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NOTES ON THE FUNCTIONS OF TWO COMPLEX VARIABLES

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In this note,*) we want to try some extension of the theorem of Hartogs in connection with the problem which was presented by Professor Hukuhara long ago.

Hartogs' Theorem: If a function $f(x, y)$ is regular in each variable, then it is regular with respect to (x, y).

Professor Hukuhara asked if a function $f(x, y)$ was yet regular or not when $f(x, y)$ was regular with respect to y for only x_n which converged to an inner point x_0.

He extended Osgood's Theorem as follows[1]: *If a function $f(x, y)$ is regular with respect to x for a fixed y and regular with respect to y for x_n which tends to x_0 in D and moreover bounded, then $f(x, y)$ is regular.*

Let x and y be complex variables lying in the domain D and D' respectively. (D, D') shoes the multiple domain of D and D', which are simply connected.

Theorem 1. *If a complex valued function $f(x, y)$ defined in (D, D') is regular with respect to x in D for any fixed y in D' and regular with respect to y in D' for fixed x_m $(m=1, 2, 3, \cdots\cdots)$ which converges to x_0 in D', then there exist regular points dencely in (D, D').*

Proof. Let D_1 and D_1' be arbitrary bounded domains whose closures lying in D and D' respectively. Put

$$M(y) = \operatorname*{Max.}_{x \in D_1} |f(x, y)|, \qquad \text{for a fixed } y,$$

and

$$S_k = \mathop{E}_{y}[M(y) \leqq K],$$

that is, S_k is a set of points y satisfying $M(y) \leqq K$. Suppose that a sequence of points $\{y_n\}$ converging to y_0 are included in S_k. Put $f(x, y_n) = f_n(x)$, then $f_n(x)$ is regular in D_1 and satisfies $|f_n(x)| \leqq K$. This shoes that $\{f_n(x)\}$ is a normal familly in D_1. On the other hand, for a fixed x_m, $f(x_m, y)$ is regular with respect to y. Then we see that $\lim_{n\to\infty} f(x_m, y_n) = f(x_m, y_0)$, that is, $\lim_{n\to\infty} f_n(x_m) = f(x_m, y_0)$, where $m=1, 2, 3, \cdots\cdots$.

Appealing to the theorem of Vitali, $f_n(x)$ converges uniformly to $f_0(x)$, which is regular in D_1. Since y_0 is an inner point of D_1, $f(x, y_0)$ is regular with respect to x. Moreover, $f_0(x_m) = f(x_m, y_0)$ for $m=1, 2, 3, \cdots\cdots$. Since x_m tends to x_0 which is an inner point of D_1, we see that $f(x) \equiv f(x, y_0)$. Since $|f(x, y_n)| \leqq K$,

$$|f(x, y_0)| = \lim_{n\to\infty} |f_n(x)| \leqq K \quad \text{in} \quad D_1 .$$

The regularity of $f(x, y_0)$ on D shows that

$$|f(x, y_0)| \leqq K \quad \text{on} \quad D_1 .$$

That is, y_0 is included in S_k and we see that S_k is a closed set. Now, we must show that at least a set S_k of $\{S_k\}$ include a circle. Let an any one of $\{S_k\}$ does not include a circle. Then there exists a closed circle C_1 in D_1 such that the intersection of C_1 and S_1 is the null set. If such circle C does not exist, then there exists at least a point of S_1 in an arbitrary neighbourhood of y, which is an arbitrary point in D_1. y must lie in S_1, because S_1 is a closed set. That is, $S_1 \supset D_1$ and we see that clearly S_1 include a closed circle, contradicting to the assumption that S_1 does not include a closed circle. Thus, we see that there exist a closed circle C_1 such that $C_1 \cdot S_1 = 0$. On the same way, we have a closed circle C_2 such that $C_1 \supset C_2$, $C_2 \cdot S_2 = 0$, and so on, we have a sequence of closed circle $\{C_k\}$ such that $C_1 \supset C_2 \supset C_3 \supset \cdots \supset C_k \supset \cdots$, $C_k \cdot S_k = 0$. Clearly, $\prod_1^\infty C_k \neq 0$. Then, for an arbitrary point y in $\prod_1^\infty C_k$, we have $M(y) = +\infty$, because, if $M(y) < K$, $y \in S_k$ contradicting to the fact that $y \in C_k$. Clearly, y is an inner point of D_1 and then $M(y)$ must be finite which contradicts to $M(y) = +\infty$. Thus we see that there exists at least S_k which includes a closed circle C. Then,

$$|f(x, y)| \leqq K ,$$

when (x, y) lies on (D_1, C).

Appealing to the extended theorem of Osgood, we see that $f(x, y)$ is regular in (D_1, C). D_1 is an arbitrary closed bounded domain including $\{x_n\}$ in D, and so $f(x, y)$ is regular in (D, C)[3]. D_1' is an arbitrary bounded closed domain in D', we can select D' as a closure of a neighbourhood of an arbitrary point in D_1'. Thus, we see that $f(x, y)$ is regular almost everywhere in (D, D').

Theorem 2. *If $f(x, y)$ defined in (D, D') satisfies following conditions* 1) *$f(x, y)$ is regular with respect to x in D for an arbitrarily fixed y in D',* 2) *for a sequence $\{x_n\}$ in D converging an inner point x_0 in D, $f(x_n, y)$ is regular with respect to y in D', then D' is divided into at most denumerable simply connected domains $\{E_m\}$ such that $f(x, y)$ is regular in (D, E_m) and $E_i \cdot E_j$ is a set of one dimension.*

Proof. Appealing to Theorem 1, we see that there exists a circle C in D' such that $f(x, y)$ is regular in (D, C). Extending $f(x, y)$ analytically from the domain (D, C), we have (D, E), where E is a domain in D' and $f(x, y)$ is regular in (D, E). Let K be an arbitrary simple closed Jordan curve having the length.

Put
$$F(x, y) = \frac{1}{2\pi i} \int_K \frac{f(x, \zeta)}{\zeta - y} d\zeta ,$$

where y lies in the inside of K. Let G be a domain which lies inside of K.

Then $F(x, y)$ is regular in (D, G). On the other hand,

$$F(x_n, y) = \frac{1}{2\pi i}\int_K \frac{f(x_n, \zeta)}{\zeta - y}\, d\zeta = f(x_n, y)\,,$$

by the assumption 2). Thus we have $F(x_n, y) = f(x_n, y)$, for $n=1, 2, \cdots\cdots$, for fixed y arbitrarily in E. Then we see that $f(x, y) \equiv F(x, y)$ in (D, E) and $f(x, y)$ is extended analytically in (D, G). That is, there does not exist any singular point of $f(x, y)$ for y in E. This shoes that E is the simply connected set. Generally, such a set E are denumerable at most in D' and moreover clearly $E_i \cdot E_j$ is a set of one dimension.

References

1) Proof: Let x_n tends to x_0 in D and T be a circle such that $|x-x_0| \leqq r$ which is included in D and C be the boundary of T. Without losing generality, we may think that $\{x_n\}$ lie in T. For an arbitrary y in D', we have

$$f(x, y) = \sum_{n=0}^{\infty} f_n(y)(x-x_0)^n\,,$$

where

$$f_n(y) = \frac{1}{2\pi i}\int_c \frac{f(x, y)}{(x-x_0)^{n+1}}\,dx\,, \quad \text{for} \quad n=0, 1, 2, \cdots\cdots.$$

Since $f(x, y)$ is bounded in (D, D'), there exists a positive number M such that $|f(x, y)| \leqq M$ in (D, D'). Then we have

$$\begin{aligned} |f_n(y)| &\leqq \left| \frac{1}{2\pi i}\int_c \frac{f(x, y)}{(x-x_0)^{n+1}}\,dx \right| \\ &\leqq \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(x, y)|}{r^{n+1}}\, r d\theta \\ &\leqq \frac{M}{r^n}\,, \end{aligned}$$

where $x = x_0 + re^{i\theta}$ and $dx = ire^{i\theta}d\theta$. Let $x = x_m$,

$$f(x_m, y) = f_0(y) + \sum_{n=0}^{\infty} f_n(y)(x_m - x_0)^n\,.$$

Then

$$\begin{aligned} |f(x_m, y) - f_0(y)| &= \left| \sum_{n=1}^{\infty} f_n(y)(x_m-x_0)^n \right| \\ &\leqq M \sum_1^{\infty} \left(\frac{|x_m - x_0|}{r} \right)^n \\ &= \frac{M|x_m - x_0|}{r - |x_m - x_0|}\,. \end{aligned}$$

Let x_m tend to x_0, then we see that $f(x_m, y)$ converges uniformly to $f_0(y)$. Since $f(x_m, y)$ is regular with respect to y, $f_0(y)$ is also regular. By the mathematical induction we see that the regular function

$$\frac{1}{(x_m - x_0)^n}\{f(x_m, y) - f_0(y) - f_1(y)(x_m - x_0) - \cdots - f_{n-1}(y)(x_m - x_0)^{n-1}\}$$

converges uniformly to $f_n(y)$ and so $f_n(y)$ is regular. Thus we see that

$$f(x, y) = \sum_{n=0}^{\infty} f_n(y)(x - x_0)^n$$

converges uniformly in (T, D') and so $f(x, y)$ is regular in (T, D'). By the method of the analytical continuation, $f(x, y)$ becomes regular in (D, D').

2) S. Bochner and W. T. Martin: Several complex variables.

3) D_1 is extended to D by the usual way, since $f(x, y)$ is regular with respect to x in D for an arbitrary fixed y in D'.

*) I expect the advice of the general public, since I don't know much of this field.