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ON DARBOUX LINES CONTAINED IN A RIEMANNIAN SPACE

By

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§ 1. Miss Prvanovitch[1] recently defined a Darboux line contained in an n-dimensional sub-space V_n immersed in a Riemann space V_m, obtained differential equations of this curve and found some properties.

According to her definition, a Darboux line is a curve on each point of which the vector

$$R_1 t_2^{\alpha} + \frac{dR_1}{ds} R_2 t_3^{\alpha}$$ [2]

is normal to a sub-space V_n immersed in V_m, where t_2^{α}, t_3^{α} are the first and the second principal normal vectors of this curve, R_1, R_2 the first and the second radius of curvature respectively.

Let $a_{\alpha\beta} dy^{\alpha} dy^{\beta}$ and $g_{ij} dx^i dx^j$ be the fundamental forms of V_m and V_n, and let H^P_{ij} be the second fundamental tensor of V_n immersed in V_m, ξ^{α}_P be $(m-n)$ mutually orthogonal unit vectors in V_m normal to V_n in V_m, s be arc length, then the equation of a Darboux line is given by[3]

$$(1,1) \qquad a_{\alpha\beta} \frac{\delta^3 y^{\alpha}}{ds^3} \xi^{\beta}_R = \left(\frac{\partial H^R_{ij}}{\partial x^k} + H^R_{kl} \left\{ {l \atop ij} \right\} - \sum_P \mu_{PR|k} H^P_{ij} \right) \frac{dx^i}{ds} \frac{dx^j}{ds} \frac{dx^k}{ds} + 3H^R_{ij} \frac{d^2 x^i}{ds^2} \frac{dx^j}{ds} = 0 ,$$

where ξ^{β}_R (R is fixed) is a considering normal vector of V_n and the quantities $\mu_{PR|k}$ are given by

$$(1,2) \qquad \mu_{PR|k} = a_{\alpha\beta} \xi^{\alpha}_P \xi^{\beta}_{R,k} + [\gamma\delta, \beta]_a \, y^{\gamma}_{,k} \xi^{\delta}_R \xi^{\beta}_P .$$

Now we can rewrite this equation such as

$$(1,3) \qquad H^R_{ij;K} \frac{dx^i}{ds} \frac{dx^j}{ds} \frac{dx^k}{ds} + 3H^R_{ij} \frac{\delta^2 x^i}{ds^2} \frac{dx^j}{ds} - \sum_P \mu_{PR|k} H^P_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0 ,$$

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1) M. Prvanovitch; Lignes de Darboux dans l'espace riemannien. (Bull. Sci. Math. (2) 78 1954 p.p. 9–14)
; Hyperligne de Darboux appartenant à l'espace riemannien. (Bull. Sci. Math. (2) 78 1954 p.p. 89–97)

2) In this paragraph we shall denote by $\alpha, \beta, \gamma, \cdots$ the suffices which take the value $1, 2, \cdots, m$; by $h, i, j, k, l, \cdots$ those which take the value $\dot{1}, \dot{2}, \dot{3}, \cdots, \dot{n}$; by $P, Q, R, S, \cdots$ those which take the value $\dot{n+1}, \dot{n+2}, \cdots, \dot{m}$.

3) M. Prvanovitch; loc. cit.

or

$$\frac{\delta}{ds}\left(H^R_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right)+H^R_{ij}\frac{\delta^2x^i}{ds^2}\frac{dx^j}{ds}-\sum_P \mu_{PR|K}H^P_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}=0. \tag{1, 4}$$

In the case $m=n+1$, V_n is a hypersurface of V_{n+1}, and

$$\mu_{PR|k}=0. \tag{1, 5}$$

That is to say, if we denote the normal of V_n by ξ^α and put

$$H_{(ij;k)}=\frac{1}{3}[H_{ij;k}+H_{jk;i}+H_{ki;j}]=H_{ijk},$$

then the equation of a Darboux line is

$$H_{ijk}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}+3H_{ij}\frac{\delta^2x^i}{ds^2}\frac{dx^j}{ds}=0, \tag{1, 6}$$

or

$$3\frac{\delta}{ds}\left(H_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right)-H_{ijk}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}=0. \tag{1, 7}$$

The curves whose equations are given by

$$H_{ijk}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}=0, \tag{1, 8}$$

$$K_{ijk}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}=0. \tag{1, 9}$$

are, by Prof. Kanitani[4], called a Darboux line of the first kind and the second kind respectively, where we put

$$K_{ijk}=H_{ijk}-\frac{1}{n+2}[H_{ij}H_k+H_{jk}H_i+H_{ki}H_j],$$

$$H_i=H^{jk}H_{ijk}.$$

Also he calls the hypersurface which satisfies $K_{ijk}=0$ a hyperquadric. Taking V_n a hyperquadric, we see the equation of a Darboux line defined by Prvanovitch may be written as

$$\frac{\delta}{ds}\left(H_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right)-\frac{1}{n+2}\left(H_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right)H_k\frac{dx^k}{ds}=0. \tag{1, 10}$$

Hence, if a curve contained in a hyperquadric in V_{n+1} is an asymptotic line, it is a Darboux line defined by Prvanovitch.

As a special case, V_n being a hyperquadric which satisfies the relation $H_{ijk}=0$, we see that, as $H_k=0$, equation (1, 10) is

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4) J. Kanitani; Les equation fondamentales d'une surface plongée dans un espace à connexion projective. (Mem. of Ryojun college of Engineering. Vol. XII. No. 3 p.p. 61–88 1939)

; Sur un Espace à connexion Projective Renferment des Hyperquadriques. (Proc. of Physico- Math. Soc. of Japan. 25 p.p. 617–621 1943)

$$\frac{d}{ds}\left(H_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right)=0\,. \tag{1,11}$$

Therefore from the equation (1, 11) we see when V_n is a hyperquadric satisfying $H_{ijk}=0$, a Darboux line defined by Prvanovitch is a curve whose normal curvature $H_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}$ is constant.

From now we call a Darboux line defined by Prvanovitch the Darboux line of the third kind. In this paper we mean the Darboux line of the third kind by Darboux line.

Here, we shall find the necessary and sufficient condition that every curve contained in this hyperquadric immersed in V_{n+1} be a Darboux line.

For this purpose putting

$$H_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}=\tau\,,$$

we obtain from (1, 10)

$$\frac{d}{ds}\log\tau=\frac{1}{n+2}H_k\frac{dx^k}{ds}\,,$$

since this equation must be indeterminate, we see τ is a function of x^i and

$$\frac{1}{n+2}H_k=\frac{\partial\log\tau}{\partial x^k}\,.$$

Moreover when we put $H=(n+2)\log\tau$, we have $H_k=\frac{\partial H}{\partial x^k}$ and $e^{\frac{H}{n+2}}=H_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}$.

Of course this last result must be identical, then we have

$$H_{ij}=e^{\frac{H}{n+2}}g_{ij}\,.$$

Because of this relation we have

$$H_{ijk}=\frac{1}{3\,(n+2)}e^{\frac{H}{n+2}}(g_{ij}H_k+g_{jk}H_i+g_{ki}H_j)\quad\text{and}\quad H_i=H^{jk}H_{ijk}=\frac{1}{3}H_i\,.$$

Accordingly we have $H_i=0$, that is, H is constant.

Thus we obtain $H_{ij}=\rho g_{ij}$ (ρ is constant).

Conversly, when $H_{ij}=\rho g_{ij}$ ($\rho=$const.), we see easily $H_{ijk}=0$, $H_i=0$, that is, V_n is a hyperquadric and Darboux lines are indeterminate in it.

Consequently we obtain *the necessary and sufficient condition that every curve contained in a hyperquadric immersed in V_{n+1} be Darboux lines is this hyperquadric be properly totally umbilic hypersurface.*

§ **2.** Suppose $G_{AB}dz^Adz^B$ and $a_{\alpha\beta}dy^\alpha dy^\beta$ be the fundamental forms of V_l ($l\geqq 4$) and V_m ($l>m\geqq 3$) which is subvariety of V_l, and $B_{\alpha}^{\cdot A}=\frac{\partial z^A}{\partial y^\alpha}$ and ξ_σ^A be components of a tangent vector of V_m and $(l-m)$ mutually orthogonal unit vectors in V_l normal to V_m respectively then we have well known relations

(2, 1) $$a_{\alpha\beta}=G_{AB}B_{\alpha}^{\cdot A}B_{\beta}^{\cdot B}.$$ [4]

Moreover when we indicate by V_n a subspace V_m, by $B_i^{\cdot\alpha}=\frac{\partial y^\alpha}{\partial x^i}$ tangent vectors of V_n, by ξ_a^α $(m-n)$ mutually orthogonal unit vectors in V_m normal to V_n, we see the metric tensor g_{ij} of V_n is given by

(2, 2) $$g_{ij}=a_{\alpha\beta}B_i^{\cdot\alpha}B_j^{\cdot\beta}.$$

From the relations $B_i^{\cdot A}=B_\alpha^{\cdot A}B_i^{\cdot\alpha}$ we may put

(2, 3) $$\xi_a^A=B_\alpha^{\cdot A}\xi_a^\alpha,$$

and indicate (ξ_a^A, ξ_σ^A) by ξ_P^A. Moreover we indicate by $H_{\alpha\beta}^A$, H_{ij}^α and H_{ij}^A the components of normal curvature tensor of V_m in V_l, of V_n in V_m, and of V_n in V_l respectively, then we can represent

$$H_{\alpha\beta}^A=B_{\alpha;\beta}^{\cdot A},\quad H_{ij}^\alpha=B_{i;j}^{\cdot\alpha},\quad H_{ij}^A=B_{i;j}^{\cdot A}.$$

Since $H_{\alpha\beta}^A$ are orthogonal to B_γ^A, we can put

(2, 4) $$H_{\alpha\beta}^A=\sum_\sigma H_{\alpha\beta}^\sigma\xi_\sigma^A,\quad H_{ij}^\alpha=\sum_a H_{ij}^a\xi_a^\alpha,\quad H_{ij}^A=\sum_a H_{ij}^P\xi_P^A,$$

and obtain

$$\sum_P H_{ij}^P\xi_P^A=H_{ij}^A=(B_\alpha^{\cdot A}B_i^{\cdot\alpha})_{;j}=\sum_\sigma H_{\alpha\beta}^\sigma B_i^{\cdot\alpha}B_j^{\cdot\beta}\xi_\sigma^A+\sum_\sigma H_{ij}^a\xi_a^A.$$

Hence, from the definition of ξ_P^A, comparing the coefficients of vectors ξ_P^A, we have

(2, 5) $$H_{ij}^\tau=H_{\alpha\beta}^\tau B_i^{\cdot\alpha}B_j^{\cdot\beta},\quad [H_{ij}^a]_{V_m}=[H_{ij}^a]_{V_n}.$$

When $x^i(s)$ is a Darboux line of V_n in V_m, denoting by ξ_e^α (e is fixed) the normal vector of V_n in V_m, whose components are determined by the vector

$$R_1t_2^\alpha+R_1'R^2t_3^\alpha$$

we see the equation of Darboux line is given by

(2, 6) $$\frac{\delta}{ds}\left(H_{ij}^e\frac{dx^i}{ds}\frac{dx^j}{ds}\right)+H_{ij}^e\frac{\delta^2x^i}{ds^2}\frac{dx^j}{ds}-\sum_a[\mu_{ae|k}]_{V_m}\cdot H_{ij}^a\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}=0.$$

On the other hand when we represent by $x^i(s)$ a Darboux line in V_n in V_l concerning ξ_e^A from (2, 5) the equation of this curve is given by

(2, 7) $$\frac{\delta}{ds}\left(H_{ij}^e\frac{dx^i}{ds}\frac{dx^j}{ds}\right)+H_{ij}^e\frac{\delta^2x^i}{ds^2}\frac{dx^j}{ds}-\sum_P[\mu_{Pe|k}]_{V_l}H_{ij}^P\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}=0,$$

where $[\mu_{Pe|k}]_{V_l}$ is given by

$$[\mu_{Pe|k}]_{V_l}=G_{AB}\xi_P^A\xi_{e,k}^B+[CD,B]_{V_l}z_{,k}^C\xi_e^D\xi_P^B.$$

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4) In this paragraph we shall denote by $A, B, C, \cdots, H$ the suffices which take the value $1, 2, \cdots, l$; by $\alpha, \beta, \gamma, \delta$ those which take the value $\dot{1}, \dot{2}, \dot{3}, \cdots, \dot{m}$; by h, i, j, k those which take the value $\ddot{1}, \ddot{2}, \cdots, \ddot{n}$; by a, b, c, d, e, f those which take the value $\ddot{n}+\ddot{1}, \ddot{n}+\ddot{2}, \cdots, \ddot{m}$; by μ, σ, τ those which take the value $\dot{m}+\dot{1}, \dot{m}+\dot{2}, \cdots, \dot{l}$; bu P, Q, R, S those which take the value $\dot{n}+\dot{1}, \dot{n}+\dot{2}, \cdots, \dot{l}$.

We consider, therefore, two cases, that is, $n+1 \leqq P \leqq m$ and $m < P \leqq l$. First in the case $n+1 \leqq P \leqq m$, from the relation (2, 1), (2, 3),

$$[\mu_{ae|k}]_{V_l} = G_{AB}\xi^A_a H^B_{\alpha\gamma} B_k^{\cdot\gamma}\xi^\alpha_e + G_{AB}\xi^A_a B_\alpha^{\cdot B}\xi^\alpha_{e;k},$$

and
$$G_{AB}\xi^A_a\xi^B_\sigma = G_{AB}B_\alpha^{\cdot A}\xi^B_\sigma\xi^\alpha_a = 0,$$

we have

(2, 8)
$$[\mu_{ae|k}]_{V_l} = [\mu_{ae|k}]_{V_m}.$$

In the second case we have similarly

(2, 9)
$$[\mu_{\sigma e|k}]_{V_l} = H^\sigma_{\alpha\beta}\xi^\alpha_e\xi^\beta_k.$$

Substituting this into (2, 7) we find

(2, 10)
$$\frac{\delta}{ds}\left(H^e_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right) + H^e_{ij}\frac{\delta^2 x^i}{ds^2}\frac{dx^j}{ds} - \sum_a [\mu_{ae|k}]_{V_m}\cdot H^a_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}$$
$$-\sum_\sigma H^\sigma_{\alpha\beta}\xi^\alpha_e B_k^{\cdot\beta} H^\sigma_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0,$$

and comparing with (2, 6) we obtain *the necessary and sufficient condition that a Darboux line contained in V_n immersed in V_l concerning the vector $\xi^A_e = B_\alpha^{\cdot A}\xi^\alpha_e$ of V_n be a Darboux line contained in V_n immersed in V_m concerning the vector ξ^x is that along this curve the relation.*

(2, 11)
$$\sum_\sigma H^\sigma_{\alpha\beta} H^\sigma_{ij}\xi^\alpha_e B^{\cdot\beta}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

be satisfied.

In particular, if we take $m=n+1$ and $l=n+2$, we have from (2, 8)

$$[\mu_{ae|k}]_{V_l} = [\mu_{ae|k}]_{V_m} = 0.$$

When we put $\sigma = \mathrm{I}$, $e = \mathrm{II}$ and denote by η^α components of a unit vector in V_{n+1} normal to V_n, by ξ^A_{I}, $\xi^A_{\mathrm{II}} = B_\alpha^{\cdot A}\eta^\alpha$ components of two mutually orthogonal unit vectors in V_{n+2} normal to V_n, we have

$$H^{\mathrm{I}}_{ij} = H_{\alpha\beta}B_i^{\cdot\alpha}B_j^{\cdot\beta}, \quad H^{\mathrm{II}}_{ij} = H_{ij},$$

then the equation (2, 11) is

$$H_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\cdot\mu_{\mathrm{I\,II}|k}\frac{dx^k}{ds} = 0,$$

and the quantities $\mu_{\mathrm{I\,II}|k}$ are given by

$$\mu_{\mathrm{I\,II}|k} = G_{AB}\xi^A_{\mathrm{I}}\xi^B_{\mathrm{II};k} = H_{\alpha\beta}B_k^{\cdot\beta}\eta^\alpha.$$

Hence we obtain *the necessary and sufficient condition that a Darboux line contained in V_n immersed in V_{n+2} concerning the normal vector $\xi^A_{\mathrm{II}} = B_\alpha^{\cdot A}\eta^\alpha$, where η^α is a normal vector to V_n in V_{n+1}, be a Darboux line contained in V_n immersed in V_{n+1} is that along this curve one of the following two conditions be satisfied*

(2, 12) $$H_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}=0\,,$$

(2, 13) $$H_{\alpha\beta}\eta^\alpha\frac{dy^\beta}{ds}=0\,.$$

§ **3.** Using the same notations in preceding paragraph we consider V_n in V_{n+1} immersed in V_{n+2}. Taking a notice that the equations of a Darboux line contained in V_{n+1} immersed in V_{n+2} is given by

$$G_{AB}\frac{\delta^3 z^A}{ds^3}\xi^B_{\mathrm{I}}=0\,,$$

we call V_n a Darboux variety of V_{n+1} immersed in V_{n+2} when, in V_n, for all indices i, j, k the equation

(3, 1) $$G_{AB}z^A_{;i;j;k}\xi^B_{\mathrm{I}}=0$$

are satisfied.

Since

$$z^A_{;i;j;k}=H_{\alpha\beta;\gamma}B_i^{\cdot\alpha}B_j^{\cdot\beta}B_k^{\cdot\gamma}\xi^A_{\mathrm{I}}+2H_{\alpha\beta}B_i^{\cdot\alpha}H_{jk}\eta^\beta\xi^A_{\mathrm{I}}+H_{\alpha\beta}B_i^{\cdot\alpha}B_j^{\cdot\beta}\xi^A_{\mathrm{I};k}$$
$$+H_{ij;k}\eta^\alpha B_\alpha^{\cdot A}+H_{ij}H_{\alpha\beta}B_k^{\cdot\beta}\eta^\alpha\xi^A_{\mathrm{I}}+H_{ij}\eta^\alpha_{;k}B_\alpha^{\cdot A}\,.$$

and $$G_{AB}\xi^A_{\mathrm{I};k}\xi^B_{\mathrm{I}}=0\,.$$

The equations (3, 1) can be written as

(3, 2) $$H_{\alpha\beta;\gamma}B_i^{\cdot\alpha}B_j^{\cdot\beta}B_k^{\cdot\gamma}+H_{\alpha\beta}\eta^\beta[2B_i^{\cdot\alpha}H_{jk}+B_k^{\cdot\alpha}H_{ij}]=0\,.$$

However, (3, 2) are written as

(3, 3) $$M_{ijk}=0\,,$$

where we put

(3, 4) $$M_{ijk}=H_{\alpha\beta\gamma}B_i^{\cdot\alpha}B_j^{\cdot\beta}B_k^{\cdot\gamma}+3H_{\alpha\beta}\eta^\beta B^\alpha_{[i}H_{jk]}\,.$$

Hence we obtain the necessary and sufficient condition that V_n immersed in V_{n+1} immersed in V_{n+2} be a Darboux variety is

$$M_{ijk}=0\,.$$

Especially if we take $n=1$ they evidently become equations of a Darboux line. Hence we can consider that a Darboux variety is a extension of a Darboux line contained in V_2 immersed in V_3 to n-dimensional sub-space.