Journal of Gakugei, Tokushima University
Vol. VIII. 1957

ON p-VALENT FUNCTIONS

By

Hitosi ABE

(Received September 30, 1957)

1. Introduction.

Let $f(z)$ *be a function of the class of ones*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n ,$$

which are regular and schlicht in $|z|<1$. *Then*

$$|a_n| < en .$$

This result is well known as Littlewood's theorem. In this paper first we shall extend this result to the case of weakly p–valent functions defined by Hayman [1], which contain p–valent functions, and solve Hayman's conjecture on the coefficient of weakly p–valent functions [1].

Secondly we consider the following functions

$$w(z) = z^{-p}(1 + a_1 z + \cdots\cdots) ,$$

which are regular and p–valent in $0<|z|<1$. These functions were studied first by Prof. Kobori [2]. We shall study the values taken by $w(z)$ and the distortion theorem.

Lastly we shall remark that we can extend the following theorem of Hayman

Suppose that $w = f(z) = 1/z + a_0 + a_1 z + \cdots\cdots$ *is meromorphic in* $|z|<1$ *and has a simple pole of residue* 1 *at the origin. Let* D_f *be the domain of all values* w *taken by* $f(z)$ *in* $|z|<1$, *and let* E_f *be the complement of* D_f *in the closed plane. Then*

$$d(E_f) \leqq 1 ,$$

where $d(E_f)$ *denotes the transfinite diameter of* E_f. *Equality hold if and only if* $f(z)$ *is univalent.*

and derive some analogous results to the ones derived by Hayman [1] from this theorem.

2.

According to Hayman's definition [1] we say that $f(z)$ is weakly p–valent, if for every $r>0$ the equation $f(z) = w$ either

(i) has exactly p roots in the unit circle for every value on the circle $|w|=r$ or

(ii) has less than p roots in the unit circle for some w on the circle $|w|=r$.

Of course p-valent or mean p-valent functions defined by Biernacki are weakly p-valent. We will begin with the proof of the following lemma and prove it by Mandelbrojt's method [3].

Lemma 1. *Let*

$$f(z)=z^p+\sum_{n=2}^{\infty} a_{n+p-1}z^{n+p-1},$$

be regular and weakly p-valent in $|z|<1$, then

$$\frac{1}{2\pi}\int_0^{2\pi}|f(z)|\,d\theta<\frac{r^p}{(1-r)^{2p-1}}, \qquad (|z|=r<1).$$

Proof. We put

$$f(z)=Re^{i\Theta}, \qquad z=re^{i\theta}.$$

$f(z)\neq 0$ in $|z|<1$ except $z=0$. Therefore $\log f(z)$ is a regular function of $\log z$ in $0<r_1\leqq|z|\leqq r_2<1$. According to Cauchy-Riemann equation we have

$$\frac{\partial R}{\partial r}=\frac{R}{r}\frac{\partial\Theta}{\partial\theta}. \tag{1}$$

Then the following relations are got by means of the above equation, where C is the image curve of the circle $|z|=r$.

$$\begin{aligned}\frac{d}{dr}\int_0^{2R}|f(re^{i\theta})|\,d\theta&=\frac{d}{dr}\int_0^{2\pi}R\,d\theta\\&=\int_0^{2\pi}\frac{\partial R}{\partial r}d\theta=\frac{1}{r}\int_C R\,d\Theta.\end{aligned} \tag{2}$$

On the other hand $f(z)$ has neither zero points or poles except only one zero point at the origin because of weak p-valence of $f(z)$.

Hence according to the argument principle

$$\int_C d\arg f(z)=\int_C d\Theta=2\pi p. \tag{3}$$

By integrating the formula (2)

$$\int_{r_1}^{r_2}\frac{dr}{r}\int_C R\,d\Theta=\int_0^{2\pi}|f(r_2e^{i\theta})|\,d\theta-\int_0^{2\pi}|f(r_1e^{i\theta})|\,d\theta.$$

Because of $f(0)=0$ we have the next relation by tending r_1 to zero and substituting r for r_2

$$\int_0^{2\pi}|f(re^{i\theta})|\,d\theta=\int_0^{r}\frac{dr}{r}\int_C R\,d\Theta.$$

By Hayman's result [1]

$$|f(z)|=R\leqq\frac{r^p}{(1-r)^{2p}}, \qquad (|z|=r<1).$$

Hence we have by (3)

$$\int_C R d\Theta < \int_C \frac{r^p}{(1-r)^{2p}} d\Theta = \frac{2\pi p r^p}{(1-r)^{2p}}.$$

Therefore

$$\int_0^{2\pi} |f(re^{i\theta})| d\theta < 2\pi p \int_0^r \frac{r^{p-1}}{(1-r)^{2p}} dr.$$

On the other hand

$$p \int_0^r \frac{r^{p-1}}{(1-r)^{2p}} dr \leq \frac{r^p}{(1-r)^{2p-1}},$$

because

$$\frac{d}{dr}\left(\frac{r^p}{(1-r)^{2p-1}}\right) - \frac{p r^{p-1}}{(1-r)^{2p}} = \frac{(p-1) r^p}{(1-r)^{2p}} \geq 0.$$

This completes the proof.

Theorem 1. *Let $f(z)$ satisfy the condition in lemma* 1. *Then*

$$|a_{n+p-1}| < e^{2p-1}\left(1+\frac{n-1}{2p-1}\right)^{2p-1} = O(n^{2p-1}).$$

Proof. By lemma 1

$$|a_{n+p-1}| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(z)|}{r^{n+p-1}} d\theta < \frac{1}{r^{n-1}(1-r)^{2p-1}}.$$

$r^{n-1}(1-r)^{2p-1}$ takes the greatest value when $r=(n-1)(n+2p-2)^{-1}$.

Hence we have theorem 1.

Remark. Hayman [1] conjectured the order of the coefficients in theorem 1.

3.

Let $w=f(z)=z^{-p}(1+a_1 z+\cdots\cdots)$ be regular and p-valent in $0<|z|<1$. Hereafter this family of functions will be denoted by F.

First we consider the case where $f(z)$ has no zero point in $0<|z|<1$. Then

$$\frac{1}{f(z)} = z^p(1-a_1 z+\cdots\cdots)$$

is regular and p-valent in $|z|<1$, and has only one zero point of order p at the origin. Hence we have the following theorem by means of Hayman's result [1] and Biernacki's one [4].

Theorem 2. *Let $f(z) \in F$ and has no zero point in $|z|<1$. Then*

(i) *The image by $f(z)$ covers the circle $|w|>4$, and*

(ii) *moreover covers $|w|>4^p$ exactly p times.*

(iii) $$r^{-p}(1-r)^{2p} \leq |f(z)| \leq r^{-p}(1+r)^{2p}, \qquad (|z|=r<1).$$

These estimates are sharp as is shown by $z^{-p}(1-z)^{2p}$ and $z^{-p}(1-z^p)^2$. The results (ii) *and* (iii) *hold still when we substitute the condition of weak p-valence for the one of p-valence.*

Secondly we consider the case where $f(z) \in F$ has zero points in $|z|<1$. It is sufficient to extend Montel-Bieberbach's theorem [5] as follows in order to have a result on the values by $f(z)$.

Lemma 2. *Let* $w(z)=z^p+b_{p+1}z^{p+1}+\cdots\cdots$ *be* p*-valent and meromorphic, then the image by* $w(z)$ *covers the circle* $|w|<\delta=\sqrt{5}-2$ *or the circle* $|w|>\frac{1}{\delta}=\sqrt{5}+2$. *This result is sharp as is shown by*

$$w_0(z)=\delta\frac{z^p}{(1-z^p)^2}\Big/\left(\frac{z^p}{(1-z^p)^2}+\delta\right)=\frac{\delta z^p}{z^p+\delta(1-z^p)^2}=z^p+\cdots\cdots.$$

Proof. Let α be one of the points on the boundary of the domain mapped by $w(z)$, which are nearest from the origin. We may suppose without loss of generality that α is positive. We put

$$\zeta(z)=\frac{\alpha w(z)}{\alpha-w(z)}=z^p+\cdots\cdots.$$

$\zeta(z)$ is regular and p-valent. Therefore we see that the image by $f(z)$ covers the circle $|\zeta|<1/4$ by means of Biernacki's theorem [4].

On the other hand

$$w(z)=\frac{\alpha\zeta}{\alpha+\zeta}.$$

Hence the image by $w(z)$ covers the exterior of the circle which has the segment $(\alpha/(1+4\alpha),\ \alpha/(1-4\alpha))$ on the real axis. And $\delta/(1-4\delta)=1/\delta$, $(\delta=\sqrt{5}-2)$. Therefore the image by $w(z)$ covers the circle $|w|>1/\delta$ when $\alpha<\delta$, or the circle $|w|<\delta$ when $\alpha\geqq\delta$. This estimate is sharp clearly, because

$$w_0(1)=\delta,\qquad w_0(e^{\pi i/p})=\frac{\delta}{1-4\delta}.$$

Here we have the following theorem directly.

Theorem 3. *Let* $f(z)\in F$ *and have zero points. Then the image by* $f(z)$ *covers the circle* $|w|<\delta$ *or the circle* $|w|>\frac{1}{\delta}$. *This estimate is best possible as is shown by* $f_0(z)=(z^p+\delta(1-z^p)^2)/\delta z^p=z^{-p}+\cdots\cdots$. $(\delta=\sqrt{5}-2)$.

Now we shall study the circle covered by $f(z)$ p times under the condition thaf $f(z)$ has only one zero point of order p.

First we will prove the following lemma.

Lemma 3. *Let* $w(z)=z+a_2z^2+\cdots\cdots$ *be meromorphic and weakly 1-valent. Then the image by* $w(z)$ *covers the circle* $|w|<\delta$ *or the circle* $|w|>\frac{1}{\delta}$, $(\delta=\sqrt{5}-2)$. *This estimate is sharp as is shown by*

$$w_0(z)=\frac{\delta z}{(1-z)^2}\Big/\left(\frac{z}{(1-z)^2}+\delta\right)=z+\cdots\cdots.$$

Proof. This proof is quite analogous to lemma 2, that is, we may use Hayman's one-quarter theorem of weakly 1-valent functions for Koebe's one with respect to schlicht functions.

Theorem 4. *Let* $f(z)\in F$ *and has one zero point of order* p *in* $|z|<1$. *Then the image by* $f(z)$ *covers the circle* $|w|<\delta^p$ *or the circle* $|w|>\frac{1}{\delta^p}$, p *times.*

These bounds are best possible as is shown by

$$f_0(z)=\left(\frac{\delta z}{z+\delta(1-z)^2}\right)^{-p}=z^{-p}+\cdots\cdots. \quad (\delta=\sqrt{5}-2).$$

Proof. We can prove this theorem even when $f(z)$ is weakly p-valent.

$$\frac{1}{f(z)}=z^p+\cdots\cdots$$

is weakly p-valent and regular in $|z|<1$ except one pole of order p. We consider $(1/f(z))^{1/p}$ and if we use the slight modification of Hayman's lemma [1], we see that this function is weakly 1-valent, and therefore we can use lemma 3. This completes the proof.

Theorem 5. *Let $f(z)$ satisfy the same conditions in theorem 4. Then we have one of the following estimates for all $|z|=r$ $(r<1)$.*

$$|f(z)|\leq\frac{1}{(4\delta)^p}\times r^{-p}(1+r)^{2p},$$

or

$$|f(z)|\geq\left(\frac{\delta}{4}\right)^p\times r^{-p}(1+r)^{2p},$$

where $\delta=\sqrt{5}-2$.

Proof. Without loss of generality we can assume $z=-|z|=-r<0$ and therefore it is sufficient for this proof to evaluate $|f(-r)|$. We remark that the function $h(z)$ mapping univalently the circle $|z|<1$ to the unit circle slitted by the segment $(-1, -r)$ under the conditions $h(0)=0$ and $h'(0)>0$ is given uniquely as follows [1] or [5].

$$\frac{h(z)}{(1-h(z))^2}=q\frac{z}{(1-z)^2}, \qquad q=\frac{4r}{(1+r)^2}, \qquad h'(0)=q.$$

$$g(h(z))=z+\cdots\cdots, \quad \left(g(z)=\left(\frac{1}{f(z)}\right)^{\frac{1}{p}}\right)$$

is weakly 1-valent because of weak 1-valence of $g(z)$ and meromorphic in $|z|<1$, and therefore we can use lemma 3 for this function. On the other hand the value $g(-r)/q$ which corresponds to $z=-1$, is not taken by this function. Hence we have

$$|g(-r)|\geq q\delta=4\delta\times\frac{r}{(1+r)^2}$$

or

$$|g(-r)|\leq q\times\frac{1}{\delta}=\frac{4}{\delta}\times\frac{r}{(1+r)^2}.$$

This completes the proof.

4.

First we will extend Hayman's theorem showed in the introduction. Each of E_f, D_f and $d(E_f)$ denotes the one indicated in the introduction.

Theorem 6. *Let $f(z)=z^{-p}(1+c_1z+\cdots\cdots)$ be meromorphic in $|z|<1$, where p is a positive integer, and E_f denote the complement of D_f which is the domain of the values taken by $f(z)$. Then $d(E_f)\leqq 1$. Equality occurs only when $f(z)=g(z^p)$, where $g(z)=z^{-1}(1+a_1z+\cdots\cdots)$ is an univalent function.*

Proof. We may do the slight modification of Hayman's proof [1]. Let $G(w)=G(w, D_f)$ denote the Green function of D_f which has a pole at ∞. Then $G(w)-\log|w|\to -\log d(E_f)$ as $w\to\infty$. We put

$$u(z) = G(f(z))-\log\frac{1}{|z|^p}.$$

$u(z)$ is harmonic in $|z|<1$ except at the points where $f(z)$ has a pole other than $z=0$. And by means of the above stated property of $G(w)$ we have the following equality in the neighbourhood of the origin.

$$u(z) = \log|z^{-p}(1+c_1z+\cdots\cdots)|-\log d(E_f)-\log\frac{1}{|z|^p}+o(1),$$

From this

$$u(z) = \log d(E_f)+o(1).$$

Hence $u(z)$ is bounded and therefore harmonic at $z=0$.

On the other hand

$$\lim_{|z|\to 1} u(z) = \lim_{|z|\to 1} G(f(z))\geqq 0.$$

Therefore $u(z)$ is non-negative in $|z|<1$ and $-\log d(E_f)\geqq 0$, that is,

$$d(E_f)\leqq 1.$$

Equality occurs only when $u(z)\equiv 0$ in $|z|<1$. In this case $f(z)$ must approach the boundary of D_f as $|z|\to 1$ by the property of Green function and $f(z)$ is able to have only one pole of order p at the origin also. Here we use the following lemma of Heins-Hayman [1].

Suppose that $F(z)$ is meromorphic in a domain Δ, that the values which $F(z)$ takes in Δ lie in a domain D, and that as z tends to the boudary of Δ in any manner, $F(z)$ always approaches the boundary of D. Then $F(z)$ takes every value of D an equal finite number of times in Δ.

Hence we see that $f(z)$ takes every value exactly p times, and furthermore $w=f(z)$ must be a function in the form of $g(z^p)$, where $g(z)=z^{-1}(1+a_1z+\cdots\cdots)$ is an univalent function.

At this time the Green function is given by

$$\log\frac{1}{|g^{-1}(w)|} = \log\frac{1}{|z|^p}.$$

Theorem 7. *Let e be a bounded closed set of real numbers x whose Lesbesgue measure is at least 4. For each x in e let $C(x)$ be a closed set of points in the w-plane such that, if w_1, w_2 be any points on $C(x_1)$, $C(x_2)$ respectively, we have always*

$$|w_2-w_1|\geqq|x_2-x_1|.$$

Then with the hypotheses of theorem 6, D_f contains at least one of the set of $C(x)$, except possibly when e, E_f are intervals of length 4, and

$$f(z) = \frac{1}{z^p} + a_0 + z^p e^{i\lambda},$$

a_0 arbitrary, λ real arbitrary.

Proof. We can prove this theorem in the same method with Hayman's one [1]. We remark that $f(z)$ with respect to the exceptional case must be $g(z^p) = \frac{1}{z^p} + a_0 + \cdots\cdots$, where $g(z)$ is an univalent function, and E_f must contain two points w_1, w_2, such that $|w_2 - w_1| \geqq 4$, and therefore

$$f(z) = z^{-p} + a_0 + z^p e^{i\lambda}.$$

Lemma 4. *Let $f(z) = a_0 + a_p z^p + \cdots\cdots$ be meromorphic in $|z| < 1$ and let E be the set of all real positive r for which the circle $|w| = r$ meets E_f. Then we have*

$$|a_p| \int_{E_f} \frac{dr}{(|a_0| + r^2)} \leqq 4.$$

Proof. We may consider

$$w = \varphi(z) = \frac{a_p}{f(z) - a_0} = \frac{1}{z^p} + \cdots\cdots.$$

and use theorem 7 in the same way with Hayman's one [1].

Theorem 8. *Let $w = f(z) = a_0 + a_p z^p + \cdots\cdots$ is regular in $|z| < 1$. Then we have $|a_p| \leqq 4(|a_0| + l_f)$, where l_f denotes the Lebesgue measure of the set of all positive r, for which the circle $|w| = r$ lies entirely inside D_f, and it is assumed that l_f is finite. Equality occurs only when*

$$f(z) = \frac{a_p z^p}{(1 - e^{i\alpha} z^p)}, \qquad a_0 = 0$$

$$f(z) = a_0 + \frac{a_0 \lambda z^p e^{i\alpha}}{(1 - z^p e^{i\alpha})^2}, \qquad a_0 \neq 0.$$

Proof. Let

$$I = \int_E \frac{dr}{(|a_0| + r^2)}.$$

Then Hayman proved the following inequalities [1].

$$|a_0| + l_f \geqq \frac{1}{I}$$

Therefore by means of lemma 4 we have this theorem.

Accordingly we can derive the following theorem clearly from this.

Theorem 9. *Let $f(z) = z^p + a_{p+1} z^{p+1} + \cdots\cdots$ be regular in $|z| < 1$. D_f contains a circle $|w| = r$ with $r > \frac{1}{4}$ except when*

$$f(z) = \frac{z^p}{(1 - e^{i\alpha} z^p)^2}.$$

References

[1] W. K. Hayman, Some applications of the transfinite diameter to the theory of functions, Journal d'Analyse Mathématique vol. (1951), pp. 155–179.

[2] A. Kobori, Une remarque sur les fonction multivalentes, Mem. Coll. Sci. Kyoto (A) vol. 27 (1952), pp. 1–5.

[3] S. Mandelbrojt, Quelques remarques sur les fonction univalentes, Bull. des Sci. Math. (1934), pp. 185–200.

or M. Tsuji, Theory of Functions II, (in Japanese) pp. 88–89.

[4] M. M. Biernacki, Sur les domaines couverts par les fonctions, Bull. des Sci. Math. vol. 70 (1946), pp. 45–51.

[5] Y. Komatu, Theory of conformal mapping I, (in Japanese) pp. 200–204.