Journal of Gakugei, Tokushima University
Vol. VIII. 1957

COMMUTATIVE NONPOTENT ARCHIMEDEAN SEMIGROUP WITH CANCELATION LOW I

By

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(Received September 30, 1957)

We established in the paper [1] that a commutative semigroup is decomposed into the class sum of unipotent or nonpotent semigroups. In the present paper we shall investigate the structure of a commutative nonpotent archimedean semigroup admitting cancelation law. We shall see that such a semigroup will be determined by the additive semigroup of non-negative integers and the indexed group.

If, for any elements a and b of a commutative semigroup S, there exist a positive integer m and an element c of S such that

$$a^m = bc\,,$$

then S is called archimedean. By "nonpotent" we mean "without idempotent".

§1. Unique Factorization.

Lemma 1. *Let S be a commutative nonpotent archimedean semigroup. Then*

$$\bigcap_{n=1}^{\infty} a^n S = \emptyset \quad \text{for every} \quad a \in S\,.$$

Proof. Let $D = \bigcap_{n=1}^{\infty} a^n S$. Suppose that D is not empty. Then we shall prove the following (1.1), (1.2), (1.3), and (1.4) step by step.

(1.1) *D is an ideal of S.*

Since any y in D is expressed as $y = a^n t$ where $t \in S$, we get

$$yx = (a^n t)x = a^n(tx) \in a^n S \quad \text{for all } n$$

whence $yx \in D$ and so $Dx \subset D$.

(1.2) *$D \subset zS$ for any $z \in S$.*

Since S is archimedean, there are $m > 0$ and $x \in S$ such that $a^m = zx$. Then any $d \in D$ is expressed as $d = a^m y = (zx)y = z(xy) \in zS$. Therefore $D \subset zS$.

(1.3) *D is the least ideal of S.*

Let I be any ideal of S, that is, $IS \subset I$. For $y \in I$, $D \subset yS \subset IS \subset I$ by (1.2).

(1.4) *$D = dD$ for any $d \in D$.*

Using (1.1), $(dD)S = d(DS) \subset dD$, and so dD is an ideal of S. According to

(1.3), we have $D \subset dD$, while, of course, $dD \subset D$. At last we have $D = dD$.

Since D is commutative, it follows that D is a group. Consequently D contains an idempotent, contradicting with the assumption. Thus the proof of the lemma has been finished.

Denote $T_0 = S - aS$, $T_i = a^i S - a^{i+1} S$ $(i = 1, 2, \cdots)$. Then T_i $(i = 1, 2, \cdots)$ are not empty. Because, if T_i is emtpy, we get

$$a^i S = a^{i+1} S = \cdots$$

which leads to $D \neq \emptyset$, contradicting with Lemma 1.

Corollary 1. $S = \sum_{i=0}^{\infty} T_i$, $T_i \neq \emptyset$, $T_i \cap T_j = \emptyset$ $(i \neq j)$.

Lemma 2. *In a commutative archimedean semigroup S, S is nonpotent if and only if $a \neq ab$ for every $a, b \in S$.*

Proof. Suppose $a = ab$ in spite of nonpotentness of S. Then we have

$$a = ab = ab^2 = \cdots = ab^n = \cdots$$

whence $\bigcap_{n=1}^{\infty} b^n S \neq \emptyset$, contradicting with Lemma 1. Thus we see that if S is nonpotent, $a \neq ab$ for any $a, b \in S$. The converse is clear: if S has an idempotent e, then $e = ee$. q. e. d.

Hereafter a denotes a fixed element of a commutative nonpotent archimedean semigroup S with cancelation.

According to Corollary 1, for an element x of aS, a positive integer n is uniquely determined such that

$$x \in T_n \quad \text{that is,} \quad x = a^n z \,.$$

Further we can see that z lies in $S - aS$. Indeed, if $z = au$, then $x = a^{n+1} u \in T_{n+1}$ which conflicts with $x \in T_n$ and $T_n \cap T_{n+1} = \emptyset$. Uniqueness of z is assured by the cancelation law.

Let us introduce a symbol a°:

$$a^\circ b \quad \text{means} \quad b,$$

in words, a° is not an element, but is considered as a symbolical operation. Then, if $x \in S - aS$, x is expressed as $x = a^\circ x$. We can summarize the above description as follows.

Theorem 1. *An element x of S determines uniquely a non-negative integer n and an element z of $S - aS$ such that $x = a^n z$.*

§ 2. Homomorphism to a Group.

Now let us introduce a relation $x \sim y$ among all the elements of a commutative nonpotent archimedean semigroup with cancelation. Denote $x \sim y$ if there is a non-negative integer n such that either $x = a^n y$ or $y = a^n x$. This relation is an equivalence relation. Indeed $x \sim x$ since $x = a^\circ x$; the symmetric law is obvious. We shall prove only the transitive law in the four cases:

(2. 1) $x=a^n y$, $y=a^m z$, (2. 2) $x=a^n y$, $z=a^m y$,

(2. 3) $y=a^n x$, $y=a^m z$, (2. 4) $y=a^n x$, $z=a^m y$.

Then we have

in the case (2. 1) $x=a^{n+m}z$,

in the case (2. 2) $\begin{cases} x=a^{n-m}z & \text{if } n>m, \\ z=a^{m-n}x & \text{if } n<m, \\ x=z & \text{if } n=m, \end{cases}$

in the case (2. 3) $\begin{cases} z=a^{n-m}x & \text{if } n>m, \\ x=a^{m-n}z & \text{if } n<m, \\ x=z & \text{if } n=m, \end{cases}$

in the case (2. 4) $z=a^{n+m}x$.

Hence the transitive law holds. Further we see easily that $x\sim y$ implies $xu\sim yu$. Thus we get

Lemma 4. (2. 5) $a^n\sim a^m$. $(n, m=1, 2, \cdots)$

(2. 6) $x\sim a^n x$. $(n=1, 2, \cdots)$

(2. 7) *For any x, there is y such that $xy\sim a$.*

(2. 8) *If $x, y\in S-aS$, and $x\neq y$, then $x\not\sim y$.*

Proof. (2. 5), (2. 6), and (2. 8) are obvious by the definition of the equivalence relation; (2. 7) is led from archimedeaness as follows. For any x, there is y such that $xy=a^m\sim a$. q. e. d.

Now all the elements of S is classified by the relation $x\sim y$. S is the set union of S_α where we denote by S^* the set of all indices α. $S=\sum_{\alpha\in S^*} S_\alpha$ $S_\alpha\cap S_\beta=\emptyset$ $(\alpha\neq\beta)$.

In particular, denote by S_ε the class containing a:

$$S_\varepsilon=\{a^n;\ n=1,\ 2\ \cdots\}.$$

Since the relation is a congruence relation, $S_\alpha S_\beta\subset S_\gamma$ for some γ by which the product $\alpha\beta$ of elements α and β is defined as $\gamma=\alpha\beta$. By Lemma 4, we have

Theorem 2. *S^* is a group, and S is homomorphic onto S^*.*

§ 3. Linear Order in S_α.

We shall define an ordering between the elements of a class S_α as follows.

$$x>y \quad (x, y\in S_\alpha)$$

if and only if $x\neq y$ and there is a positive integer n such that $x=a^n y$ where a is the fixed element.

Lemma 5. (3. 1) $x\not> x$ (3. 2) *$x>y$ and $y>x$ are incompatible.* (3. 3) *$x>y$ and $y>z$ imply $x>z$.*

Proof. If $x>x$, then $x=a^n x$ for some n; if $x>y$ and $y>x$, then we have $x=a^m x$ for some m. These are impossible according to Lemma 2. Thus (3.1) and (3.2) have been proved. (3.3) is also obtained as follows:

$$x=a^n y, \text{ and } y=a^m z \text{ imply } x=a^{n+m}z.$$

Lemma 6. *Suppose that* $x>y$ *or* $x=a^n y$, $n\geqq 1$, $x, y\in S_\alpha$. *Then* $x\geqq u\geqq y$ *implies* $u=a^i y$ $(0\leqq i\leqq n)$.

Proof. $u=a^k y$, and $x=a^l u$ (for certain $k, l\geqq 0$) follow from $u\geqq y$ and $x\geqq u$ respectively; and so $x=a^{k+l}y=a^n y$. By Theorem 1, we have $k+l=n$. Hence $0\leqq k\leqq n$, q. e. d.

Consequently the interval between x and y is composed of $x_i=a^i y$ $(i=0, 1, \cdots, n)$ such that

$$x=a^n y>a^{n-1}y>\cdots>ay>y.$$

Lemma 7. S_α *satisfies the descending chain condition, that is, a sequence* $x_1>x_2>\cdots>x_n>\cdots$ *ceases at finite term.*

Proof. Suppose that there is an infinite sequence.

$$x_1>x_2>\cdots>x_n>\cdots$$

where $x_i=a^{m_i}x_{i+1}$, $(i=1, 2, \cdots, n, \cdots)$ and $m_i>0$. Letting $k_n=m_1+m_2+\cdots+m_n$, $k_1<k_2<\cdots<k_n<\cdots$ and $x_1=a^{k_1}x_2=a^{k_2}x_3=\cdots=a^{k_n}x_{n+1}=\cdots$ which arrives at $x_1\in\bigcap_{i=1}^{\infty} a^{k_i}S\neq\emptyset$ contradicting with Lemma 1. q. e. d.

According to Lemmas 6 and 7, we see that there is a minimal element in S_α. Denote $T_1=S-aS$.

Lemma 8. *A minimal element of* S_α *lies in* T_1, *and conversely an element of* T_1 *is minimal in certain* S_α.

Proof. If a minimal element z of S_α belongs to aS, then $z=au$, $u\in S$, where $au\sim u$ by Lemma 4, and hence $u\in S_\alpha$, $u<z$. This contradicts with the fact that z is minimal in S_α. Therefore $z\bar{\in} aS$. Conversely if $z\in S-aS$ and $z\in S_\alpha$; then there is no $u<z$.

By the definition of the relation $x\sim y$ and the ordering $x>y$, $T_1\cap S_\alpha$ consists of only one element denoted by x_α.

Theorem 3. *Each* S_α *is a linearly ordered set with respect to the ordering* $x>y$, *and any element* x *of* S_α *is expressed as* $x=a^n x_\alpha$ *where* $n\geqq 0$, *and* x_α *is a unique element of* T_1 *contained in* S_α.

§ 4. Construction.

Since S is homomorphic onto S^* by Theorem 2, $x_\alpha\in S_\alpha\cap T_1$ and $x_\beta\in S_\beta\cap T_1$ determine $\gamma\in S^*$ and a non-negative integer n such that $x_\alpha x_\beta=a^n x_\gamma$ where $x_\gamma\in S_\gamma\cap T_1$. This n is called the index of a pair of x_α and x_β, which is denoted by $n=I(\alpha, \beta)$. Of course $I(\alpha, \beta)=I(\beta, \alpha)\geqq 0$.

Let $(x_\alpha x_\beta)x_\gamma=x_\alpha(x_\beta x_\gamma)\in S_\pi$ and let $I(\alpha, \beta)=n$, $I(\alpha\beta, \gamma)=p$, $I(\alpha, \beta\gamma)=q$, $I(\beta, \gamma)=m$. Then $(x_\alpha x_\beta)x_\gamma=a^{n+p}x_\pi$, $x_\alpha(x_\beta x_\gamma)=a^{q+m}x_\pi$, so that we have $n+p=q+m$ by Theorem 1, or $$I(\alpha, \beta)+I(\alpha\beta, \gamma)=I(\alpha, \beta\gamma)+I(\beta, \gamma) .$$

Since the minimal element of $S_\varepsilon=\{a^i ; i=1, 2, \cdots\}$ is a, we get $I(\varepsilon, \varepsilon)=1$. Because of archimedaness, there is $m+1>1$ such that $x_\alpha^{m+1}=x_\alpha^m x_\alpha \in aS$, therefore $I(\alpha^m, \alpha)>0$ for some $m>0$. Thus a group S^* with an index is determined from S. The group S^* with an index is called "the fundamental group" of S.

Conversely, consider an abstract commutative group G and a non-negative integer-valued function $I(x, y)$ defined on all the pairs of elements of G satisfying the following conditions:

(4.1) $I(x, y)=I(y, x)$ for any $x, y \in G$.

(4.2) $I(x, y)+I(xy, z)=I(x, yz)+I(y, z)$ for any $x, y, z \in G$.

(4.3) For any $x \in G$, there is $m>0$ (depending on x) such that $I(x^m, x)>0$.

(4.4) $I(e, e)=1$ where e is an identity of G.

This I is called "index" again, and G with I is called "an indexed group"

Lemma 9. $I(e, x)=I(e, e)=1$ *for all* $x \in G$.

Proof. Setting x, y, z as e, e, x respectively in (4.2),

$$I(e, e)+I(e, x)=I(e, x)+I(e, x)$$

from which $I(e, x)=I(e, e)$ is derived.

Theorem 4. *For a commutative group G with an index I satisfying the conditions* (4.1), (4.2), (4.3), *and* (4.4), *there is a commutative nonpotent archimedean semigroup S' with cancelation law, the fundamental group of which is isomorphic to the indexed group G.*

Remark. We say that G_1 with I_1 is isomorphic to G_2 with I_2 if the isomorphism f of a group G_1 to G_2 satisfies $I_1(x, y)=I_2(f(x), f(y))$.

Proof. Consider the set S' of all ordered pairs (n, x) of non-negative integer and an element of G: $S'=\{(n, x) ; n=0, 1, 2, \cdots, x \in G\}$. Equality of elements of S' is defined as

$$(n_1, x_1)=(n_2, x_2) \text{ if and only if } n_1=n_2,\ x_1=x_2 ;$$

the product of (n, x) and (m, y) is defined as

$$(n, x)(m, y)=(k, z)$$

where $k=n+m+I(x, y)$, $z=xy$ in G.

S' is a semigroup, for

$$\{(n, x)(m, y)\}(l, z)=(n+m+I(x, y),\ xy)(l, z)$$
$$=(n+m+l+I(x, y)+I(xy, z),\ (xy)z),$$
$$(n, x)(\{m, y)(l, z)\}=(n, x)(m+l+I(y, z),\ yz)$$
$$=(n+m+l+I(x, yz)+I(y, z),\ x(yz)).$$

By the condition (4.2), we obtain

$$\{(n, x)(m, y)\}(l, z)=(n, x)\{(m, y)(l, z)\}.$$

It goes without saying that S' is commutative.

Let us prove that S' is nonpotent. Suppose that there is an idempotent (n, x), $(n, x)(n, x)=(2n+I(x, x), x^2)=(n, x)$. From $x^2=x$, we have $x=e$; from $2n+I(e, e)=n$, we have $n+I(e, e)=0$. This is impossible by (4.4). Hence S' is nonpotent.

Proof of Archimedeaness. We shall show that for (n, x) and (m, y), there are $p>0$ and (l, u) such that $(n, x)^p=(m, y)(l, u)$.

i) In the case $n\geqq 1$. Since $(n, x)=(0, e)(n-1, x)$, we may show the existence of p and (k, z) such that $(0, e)^p=(m, y)(k, z)$. Choose p such that $p-1>m+I(y, y^{-1})$ and let $k=p-1-m-I(y, y^{-1})$, and let $z=y^{-1}$. Then we get $(m, y)(k, z)=(m+k+I(y, y^{-1}), e)=(p-1, e)$, while $(0, e)^p=(I(e^{p-1}, e)+\cdots+I(e, e), e^p)=(p-1, e)$. Accordingly we have $(0, e)^p=(m, y)(k, z)$. At last $(n, x)^p=(0, e)^p(n-1, x)^p=(m, y)(k, z)(n-1, x)^p$. Hence we may adopt $(k, z)(n-1, x)^p$ as (l, u).

ii) In the case $n=0$. Due to the condition (4.3), there is $m>0$: $I(x^m, x)>0$. Choose q such that $q\geqq m$, then $(0, x)^q=(s, x^q)$, for some $s\geqq 1$. For (s, x^q), we find p and (k, z) for (m, y) such that

$$(s, x^q)^p=(m, y)(k, z)$$

and hence $(0, x)^{qp}=(m, y)(k, z)$.

Proof of Cancelation. From $(n, x)(m, y)=(n, x)(k, z)$ or $(n+m+I(x, y), xy)=(n+k+I(x, z), xz)$, we get $xy=xz$, hence $y=z$; further from $n+m+I(x, y)=n+k+I(x, y)$, we have $m=k$. Thus it has been proved that $(n, x)(m, y)=(n, x)(k, z)$ implies $(n, y)=(k, z)$.

Consider the mappping $(n, x)\rightarrow x$. From the definition of multiplication in S', it follows that S' is homomorphic onto G under the mapping. Let us consider the relation with respect to $(0, e)$, which is defined at the beginning of §2. Then there is $n\geqq 0$ such that $(k, x)=(0, e)^n(l, y)$, if and only if $k\geqq l+1$ and $x=y$. Accordingly we have $(k, x)\sim(l, y)$ if and only if $x=y$, so that S'^* corresponds to G one to one. Further,

$$T_0=S'-(0, e)\cdot S'=\{(0, x)\ ;\ x\in G\}$$

and we have $(0, x)(0, y)=(I(x, y), xy)=(0, e)^{I(x,y)}(0, xy)$ from which we see that the fundamental group S'^* is isomorphic to the given indexed group G.

The following theorem is clear.

Theorem 5. *Let S^* be the fundamental group of a commutative non potent archimedean semigroup S with cancelation. Suppose that there is given an indexed group G which is isomorphic to S^*. If we construct the semigroup S' from G by the method of Theorem 4, then S is isomorphic to S'.*

Proof. S is isomorphic to S' under the mapping $a^n x\rightarrow(n, x)$.

Remark. In the present paper, we leave the following problems unsolved.

(1) what is the relation between the fundamental group as to $a\in S$ and the fundamental group as to $b\in S$?

(2) Under what condition, is S_1' constructed from G_1 with I_1 isomorphic to S_2' from G_2 with I_2?

These problems will be discussed in the continued paper II.

References

[1] T. Tamura and N. Kimura: On decompositions of a commutative semigroup, Kōdai Math. Sem. Rep., No. 4. Dec. 1954, 109–112.

Remark

In this paper, the notation $A \subseteq B$ means that A is a proper subset of B or $A = B$,