\pi/4}^{5\pi/4} = \frac{1}{\sqrt{\alpha\beta}}\left[\cdots\Big|_{\pi/4}^{\pi/2} + \cdots\Big|_{\pi/2+0}^{\pi-0} + \cdots\Big|_{\pi}^{5\pi/4}\right] = \frac{\pi}{\sqrt{\alpha\beta}}.$$

Thus

(8.6)
$$K_{00}^{\alpha,\beta} = J_{00}^{\alpha,\beta} = \frac{1}{2\sqrt{\alpha\beta}\,\sqrt{2\pi}^{\,\alpha+\beta-2}} = \frac{1}{2} c_\alpha c_\beta\,.$$

(8.6.1)

α	1	1	2	1	3	2
β	1	2	1	3	1	2
$K_{00}^{\alpha,\beta}$	$\frac{1}{2}$	$\frac{1}{4\sqrt{\pi}}$	$\frac{1}{4\sqrt{\pi}}$	$\frac{1}{4\pi\sqrt{3}}$	$\frac{1}{4\pi\sqrt{3}}$	$\frac{1}{8\pi}$ &c.

2° $\mu=1$, $\nu=0$. Since

$$\int^t \varphi_1^\beta dt_1 = \int^0 + \int_0^t = \frac{1}{2\sqrt{\beta}\,\sqrt{2\pi}^{\,\beta-1}} + \int_0^t \varphi_1^\beta dt_1\,,$$

and U is odd, it yields that

$$K_{10}^{\alpha,\beta} = \int U\varphi^\alpha dt \int^t \varphi_1^\beta dt_1 = \int \varphi^\alpha dt \int_0^t \varphi_2 dt_2 \int_0^t \varphi_1^\beta dt_1 = \int \varphi^\alpha I(t)\, dt\,,$$

of which the inner integral $I(t)$ is to be calculated below: Transforming again into polar co-ordinates (r, ψ), as in (7.6), but now observing that here the contributions from domains $0 \leq \psi \leq \frac{\pi}{4}$ and $\frac{\pi}{4} \leq \psi \leq \frac{\pi}{2}$ are different, we get

$$\sqrt{2\pi}^{\,\beta+1} I(t) = \int_0^t\!\!\int_0^t \exp\left\{-\frac{1}{2}(t_2^2 + \beta t_1^2)\right\} dt_1 dt_2$$

$$= \int_0^{\pi/4} d\psi \int_0^{t\sec\psi} \exp\left\{-\frac{r^2}{2}(\sin^2\psi + \beta\cos^2\psi)\right\} r\,dr$$

$$+ \int_{\pi/4}^{\pi/2} d\psi \int_0^{t\,\mathrm{cosec}\,\psi} \exp\left\{-\frac{r^2}{2}(\sin^2\psi + \beta\cos^2\psi)\right\} r\,dr\,.$$

After integrating about r and writing $\tan\psi = u$, we obtain

(8.7)
$$I(t) = \frac{1}{\sqrt{2\pi}^{\,\beta+1}} \int_0^1 \frac{1 - \exp\left\{-\frac{t^2}{2}(u^2+\beta)\right\}}{u^2+\beta} du + \frac{1}{\sqrt{2\pi}^{\,\beta+1}} \int_1^\infty \frac{1 - \exp\left\{-\frac{t^2}{2}\left(1 + \frac{\beta}{u^2}\right)\right\}}{u^2+\beta} du$$

Here we have assumed to be $t > 0$; however it remains the same with $t < 0$ because $I(t)$ is even. On substituting (8.7) in $K_{10}^{\alpha,\beta}$, we have

$$\sqrt{2\pi}^{\,\alpha+\beta} K_{10}^{\alpha,\beta} = \frac{1}{\sqrt{2\pi}} \int_0^1 \frac{du}{u^2+\beta} \int \left[\exp\left\{-\frac{\alpha}{2}t^2\right\} - \exp\left\{-\frac{t^2}{2}(u^2+\alpha+\beta)\right\}\right] dt$$

$$+ \frac{1}{\sqrt{2\pi}} \int_1^\infty \frac{du}{u^2+\beta} \int \left[\exp\left\{-\frac{\alpha}{2}t^2\right\} - \exp\left\{-\frac{t^2}{2}\left(\alpha + 1 + \frac{\beta}{u^2}\right)\right\}\right] dt$$

$$=\int_0^1 \frac{du}{u^2+\beta}\left[\frac{1}{\sqrt{\alpha}}-\frac{1}{\sqrt{u^2+\alpha+\beta}}\right]+\int_1^\infty \frac{du}{u^2+\beta}\left[\frac{1}{\sqrt{\alpha}}-\frac{u}{\sqrt{(\alpha+1)u^2+\beta}}\right]$$

$$=\frac{1}{\sqrt{\alpha}}\int_0^\infty \frac{du}{u^2+\beta}-\int_0^1 \frac{du}{(u^2+\beta)\sqrt{u^2+\alpha+\beta}}-\int_1^\infty \frac{udu}{(u^2+\beta)\sqrt{(\alpha+1)u^2+\beta}}.$$

On performing integrations we get

$$\text{(8.8)}\quad K_{10}^{\alpha,\beta}=\frac{c_\alpha c_\beta}{2\pi}\left[\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}-\sin^{-1}\sqrt{\frac{\alpha}{(\alpha+\beta)(\beta+1)}}\right]=-K_{01}^{\beta,\alpha}.$$

In particular, if $\beta=\alpha$,

$$\text{(8.8.1)}\quad K_{10}^{\alpha,\alpha}=\frac{c_\alpha^2}{2\pi}\left[\sin^{-1}\frac{\sqrt{2\alpha+1}}{\alpha+1}-\sin^{-1}\frac{1}{\sqrt{2(\alpha+1)}}\right]$$

$$=\frac{c_\alpha^2}{2\pi}\sin^{-1}\frac{1}{\sqrt{2(\alpha+1)}}=\frac{1}{2}c_\alpha^2 S\left(\frac{1}{\alpha}\right)=-K_{01}^{\alpha,\alpha};$$

and more particularly e. g.

$$\text{(8.8.1.1)}\quad K_{10}^{2,2}=\frac{1}{8\pi}S\left(\frac{1}{2}\right)=\frac{1}{8\pi^2}\sin^{-1}\frac{1}{\sqrt{6}}=-K_{01}^{2,2}.$$

Accordingly

$$\text{(8.9)}\quad J_{10}^{\alpha,\beta}=\int\left(\frac{1}{2}+U\right)\varphi^\alpha dt\int^t \varphi_1^\beta dt_1=\frac{1}{2}K_{00}^{\alpha,\beta}+K_{10}^{\alpha,\beta}$$

$$=\frac{1}{2}c_\alpha c_\beta\left[\frac{1}{2}+\frac{1}{\pi}\left(\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}-\sin^{-1}\sqrt{\frac{\alpha}{(\alpha+\beta)(\beta+1)}}\right)\right],$$

$$\text{(8.9.1)}\quad J_{10}^{\alpha,\alpha}=\frac{1}{2}c_\alpha^2\left[\frac{1}{2}+S\left(\frac{1}{\alpha}\right)\right],\quad J_{10}^{2,2}=\frac{1}{8\pi}\left[\frac{1}{2}+S\left(\frac{1}{2}\right)\right].$$

Also

$$\text{(8.10)}\quad J_{01}^{\alpha,\beta}=J_0^{(\alpha)}J_1^{(\beta)}-J_{10}^{\beta,\alpha}=\frac{1}{2}c_\alpha c_\beta-J_{10}^{\beta,\alpha}$$

$$=\frac{1}{2}c_\alpha c_\beta\left[\frac{1}{2}-\frac{1}{\pi}\left(\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}-\sin^{-1}\sqrt{\frac{\beta}{(\alpha+\beta)(\alpha+1)}}\right)\right].$$

$$\text{(8.10.1)}\quad J_{01}^{\alpha,\alpha}=\frac{1}{2}c_\alpha^2\left[\frac{1}{2}-S\left(\frac{1}{\alpha}\right)\right],\quad J_{01}^{2,2}=\frac{1}{8\pi}\left[\frac{1}{2}-S\left(\frac{1}{2}\right)\right].$$

3° $\mu=\nu=1$.

$$K_{11}^{\alpha,\beta}=\int U\varphi^\alpha dt\int^t U_1\varphi_1^\beta dt_1.$$

Here evidently the inner integral becomes even and the whole integrand odd, so that

$$\text{(8.11)}\quad K_{11}^{\alpha,\beta}=0.$$

However

$$\text{(8.12)}\quad J_{11}^{\alpha,\beta}=\int\left(\frac{1}{2}+U\right)\varphi^\alpha dt\int^t\left(\frac{1}{2}+U_1\right)\varphi_1^\beta dt_1=\frac{1}{4}K_{00}^{\alpha,\beta}=\frac{1}{8}c_\alpha c_\beta.$$

4° $\mu=2,\ \nu=0$. Observing that $\int_0^t \varphi_1^\beta dt$ is odd, we get readily

(8.13) $K_{20}^{\alpha,\beta}=\int U^2\varphi^\alpha dt\left[\int^0 \varphi_1^\beta dt_1+\int_0^t \varphi_1^\beta dt_1\right]=\frac{c_\beta}{2}K_2^{(\alpha)}=\frac{1}{2}c_\alpha c_\beta\left[\frac{1}{4}-S(\alpha)\right].$

(8.14) $J_{20}^{\alpha,\beta}=\frac{1}{4}K_{00}^{\alpha,\beta}+K_{10}^{\alpha,\beta}+K_{20}^{\alpha,\beta}$

$$=\frac{1}{2}c_\alpha c_\beta\left[\frac{1}{2}-S(\alpha)+\frac{1}{\pi}\left(\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}-\sin^{-1}\sqrt{\frac{\alpha}{(\alpha+\beta)(\beta+1)}}\right)\right].$$

(8.14.1) $J_{20}^{\alpha,\alpha}=\frac{1}{2}c_\alpha^2\left[\frac{1}{2}-S(\alpha)+S\left(\frac{1}{\alpha}\right)\right],\quad J_{20}^{2,2}=\frac{1}{8\pi}\left[\frac{1}{2}-S(2)+S\left(\frac{1}{2}\right)\right].$

Also for $\mu=0,\ \nu=2$:

(8.15) $K_{02}^{\alpha,\beta}=K_0^{(\alpha)}K_2^{(\beta)}-K_{20}^{\beta,\alpha}=\frac{1}{2}c_\alpha c_\beta\left[\frac{1}{4}-S(\beta)\right].$

(8.16) $J_{02}^{\alpha,\beta}=J_0^{(\alpha)}J_2^{(\beta)}-J_{20}^{\beta,\alpha}$

$$=\frac{1}{2}c_\alpha c_\beta\left[\frac{1}{2}-S(\beta)-\frac{1}{\pi}\left(\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}-\sin^{-1}\sqrt{\frac{\beta}{(\alpha+\beta)(\alpha+1)}}\right)\right],$$

(8.16.1) $J_{02}^{\alpha,\alpha}=\frac{1}{2}c_\alpha^2\left\{\frac{1}{2}-S(\alpha)-S\left(\frac{1}{\alpha}\right)\right\},\quad J_{02}^{2,2}=\frac{1}{8\pi}\left\{\frac{1}{2}-S(2)-S\left(\frac{1}{2}\right)\right\}.$

5° $\mu=2,\ \nu=1$.

$$K_{21}^{\alpha,\beta}=\int U^2\varphi^\alpha dt\int^t U_1\varphi_1^\beta dt_1,\quad \text{where}\quad U_1=\int_0^{t_1}\varphi(\tau)\,d\tau.$$

The inner integral is

$$\int^t U_1\varphi_1^\beta dt_1=\int^0+\int_0^t=\text{(i)}+\text{(ii)}.$$

Here

$$\sqrt{2\pi}^{\beta+1}\ \text{(i)}=\int^0 dt_1\int_0^{t_1}\exp\left\{-\frac{1}{2}(\beta t_1^2+\tau^2)\right\}d\tau$$

$$=-\int_{5\pi/4}^{3\pi/2}d\psi\int_0^\infty \exp\left\{-\frac{r^2}{2}(\beta\sin^2\psi+\cos^2\psi)\right\}r\,dr$$

$$=-\int_1^\infty\frac{du}{1+\beta u^2}\ (u=\tan\psi)=-\left[\frac{1}{\sqrt{\beta}}\tan\sqrt{\beta}u\right]_1^\infty=-\frac{1}{\sqrt{\beta}}\tan\frac{1}{\sqrt{\beta}},$$

and accordingly we have

$$\text{(i)}=-\frac{1}{\sqrt{\beta}\sqrt{2\pi}^{\beta+1}}\tan^{-1}\frac{1}{\sqrt{\beta}}.$$

Hence the corresponding whole integral becomes

$$\text{(I)}=-\frac{c_\beta}{2\pi}\tan^{-1}\frac{1}{\sqrt{\beta}}\cdot K_2^{(\alpha)}=-\frac{c_\alpha c_\beta}{2\pi}\tan^{-1}\frac{1}{\sqrt{\beta}}\left[\frac{1}{4}-S(\alpha)\right].$$

On the other hand

$$\sqrt{2\pi}^{\beta+1}\ (\text{ii}) = \int_0^t \exp\left\{-\frac{\beta}{2}t_1^2\right\}dt_1 \int_0^{t_1}\exp\left\{-\frac{1}{2}\tau^2\right\}d\tau$$
$$= \int_{\pi/4}^{\pi/2} d\psi \int_0^{t\,\mathrm{cosec}\,\psi} \exp\left\{-\frac{r^2}{2}(\beta\sin^2\psi+\cos^2\psi)\right\} r\,dr$$
$$= \int_1^\infty \frac{dv}{1+\beta v^2}\left[1-\exp\left\{-\frac{t^2}{2}\left(\beta+\frac{1}{v^2}\right)\right\}\right] \quad (v=\tan\psi)\,.$$

Here we have assumed $t>0$. However for $t<0$ it results the same. Hence the contribution from (ii) is, making use of the above and (7.6),

$$(\text{II}) = \int U^2\varphi^\alpha dt \int_0^t U_1\varphi_1^\beta dt_1$$
$$= \frac{1}{\sqrt{2\pi}^{\alpha+\beta+1}}\int \exp\left\{-\frac{1}{2}\alpha t^2\right\}\left[\frac{1}{4}-\frac{1}{\pi}\int_0^{\pi/4}\exp\left\{-\frac{t^2}{2}\sec^2\theta\right\}d\theta\right]dt$$
$$\times\int_1^\infty \frac{dv}{1+\beta v^2}\left[1-\exp\left\{-\frac{t^2}{2}\left(\beta+\frac{1}{v^2}\right)\right\}\right]$$
$$= \frac{1}{\sqrt{2\pi}^{\alpha+\beta+1}}\int_1^\infty \frac{dv}{1+\beta v^2}\int\left[\exp\left\{-\frac{\alpha}{2}t^2\right\}-\exp\left\{-\frac{t^2}{2}\left(\alpha+\beta+\frac{1}{v^2}\right)\right\}\right]$$
$$\times\left[\frac{1}{4}-\frac{1}{\pi}\int_0^{\pi/4}\exp\left\{-\frac{t^2}{2}\sec^2\theta\right\}d\theta\right]dt$$
$$= (\text{II})'-(\text{II})''\,.$$

where (II)′ and (II)″ correspond to two terms in the latest square brackets.

Integrating (II)′ in regards to t,

$$(\text{II})' = \frac{1}{4\sqrt{2\pi}^{\alpha+\beta}}\left[\frac{1}{\sqrt{\alpha}}\int_1^\infty\frac{dv}{1+\beta v^2}-\int_1^\infty\frac{v\,dv}{(1+\beta\gamma^2)\sqrt{1+(\alpha+\beta)v^2}}\right]$$
$$= \frac{c_\alpha c_\beta}{8\pi}\left[\sin^{-1}\sqrt{\frac{\beta(1+\alpha+\beta)}{(\alpha+\beta)(1+\beta)}}-\tan^{-1}\sqrt{\beta}\right].$$

Also (7.7) being applied,

$$(\text{II})'' = \frac{1}{\pi\sqrt{2\pi}^{\alpha+\beta+1}}\int_1^\infty\frac{dv}{1+\beta v^2}\int_0^{\pi/4}d\theta\int\left[\exp\left\{-\frac{t^2}{2}(\alpha+\sec^2\theta)\right\}\right.$$
$$\left.-\exp\left\{-\frac{t^2}{2}\left(\alpha+\beta+\frac{1}{v^2}+\sec^2\theta\right)\right\}\right]dt$$
$$= \frac{1}{\pi\sqrt{2\pi}^{\alpha+\beta}}\int_1^\infty\frac{dv}{1+\beta v^2}\int_0^{\pi/4}\left[\frac{1}{\sqrt{\alpha+\sec^2\theta}}-\frac{1}{\sqrt{\alpha+\beta+\frac{1}{v^2}+\sec^2\theta}}\right]d\theta$$
$$= \frac{1}{2}c_\alpha c_\beta\left[\frac{1}{\pi}\tan^{-1}\frac{1}{\sqrt{\beta}}\cdot S(\alpha)\right.$$
$$\left.-\frac{\sqrt{\alpha\beta}}{\pi^2}\int_1^\infty\frac{v\,dv}{(1+\beta v^2)\sqrt{1+(\alpha+\beta)v^2}}\sin^{-1}\sqrt{\frac{1+(\alpha+\beta)v^2}{2[1+(1+\alpha+\beta)v^2]}}\right].$$

Therefore

$$(8.17)\quad K_{21}^{\alpha,\beta} = (\text{I})+(\text{II})'-(\text{II})'' = \frac{1}{2}c_\alpha c_\beta\left[-\frac{1}{4\pi}\sin^{-1}\sqrt{\frac{\alpha}{(\alpha+\beta)(1+\beta)}}+I_{21}^{\alpha,\beta}\right],$$

where

(8.18) $$I_{21}^{\alpha,\beta}=\frac{\sqrt{\alpha\beta}}{\pi^2}\int_1^\infty \frac{vdv}{(1+\beta v^2)\sqrt{1+(\alpha+\beta)\,v^2}}\sin^{-1}\sqrt{\frac{1+(\alpha+\beta)\,v^2}{2[1+(1+\alpha+\beta)\,v^2]}}$$
$$=\frac{\sqrt{\alpha\beta}}{\pi^2}\int_0^1 \frac{du}{(u^2+\beta)\sqrt{u^2+\alpha+\beta}}\sin^{-1}\sqrt{\frac{u^2+\alpha+\beta}{2(u^2+\alpha+\beta+1)}}\,.$$

(8.18.1) $$I_{21}^{\alpha,\alpha}=\frac{\alpha}{\pi^2}\int_0^1 \frac{du}{(u^2+\alpha)\sqrt{u^2+2\alpha}}\sin^{-1}\sqrt{\frac{u^2+2\alpha}{2(u^2+2\alpha+1)}}$$
$$=\frac{\alpha}{2\pi^2}\int_0^1 \frac{\sec^{-1}(u^2+2\alpha+1)}{(u^2+\alpha)\sqrt{u^2+2\alpha}}\,du\,.$$

(8.17.1) $$K_{21}^{\alpha,\alpha}=\frac{1}{2}\,c_\alpha^2\left[-\frac{1}{4}\,S\left(\frac{1}{\alpha}\right)+I_{21}^{\alpha,\alpha}\right],$$

(8.17.1.1) $$K_{21}^{2,2}=\frac{1}{8\pi}\left[-\frac{1}{4}\,S\left(\frac{1}{2}\right)+I_{21}^{2,2}\right].$$

And

(8.19) $$J_{21}^{\alpha,\beta}=\int\left(\frac{1}{2}+U\right)^2\varphi^\alpha dt\int^t\left(\frac{1}{2}+U_1\right)\varphi_1^\beta dt_1$$
$$=\frac{1}{8}\,K_{00}^{\alpha,\beta}+\frac{1}{2}\,K_{10}^{\alpha,\beta}+\frac{1}{4}\,K_{01}^{\alpha,\beta}+\frac{1}{2}\,K_{20}^{\alpha,\beta}+K_{21}^{\alpha,\beta}$$
$$=\frac{c_\alpha c_\beta}{2}\left[\frac{1}{4}-\frac{1}{2}\,S(\alpha)+\frac{1}{4\pi}\left(\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}\right.\right.$$
$$\left.\left.-3\sin^{-1}\sqrt{\frac{\alpha}{(\alpha+\beta)(\beta+1)}}+\sin^{-1}\sqrt{\frac{\beta}{(\alpha+\beta)(\alpha+1)}}\right)+I_{21}^{\alpha,\beta}\right].$$

(8.19.1) $$J_{21}^{\alpha,\alpha}=\frac{1}{2}\,c_\alpha^2\left[\frac{1}{4}-\frac{1}{2}\,S(\alpha)+I_{21}^{\alpha,\alpha}\right],$$

here the relation $\sin^{-1}\dfrac{\sqrt{1+2\alpha}}{1+\alpha}=2\sin^{-1}\dfrac{1}{\sqrt{2(1+\alpha)}}$ has been used. And

(8.19.1.1) $$J_{21}^{2,2}=\frac{1}{8\pi}\left[\frac{1}{4}-\frac{1}{2}\,S(2)+I_{21}^{2,2}\right]\quad\text{with}$$

(8.19.1.2) $$I_{21}^{2,2}=\frac{1}{\pi^2}\int_0^1\frac{\sec^{-1}(u^2+5)}{(u^2+2)\sqrt{u^2+4}}\,du\,.$$

Also for $\mu=1$, $\nu=2$,

$$K_{12}^{\alpha,\beta}=-K_{21}^{\beta,\alpha}\quad\text{and}$$

(8.20) $$J_{12}^{\alpha,\beta}=J_1^{(\alpha)}J_2^{(\beta)}-J_{21}^{\beta,\alpha}$$
$$=\frac{c_\alpha c_\beta}{2}\left[\frac{1}{4}-\frac{1}{2}\,S(\beta)-\frac{1}{4\pi}\left(\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}\right.\right.$$
$$\left.\left.-3\sin^{-1}\sqrt{\frac{\beta}{(\alpha+\beta)(\alpha+1)}}+\sin^{-1}\sqrt{\frac{\alpha}{(\alpha+\beta)(\beta+1)}}\right)-I_{21}^{\beta,\alpha}\right].$$

(8.20.1) $$J_{12}^{\alpha,\alpha}=\frac{1}{2}c_\alpha^2\left[\frac{1}{4}-\frac{1}{2}S(\alpha)-I_{21}^{\alpha,\alpha}\right],$$

(8.20.1.1) $$J_{12}^{2,2}=\frac{1}{8\pi}\left[\frac{1}{4}-\frac{1}{2}S(2)-I_{21}^{2,2}\right].$$

6° $\mu=3$, $\nu=0$. $U^3(t)$ being odd,

$$K_{30}^{\alpha,\beta}=\int U^3\varphi^\alpha dt\int^t\varphi_1^\beta dt_1=\int U^3\varphi^\alpha dt\int_0^t\varphi_1^\beta dt_1=\int U^2\varphi^\alpha dt\int_0^t\varphi_2 dt_2\int_0^t\varphi_1^\beta dt_1$$
$$=\int\left[\frac{1}{4}-\frac{1}{\pi}\int_0^{\pi/4}\exp\left\{-\frac{1}{2}t^2\sec^2\theta\right\}d\theta\right]\varphi^\alpha\cdot I(t)\,dt\quad\text{(availed (7.6))}$$
$$=(\mathrm{I})-(\mathrm{II}),$$

where (I) and (II) correspond to the two terms in the square brackets, and $I(t)$ is the same as in (8.7). Hence by (8.8)

$$(\mathrm{I})=\frac{1}{4}K_{10}^{\alpha,\beta}=\frac{c_\alpha c_\beta}{8\pi}\left[\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}-\sin^{-1}\sqrt{\frac{\alpha}{(\alpha+\beta)(\beta+1)}}\right].$$

As to (II), still availing (8.7) and integrating about t, we obtain

$$(\mathrm{II})=\frac{1}{\pi\sqrt{2\pi}^{\,\alpha+\beta}}\int_0^{\pi/4}d\theta\left[\int_0^\infty\frac{du}{(u^2+\beta)\sqrt{\alpha+\sec^2\theta}}-\int_0^1\frac{du}{(u^2+\beta)\sqrt{u^2+\alpha+\beta+\sec^2\theta}}\right.$$
$$\left.-\int_1^\infty\frac{du}{(u^2+\beta)\sqrt{1+\alpha+\sec^2\theta+\dfrac{\beta}{u^2}}}\right],$$

which, on interchanging the order of integrations and applying (7.8), yields

(8.21) $$(\mathrm{II})=\frac{1}{2}c_\alpha c_\beta\left[\frac{1}{2}S(\alpha)-\frac{\sqrt{\alpha\beta}}{\pi^2}\left\{\int_0^1\frac{du}{(u^2+\beta)\sqrt{u^2+\alpha+\beta}}\sin^{-1}\sqrt{\frac{u^2+\alpha+\beta}{2(u^2+\alpha+\beta+1)}}\right.\right.$$
$$\left.\left.+\int_1^\infty\frac{u\,du}{(u^2+\beta)\sqrt{(1+\alpha)u^2+\beta}}\sin^{-1}\sqrt{\frac{(1+\alpha)u^2+\beta}{2[(2+\alpha)u^2+\beta]}}\right\}\right].$$

Hence we obtain

(8.22) $$K_{30}^{\alpha,\beta}=\frac{c_\alpha c_\beta}{2}\left[\frac{1}{4\pi}\left\{\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}-\sin^{-1}\sqrt{\frac{\alpha}{(\alpha+\beta)(\beta+1)}}\right\}\right.$$
$$\left.-\frac{1}{2}S(\alpha)+I_{30}^{\alpha,\beta}\right],$$

where, according to (8.21) and (8.18),

(8.23) $$I_{30}^{\alpha,\beta}=I_{21}^{\alpha,\beta}+\frac{\sqrt{\alpha\beta}}{\pi^2}\int_1^\infty\frac{u\,du}{(u^2+\beta)\sqrt{(1+\alpha)u^2+\beta}}\sin^{-1}\sqrt{\frac{(1+\alpha)u^2+\beta}{2[(2+\alpha)u^2+\beta]}}$$
$$=I_{21}^{\alpha,\beta}+\frac{\sqrt{\alpha\beta}}{\pi^2}\int_0^1\frac{dv}{(1+\beta v^2)\sqrt{1+\alpha+\beta v^2}}\sin^{-1}\sqrt{\frac{1+\alpha+\beta v^2}{2[2+\alpha+\beta v^2]}}.$$

In particular

(8.23.1) $$K_{30}^{\alpha,\alpha}=\frac{c_\alpha^2}{8}\left\{S\left(\frac{1}{\alpha}\right)-2S(\alpha)+4I_{30}^{\alpha,\alpha}\right\},$$

(8.23.1.1) $$K_{30}^{2,2}=\frac{1}{8\pi}\left\{\frac{1}{4}S\left(\frac{1}{2}\right)-\frac{1}{2}S(2)+I_{30}^{2,2}\right\}=-K_{03}^{2,2}.$$

Consequently

(8.24) $$J_{30}^{\alpha,\beta}=\int\left(\frac{1}{2}+U\right)^3\varphi^\alpha dt\int^t\varphi_1^\beta dt_1=\frac{1}{8}K_{00}^{\alpha,\beta}+\frac{3}{4}K_{10}^{\alpha,\beta}+\frac{3}{2}K_{20}^{\alpha,\beta}+K_{30}^{\alpha,\beta}$$
$$=\frac{1}{2}c_\alpha c_\beta\left[\frac{1}{2}-2S(\alpha)+\frac{1}{\pi}\left\{\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}\right.\right.$$
$$\left.\left.-\sin^{-1}\sqrt{\frac{\alpha}{(\alpha+1)(\beta+1)}}\right\}+I_{30}^{\alpha,\beta}\right],$$

(8.24.1) $$J_{30}^{\alpha,\alpha}=\frac{c_\alpha^2}{2}\left[\frac{1}{2}-2S(\alpha)+S\left(\frac{1}{\alpha}\right)+I_{30}^{\alpha,\alpha}\right],$$

(8.24.1.1) $$J_{30}^{2,2}=\frac{1}{8\pi}\left[\frac{1}{2}-2S(2)+S\left(\frac{1}{2}\right)+I_{30}^{2,2}\right]\quad\text{with}$$

(8.24.1.2) $$I_{3,0}^{2,2}=I_{2,1}^{2,2}+\frac{2}{\pi^2}\int_0^1\frac{dv}{(1+2v^2)\sqrt{3+2v^2}}\sin^{-1}\sqrt{\frac{3+2v^2}{8+4v^2}}$$
$$=I_{2,1}^{2,2}+\frac{1}{\pi^2}\int_0^1\frac{\sec^{-1}(4+2v^2)}{(1+2v^2)\sqrt{3+2v^2}}dv.$$

Lastly $K_{03}^{\alpha,\beta}=-K_{30}^{\beta,\alpha}$, and

(8.25) $$J_{03}^{\alpha,\beta}=J_0^{(\alpha)}J_3^{(\beta)}-J_{30}^{\beta,\alpha}=\frac{c_\alpha c_\beta}{2}\left[\frac{1}{2}-S(\beta)-\frac{1}{\pi}\left\{\sin^{-1}\sqrt{\frac{\alpha+\beta+1}{(\alpha+1)(\beta+1)}}\right.\right.$$
$$\left.\left.-\sin^{-1}\sqrt{\frac{\beta}{(\alpha+\beta)(\alpha+1)}}\right\}-I_{3,0}^{\beta,\alpha}\right].$$

(8.25.1) $$J_{03}^{\alpha,\alpha}=\frac{1}{2}c_\alpha^2\left[\frac{1}{2}-S(\alpha)-S\left(\frac{1}{\alpha}\right)-I_{30}^{\alpha,\alpha}\right]$$

(8.25.1.1) $$J_{0,3}^{2,2}=\frac{1}{8\pi}\left[\frac{1}{2}-S(2)-S\left(\frac{1}{2}\right)-I_{30}^{2,2}\right].$$

Although it could be computed similarly for

$$J_{\lambda,\mu,\nu}^{\alpha,\beta,\gamma}=\int\Phi^\lambda\varphi^\alpha dt\int^t\Phi_1^\mu\varphi_1^\beta dt_1\int^{t_1}\Phi_2^\nu\varphi_2^\gamma dt_2\quad(t\geqq t_1\geqq t_2)\quad\&c.,$$

which are requisite for calculations of $E(t_{i|n}^p t_{j|n}^q t_{k|n}^r)$ &c., it is postponed for future.

For purpose of later references we have tabulated the values of $K_\lambda^{(\alpha)}$, $J_\lambda^{(\alpha)}$, for $\alpha=2,3$ (Table I) and those of $K_{\mu\nu}$, $J_{\mu\nu}$ (Table II)[15]:

15) For calculations of the 2nd moments, $K_{\mu,\nu}^{2,2}$ and $J_{\mu,\nu}^{2,2}$ being only of use, these are below simply expressed as $K_{\mu\nu}$, $J_{\mu\nu}$.

Table I

λ	$K_\lambda^{(2)}$	$K_\lambda^{(3)}$	$J_\lambda^{(2)}$	$J_\lambda^{(3)}$
0	$\frac{1}{2\sqrt{\pi}}=c_2$	$\frac{1}{2\pi\sqrt{3}}=c_3$	$\frac{1}{2\sqrt{\pi}}=c_2$	$\frac{1}{2\pi\sqrt{3}}=c_3$
1	0	0	$\frac{1}{4\sqrt{\pi}}=\frac{1}{2}c_2$	$\frac{1}{4\pi\sqrt{3}}=\frac{1}{2}c_3$
2	$\frac{1}{2\sqrt{\pi}}\left(\frac{1}{4}-S(2)\right)$	$\frac{1}{2\pi\sqrt{3}}\left(\frac{1}{4}-S(3)\right)$	$\frac{1}{4\sqrt{\pi}}[1-2S(2)]$	$\frac{1}{4\pi\sqrt{3}}[1-2S(3)]$
3	0	0	$\frac{1}{4\sqrt{\pi}}[1-3S(2)]$	$\frac{1}{4\pi\sqrt{3}}[1-3S(3)]$
4	$\frac{1}{4\sqrt{\pi}}\left[\frac{1}{8}-S(2)+2I_4^{(2)}\right]$	$\frac{1}{4\pi\sqrt{3}}\left[\frac{1}{8}-S(3)+2I_4^{(3)}\right]$	$\frac{1}{4\sqrt{\pi}}[1-4S(2)+2I_4^{(2)}]$	$\frac{1}{4\pi\sqrt{3}}[1-4S(3)+2I_4^{(3)}]$
5	0	0	$\frac{1}{4\sqrt{\pi}}[1-5S(2)+5I_4^{(2)}]$	$\frac{1}{4\pi\sqrt{3}}[1-5S(3)+5I_4^{(3)}]$

Table II

μ	ν	$K_{\mu\nu}$	$J_{\mu\nu}$
0	0	$\frac{1}{8\pi}$	$\frac{1}{8\pi}$
1	0	$\frac{1}{8\pi}S\left(\frac{1}{2}\right)$	$\frac{1}{8\pi}\left[\frac{1}{2}+S\left(\frac{1}{2}\right)\right]$
0	1	$-K_{10}$	$\frac{1}{8\pi}\left[\frac{1}{2}-S\left(\frac{1}{2}\right)\right]$
1	1	0	$\frac{1}{32\pi}$
2	0	$\frac{1}{8\pi}\left[\frac{1}{4}-S(2)\right]$	$\frac{1}{8\pi}\left[\frac{1}{2}-S(2)+S\left(\frac{1}{2}\right)\right]$
0	2	K_{20}	$\frac{1}{8\pi}\left[\frac{1}{2}-S(2)-S\left(\frac{1}{2}\right)\right]$
2	1	$\frac{1}{8\pi}\left[-\frac{1}{4}S\left(\frac{1}{2}\right)+I_{2,1}\right]$	$\frac{1}{8\pi}\left[\frac{1}{4}-\frac{1}{2}S(2)+I_{2,1}\right]$
1	2	$-K_{21}$	$\frac{1}{8\pi}\left[\frac{1}{4}-\frac{1}{2}S(2)-I_{2,1}\right]$
3	0	$\frac{1}{8\pi}\left[\frac{1}{4}S\left(\frac{1}{2}\right)-\frac{1}{2}S(2)+I_{3,0}\right]$	$\frac{1}{8\pi}\left[\frac{1}{2}-2S(2)+S\left(\frac{1}{2}\right)+I_{3,0}\right]$
0	3	$-K_{30}$	$\frac{1}{8\pi}\left[\frac{1}{2}-S(2)-S\left(\frac{1}{2}\right)-I_{3,0}\right]$

§ 9. Calculations of $E(t_{i|n})$, $E(t_{i|n}^2)$ and $E(t_{i|n}t_{k|n})$.

By means of foumulas in Part I, and the above Table I, II, we have calculated $E(t_{i|n})$, $E(t_{i|n}^2)$ as well as $E(t_{i|n}t_{k|n})$ up to $n=7$, the results of which are tabulated below in Table III and IV. (As to numerical approximations, cf. § 10).

Table III (Expectations of $t_{i|n}$ and $t_{i|n}^2$) [16]

n	i	$n-i+1$	$E(t_{i\|n})=-E(t_{n-i+1\|n})$	$E(t_{i\|n}^2)=E(t_{n-i+1}^2)$
2	2	1	$\frac{1}{\sqrt{\pi}}=0.5641896$	1
3	3	1	$\frac{3}{2\sqrt{\pi}}=0.8462844$	$1+\frac{\sqrt{3}}{2\pi}=1.2756644$
	2		0	$1-\frac{\sqrt{3}}{\pi}=0.4486711$
4	4	1	$\frac{3}{\sqrt{\pi}}(1-2S(2))=1.0293754$	$1+\frac{\sqrt{3}}{\pi}=1.5513289$
	3	2	$\frac{3}{\sqrt{\pi}}(-1+6S(2))=0.2970114$	$1-\frac{\sqrt{3}}{\pi}=0.4486711$
5	5	1	$\frac{5}{\sqrt{\pi}}(1-3S(2))=1.1629645$	$1+\frac{5\sqrt{3}}{2\pi}[1-2S(3)]=1.8000204$
	4	2	$\frac{5}{\sqrt{\pi}}(-1+6S(2))=0.4950190$	$1+\frac{5\sqrt{3}}{\pi}[-1+4S(3)]=0.5565627$
	3		0	$1+\frac{5\sqrt{3}}{\pi}[1-6S(3)]=0.2868337$
6	6	1	$\frac{15}{2\sqrt{\pi}}[1-4S(2)+2I_4^{(2)}]=1.2672064$	$1+\frac{5\sqrt{3}}{\pi}[1-3S(3)]=2.0217391$
	5	2	$\frac{15}{2\sqrt{\pi}}[-1+8S(2)-10I_4^{(2)}]=0.6417550$	$1+\frac{5\sqrt{3}}{\pi}[-2+9S(3)]=0.6914273$
	4	3	$\frac{30}{\sqrt{\pi}}[0-S(2)+5I_4^{(2)}]=0.2015468$	$1+\frac{5\sqrt{3}}{\pi}[1-6S(3)]=0.2868337$
7	7	1	$\frac{21}{2\sqrt{\pi}}[1-5S(2)+5I_4^{(2)}]=1.3521784$	$1+\frac{35\sqrt{3}}{\pi}\left[\frac{1}{4}-S(3)+\frac{1}{2}I_4^{(3)}\right]=2.2203043$
	6	2	$\frac{21}{2\sqrt{\pi}}[-1+10S(2)-20I_4^{(2)}]=0.7573743$	$1+\frac{35\sqrt{3}}{\pi}\left[-\frac{1}{2}+3S(3)-3I_4^{(3)}\right]=0.8303490$
	5	3	$\frac{105}{2\sqrt{\pi}}[0-S(2)+5I_4^{(2)}]=0.3527069$	$1+\frac{35\sqrt{3}}{\pi}\left[\frac{1}{4}-3S(3)+\frac{15}{2}I_4^{(3)}\right]=0.3441234$
	4		0	$1+\frac{35\sqrt{3}}{\pi}\left[0+2S(3)-10I_4^{(3)}\right]=0.2104481$

Where $S(\alpha)=\frac{1}{\pi}\sin^{-1}\sqrt{\frac{\alpha}{2(1+\alpha)}}$, so that $S(2)=\frac{1}{\pi}\sin^{-1}\frac{1}{\sqrt{3}}=0.1959132760$, $S(3)=\frac{1}{\pi}\sin^{-1}\sqrt{\frac{3}{8}}=0.2097846884$, and $S\left(\frac{1}{2}\right)=\frac{1}{\pi}\sin^{-1}\frac{1}{\sqrt{6}}=0.1338602364$. Also $I_4^{(\alpha)}=\frac{1}{\pi^2}\int_0^{\pi S(\alpha)}\sin^{-1}\sqrt{\frac{\alpha(\alpha+1)/2}{\alpha(\alpha+2)-\tan^2\psi}}d\psi$, so that $I_4^{(2)}=0.04156\ 42048$, and $I_4^{(3)}=0.04604\ 86206$.

16) Although some of our results apparently differ in form from those of Godwin, loc. cit. p. 284, all of them do really coincide, as e.g. $E(t_{3|4})=\frac{3}{\sqrt{\pi}}\left[\frac{6}{\pi}\sin^{-1}\frac{1}{\sqrt{3}}-1\right]=\frac{3}{\sqrt{\pi}}\left[\frac{1}{2}-\frac{3}{\pi}\sin^{-1}\frac{1}{3}\right]$, because $2\sin^{-1}\frac{1}{\sqrt{3}}+\sin^{-1}\frac{1}{3}=\frac{\pi}{2}$, &c. The same can be said about Table IV below.

Table IV (Expectations of Products $t_{i|n}t_{k|n}$ for $i\neq k$)

n	$t_i t_k$	$t_{n-k+1}t_{n-i+1}$	$E(t_{i\|n}t_{k\|n})=E(t_{n-k+1\|n}t_{n-i+1\|n})$
2	t_1t_2		0
3	t_1t_2	t_2t_3	$\frac{\sqrt{3}}{2\pi}=0.2756644$
	t_1t_3		$-\frac{\sqrt{3}}{\pi}=-0.5513289$
4	t_1t_2	t_3t_4	$\frac{\sqrt{3}}{\pi}=0.5513289$
	t_1t_3	t_2t_4	$\frac{3-2\sqrt{3}}{\pi}=0.1477281$
	t_1t_4		$-\frac{3}{\pi}=-0.9549297$
	t_2t_3		$\frac{2\sqrt{3}-3}{\pi}=0.1477281$
5	t_1t_2	t_4t_5	$\frac{5\sqrt{3}}{\pi}\left[\frac{1}{2}-S(3)\right]=0.8000204$
	t_1t_3	t_3t_5	$\frac{15}{\pi}\left[\frac{1}{2}-S\left(\frac{1}{2}\right)\right]+\frac{5\sqrt{3}}{\pi}[-1+2S(3)]=0.1481477$
	t_1t_4	t_2t_5	$\frac{15}{\pi}\left[-\frac{1}{2}+3S\left(\frac{1}{2}\right)\right]=-0.4699175$
	t_1t_5		$\frac{15}{\pi}\left[0-2S\left(\frac{1}{2}\right)\right]=-1.2782711$
	t_2t_3	t_3t_4	$\frac{15}{\pi}\left[-\frac{1}{2}+S\left(\frac{1}{2}\right)\right]+\frac{5\sqrt{3}}{\pi}\left[\frac{1}{2}+S(3)\right]=0.2084354$
	t_2t_4		$\frac{15}{\pi}\left[1-4S\left(\frac{1}{2}\right)\right]+\frac{5\sqrt{3}}{\pi}[0-4S(3)]=-0.0951011$
6	t_1t_2	t_5t_6	$\frac{5\sqrt{3}}{\pi}[1-3S(3)]=1.0217391$
	t_1t_3	t_4t_6	$\frac{45}{\pi}\left[\frac{1}{2}-S(2)-S\left(\frac{1}{2}\right)\right]+\frac{5\sqrt{3}}{\pi}[-2+6S(3)]=0.3948367$
	t_1t_4	t_3t_6	$\frac{45}{\pi}\left[-1+3S(2)+3S\left(\frac{1}{2}\right)\right]=-0.1529720$
	t_1t_5	t_2t_6	$\frac{45}{\pi}\left[1-4S(2)-2S\left(\frac{1}{2}\right)\right]=-0.7358723$
	t_1t_6		$\frac{45}{\pi}\left[-\frac{1}{2}+2S(2)\right]=-1.5494705$
	t_2t_3	t_4t_5	$\frac{45}{\pi}\left[-\frac{1}{2}+S(2)+S\left(\frac{1}{2}\right)\right]+\frac{5\sqrt{3}}{\pi}[1+0]=0.3183300$
	t_2t_4	t_3t_5	$\frac{45}{\pi}\left[\frac{3}{2}-3S(2)-5S\left(\frac{1}{2}\right)\right]+\frac{5\sqrt{3}}{\pi}[0-6S(3)]=0.0103204$
	t_2t_5		$\frac{45}{\pi}\left[-2+6S(2)+6S\left(\frac{1}{2}\right)\right]=-0.3059441$
	t_3t_4		$\frac{45}{\pi}\left[-\frac{1}{2}+0+2S\left(\frac{1}{2}\right)\right]+\frac{5\sqrt{3}}{\pi}[0+6S(3)]=0.1426517$

Table IV (Continued)

n	$t_i t_k$	$t_{n-k+1} t_{n-i+1}$	$E(t_{i\mid n} t_{k\mid n}) = E(t_{n-k+1\mid n} t_{n-i+1\mid n})$	
7	$t_1 t_k$	$t_6 t_7$	$\frac{35\sqrt{3}}{\pi}\left[\frac{1}{4} - S(3) + \frac{1}{2} I_4^{(3)}\right]$	$=1.2203041$
	$t_1 t_3$	$t_5 t_7$	$\frac{105}{\pi}\left[\frac{1}{2} - S(2) - S\left(\frac{1}{2}\right) - I_{3,0} + 0\right] + \frac{35\sqrt{3}}{\pi}\left[-\frac{1}{2} + 2S(3) - I_4^{(3)}\right]$	$=0.6090384$
	$t_1 t_4$	$t_4 t_7$	$\frac{105}{\pi}\left[-\frac{5}{4} + \frac{5}{2} S(2) + 4S\left(\frac{1}{2}\right) + 4I_{3,0} - 3I_{2,1}\right]$	$=0.0984870$
	$t_1 t_5$	$t_3 t_7$	$\frac{105}{\pi}\left[\frac{3}{2} - 3S(2) - 6S\left(\frac{1}{2}\right) - 6I_{3,0} + 12I_{2,1}\right]$	$=-0.4003630$
	$t_1 t_6$	$t_2 t_7$	$\frac{105}{\pi}\left[-\frac{3}{4} + \frac{1}{2} S(2) + 5S\left(\frac{1}{2}\right) + 5I_{3,0} - 15I_{2,1}\right]$	$=-0.9641864$
	$t_1 t_7$		$\frac{105}{\pi}\left[0 + S(2) - 2S\left(\frac{1}{2}\right) - 2I_{3,0} + 6I_{2,1}\right]$	$=-1.7835842$
	$t_2 t_3$	$t_5 t_6$	$\frac{105}{\pi}\left[-\frac{1}{2} + S(2) + S\left(\frac{1}{2}\right) + I_{3,0} + 0\right] + \frac{35\sqrt{3}}{\pi}\left[\frac{1}{4} + 0 - \frac{3}{2} I_4^{(3)}\right]$	$=0.4416147$
	$t_2 t_4$	$t_4 t_6$	$\frac{105}{\pi}\left[2 - 4S(2) - 7S\left(\frac{1}{2}\right) - 4I_{3,0} + 6I_{2,1}\right] + \frac{35\sqrt{3}}{\pi}\left[0 - 2S(3) + 4I_4^{(3)}\right]$	$=0.1307293$
	$t_2 t_5$	$t_3 t_6$	$\frac{105}{\pi}\left[-\frac{15}{4} + \frac{21}{2} S(2) + 15S\left(\frac{1}{2}\right) + 6I_{3,0} - 27I_{2,1}\right]$	$=-0.1651763$
	$t_2 t_6$		$\frac{105}{\pi}\left[3 - 8S(2) - 14S\left(\frac{1}{2}\right) - 8I_{3,0} + 36I_{2,1}\right]$	$=-0.4936345$
	$t_3 t_4$	$t_4 t_5$	$\frac{105}{\pi}\left[-\frac{3}{4} + \frac{3}{2} S(2) + 3S\left(\frac{1}{2}\right) + 0 - 3I_{2,1}\right] + \frac{35\sqrt{3}}{\pi}\left[0 + S(3) + I_4^{(3)}\right]$	$=0.1655597$
	$t_3 t_5$		$\frac{105}{\pi}\left[3 - 9S(2) - 12S\left(\frac{1}{2}\right) + 0 + 18I_{2,1}\right] + \frac{35\sqrt{3}}{\pi}\left[0 + 0 - 6I_4^{(3)}\right]$	$=0.0052035$

Where
$$I_{2,1} = \frac{1}{\pi^2}\int_0^1 \frac{\sec^{-1}(u^2+5)}{(u^2+2)\sqrt{u^2+4}}\,du = 0.02940\ 08395\,,$$
$$I_{3,0} = I_{2,1} + \frac{1}{\pi^2}\int_0^1 \frac{\sec^{-1}(2v^2+4)}{(2v^2+1)\sqrt{2v^2+3}}\,dv = 0.07898\ 12767\,.$$

§10. Checks by Numerical Integrations. We have so far obtained explicit forms of all $E(t_{i\mid n})$, $E(t_{i\mid n}^2)$ and $E(t_{i\mid n} t_{k\mid n})$ up to $n=7$. They are expressed by the combinations of constants $S(2)$, $S(3)$, $S\left(\frac{1}{2}\right)$ and integrals $I_4^{(\alpha)}$, $I_{2,1}$, $I_{3,0}$. But, to check our results e.g. by comparing with Godwin's, their numerical values are requisite.

First to compute

(10. 1) $$S(\alpha) = \frac{1}{\pi}\sin^{-1}\sqrt{\frac{\alpha}{2(1+\alpha)}} = \frac{1}{2\pi}\sec^{-1}(1+\alpha)\,,$$

we need to know inverse-secants, which would be found e.g. from Chamber's seven-figures mathematical tables. In fact, by making use of the law of P. P., we found

$$S(2) = 0.19591\ 32677,\ S(3) = 0.20978\ 46759,\ S\left(\frac{1}{2}\right) = 0.13386\ 02324\,.$$

To ascertain the precision of these figures, we have alternatively evaluated them by expanding $\sec^{-1}X$ into series:

$$\text{(10. 2)}\quad \sec^{-1}X=\frac{\pi}{2}-\sum_{\nu=0}^{\infty}\frac{1.3.\cdots(2\nu-1)}{2.4.\cdots 2\nu}\frac{1}{2\nu+1}\frac{1}{X^{2\nu+1}}=\frac{\pi}{2}-\frac{1}{X}\sum_{\nu=0}^{\infty}\frac{\underline{|2\nu}}{(2^{\nu}\underline{|\nu})^2}\frac{X^{-2\nu}}{2\nu+1}$$

$$=\frac{\pi}{2}-\left[\frac{1}{X}+\frac{1}{6X^3}+\frac{3}{40}\frac{1}{X^5}+\frac{5}{112}\frac{1}{X^7}+\frac{35}{1152}\frac{1}{X^9}+\frac{63}{2816}\frac{1}{X^{11}}+\cdots\right]$$

and obtained, by taking a sufficiently many number of terms,

$$\text{(10. 3)}\quad S(2)=0.19591\ 32760,\ S(3)=0.20978\ 46884,\ S\left(\frac{1}{2}\right)=0.13386\ 02364\,.$$

Thus even those before obtained from simple P. P. already agree with true values up to the seventh decimal place, so that by the following numerical integrations the values of inverse-secants were frequently evaluated simply by aid of the law of P. P. from Chamber's tables, the results of which are however reliable to seven efficient figures.

Next to evaluate

$$\text{(10. 4)}\quad I_4^{(\alpha)}=\frac{1}{\pi^2}\int_0^{\pi S(\alpha)}\sin^{-1}\sqrt{\frac{\alpha(\alpha+1)/2}{\alpha(\alpha+2)-\tan^2\psi}}\,d\psi=\frac{1}{2\pi^2}\int_0^{\pi S(\alpha)}\sec^{-1}\left\{1+\frac{\alpha(\alpha+1)}{\alpha-\tan^2\psi}\right\}d\psi$$

$$(\alpha=2,\ 3)\,.$$

After Gauss, putting $\psi=\frac{1}{2}(1+t)\pi S(\alpha)$ and $\sec^{-1}\left\{1+\frac{\alpha(\alpha+1)}{\alpha-\tan^2\psi}\right\}=f(t)$, we have

$$I_4^{(\alpha)}=\frac{1}{2\pi}S(\alpha)\sum_{\nu=1}^{n}R_\nu f(t_\nu)\,,\qquad (\alpha=2,\ 3)\,.$$

Since $\tan S(2)=0.198\cdots\ll\sqrt{2}$ and $\tan S(3)=0.212\cdots\ll\sqrt{3}$, the integrand of (10. 4) is surely regular in $|t|\leqq 1$, so that its Maclaurin's expansion $\sum_{\nu=0}^{\infty}c_\nu t^\nu$ converges absolutely and uniformly in $|t|\leqq 1$. Therefore, if m be taken appropriately large, the integrand may be approximated by $\sum_{\nu=0}^{m}c_\nu t^\nu$, a polynomial of degree m. Hence, Gauss' method of numerical integrations by n selected ordinates would certainly give a good approximation for the integral, if $m\leqq 2n-1$. We have taken as $n=5$, $m=9$, and obtained

$$\text{(10. 5)}\qquad I_4^{(2)}=0.04156\ 420,\quad I_4^{(3)}=0.04604\ 862\,.$$

To secure how many figures are correct, we have after Gauss to find the upper bound of errors

$$F_n=c_{2n}\Omega_n\,,\qquad \text{where } c_{2n}=\frac{1}{\underline{|2n}}f^{(2n)}(0)\,.$$

However, as it is utterly cumbersome to get the $2n$-th (here tenth) derivative, so we proceed to compute Gauss' approximations at any rate, and check them for some known results. Really, we calculated $I_4^{(2)}$, $I_4^{(3)}$ from our Table III, using some explicit expressions of $E(t_{i|n})$, $E(t_{i|n}^2)$, say $E(t_{1|7})$, $E(t_{4|7}^2)$, whose numerical values are given in Godwin's paper, and found

$$I_4^{(2)}=0.04156\ 420,\quad I_4^{(3)}=0.04604\ 862\,.$$

These coincide precisely with our results (10. 5).

Lastly to evaluate

$$\pi^2 I_{2,1}=\int_0^1 \frac{\sec^{-1}(5+x^2)}{(2+x^2)\sqrt{4+x^2}}\,dx=\int_{-1}^0=\frac{1}{2}\int_{-1}^1 f(x)dx\,. \tag{10. 6}$$

Here the integrand $f(x)$ is also regular in $(-1-\varepsilon, 1+\varepsilon)$ and expansible in a Maclaurin's series, which is uniformly and absolutely convergent in $(-1, 1)$ Hence, on taking the partial sum $\sum_{\nu=0}^{m} c_\nu x^\nu$ adequately, again Gauss' method is applicable. However, now that

$$\left(1+\frac{1}{2}x^2\right)^{-1}=1-0.5x_2+0.25x^4-0.125x^6+0.0625x^8-0.03125x^{10}-\cdots \&c.,$$

the convergency is rather slow in the vicinity of $x=1$, and if we make $n=5$, $m=9$, the error shall possibly take place at the third decimal place or thereabouts. Also, in the integral

$$\pi^2(I_{3,0}-I_{2,1})=\int_0^1 \frac{\sec^{-1}(4+2x^2)}{(1+2x^2)\sqrt{3+2x^2}}\,dx=\frac{1}{2}\int_{-1}^1 f(x)dx \tag{10. 7}$$

the factor $(1+2x^2)^{-1}$ being already not regular on $|x|=1/\sqrt{2}$ in the complex x–plane, the relating Maclaurin's series cannot be uniformly convergent in $|x|\leqq 1$, so that the applicability of Gauss' method becomes even doubtful. In fact, on applying this method, as $n=7$, we obtained

$$I_{2,1}=0.02943\ 033,\quad I_{3,0}=0.07902\ 063,$$

which are inacculate, since they make

$$\mathrm{Cov}(t_{1|7}, t_{4|7})=E(t_{1|7}t_{4|7})=\frac{105}{\pi}\left[-\frac{5}{4}+\frac{5}{2}S(2)+4S\left(\frac{1}{2}\right)+4I_{3,0}-3I_{2,1}\right]=0.10079\cdots,$$

while Godwin's result informs to be 0.09849. Of course, if we put $x=\frac{1}{2}(1+t)$ and $\dfrac{\sec^{-1}\left\{5+\frac{1}{2}(1+t)^2\right\}}{[8+(1+t)^2]\sqrt{(16+(1+t)^2}}=g(t)$, we get $I_{2,1}=\frac{8}{\pi^2}\times\frac{1}{2}\int_{-1}^1 g(t)dt=\frac{8}{\pi^2}\sum_{\nu=1}^{n} R_\nu g(t_\nu)$. Whence we found, as $n=5$, $I_{2,1}=0.02940\ 083$ and similarly $I_{3,0}=0.07901\ 226$ but $\mathrm{Cov}(t_{1|7}, t_{4|7})=0.10263$. This is worse than before obtained one, in which however we took, as $n=7$.

Rather, if computed after Simpson's rule with twenty subdivisions, we obtain

$$I_{2,1}=0.02940\ 082,\quad I_{3,0}=0.07898\ 295\quad \text{and}\quad \mathrm{Cov}\,(t_{1|7}, t_{4|7})=0.09870\,,$$

which is still aberrant from Godwin's result though, yet somewhat nearer than those obtained before by Gauss' method[17]. Therefore we have contrived another method of approximation as follows:

On applying (10.2) the inverse-secant in the integrand of (10.6) is

$$\sec^{-1}(x^2+5)=\frac{\pi}{2}-\sum_{\nu=0}^{\infty}\frac{\lfloor 2\nu}{(2^\nu \lfloor \nu)^2(2\nu+1)(x^2+5)^{2\nu+1}},$$

in which, already for $\nu=5$, the corresponding term becomes $<\frac{1}{10^9}$ and also the total remainder inclusive this term $R_5<\frac{1}{10^9}\left[1+\frac{1}{5^2}+\frac{1}{5^4}+\cdots\right]<\frac{2}{10^9}$. Hence, when R_5 be substituted in (10.6) and integrated, it would be at most amount $<\frac{1}{4\pi^2}\times\frac{2}{10^9}<\frac{1}{10^{10}}$. Therefore, to obtain an approximate value of $I_{2,1}$ up to the tenth decimal place, we may neglect R_5 and take only five terms in summation. Thus it suffices to compute

(10.8) $$I_\nu=\int_0^1\frac{dx}{(x^2+2)\sqrt{x^2+4}\,(x^2+5)^{2\nu-1}}\quad\text{for}\quad \nu=\frac{1}{2},1,\cdots,5.$$

Or, putting $x=2\tan\theta$, we obtain

(10.9) $$I_\nu=\frac{1}{2}\int_0^{\tan^{-1}1/2}\frac{\cos^{4\nu-1}\theta\,d\theta}{(1+\sin^2\theta)(5-\sin^2\theta)^{2\nu-1}}=\frac{1}{2}\int_0^{1/\sqrt{5}}\frac{(1-t^2)^{2\nu-1}\,dt}{(1+t^2)(5-t^2)^{2\nu-1}}\quad (t=\sin\theta).$$

17) We have also tried Markov-Berger's method of numerical integrations; Really we have from formulas in Tables II and III

$$K_{3,0}=\int U^3\varphi^2\,dt\int^t\varphi_1^2dt_1=\frac{1}{8\pi}\left[I_{3,0}-\frac{1}{2}S(2)+\frac{1}{4}S\left(\frac{1}{2}\right)\right].$$

On the other hand after Markov and Berger

$$K_{3,0}=\frac{1}{4\pi}\int\frac{e^{-t^2}}{\sqrt{\pi}}U^3\,dt\int^t\frac{e^{-t_1^2}}{\sqrt{\pi}}dt_1=\int\frac{e^{-t^2}}{\sqrt{\pi}}f(t)\,dt=\sum_{i=1}^{n}A_if(t_i),$$

where

$$f(t)=\frac{1}{4\pi}\left\{\int_0^t e^{-t^2/2}\frac{dt}{\sqrt{2\pi}}\right\}^3\int_{-\infty}^t e^{-t^2}\frac{dt}{\sqrt{\pi}}$$

can be found from the tables of Probability Integrals. Likewise

$$K_{2,1}=\frac{1}{8\pi}\left[I_{2,1}-\frac{1}{4}S\left(\frac{1}{2}\right)\right]$$

and $$K_{2,1}=\int U^2\varphi^2\,dt\int^t U_1\varphi_1^2\,dt_1=\int\frac{e^{-t^2}}{\sqrt{\pi}}f(t)dt=\sum_{i=1}^{n}A_if(t_i),$$

if $f(t)=U^2(t)V(t)$ and $V(t)=\frac{1}{2\sqrt{\pi}}\int^t U_1\varphi_1^2\,dt_1=\int^0+\int_0^t=\text{(i)}+\text{(ii)}$,

where $$\text{(i)}=\frac{1}{2\sqrt{\pi}}\int^0 U_1\varphi_1^2\,dt_1=\frac{1}{2\sqrt{\pi}}\int_0 e^{-t_1^2}\frac{dt}{2\pi}\int_0^{t_1}e^{-\tau^2/2}\frac{d\tau}{\sqrt{2\pi}}$$

$$=\frac{1}{4\pi^2\sqrt{2}}\int_{\pi/4}^{\pi/2}d\theta\int_0\exp\left\{-\frac{r^2}{2}(\cos^2\theta+2\sin^2\theta)\right\}r\,dr=\frac{1}{8\pi^2}\tan^{-1}\frac{1}{\sqrt{2}}$$

and $$\text{(ii)}=\frac{1}{4\pi^2\sqrt{2}}\int_{\pi/4}^{\pi/2}d\theta\int_0^{t\,\mathrm{cosec}\,\theta}\exp\left\{-\frac{r^2}{2}(\cos^2\theta+2\sin^2\theta)\right\}r\,dr$$

$$=\frac{1}{4\pi^2\sqrt{2}}\int_0^1\frac{1-\exp\{-\frac{1}{2}(1+u^2)t^2\}}{2+u^2}du\qquad (u=\cot\theta)$$

which for every $t=t_i$ can be computed by Gauss. Thus when $K_{3,0}$ and $K_{2,1}$ be found, we can thereby calculate $I_{3,0}$ and $I_{2,1}$. However, the results obtained as $n=7$ were very unpleasing.

These were exactly integrated, and the required integral is nearly

(10. 10) $$I_{2,1}=\frac{1}{\pi^2}\left[\frac{\pi}{2}I_{\frac{1}{2}}-\left(I_1+\frac{1}{6}I_2+\frac{3}{40}I_3+\frac{5}{112}I_4+\frac{35}{1152}I_5\right)\right]=0.02940\ 08395.$$

Similarly with (10. 7), we have

(10. 11) $$I_{3,0}=I_{2,1}+\frac{1}{\pi^2}\left[\frac{\pi}{2}J_{\frac{1}{2}}-\left(J_1+\frac{1}{6}J_2+\frac{3}{40}J_3+\frac{5}{112}J_4+\frac{35}{1152}J_5\right)\right],$$

where

(10. 12) $$J_\nu=\int_0^1\frac{dx}{(1+2x^2)\sqrt{3+2x^2}\ (4+2x^2)^{2\nu-1}}$$
$$=\frac{1}{\sqrt{2}}\int_0^{\tan^{-1}\sqrt{2/3}}\frac{\cos^{4\nu-1}\theta\, d\theta}{(1+2\sin^2\theta)(4-\sin^2\theta)^{2\nu-1}}\quad\left(x=\sqrt{\frac{3}{2}}\tan\theta\right)$$
$$=\frac{1}{\sqrt{2}}\int_0^{\sqrt{0\cdot4}}\frac{(1-t^2)^{2\nu-1}\,dt}{(1+2t^2)(4-t^2)^{2\nu-1}}\left(t=\sin\theta,\ \nu=\frac{1}{2},1,2,\cdots,5\right).$$

Whence its numerical value was obtained as

(10. 13) $$I_{3,0}=0.0789812767,\quad\text{approximately.}$$

These values being substituted, now yields

$$\text{Cov}\ (t_{1|7},t_{4|7})=0.0984869917$$

which coincides with Godwin's result.

By making use of (10. 3) (10. 5) (10. 10) and (10.13) numerical values in Tables III and IV were obtained up to the seventh decimal place, all of which agree with those in Godwin's paper.

§ **11. Variance $\mathbf{D}^2(\zeta)$.** We have seen in §6 that, if $z=\sum c_ix_i$, $\sum c_i=1$, $x_i=m+\sigma t_i$, $\zeta=\sum c_it_i$, so $E(z)=m+\sigma E(\zeta)$, $D^2(z)=\sigma^2D^2(\zeta)$ with $E(\zeta)=\sum c_iE(t_i)$ and

(11. 1) $$D^2(\zeta)=\sum_{i=1}^n c_i^2\ \text{Var}\ (t_{i|n})+\sum_{k=1}^n\sum_{k\neq i}c_ic_k\ \text{Cov}\ (t_{i|n},t_{k|n})$$
$$=\sum_{i=1}^n c_i^2[E(t_i^2)-E(t_i)^2]+\sum_{i=1}^n\sum_{k\neq i}c_ic_k[E(t_it_k)-E(t_i)E(t_k)]=u\ \text{say.}$$

Let us consider its relative minimum under condition $\sum_{i=1}^n c_i=1$. For this we have to find the absolute minimum of the function.

(11. 2) $$w=u-2\lambda v\qquad\text{with}\qquad v=\sum c_i-1,$$

so that

(11. 3) $$\frac{1}{2}\frac{\partial w}{\partial c_j}=c_j[E(t_j^2)-E(t_j)^2]+\sum_{k\neq j}c_k[E(t_jt_k)-E(t_j)E(t_k)]-\lambda=0,$$

viz.

$$\sum_{k=1}^n c_k[E(t_jt_k)-E(t_j)E(t_k)]=\lambda\qquad(j=1,2,\cdots,n).$$

Hence we obtain by G. Cramer's formula

(11.4) $$c_k = \Delta_k/\Delta \qquad (k=1, 2, \cdots, n),$$

where

$$\Delta = \begin{vmatrix} E(t_1t_1)-E(t_1)E(t_1) & \cdots & E(t_1t_k)-E(t_1)E(t_k) & \cdots & E(t_1t_n)-E(t_1)E(t_n) \\ \cdots\cdots & \cdots & \cdots\cdots & \cdots & \cdots\cdots \\ E(t_it_1)-E(t_i)E(t_1) & \cdots & E(t_it_k)-E(t_i)E(t_k) & \cdots & E(t_it_n)-E(t_i)E(t_n) \\ \cdots\cdots & \cdots & \cdots\cdots & \cdots & \cdots\cdots \\ E(t_nt_1)-E(t_n)E(t_1) & \cdots & E(t_nt_k)-E(t_n)E(t_k) & \cdots & E(t_nt_n)-E(t_n)E(t_n) \end{vmatrix}$$

$$= \begin{vmatrix} E(t_1t_1)-E(t_1)E(t_1) & \cdots & \sum_{k=1}^{n} E(t_1t_k)-E(t_1)\sum_{k=1}^{n} E(t_k) & \cdots & E(t_1t_n)-E(t_1)E(t_n) \\ \cdots\cdots & \cdots & \cdots\cdots & \cdots & \cdots\cdots \\ E(t_it_1)-E(t_i)E(t_1) & \cdots & \sum_{k=1}^{n} E(t_it_k)-E(t_i)\sum_{k=1}^{n} E(t_k) & \cdots & E(t_it_n)-E(t_i)E(t_n) \\ \cdots\cdots & \cdots & \cdots\cdots & \cdots & \cdots\cdots \\ E(t_nt_1)-E(t_n)E(t_1) & \cdots & \sum_{k=1}^{n} E(t_nt_k)-E(t_n)\sum_{k=1}^{n} E(t_k) & \cdots & E(t_nt_n)-E(t_n)E(t_n) \end{vmatrix}.$$

But $\sum_{k=1}^{n} E(t_it_k)-E(t_i)\sum_{k=1}^{n} E(t_k)=1$ $(i=1, 2, \cdots, n)$, in view of (5.10) and (3.1), so

$$\Delta_k = \lambda\Delta \quad \text{and consequently} \quad c_k=\lambda \quad (k=1, 2, \cdots, n).$$

Hence $c_1=c_2=\cdots=c_n=\frac{1}{n}$ and $E(\zeta)=\frac{1}{n}\sum E(t_k)=0$, $E(z)=\frac{1}{n}\sum E(x_i)=\frac{1}{n}\sum [m+\sigma E(t_i)]=m$. Thus, the theorem that the A. M. $\bar{x}=\frac{1}{n}\sum x_i$ is the efficient estimate of population mean, which is the case for any unordered sample with independent individuals, still remains true for any ordered sample with non-independent individuals also. In fact, $E(\zeta)=0$, $E(\bar{x})=m$, so $\bar{x}$ is unbiased, and besides, in virtue of (5.3) and (3.1)

$$D^2(\bar{t}) = \frac{1}{n^2}\left[\sum_{i=1}^{n} E(t_i^2)+\sum_{i=1}^{n-1}\sum_{k>i}^{n} E(t_it_k)-\left\{\sum_{i=1}^{n} E(t_i)\right\}^2\right] = \frac{1}{n^2}[n+0+0] = \frac{1}{n}.$$

Hence

(11.5) $$D^2(\bar{x}) = \sigma^2/n.$$

More specially, if we consider the case that $c_{n-i+1}=c_i$ for $i=1, 2, \cdots, [n/2]$ with $\sum_{i=1}^{n} c_i=1$, and if $n=2p+1$, making $c_{p+1}=1-2\sum_{i=1}^{p} c_i$, we see that by the above theorem that every $D^2(\zeta)$ cannot be smaller than $1/n$. This can be directly shown by formation of actual expressions, on substituting those values $E(t_{i|n})$, $E(t_{i|n}^2)$ and $E(t_{i|n}t_{k|n})$ of Tables III and IV in (11.1):

1° $n=3$ (Cramér's example): $c_1=c_3=c$, $c_2=1-2c$. We get readily

(11.6) $$D^2(\zeta) = \frac{1}{3}+\frac{3}{\pi}(2\pi-3\sqrt{3})\left(c-\frac{1}{3}\right)^2, \quad D^2(z)=\sigma^2D^2(\zeta) \geqq \frac{\sigma^2}{3}.$$

2° $n=4$: $c_1=c_4=c$, $c_2=c_3=\frac{1}{2}-c$.

(11.7) $$D^2(\zeta)=\frac{1}{4}+\frac{1}{\pi}(2\pi+4\sqrt{3}-3)\left(c-\frac{1}{4}\right)^2,\quad D^2(z)\gtrsim\frac{\sigma^2}{4}.$$

3° $n=5$: $c_1=c_5=c$, $c_2=c_4=c'$, $c_3=1-2c-2c'$. We assume $D^2(\zeta)$ to be of the form:

(11.8) $$D^2(\zeta)=\frac{1}{5}+A\left(c+c'-\frac{2}{5}\right)^2+B\left(c-\frac{1}{5}\right)^2+C\left(c'-\frac{1}{5}\right)^2.$$

To find A, we ask the coefficient of $2cc'$ in the obtained expression of $D^2(\zeta)$. This is really

$$A=4+\frac{5\sqrt{3}}{\pi}[7-38S(3)]+\frac{15}{\pi}\left[6S\left(\frac{1}{2}\right)-1\right]>0.$$

On subtractidg $\frac{1}{5}+A\left(c+c'-\frac{2}{5}\right)^2$ from $D^2(\zeta)$, the remainder decomposes into a sum of squares whose coefficients are

$$B=2+\frac{10\sqrt{3}}{\pi}[3-2S(3)]-\frac{15}{\pi}\left[3+2S\left(\frac{1}{2}\right)\right]>0,$$

$$C=2-\frac{15\sqrt{3}}{\pi}[3-2S(3)]+\frac{15}{\pi}\left[7-11S\left(\frac{1}{2}\right)\right]>0.$$

4° $n=6$: $c_1=c_6=c$, $c_2=c_5=c'$, $c_3=c_4=\frac{1}{2}-c-c$. Assuming again as before

(11.9) $$D^2(\zeta)=\frac{1}{6}+A\left(c+c'-\frac{1}{3}\right)^2+B\left(c-\frac{1}{6}\right)^2+C\left(c'-\frac{1}{6}\right)^2,$$

and asking the coefficient of $2cc'$, we find

$$A=2+\frac{30\sqrt{3}}{\pi}[1-S(3)]+\frac{180}{\pi}\left[-2S(2)+S\left(\frac{1}{2}\right)\right]>0.$$

Subtracting $\frac{1}{6}+A\left(c+c'-\frac{1}{3}\right)^2$ from the expression of $D^2(\zeta)$, the remainder decomposes into a sum of squares whose coefficients are

$$B=2+\frac{30\sqrt{3}}{\pi}[1-4S(3)]+\frac{180}{\pi}\left[S(2)-2S\left(\frac{1}{2}\right)\right]>0,$$

$$C=2+\frac{60\sqrt{3}}{\pi}[4S(3)-1]+\frac{45}{\pi}\left[-9+28S(2)+28S\left(\frac{1}{2}\right)\right]>0.$$

5° $n=7$: $c_1=c_7=c$, $c_2=c_6=c'$, $c_3=c_5=c''$, $c_4=1-2c-2c'-2c''$. Assuming

(11.10) $$D^2(\zeta)=\frac{1}{7}+A\left(c'+c''-\frac{2}{7}\right)^2+B\left(c+c''-\frac{2}{7}\right)^2+C\left(c+c'-\frac{2}{7}\right)^2$$
$$+H\left(c-\frac{1}{7}\right)^2+K\left(c'-\frac{1}{7}\right)^2+L\left(c''-\frac{1}{7}\right)^2,$$

and asking the coefficients of $2c'c''$, $2cc''$ and $2cc'$, we find

$$A=4+\frac{35\sqrt{3}}{\pi}\left[\frac{1}{2}+12S(3)-63I_4^{(3)}\right]+\frac{105}{\pi}\left[-\frac{27}{2}+33S(2)+48S\left(\frac{1}{2}\right)\right.$$
$$\left.-66I_{2,1}+30I_{3,0}\right]>0,$$

$$B=4+\frac{35\sqrt{3}}{\pi}\left[-1+8S(3)-46I_4^{(3)}\right]+\frac{105}{\pi}\left[12-24S(2)-42S\left(\frac{1}{2}\right)\right.$$
$$\left.+48I_{2,1}-30I_{3,0}\right]>0,$$

$$C=4+\frac{35\sqrt{3}}{\pi}\left[\frac{1}{2}+14S(3)-55I_4^{(3)}\right]+\frac{105}{\pi}\left[-\frac{9}{2}+7S(2)+22S\left(\frac{1}{2}\right)\right.$$
$$\left.-42I_{2,1}+10I_{3,0}\right]>0.$$

Subtracting $\frac{1}{7}+A\left(c'+c''-\frac{2}{7}\right)^2+B\left(c+c''-\frac{2}{7}\right)^2+C\left(c+c'-\frac{2}{7}\right)^2$ from the whole expression of $D^2(\zeta)$, we see that the remainder decomposes into a sum of three squares whose coefficients are

$$H=-2+\frac{35\sqrt{3}}{\pi}[1-16S(3)+62I_4^{(3)}]+\frac{105}{\pi}\left[\frac{5}{2}-S(2)-16S\left(\frac{1}{2}\right)\right.$$
$$\left.+30I_{2,1}-16I_{3,0}\right]>0,$$

$$K=-2+\frac{35\sqrt{3}}{\pi}[-2+4S(3)+62I_4^{(3)}]+\frac{105}{\pi}\left[8-24S(2)-42S\left(\frac{1}{2}\right)\right.$$
$$\left.+132I_{2,1}-24I_{3,0}\right]>0,$$

$$L=-2+\frac{35\sqrt{3}}{\pi}[1-26S(3)+64I_4^{(3)}]+\frac{105}{\pi}\left[\frac{27}{2}-39S(2)-54S\left(\frac{1}{2}\right)+78I_{2,1}\right]>0.$$

If we take a single $x_{i|n}$ as z, we get $E(z)=m+\sigma E(t_{i|n})$. Thus there gives rise to a biase $\sigma E(t_{i|n})$, which is $\neq 0$, except the case of median, and $D^2(z)=\sigma^2$ Var $(t_{i|n})$. All values of $E(t_{i|n})$, Var $(t_{i|n})$ as well as Cov $(t_{i|n}, t_{k|n})$ for $n=2, 3, \cdots, 10$ are given to five decimal places in Godwin's paper loc. cit., pp. 281–2. Only when $n=2p+1$, $i=p+1$, we have $E(t_{p+1|2p+1})=0$ and $x_{p+1|2p+1}$, as single observation, renders an unbiased estimate of the population mean with efficiency

$$\text{(11. 11)} \qquad \text{eff.}=\frac{1}{n}:\ \text{Var}\ (t_{p+1|2p+1})=1/nE(t^2_{p+1|2p+1})\,.$$

Really, using Godwin's Table, we get these efficiencies as those starred in Table V. Lastly, every $z=\frac{1}{2}(x_{i|n}+x_{n-i+1|n})$ or $\zeta=\frac{1}{2}(t_{i|n}+t_{n-i+1|n})$ gives an unbiased estimate, its variance being

$$D^2(\zeta)=E(\zeta)^2-E(\zeta)^2=\frac{1}{4}\{E(t_i^2+t_{n-i+1}^2+2t_it_{n-i+1})-[E(t_i)+E(t_{n-i+1})]^2\}$$
$$=\frac{2}{4}\{E(t_i^2)-E(t_i)^2+E(t_it_{n-i+1})-E(t_i)E(t_{n-i+1})\}=\frac{1}{2}[\text{Var}(t_i)+\text{Cov}(t_i, t_{n-i+1})].$$

Hence its efficiency is

$$\text{(11. 12)} \qquad \text{eff.}=2/n[\text{Va}(t_{i|n})+\text{Cov}(t_{i|n}t_{n-i+1|n})]\,.$$

These are also calculated from Godwin's Table and tabulated in Table V, below.

Table V. (Efficiencies of estiamates $\frac{1}{2}(t_{i|n}+t_{n-i+1|n})$)

n	i	1	2	3	4	5
2	2	1				
3	2 3	0.92038	0.74294*			
4	3 4	0.83836	0.83836			
5	3 4 5	0.76665	0.86681	0.69728*		
6	4 5 6	0.70581	0.86470	0.77613		
7	4 5 6 7	0.65423	0.84855	0.81792	0.67882*	
8	5 6 7 8	0.61014	0.82604	0.83727	0.74323	
9	5 6 7 8 9	0.57213	0.80106	0.84300	0.78460	0.66894*
10	6 7 8 9 10	0.53895	0.77555	0.84027	0.81008	0.72294

§ **12. Truncated Samples.** If a random variable ξ distribute logarithmico-normally, viz. its fr. f. be

$$\frac{1}{\sqrt{2\pi}\,\sigma(\xi-a)}\exp\left\{-\frac{1}{2\sigma^2}[\log(\xi-a)-m]^2\right\}, \qquad (\xi>a)$$

the variable $x=\log(\xi-a)$ distributes normally, viz. its fr. f. becomes

$$\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left\{-\frac{1}{2\sigma^2}(x-m)^2\right\}, \qquad (-\infty<x<\infty).$$

Hence, putting again $x=m+\sigma t$, the problem reduces to our case.

We are now interested in the so-called truncated sample[18]: namely, when only $k(<n)$ values $\xi_1\leqq\xi_2\leqq\cdots\leqq\xi_k$ are observed, but there experiment being stopped, the remaining values $\xi_{k+1}\leqq\cdots\leqq\xi_n$ left unmeasured (missed), required is how to estimate the mean and variance of the population? To determine

18) Cf. e.g. A. C. Cohen, Estimating the mean and variance of normal populations from simply truncated and doubly truncated sample, Math. Statis., Vol. 21 (1950), pp. 557–569.

them we are used to avail the so-called likelihood function, which is obtainable as follows.

The probability element to obtain $\{\xi_1, \xi_2, \cdots, \xi_k\}$ or $\{t_1, t_2, \cdots, t_k\}$ is

$$n!\, d\Phi_1 d\Phi_2 \cdots d\Phi_k \int_k d\Phi_{k+1} \cdots \int_{n-1} d\Phi_n$$
$$= \frac{n!}{(n-k)!}(1-\Phi_k)^{n-k} d\Phi_1 d\Phi_2 \cdots d\Phi_k, \qquad (1<k<n)^{19)}$$

where

$$d\Phi_i = \varphi_i dt_i = \frac{1}{\sqrt{2\pi}\sigma} e^{-t_i^2/2} dt_i,\ t_i = \frac{x_i - m}{\sigma} \quad \text{and} \quad 1-\Phi_k = \frac{1}{\sqrt{2\pi}} \int_{t_k} e^{-t^2/2} dt.$$

Hence the required likelihood function is

$$L(x_1, \cdots, x_k;\, m, \sigma) = \frac{n!}{(n-k)!} \frac{1}{(\sqrt{2\pi}\,\sigma)^k} \exp\left\{-\frac{1}{2\sigma^2}\sum_{\nu=1}^{k}(x_\nu - m)^2\right\} \cdot (1-\Phi_k)^{n-k},$$

and consequently

$$\log L = -\frac{1}{2\sigma^2}\sum_{\nu=1}^{k}(x_\nu - m)^2 - k\log\sigma + (n-k)\log(1-\Phi_k) + \log\frac{n!}{(n-k)!\sqrt{2\pi^k}}.$$

According to the Principle of Maximum Likelihood,

$$(12.1) \qquad \frac{\partial \log L}{\partial m} = \frac{1}{\sigma^2}\sum_{\nu=1}^{k}(x_\nu - m) + \frac{n-k}{(1-\Phi_k)\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x_k - m)^2}{2\sigma^2}\right\} = 0,$$

$$(12.2) \qquad \frac{\partial \log L}{\partial \sigma} = \frac{1}{\sigma^3}\sum_{\nu=1}^{k}(x_\nu - m)^2 - \frac{k}{\sigma} + \frac{(n-k)(x_k - m)}{(1-\Phi_k)\sqrt{2\pi}\sigma^2}\exp\left\{-\frac{(x_k - m)^2}{2\sigma^2}\right\} = 0.$$

On multiplying the first equation by $m - x_k$, the second by σ, respectively, and adding them, we obtain,

$$\sum_{\nu=1}^{k}(x_\nu - m)^2 - (m - x_k)\sum_{\nu=1}^{k}(m - x_\nu) = k\sigma^2.$$

Therefore

$$(12.3) \qquad \sigma^2 = \frac{1}{k}\sum_{\nu=1}^{k}(m - x_\nu)(x_k - x_\nu).$$

Also (12.1) being rewritten,

$$(12.4) \qquad \frac{n-k}{\sqrt{2\pi}}\exp\left\{-\frac{(m - x_k)^2}{2\sigma^2}\right\} = (1-\Phi_k)\sum_{\nu=1}^{k}\frac{m - x_\nu}{\sigma}.$$

If the value of σ^2, (12.3) be substituted in (12.4), we obtain an equation containing m only. Now, if a first approximation for m be m_1, we may put $m = m_1 + \varepsilon$, and seek the correction ε. We assume that m_1 being already fairly fitting, ε is so small in magnitude, that all its powers of higher degree than 1 are negligible. So, we get from (12.3)

$$\sigma^2 = \frac{1}{k}\left[\sum_{\nu=1}^{k}(m_1 - x_\nu)(x_k - x_\nu) + \varepsilon\sum_{\nu=1}^{k}(x_k - x_\nu) + 0(\varepsilon^2)\right].$$

19) Evidently it is impossible to determine two unknown m and σ from only one observation. Also, if $k=n$, the sample becomes complete. Hence we assume to be $1<k<n$.

For the sake of abbreviations, we put[20]

(12.5)
$$\begin{cases} \bar{x}=\sum_{\nu=1}^{k} x_\nu/k, \quad S_1=\sum_{\nu=1}^{k}(x_k-x_\nu)=k(x_k-\bar{x}), \quad S_2=\sum_{\nu=1}^{k} x_\nu(x_k-x_\nu), \\ \sum_0=\sum_{\nu=1}^{k}(m_1-x_\nu)(x_k-x_\nu), \quad \sigma_0^2=\sum_0/k=m_1S_1-S_2, \\ S=\sum_{\nu=1}^{k}(m_1-x_\nu)=k(m_1-\bar{x}), \quad S_1/\sum_0=(x_k-\bar{x})/\sigma_0^2. \end{cases}$$

And thus

(12.6)
$$\frac{1}{\sigma^2}=\frac{1}{\sigma_0^2}\left(1-\frac{S_1}{\sum_0}\varepsilon\right), \quad \frac{1}{\sigma}=\frac{1}{\sigma_0}\left[1-\frac{S_1}{2\sum_0}\varepsilon\right].$$

So that

(12.7)
$$t_\nu=\frac{x_\nu-m}{\sigma}=\frac{1}{\sigma_0}\left[x_\nu-m_1-\left(1+\frac{x_\nu-m_1}{2}\frac{S_1}{\sum_0}\right)\varepsilon\right].$$

And in particular

(12.8)
$$t_k=\frac{x_k-m}{\sigma}=\frac{1}{\sigma_0}\left[x_k-m_1-\left(1+\frac{x_k-m_1}{2}\frac{S_1}{\sum_0}\right)\varepsilon\right]=A-B\varepsilon,$$

where

(12.9)
$$A=\frac{x_k-m_1}{\sigma_0}, \quad B=\frac{1}{\sigma_0}\left[1+\frac{x_k-m_1}{2}\frac{S_1}{\sum_0}\right].$$

Also from (12.7) and (12.5) yields

(12.10)
$$\sum_{\nu=1}^{k}\frac{m-x_\nu}{\sigma}=\frac{1}{\sigma_0}\left[S+\left(k-\frac{SS_1}{2\sum_0}\right)\varepsilon\right].$$

As to Φ_k we have by (12.8)

(12.11)
$$\Phi_k=\int^{t_k}\varphi dt=\int^{A-B\varepsilon}=\int^{A}+\int_{A}^{A-B\varepsilon}=\Phi(A)-B\varphi(A)\varepsilon, \quad \text{nearly}.$$

All these substituted in (12.4) and after neglection of higher power of ε, solved for ε, we attain finally

(12.12)
$$\varepsilon=\varepsilon_1=\frac{(n-k)\sigma_0\varphi(A)-S[1-\varphi(A)]}{BS\varphi(A)+[k-ABS-SS_1/2\sum_0][1-\Phi(A)]}, \quad \text{approximately}.$$

Thus $\varepsilon=\varepsilon_1$ being found, we recompute σ_0, S, A, B with the corrected $m_1+\varepsilon_1=m_2$ (the second approximation), and again calculate a new correction ε_2; and over again using $m_2+\varepsilon_2=m_3$ recompute $\sum_0$ &c., find third correction ε_3, and so on (successive approximations).

However, if we could observe every value x_ν several times (each N times, say) we may utilize Gauss' Method of Least Squares. Really from $x_\nu=m+\sigma t_\nu$, we have

$$E(x_\nu)=m+\sigma E(t_\nu), \quad \nu=1, 2, \cdots, k.$$

20) Here Σ_0 shall be positive, because, assumed that $\sigma_0>0$, such m_1 as makes $\Sigma_0\leqq 0$ is previously to be rejected.

Especially, if N truncated observations be repeated, and N be a pretty large, $\bar{x}_\nu = \sum_{j=1}^{N} x_{\nu j}/N$ would be nearly $E(x_\nu)$, and we have

$$m+\sigma E(t_\nu) = \bar{x}_\nu \ (\nu = 1, 2, \cdots, k) . \tag{12.13}$$

Or, even when $N=1$, we may roughly consider every singly observed value x_ν as $\bar{x}_\nu$; but yet if $k>2$, we have a number of equations, more than the number of unknowns. Thus equations (12.13) give the so-called observation equations:

$$a_\nu m + b_\nu \sigma = c_\nu \quad (\nu = 1, 2, \cdots, k>2) , \tag{12.14}$$

where $a_\nu=1$, $b_\nu=E(t_\nu)$ and $c_\nu=\bar{x}_\nu$ are all known, theoretically or experimentally. We form Gaussian brackets (sums of products): i.e.

$$\left\{ \begin{array}{l} [aa] = k , \quad [ab] = \sum_{\nu=1}^{k} E(t_{\nu|n}) , \quad [ac] = \sum_{\nu=1}^{k} \bar{x}_\nu , \\ \qquad [bb] = \sum_{\nu=1}^{k} E(t_{\nu|n})^2, \quad [bc] = \sum_{\nu=1}^{k} E(t_{\nu|n})\bar{x}_\nu , \end{array} \right. \tag{12.15}$$

and write the normal equations

$$[aa]m + [ab]\sigma = [ac] ,$$
$$[ab]m + [bb]\sigma = [bc] .$$

Whence

$$m = \frac{[bb][ac]-[ab][bc]}{[aa][bb]-[ab]^2} , \quad \sigma = \frac{[aa][bc]-[ab][ac]}{[aa][bb]-[ab]^2} . \tag{12.16}$$

These would probably afford better estimates than those obtained by method of maximum likelihood, if N large.

Furthermore, if beginning i values $x_1 \leqq x_2 \leqq \cdots \leqq x_i$ were ignored, besides missed measurements $x_{i+k+1} \leqq \cdots \leqq x_n$, the actually known values are only the intermediate values: $x_{i+1} \leqq x_{i+2} \leqq \cdots \leqq x_{i+k}$. In this doubly truncated sample we may also use the above mentioned Gauss' method of least squares, especially if repeatedly observed. Here we have, as before, obseravation-equations:

$$m+\sigma E(t_{i+\nu|n}) = \bar{x}_{i+\nu} \quad (\nu = 1, 2, \cdots, k>2) ,$$

normal equations of which determine the most probable values of m and σ. However, if the experiments are not repeated or few, we should again proceed by method of maximum likelihood. Now the probability element being

$$\frac{n!}{i!\,(n-i-k)!} \Phi_{i+1}^i (1-\Phi_{i+k})^{n-i-k} d\Phi_{i+1} d\Phi_{i+2} \cdots d\Phi_{i+k} ,$$

where

$$\Phi_{i+1} = \frac{1}{\sqrt{2\pi}} \int^{t_{i+1}} e^{-t^2/2} dt \quad \text{and} \quad 1-\Phi_{i+k} = \frac{1}{\sqrt{2\pi}} \int_{t_{i+k}} e^{-t^2/2} dt \quad \text{with} \quad t_j = \frac{x_j - m}{\sigma} ,$$

the likelihood function is given by

$$L=\frac{n!}{i!\,(n-i-k)!}\cdot\frac{1}{(\sqrt{2\pi}\sigma)^k}\exp\left\{-\frac{1}{2\sigma^2}\sum_{\nu=1}^{k}(x_{i+\nu}-m)^2\right\}\Phi_{i+1}^{i}(1-\Phi_{i+k})^{n-i-k},$$

so that

$$\log L=-\frac{1}{2\sigma^2}\sum_{\nu=1}^{k}(x_{i+\nu}-m)^2-k\log\sigma+i\log\Phi_{i+1}+(n-i-k)\log(1-\Phi_{i+k})$$
$$+\log\frac{n!}{i!\,(n-i-k)!\,\sqrt{2\pi}^k}.$$

Hence the likelihood equations are

$$\frac{\partial\log L}{\partial m}=\frac{1}{\sigma^2}\sum_{\nu=1}^{k}(x_{i+\nu}-m)-\frac{i}{\sqrt{2\pi}\,\sigma\Phi_{i+1}}\exp\left\{-\frac{(x_{i+1}-m)^2}{2\sigma^2}\right\}$$
$$+\frac{n-i-k}{\sqrt{2\pi}\,\sigma(1-\Phi_{i+k})}\exp\left\{-\frac{(x_{i+k}-m)^2}{2\sigma^2}\right\}=0,\tag{12.17}$$

$$\frac{\partial\log L}{\partial\sigma}=\frac{1}{\sigma^3}\sum_{\nu=1}^{k}(x_{i+\nu}-m)^2-\frac{k}{\sigma}-\frac{i(x_{i+1}-m)}{\sqrt{2\pi}\,\sigma^2\Phi_{i+1}}\exp\left\{-\frac{(x_{i+1}-m)^2}{2\sigma^2}\right\}$$
$$+\frac{(n-i-k)(x_{i+k}-m)}{\sqrt{2\pi}\,\sigma^2(1-\Phi_{i+k})}\exp\left\{-\frac{(x_{i+k}-m)^2}{2\sigma^2}\right\}=0.\tag{12.18}$$

From these two equations, firstly eliminating Φ_{i+k} and secondly Φ_{i+1}, we obtain

$$\frac{i(x_{i+k}-x_{i+1})}{\sqrt{2\pi}\,\sigma}\exp\left\{-\frac{(x_{i+1}-m)^2}{2\sigma^2}\right\}=\left[k-\frac{1}{\sigma^2}\sum_{\nu=1}^{k}(m-x_{i+\nu})(x_{i+k}-x_{i+\nu})\right]\Phi_{i+1},$$

$$\frac{(n-i-k)(x_{i+k}-x_{i+1})}{\sqrt{2\pi}\,\sigma}\exp\left\{-\frac{(x_{i+k}-m)^2}{2\sigma^2}\right\}=\left[k-\frac{1}{\sigma^2}\sum_{\nu=1}^{k}(m-x_{i+\nu})(x_{i+1}-x_{i+\nu})\right][1-\Phi_{i+k}].$$

These two equations can together be denoted by

$$\frac{R}{\sqrt{2\pi}\,\sigma}\exp\left\{-\frac{(x_{i+j}-m)^2}{2\sigma^2}\right\}=\left[k-\frac{1}{\sigma^2}\sum_{\nu=1}^{k}(m-x_{i+\nu})(x_{i+j'}-x_{i+\nu})\right]\Psi_{jj'}\tag{12.19}$$

where

$$\begin{cases} R=x_{i+k}-x_{i+1}\quad\text{(range) and}\quad (j,j')=(1,k)\text{ or }(k,1)\quad\text{with}\\ \Psi_{1k}=\dfrac{\Phi_{i+1}}{i}=\dfrac{1}{i}\displaystyle\int^{i+1}\frac{e^{-t^2/2}}{\sqrt{2\pi}}dt,\qquad \Psi_{k1}=\dfrac{1-\Phi_{i+k}}{n-i-k}.\end{cases}\tag{12.20}$$

The first approximation being m_1 and σ_1, let us find their corrections ε and η i.e. such quantities as

$$m=m_1+\varepsilon,\qquad\sigma=\sigma_1+\eta.$$

Assuming m_1, σ_1 to be already fair approximations, ε, η will be so small that their powers with exponents greater than 1 may be neglected. Consequently

$$t_{i+j}=\frac{x_{i+j}-m}{\sigma}=\frac{x_{i+j}-m_1}{\sigma_1}-\frac{\varepsilon}{\sigma_1}-\frac{x_{i+j}-m}{\sigma_1^2}\eta\quad\text{(nearly)}$$
$$=\alpha_j-\beta\varepsilon-\alpha_j\beta\eta,\qquad (j=1\text{ or }k)$$

where

$$\alpha_j=\frac{x_{i+j}-m_1}{\sigma_1},\qquad\beta=\frac{1}{\sigma_1}.\tag{12.21}$$

Accordingly we have also approximately

$$\exp\left\{-\frac{1}{2}t_{i+j}^2\right\}=\exp\left\{-\frac{(x_{i+j}-m)^2}{2\sigma^2}\right\}=\exp\left\{-\frac{1}{2}\alpha_j^2\right\}[1+\alpha_j\beta\varepsilon+\alpha_j^2\beta\eta]\,,$$

i. e.

$$\varphi(t_{i+j})=\varphi(\alpha_j)[1+\alpha_j\beta\varepsilon+\alpha_j^2\beta\eta]$$

as well as

$$\Phi_{i+j}=\int^{t_{i+j}}=\int^{\alpha_j}+\int_{\alpha j}^{t_{i+j}}=\Phi(\alpha_j)-(\beta\varepsilon+\beta\alpha_j\eta)\,\varphi(\alpha_j)\,.$$

As to the summation in the right handed side of (12.19), we have

$$\sum_{\nu=1}^{k}(m-x_{i+\nu})(x_{i+j'}-x_{i+\nu})=\sum_{\nu=1}^{k}(m_1-x_{i+\nu})(x_{i+j'}-x_{i+\nu})+\varepsilon\sum_{\nu=1}^{k}(x_{i+j'}-x_{i+\nu})\,.$$

Hence, upon putting

$$\text{(12.22)}\qquad \sum_{\nu=1}^{k}(m_1-x_{i+\nu})(x_{i+j'}-x_{i+\nu})=\textstyle\sum_{j'}\quad\text{and}\quad \displaystyle\sum_{\nu=1}^{k}x_{i+\nu}=k\bar{x}\,,$$

we obtain

$$\frac{1}{\sigma^2}\sum_{\nu=1}^{k}(m-x_{i+\nu})(x_{i+j'}-x_{i+\nu})=\beta^2[\textstyle\sum_{j'}+k(x_{i+j'}-\bar{x})\,\varepsilon-2\beta\eta\sum_{j'}]\,.$$

With all these approximations, (12.19) yields, when each of j, j' denotes one and the other of $1, k$, respectively,

$$R\beta\varphi(\alpha_j)[1+\alpha_j\beta\varepsilon+(\alpha_j^2-1)\,\beta\eta]=[k-\beta^2\{\textstyle\sum_{j'}+k(x_{i+j'}-\bar{x})\,\varepsilon-2\beta\eta\sum_{j'}\}]\,\Psi_{jj'}$$

where $\Psi_{jj'}$ denotes either one of

$$\Psi_{1k}=\frac{1}{i}[\Phi(\alpha_1)-(\beta\varepsilon+\beta\alpha_1\eta)\,\varphi(\alpha_1)]\,,\quad \Psi_{k1}=\frac{1}{n-i-k}[1-\Phi(\alpha_k)+(\beta\varepsilon+\beta\alpha_k\eta)\,\varphi(\alpha_k)]$$

approximately.

We rewrite these equations in detail, according as (j, j') is $(1, k)$ or $(k, 1)$: For $j=1$, $j'=k$

$$\text{(12.23)}\qquad \begin{aligned}&[(k-\beta^2\textstyle\sum_k+iR\alpha_1\,\beta)\,\beta\varphi(\alpha_1)+k\beta^2(x_{i+k}-\bar{x})\,\Phi(\alpha_1)]\,\varepsilon\\&+[\{(k-\beta^2\textstyle\sum_k)\,\alpha_1+iR(\alpha_1^2-1)\,\beta\varphi(\alpha_1)-2\beta^3\Phi(\alpha_1)]\,\eta\\&=(k-\beta^2\textstyle\sum_k)\,\Phi(\alpha_1)-iR\beta\varphi(\alpha_1)\,,\end{aligned}$$

and for $j=k$, $j'=1$

$$\text{(12.24)}\qquad \begin{aligned}&[\{(n-i-k)\,R\alpha_k\beta-(k-\beta^2\textstyle\sum_1)\}\,\beta\varphi(\alpha_k)+k\beta^2(x_{i+1}-\bar{x})\{1-\Phi(\alpha_k)\}]\,\varepsilon\\&+[\{(n-i-k)\,R(\alpha_k^2-1)\,\beta-(k-\beta^2\textstyle\sum_1)\,\alpha_k\}\,\beta\varphi(\alpha_k)-2\beta^3\textstyle\sum_1\{1-\Phi(\alpha_k)\}]\,\eta\\&=(k-\beta^2\textstyle\sum_1)[1-\Phi(\alpha_k)]-(n-i-k)\,R\beta\varphi(\alpha_k)\,.\end{aligned}$$

We should solve[21] (12.23) and (12.24) simultaneously for ε and η, and

21) Here we have mainly aimed only to show the principle; For practical calculations more convenient procedures are devised, see Cohen, loc. cit.

find their roots $\varepsilon=\varepsilon_1$ and $\eta=\eta_1$. Now taking $m_2=m_1+\varepsilon_1$, $\sigma_2=\sigma_1+\eta_1$ as the second approximation, recompute all of

$$\beta=\frac{1}{\sigma_2},\quad \alpha_j=\frac{x_{i+j}-m_2}{\sigma_2},\quad \Sigma_j=\sum_{\nu=1}^{k}(m_2-x_{i+\nu})(x_{i+j}-x_{i+\nu}) \tag{12.25}$$

for $j=1$ or k;

or, in detail,

$$\Sigma_1=m_2S_1-S_2,\quad \text{where}\quad S_1=\sum_{\nu=2}^{k}(x_{i+1}-x_{i+\nu}),\quad S_2=\sum_{\nu=2}^{k}x_{i+\nu}(x_{i+1}-x_{i+\nu}),$$

$$\Sigma_k=m_2S_1'-S_2',\quad \text{where}\quad S_1'=\sum_{\nu=1}^{k-1}(x_{i+k}-x_{i+\nu}),\quad S_2'=\sum_{\nu=1}^{k-1}x_{i+\nu}(x_{i+k}-x_{i+\nu}),$$

and solve thus obtained new simultaneous equations (12. 23) and (12. 24). If ε_2, η_2 be the new roots, $m_3=m_2+\varepsilon_2$ and $\sigma_3=\sigma_2+\eta_2$ will give the third approximations and so on.

Example 1. We get the following 3 samples, each of size 10, by drawing at random from the table of random samples from $N(1, 0)$:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
1°	−1.38	−1.13	−1.07	−0.92	0.21	—	—	—	—	—
2°	−2.48	−1.58	−1.06	−0.58	0.33	—	—	—	—	—
3°	−1.33	−1.03	−0.87	−0.03	0.24	—	—	—	—	—
sum	−5.19	−3.07	−3.00	−1.47	−0.78					

Assuming that the first five in each sample were observed, but the remaining five unmeasured, it is required to estimate population mean m and S. D. σ.

I. Solved by method of least squares, : We have here 5 observation equations

$$\begin{aligned}
3(m-1.53875\,\sigma) &= -5.19\\
3(m-1.00136\,\sigma) &= -3.74\\
3(m-0.65606\,\sigma) &= -3.00\\
3(m-0.37577\,\sigma) &= -1.47\\
3(m-0.12267\,\sigma) &= 0.78
\end{aligned}$$

(typically : $am+b\sigma=c$ ······ (12.14)).

Hence, Gaussian brackets are

$$[aa]=45,\quad [ab]=-33.2514,\quad [ac]=-37.86,$$
$$[bb]=\ \ 35.6143,\quad [bc]=\ \ 42.4682.$$

Solving the normal equations, we find

$$m^*=\frac{-1348.36+1412.13}{1602.64-1105.66}=0.128,\qquad \sigma^*=\frac{1911.07-1258.90}{1602.64-1105.66}=1.31.$$

II. On the other hand, if we solve the maximum likelihood estimating equations by successive approximations, we find

	m^*	σ^*
1°	0.127	1.17
2°	0.288	1.67
3°	0.235	1.03
mean	0.215	1.29

Example 2. Again, by use of the table of random samples from $N(0, 1)$, we obtained

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
—	—	−0.72	−0.52	−0.37	−0.24	0.28	0.67	—	—

I. Analysed by method of least squares:

$$\begin{aligned} m-0.65606\,\sigma &= -0.72 \\ m-0.37577\,\sigma &= -0.52 \\ m-0.12267\,\sigma &= -0.37 \\ m+0.12267\,\sigma &= -0.24 \\ m+0.37577\,\sigma &= 0.28 \\ m+0.65606\,\sigma &= 0.67 \end{aligned}$$

$$[aa]=6, \quad [ab]=0, \quad [ac]=-0.9,$$
$$[bb]=1.1733, \quad [bc]=1.2285,$$

whence

$$m^*=-0.15, \quad \sigma^*=1.05.$$

II. Analysed by method of maximum likelihood:

$$m^*=-0.15, \qquad \sigma^*=0.86.$$

The calculations by least squares are far easier.