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NOTES ON GENERAL ANALYSIS (VI)

Singular set

By

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In this note, the set of singular points of analytic functions in complex Banach spaces is composed of a number of singular subspaces, which are, of course, closed linear subspaces and functions are not analytic there. In the preceding paper[1], we investigated the singular subspace. If x_0 and y_0 do not belong to a singular subspace L_0, and $y_0 \neq \alpha x_0 + \beta y$ for any complex number α,β and any y in L_0, then x_0 and y_0 are called "*independent mutually of L_0*." If there exist two vectors at least which are indepedent mutually of L_0, and an E_2-valued function $f(x)$ is analytic on the outside of L_0 in E_1, then $f(x)$ is analytic on whole space E_1, where E_1, E_2 are complex Banach spaces. That is, the singular subspace L_0 is removable.

Generally, the singular set of an analytic function in complex Banach spaces is not necessarily a singular subspace.

In the first chapter of this paper, we discuss the case that a sigular set of an analytic function in complex Banach spaces is composed of many singular subspaces. For each singular subspace, there exist at least two vectors which are independent mutually of it. In this case, the singular set is removable. In the second of this paper, it is described that the singular subspace L_1 is removable under some conditions. Of course, for this singular subspace L_1 there exists only one vector which is independent mutually of it. In the third of this paper, the singular set is composed of two singular subspaces, For each singular subspace, there exists only one vector being independent mutually of it. The function with this singular set is not simple as the function with one singular subspace.

We shall state theorems which we shall need in the following discussions:

Theorem A.[1] *If there exist two vectors at least which are independent mutually of L_0, a homogeneous function $f_n(x)$ of degree n is a homogeneous polynomial of degree n, where L_0 is a singular subspace of $f_n(x)$.*

Theorem B. *Let $h(x)$ be a homogeneous function of degree n whose singular subspace is L_1. The necessary and sufficient condition that $h(x)$ should be a homogeneous polynomial is that*

$$||h(x+\alpha y)|| \leqq K(x,y),$$

for a sufficiently small $|\alpha|$, in which x is an arbitrary point in L_1 and y is an arbitrary outside point *of L_1 and $K(x,y)$ is a positive constant with respect to α .*

§ 1. Removable singular set.

Let S be a sum of singular subspaces. For each singular subspace, there exist at least two vectors independent of the singular subspace. Suppose that non of the sequence of singular subspaces derived from S converges to any one of S. Then we have the next theorem.

Theorem 1. *If an E_2-valued function $f(x)$ defined on E_1 is analytic on the outside of S, then $f(x)$ is also analytic on S. That is, S is removable.*

Proof. Let x be a point of S being contained only one singular subspace L_0 of S. Since L_0 is not a limiting subspace of any sequence $\{L_n\}$ derived from S, we can find a neighbourhood $V(x)$ such that $f(x)$ is analytic on $V(x)$ excepting points of L_0. Let y be an arbitrary outside point of L_0 and β is a complex number satisfying $x+\beta y \in V(x)\cdot CL_0$, where CL_0 is a compliment of L_0. Then we have

$$f(x+\beta y)=\sum_{-\infty}^{\infty} h_n(x,y)\,\beta^n,$$

where

$$h_n(x,y)=\frac{1}{2\pi i}\int_C \frac{f(x+\alpha y)}{\alpha^{n+1}}\,d\alpha, \quad \text{for } n=0, \pm 1, \pm 2,\cdots$$

A circle C is defined by $|\alpha|=\rho$ such that $x+\alpha y$ lies in $V(x)$, if α lies on C.

We see that $h_n(x,y)$ is analytic as to y on the outside of L_0 and satisfies $h_n(x,\beta y)=\beta^n h_n(x,y)$. Then we see that $h_n(x,y)$ is analytic as to y, because L_0 is removable by Theorem *A*. This shows that $h_n(x,y)\equiv 0$ for $n<0$, and $h_n(x,y)$ is a homogeneous polynomial of degree n, if $n>0$. Then we have

$$f(x+y)=\sum_{n=0}^{\infty} h_n(x,y),$$

where $h_n(x,y)$ is a homogeneous polynomial of degree n.

Since $x+e^{i\theta}\rho y$ lies in $V(x)$ excepting L_0, there exists a neighbourhood $V(\theta)$ for each point $x+e^{i\theta}\rho y$ such that

$$\|f(z)-f(x+e^{i\theta}\rho y)\|<\varepsilon,$$

for an arbitrary positive number ε, if $z\in V(\theta)$, where $V(\theta)$ lies in $V(x)$ excepting L_0 and $V(\theta)$ is a set of points which satisfy $\|x+e^{i\theta}\rho y-z\|<\delta_\theta$ for a suitable positive number δ_θ determined by θ. Appealing to the covering theorem of Borel, we have $\theta_1,\theta_2,\cdots,\theta_k$, such that the set $\sum_{j=1}^{k} V(\theta_j,\frac{1}{2})$ covers the set $x+e^{i\theta}\rho y$ $(0\leqq\theta\leqq 2\pi)$, where $V(\theta_j,\frac{1}{2})$ is a neighbourhood of $x+e^{i\theta_j}\rho y$ such that $\|x+e^{i\theta_j}\rho y-z\|<\dfrac{\delta_{\theta_j}}{2}$.

Put $M=\underset{1\leq j\leq k}{Max.}\{\|f(x+e^{i\theta_j}\rho y)\|+\varepsilon\}$. If z lies in $\sum_{1}^{k} V(\theta_j)$, we have $\|f(z)\|\leqq M$. When

δ_0 is a positive number such that $0<\delta_0 \leqq \underset{1\leq j\leq k}{Max.} \frac{\delta_{\theta j}}{2}$, we have $x+e^{i\theta}V(y,\delta_0)\subset \sum_1^k U(\theta_j)$, for $0\leqq\theta\leqq 2\pi$, where $V(y, \delta_0)$ is a set of points which satisfy $\|y-z\|\leqq\delta_0$. Then

$$\|h_n(x,z)\| = \|\frac{1}{2\pi i}\int_C \frac{f(x+\alpha z)}{\alpha^{n+1}}d\alpha\|$$

$$\leqq \|\frac{1}{2\pi}\int_0^{2\pi}\frac{f(x+e^{i\theta}z)}{e^{in\theta}}d\theta\|$$

$$\leqq M,$$

where C is a circle whose radius is 1, for z lying in $V(y,\delta_0)$ and $n=0,1,2,\cdots$. Appealing to the lemma of Zorn[2] we see that $\|h_n(x,y)\|\leqq M$, when $\|y\|<\delta_0$, for $n=0,1,2,\cdots$. Thus we have

$$\sup_{\|y\|=1}\overline{\lim_{m\to\infty}}\sqrt[m]{\|h_m(x,y)\|}\ ^{3)} = \sup_{\|y\|=1}\overline{\lim_{m\to\infty}}\sqrt[m]{\|h_m(x,\frac{\delta y}{\delta})\|}, \text{ for } 0<\delta<\delta_0,$$

$$= \frac{1}{\delta}\sup_{\|y\|=1}\overline{\lim_{m\to\infty}}\sqrt[m]{\|h_m(x,\delta y)\|}$$

$$= \frac{1}{\delta}\sup_{\|y\|=1}\overline{\lim_{m\to\infty}}\sqrt[m]{M}$$

$$= \frac{1}{\delta}.$$

This shows that the radius of analyticity[3] of $f(x+y)=\sum_{m=0}^{\infty}h_m(x,y)$ is not smaller than δ and we see that $f(x)$ is analytic at x lying only on L_0. Now, let L_0 and L_0' be arbitrary two singular subspaces in S. Then $L_0\cap L_0'$ is a singular subspace. We can see that $f(x)$ is analytic on a point lying only on $L_0\cap L_0'$ as well as L_0. And so on, we see that $f(x)$ is analytic on S by the transcendental mathematical induction.

Corollary. *If $h(x)$ is analytic on the outside of S and satisfies $h(\alpha x)=\alpha^n h(x)$ there for an arbitrary complex number α, $h(x)$ is a homogeneous polynomial of degree n.*

Proof. We see that $h(x)$ is analytic on whole spaces by Theorem 1. Since $h(x)$ is continuous, the equality $h(\alpha x)=\alpha^n h(x)$ is held also for a point x on S. This shows that $h(x)$ is a homogeneous polynomial of degree n. This completes the proof.

Thus we see that a singular set S is removable if S is composed of singular subspaces which are at least lower two dimensions than the space. But, when there do not exist two vectors which are independent mutually of an each subspace L_1 of S, S is not generally removable.

From now on, let L_1, L_2 be singular subspaces such that there exists at least one vector being independent mutually of each singular subspace L_i but do not exist two vectors being independent mutually of L_i, where $i=1,2$.

§ 2. Removable singular subspaces.

The singular subspace L_1 is removable under some conditions.

Theorem 2. *Let an E_2-valued function $f(x)$ defined on E_1 has a singular subspace L_1. If, for an arbitrary point x on L_1, there exists a neighbourhood $V(x)$ of x and a constant $K(x)$ such that $\|f(y)\| \leqq K(x)$ for y in $V(x)$,[4] then L_1 is removable.*

Proof. For an arbitrary point x in L_1 and an arbitrary outside point y of L_1, $f(x+\alpha y)$ is an analytic function of α, when $0<|\alpha|<\infty$. For a suitable positive number δ, $x+\alpha y \in V(x)$, when $|\alpha|<\delta$. Then we have $\|f(x+\alpha y)\| \leqq K(x)$, when $0<|\alpha|<\delta$. Thus we see that $\alpha=0$ is removable and $f(x+\alpha y)$ is analytic as to α for $|\alpha|<\infty$.

Then we have $$f(x+\alpha y)=\sum_0^\infty h_n(x,y)\alpha^n,$$

where

$$h_n(x,y)=\frac{1}{2\pi i}\int_C \frac{f(x+\zeta y)}{\zeta^{n+1}}\,d\zeta, \text{ for } n=0,1,2,\cdots$$

Clearly, $h_n(x,y)$ is a homogeneous function as to y with a singular subspace L_1, because y is an arbitrary outside point of L_1. For an arbitrary point x_1 in L_1 and an arbitrary outside point y_1 of L_1,

$$h_n(x,x_1+\alpha y_1)=\frac{1}{2\pi i}\int_C \frac{f(x+\zeta(x_1+\alpha y_1))}{\zeta^{n+1}}\,d\zeta.$$

Put $\zeta=e^{i\theta}(0\leqq\theta\leqq 2\pi)$, then the point $x+x_1e^{i\theta}(0\leqq\theta\leqq 2\pi)$ lies on L_1 and so we have $\|f(x+x_1e^{i\theta}+y)\| \leqq K(\theta,)$ where $0\leqq\theta\leqq 2\pi$, for any y in a suitable neighbourhood $V(\theta)$ of $x+x_1e^{i\theta}$ and a constant $K(\theta)$ for $0\leqq\theta\leqq 2\pi$. By the covering theorem of Borel, there exist a system of neighbourhoods $V(\theta_1), V(\theta_2),\cdots, V(\theta_p)$ such that $x+x_1e^{i\theta} \subset \sum_1^p V(\theta_j)$, if $0\leqq\theta\leqq 2\pi$. Moreover, for a suitable neighbourhood $U(x_1)$ of x_1, we have $x+U(x_1)e^{i\theta} \subset \sum_{j=1} V(\theta_j)$, for $0\leqq\theta\leqq 2\pi$. Put $\underset{1\leq j\leq p}{Max.}\,K(\theta_j)=K$, then $\|f(x+U(x_1)e^{i\theta})\| \leqq K$. Let $U(x_1) \supset U(x_1,\delta)$ and $|\alpha|<\dfrac{\delta}{\|y_1\|}$, then $x+x_1e^{i\theta}+e^{i\theta}\alpha y_1 \subset x+U(x_1)e^{i\theta}$.
Then

$$\|h_n(x,x_1+\alpha y)\| \leqq \frac{1}{2\pi}\int_0^{2\pi} \|f(x+e^{i\theta}(x_1+\alpha y_1))\|d\theta$$
$$\leqq K.$$

Appealing to Theorem B, $h_n(x,y)$ is analytic as to y and we see that $h_n(x,y)$ is a homogeneous polynomial of degree n as to y. As well as the proof of Theorem 1, we see that the power series $\sum_{n=0}^\infty h_n(x,y)$ is convergent uniformly in a neighbourhood of x. Since x is arbitrary in L_1, L_1 is removable.

Thorem 3. *Let L_1 be a singular subspace of $f(x)$. If, for an arbitrary point x in L_1 and* *an arbitrary outside point y of L_1,*

$$\overline{\lim_{\alpha\to 0}} \|f(x+\alpha y)\| \leqq K(y),$$

where $K(y)$ is a constant depending upon y, then L_1 is removable.

Proof. Let x_1 be an arbitrary point of L_1 and y_1 be an arbitrary outside point of L_1. Since $\overline{\lim\limits_{\alpha\to 0}} \|f(x_1+\alpha y_1)\| \leqq K(y_1)$, there exists a positive number δ for a given positive number ε such that $\|f(x_1+\alpha y_1)\| \leqq K(y_1)+\varepsilon$ for $|\alpha| \leqq \dfrac{\delta}{\|y_1\|}$. If y is an arbitrary outside point of L_1, we have $y = x_0 + \alpha_0 y_1$, for a suitable x_0 in L_1 and a complex number α_0, because, there exists only one vector essentially being independent of L_1. Then

$$x_1+\alpha y = x_1+\alpha(x_0+\alpha_0 y_1) = (x_1+\alpha x_0)+\alpha\alpha_0 y_1.$$

Since $K(y)$ is independent of x,

$$\|f(x_1+\alpha y)\| \leqq K(y_1)+\varepsilon, \text{ when } \|\alpha\alpha_0 y_1\| \leqq \delta.$$

Let $|\alpha_1| = \dfrac{\delta}{\|y_1\|}$ and d be a distance between $\alpha_1 y_1$ and the singular subspace L_1. Clearly, $d>0$. If $d=0$, $\alpha_1 y_1$ is a limiting point of points derived from L_1 and so $\alpha_1 y_1$ must be a point of L_1 contradicting to the fact that y_1 is an outside point, since L_1 is closed. Let $d>\|y\|$. Since $\|y\| = \|x_0+\alpha_0 y_1\| \geqq \text{Dis. } (\alpha_0 y_1, L_1) \geqq |\alpha_0| \cdot \text{Dis. } (y_1, L_1) = |\alpha_0| \cdot \dfrac{d}{|\alpha_1|}$, $|\alpha| \geqq |\alpha_0|$. Then, $\|\alpha_0 y_1\| \leqq \|\alpha_1 y_1\| = \delta$ (the cace of $\alpha=1$). That is, $\|f(x_1+y)\| \leqq K(y_1)+\varepsilon$, when $\|y\| \leqq d$. Appealing to Theorem 2, L_1 is removable, since x_1 is an arbitrary point of L_1.

§ 3. Reciprocal homogeneous function.

If a singular set S of $f(x)$ is composed of some singlar subspaces such as L_1, the characters of $f(x)$ are not simple. Prior to the discussion of this chapter, we must define some functions.

Definition 1. *If $P(x)$ is a (reciprocal) homogeneous function of degree n with the singular subspace L_1 whose orders of singularity is m, $P(x)$ is called (n,m)-function with the singular subspace L_1. (If n is a negative integer, $P(x)$ is a reciprocal homogeneous function.)*

For example, put $x=(x_1,x_2)$, where x_1 and x_2 are complex numbers, and $P(x) = \dfrac{x_1^{n+m}}{x_2^m}$. Then $P(x)$ is a (n,m)-function with a singular subspace L_1, which is defined as $x_2=0$.

Definition 2. *Let S be composed of L_1 and L_2, and $R_n(x)$ be analytic at outside points of S and satisfies $R_n(\alpha x) = \dfrac{1}{\alpha^n} R_n(x)$ there.*

Moreover, $\overline{\lim\limits_{\alpha\to\infty}} \|R_n(\alpha X+y)\| \leqq K(L_i, y)$, for an arbitrary x in L_i and an arbitrary outside point y of L_1. Of course, $\alpha x+y$ lies in the outside of S and $i=1,2$. Then, $R_n(x)$ is called R-function of degree n.

As well as $R_n(x)$, we can define $P_n(x)$, which is called *P-function of degree n* with the singular set S. That is, (1) *$P_n(x)$ is analytic on the outside of S*, (2) *$P_n(\alpha x)=\alpha^n P(x)$ for any complex number α and an arbitrary point x in the outside of S*, (3) $\overline{\lim_{\alpha\to 0}} \|P_n(x+\alpha y)\| \frac{1}{|\alpha|^n} \leqq K(L_i,y)$ *for an arbitrary point x in L_i and an arbitrary outside point y of L_i, where $i=1,2$.*

Theorem 4. *Let a point x on L_i lie on the outside of L_2. Then, for an arbitrary outside point y of L_i, we have*

$$R_n(x+y)=\sum_{m=-n}^{\infty} R_{m,n}(x,y),$$

where $R_{m,n}(x,y)$ is (m,n)- function with respect to y.

Proof. Since $R_n(x)$ is anaylytic on the outside of S and $x+\alpha y$ lies in the outside of S for a suitable α, we have

$$R_n(x+y)=\sum_{-\infty}^{\infty} R_{m,n}(x,y),$$

where $R_{m,n}(x,y)=\frac{1}{2\pi i}\int_C R_n(x+\alpha y)\alpha^{-m-1}d\alpha$, for $m=0, \pm 1, \pm 2,\cdots$ C is a circe $|\alpha|=r$, which satifies $0<r\cdot\|y\|<d(x,L_2)$, where $d(x,L_2)$ is the distance between x and L_2. If y lies on the outside of L_2, there exists z on L_2 such that $x=\lambda y+\mu z$ for suitable complex numbers λ and μ. Then $x+\alpha y=(\lambda+\alpha)y+\mu z$. This shows that $x+\alpha_0 y$ lies on L_2, if $\alpha_0 =-\lambda$. $x+\alpha_0 y$ is an only point lying on L_2 for $|\alpha|<\infty$, because if there exist α such that $x+\alpha y\in L_2$, $(\alpha-\alpha_0)y=x+\alpha y-(x+\alpha_0 y)\in L_2$ contradicting to the fact that $y\overline{\in} L_2$.
Since $\frac{x_0}{\alpha}+y=\frac{1}{\alpha_0}(x+\alpha_0 y)\in L_2$, we see that $\frac{x}{\alpha}+y\overline{\in} L$, if $|\alpha|\leqq r$, which is naturally smaller than $|\alpha_0|$. Thus we see that $R_n(x)$ is analytic at $\frac{x}{\alpha}+y$ for $|\alpha|\leqq r$ and we have

$$R_{m,n}(x,y)=\frac{1}{2\pi i}\int_C R_n(x+\alpha y)\alpha^{-m-1}d\alpha.$$

$$=\frac{1}{2\pi i}\int_C R_n\left(\frac{x}{\alpha}+y\right)\alpha^{-m-n-1}d\alpha$$

Put $\frac{1}{\alpha}=\beta$, and $\beta=\rho e^{i\theta}$, then $d\beta=i\rho e^{i\theta}d\theta$ and so

$$R_{m,n}(x,y)=\frac{-1}{2\pi}\int_0^{2\pi} R_n(\rho e^{i\theta}x+y)(\rho e^{i\theta})^{n+m}d\theta.$$

Then,

$$\|R_{m,n}(x,y)\|\leqq\frac{1}{2\pi}\int_0^{2\pi}\|R_n(\rho e^{i\theta}x+y)\|\rho^{n+m}d\theta, \text{ and so we have}$$

$$\|R_{m,n}(x,y)\|\leqq\frac{1}{2\pi}\int_0^{2\pi}\overline{\lim_{\beta\to\infty}}\|R_n(\beta x+y)\beta^{n+m}\|d\theta$$

$$=0, \text{ if } m<-n,$$

since $\overline{\lim_{\beta\to\infty}}\|R_n(\beta x+y)\|\leqq K(L_1,y)$ by the definition.

If y lies on L_2, $\dfrac{\alpha}{x}+y$ does not lie on L_1 nor L_2 and so we see that $R_{m,n}(x,y)=0$, when y lies on L_2 and $m<-n$.

Since y is arbitrary, we can easily see that $R_{m,n}(x,y)\equiv 0$, if $m<-n$. $R_{m,n}(x,y)$ is clearly analytic as to y on the outside of L_1 and satisfies $R_{m,n}(x,\alpha y)=\alpha^m R_{m,n}(x,y)$. Now, let x' be a point on L_1 and y be an outside point of L_1. Then

$$R_{m,n}(x,x'+\beta y)=\frac{1}{2\pi i}\int_C R_n(x+\alpha(x'+\beta y))\alpha^{-m-1}d\alpha$$

and so we have

$$\overline{\lim_{\beta\to 0}}\,|\beta|^n\cdot\|R_{m,n}(x,x'+\beta y)\|$$

$$\leqq \overline{\lim_{\beta\to 0}}\frac{1}{2\pi}\int_0^{2\pi}\|R_n(\frac{x}{\beta}+\alpha(\frac{x'}{\beta}+y))\|\,|\alpha|^{-m}d\theta$$

$$\leqq \frac{1}{2\pi}\int_0^{2\pi}\overline{\lim_{\beta\to 0}}\|R_n(\frac{1}{\beta}(\frac{x}{\alpha}+x')+y)\|\cdot|\alpha|^{-m-n}d\theta$$

$$\leqq \frac{1}{2\pi}\int_0^{2\pi}K(L_1,y)\,|\alpha|^{-m-n}d\theta$$

$$=K(L_1,y)\,|\alpha|^{-m-n}.$$

This shows that $R_{m,n}(x,y)$ has a singular subspace L_1 of degree n generally and so $R_{m,n}(x,y)$ is the (m,n)-function generally. This completes the proof.

The follwing example shows exactly this fact. Put $x=(x_1,x_2)$ and $\|x\|=\text{Max.}\,(|x_1|,|x_2|)$, where x_1 and x_2 are complex numbers. Then the set of x forms complex-Banach-spaces Ω. Let $f(x)=\dfrac{1}{x_1x_2}$ and $S=L_1\cup L_2$, where L_i is a set of points such that $x_i=0$ in Ω for $i=1,2$. Then $f(x)$ is analytic on the outside of S and satisfies there $f(\alpha x)=\dfrac{1}{\alpha^2}f(x)$, for an arbitrary complex number α. That is, $f(x)$ is the R-function of degree 2. Put $x=(0,x_2,)$ where $x_2\neq 0$ and $y=(y_1,y_2)$. Then we have

$$f(x+\alpha y)=\frac{1}{\alpha y_1(x_2+\alpha y_2)}=\sum_0^\infty\frac{(-1)^n}{x_2^{n+1}}\cdot\frac{y_2^n}{y_1}\alpha^{n-1}.$$

That is $f_{n,1}(x,y)=\dfrac{(-1)^{n+1}}{x_2^{n+1}}\cdot\dfrac{y_2^{n+1}}{y_1}$, which has a singular subspace L_1 of degree 1.

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2). M. A. Zorn: Characterization of analytic functions in Banach spaces. Annals of Math. (2). 46(1945).

3). E. Hille: Functional analysis and semigroups, 1948. I. Shimoda: On power series in abstract spaces, 1948. If $\frac{1}{r} = \text{Sup.}_{\|x\|=1} \overline{\lim_{n\to\infty}} \sqrt[m]{\|h_n(x)\|}$, r is called "radius of analyticity".

4) $f(x)$ is locally bounded on L_1.

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ON A SPECIAL SEMILATTICE WITH A MINIMAL CONDITION.

By

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By a semilattice we mean a commutative idempotent semigroup, namely, a partly ordered set which has a least upper bound of any two elements[1]. In the present paper we shall discuss the structure of a special semilattice, which will be called an unbounded dispersed semilattice with a certain minimal condition.

§1. Flowing Semilattice.

Let S be a semilattice and let $a, b, c, \cdots, x, y, \cdots$ be elements of S. At first we explain the notations $[b,c]$ $[a,*)$ $(*,a]$ $(*,a)$ as following.

For $b \leqq c$
$$[b,c]=\{x;\, b\leqq x\leqq c,\ x\in S\},$$
$$[a,*)=\{x;\, a\leqq x,\ x\in S\},$$
$$(*,a]=\{x;\, x\leqq a,\ x\in S\},$$
$$(*,a)=\{x;\, x< a,\ x\in S\}.$$

Lemma 1. *The following conditions are all equivalent.*

(1.1) *If $b<c$, then $[b,c]$ is a chain in S.*

(1.2) *For any $a\in S$, $[a,*)$ froms a chain in S.*

(1.3) *For any a, x, $y\in S$, either $ax\geqq ay$ or $ax\leqq ay$.*

(1.4) *There are no x,y,z such that $z<x$, $z<y$, and $x \gtreqless\!\!\!\!/\, y$ hold simultaneously. In other words, there is no lower bound of incomparable x and y.*

Proof. (1.1)→(1.2). If $[a,*)$ is not a chain for some a, there are incomparable $x,y\in[a,*)$. Let z be a least upper bound of x and y. Then both x and y belong to $[a,z]$. This contradicts with (1.1).

(1.2)→(1.3.) Suppose that ax and ay are incomparable. Considering $a\leqq ax$, $a\leqq ay$, that is, $[a,*)$ is not a chain. This conflicts with (1.2).

(1.3)→(1.4). If (1.4) is false, there are x, y, z such that $z\leqq x$, $z\leqq y$ and $x \gtreqless\!\!\!\!/\, y$, in other words, $x=zx'$, $y=zy'$, and $zx' \gtreqless\!\!\!\!/\, zy'$, contradicting with (1.3).

(1.4)→(1.1). Suppose that (1.1) is not valid. A certain set $[b,c]$ is not a chain.

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1) See G. Birkhoff, Lattice theory. In the partly ordered set, $a\geqq b$ is defined as $ab=a$. Accordingly xy is a least upper bound of x and y. $a>b$ means $a\geqq b$ but $a\neq b$.

Then there are incomparable x and y such that $b<x<c$ and $b<y<c$. This conflicts with (1.4).

If a semilattice S satisfies the condition of Lemma 1, S is called a *flowing semilattice.*

Lemma 2. *In a flowing semilattice S, if $ab>ac$, then $ab=bc$. Conversely if $ab=bc$ and $bc\neq ac$, then $ab>ac$.*

Proof. According to (1.3) of Lemma 1, any two of the three elements ab, bc, and ca are comparable so that they form a chain. We consider the three cases in the present Lemma.

(i) $ab>ac>bc$, (ii) $ab\geqq bc\geqq ac$,

(iii) $bc>ab>ac$.

However we can show that (i) and (iii) are impossible in the following manner. If (i) holds, $ac\geqq a$, $ac>bc\geqq b$, hence $ac\geqq ab$, contradicting with the assumption $ab>ac$. In the case of (iii), $ab\geqq b$, $ab>ac\geqq c$, hence $ab\geqq bc$. This also conflicts with the inequality (iii) $bc>ab$. Thus we have proved possibility of (ii). Now, from $bc\geqq b$, $bc\geqq ac\geqq a$, it follows that $bc\geqq ab$ and so $ab=bc$. We shall prove the latter half of this lemma. Since $ab=bc\neq ac$, either $ab>ac$ or $ac>ab$ by (1.3) of Lemma 1. If $ac>ab$, we obtain $ac=bc$ by ther former half of of this lemma. This contradicts with the assumption. Hence we have only $ab>ac$.

Theorem 1. *Let S be a flowing semilattice and let a,b,c be any elements of S. Then two at least of the three elements ab,bc, and ca are equal. Conversely if a semilattice S satisfies this condition, it is a flowing semilattice.*

Proof. As far as the former half of the theorem is concerned, we may show that only one of the following four cases arises:

(1) $ab=bc=ca$, (2) $ca<ab=bc$,

(3) $ab<ca=bc$, (4) $bc<ab=ca$.

According to (1.3), ab and ca are comparable. By Lemma 2, $ab>ca$ implies $ab=bc$, $ab<ca$ implies $ac=bc$; if $ab=ca$ and $ac\neq bc$, then $ab>bc$. Therefore the above four cases are obtained.

Conversely suppose that S satisfies the above conditions, neverthless, that S is not flowing. By Lemma 1, there are incomparable x,y and their lower bound z: $z<x$, $z<y$, $x\neq y$. Of course $zx=x$, $zy=y$, and $xy\neq x$, $xy\neq y$, because x and y are incomparable. Therefore any two of the three elements zx, zy, and xy are not equal. This contradicts with the assumption. Thus the theorem has been proved.

§ 2. Dispersed Semilattice.

Lemma 3. *In a flowing semilattice S, the following conditions ase all equivalent.*

(2.1) *For any $b, c\in S$, $b<c$, $[b,c]$ is finite.*

(2.2) *For any $a\in S$, $[a,*)$ is mapped isomorphically into the chain composed of all positive* *integers.*

(2.3) *Any maximal chain of S is mapped isomorphically into the chain of all integers.*

Proof. (2.1)→(2.2) Let x be an element of $(a,*)$. Since $[a,x]$ is finite, a positive integer k is determined such that

$$a=x_0<x_1<\cdots<x_k=x.$$

It is clear that the correspondence $x\leftrightarrow k$ is one to one and preserves the ordering.

(2.2)→(2.3). Let C be a maximal chain of S, and a be any fixed element of C. The subset $[a,*)$ of C consists of

$$a=x_0<x_1<x_2<\cdots<x_n<\cdots.$$

If the subset $(*,a)$ of C is not empty, then, for any $z\in(*,a)$, $[z,*)$ is isomorphic into the set of all positive integers, and a is certainly contained in $[z,*)$,

$$z=z_0<z_1<\cdots<z_l=a,$$

that is, $[z,a]$ is finite. We rewrite them as follows:

$$z=z_0=x_{-l},\ z_1=x_{-l+1},\cdots,\ z_{l-1}=x_{-1},\ a=z_l=x_0,$$

and then $z=x_{-l}<x_{-l+1}<\cdots<x_{-1}<x_0=a$,
so $x\to -l$ preserves the ordering.

(2.3)→(2.1). This is obvious.

If a flowing semilattice S satisfies the conditions of Lemma 3, then S is called a *dispersed semilattice.*

Lemma 4. *If S is a dispersed semilattice, then S is a complete semilattice, in other words, S contains the least upper bound of any subset.*

Proof. Let b be an upper bound of any subset T, and a an element of T. Since $[a,b]$ is finite by Lemma 3, we can find the least p_0 of elements which are upper bounds of T and belong to $[a,b]$. In the following manner, it is proved that this p_0 is required one. Let u be any upper bound of T. According to Theorem 1, the two at least of the three elements ab, au, and bu are equal, that is, one of the following identities (1), (2), and (3) holds.

(1) $b=ab=au=u,$ (2) $bu=au=u,$

(3) $bu=ab=b.$

In all cases, it is concluded that either $u\in[a,b]$ or $b\leqq u$, consequently we have $p_0\leqq u$ i.e. p_0 is the least upper bound of T.

Now we shall define a terminology, the *length of an element.*
Since $[a,x]$, for $a<x$, is finite, a non-negative integer k is determined such that

$$a=x_0<x_1<\cdots<x_k=x.$$

This k is called the length of an element x to a, and k is denoted by $k=l_a(x)$. We make a promise $l_a(a)=0$.

Hereafter S denotes a dispersed semilattice.

Lemma 5. *Let* $a\leqq x$, $a\leqq y$.

(2.4) $x=y$ *if and only if* $l_a(x)=l_a(y)$.

(2.5) $x>y$ *if and only if* $l_a(x)>l_a(y)$.

(2.6) *If* $a\leqq b\leqq c$, *then* $l_a(b)+l_b(c)=l_a(c)$.

Proof. By Lemmas 1 and 3, $[a,*)$ is a chain and $[a, ax]$, $[a, ay]$ are finite. This lemma is obvious.

Lemma 6. $ab=ac$ *if and only if*

(2.5) $l_b(ab)+l_c(bc)=l_b(bc)+l_c(ca)$.

Proof. Necessity of (2.5). Let $ab=ac=p$.

$p\geqq b$, and $p\geqq c$ imply $p\geqq bc\geqq b$ and $p\geqq bc\geqq c$. By Lemma 5,

$$l_b(p)=l_b(bc)+l_{bc}(p), \qquad l_c(p)=l_c(bc)+l_{bc}(p).$$

From the two identities, we get (2.5) directly.

Sufficiency of (2.5). Suppose $ab>ac$ under (2.5). According to Lemmas 2 and 5, $ab=bc$, $l_b(ab)=l_b(bc)$. By (2.5), we have $l_c(bc)=l_c(ca)$, so $bc=ca$; consequently $ab=bc=ca$, contradicting with $ab>ac$. This leads to $ab\not> ac$, similarly $ab\not< ac$. Hence $ab=ac$.

Gathering Theorem 1 and Lemma 6 into together,

Lemma 7. *If S is a dispersed semilattice, then, for any* a,b, $c\in S$, *one of the following identities holds.*

(2.5) $l_b(ab)+l_c(bc)=l_b(bc)+l_c(ca)$,

(2.6) $l_c(bc)+l_a(ca)=l_c(ca)+l_a(ab)$

(2.7) $l_a(ca)+l_b(ab)=l_a(ab)+l_b(bc)$.

In detail,

(2.5), (2.6), *and* (2.7) *hold at the same time if and only if* $ab=bc=ca$.

(2.5) *holds if and only if* $bc<ab=ac$.

(2.6) *holds if and only if* $ca<bc=ba$.

(2.7) *holds if and only if* $ab<ca=cb$.

Hereafter we shall provide a dispersed semilattice S with a minimal condition as follows.

For any $x\in S$, there is a minimal element a of S such that $a\leqq x$.

Let M be the set of all minimal elements of S. Naturally distinct minimal elements are incomparable. Let $M\times M$ be the set of all pairs (a,b) of elemements a,b of M.

Consider a mapping which associates $(a,b)\in M\times M$ with a pair $(l_a(ab), l_b(ab))$ of non-negative integers $l_a(ab)$, $l_b(ab)$, where we rewrite $f(a;(a,b))=l_a(ab)$, $f(b;(a,b))=l_b(ab)$. These satisfy the following conditions.

(2.8) $f(a;(a,b))\geqq 0$, and $f(a;(a,b))=0$ if and only if $a=b$.

(2.9) $f(a;(a,b))=f(a;(b,a))$.

(2.10) For any a,b,c, one least of the following three identities holds.

$$f(b;(a,b))+f(c;(b,c))=f(b;(b,c))+f(c;(c,a)),$$
$$f(c;(b,c))+f(a;(c,a))=f(c;(c,a))+f(a;(a,b)),$$
$$f(a;(c,a))+f(b;(a,b))=f(a;(a,b))+f(b;(b,c)).$$

On the other hand, we shall see that a dispersed semilattice with a minimal condition is characterized by a mapping

$$(a,b)\rightarrow(f(a;(a,b)),\ f(b;(a,b)))$$

If S is a dispersed semilattice with the minimal condition, any element x of S determines one minimal element a at least and a non-negative integer $l_a(x)$ such as above mentioned.

Lemma 8. *Let a and b be minimal element of S. $l_a(x)$ is bounded for fixed a and varying x, if and only if $l_b(x)$ is bounded for fiexed b and varying x.*

Proof. If $l_a(x)$ is unbounded, then $[a,*)$ is infinite. Since $[ab,*)\subset[a,*)$, $[ab,*)$ is infinite and also $[b,*)$ is so, which means that $l_b(x)$ is unbouded.

Lemma 9. *$l_a(x)$ is bounded for varying x if only if S has the greatest element.*

Proof. If S has the greatest element g, $[a,g]$ is finite and any element $x\geqq a$ is included in $[a,g]$ because S is a dispersed semilattice. Hence $l_a(x)$ is bounded. Conversely if S has not the greatest element, then, for any x, there is y such that $y\geqq x$. Accordingly we have an infinite chain

$$a=x_0<x_1<x_2<\cdots<x_n<\cdots,$$

so $l_a(x)$ is unbounded. When S has not the greatest element, S is called *unbounded.*

In this paper we shall treat construction of unbounded dispersed semilattice with the minimal condition, in which $l_a(x)$ is unbounded, that is, $l_a(x)$ is valued throughout non-negative integers.

§ 3. Construction of an Unbounded Dispersed Semilattice with a Minimal Condition.

In this paragraph, consider M as an abstract set, and suppose that a mapping of $(a,b)\in M\times M$ to a pair of non-negative integers: $(f(a;(a,b)), f(b;(a,b)))$ where $f(a;(a,b))$ and $f(b;(a,b))$ satisfy the following conditions.

(3.1) $f(a;(a,b))\geqq 0$, and $f(a;(a,b))=0$ if and only if $a=b$.

(3.2) $f(a;(a,b))=f(a;(b,a))$ for any $a,b\in M$.

(3.3) For any $a,b,c\in M$, one at least of (3.3.1), (3.3.2), and (3.3.3.) holds.

$$(3.3.1)\quad \begin{cases} f(a;(a,b))=f(a;(a,c)) \\ f(c;(c,a))\geqq f(c;(c,b)) \\ f(b;(a,b))+f(c;(b,c))=f(b;(b,c))+f(c;(c,a)) \end{cases}$$

$$(3.3.2)\quad \begin{cases} f(b;(b,c))=f(b;(a,b)) \\ f(a;(a,b))\geqq f(a;(a,c)) \\ f(c;(b,c))+f(a;(c,a))=f(c;(c,a))+f(a;(a,b)) \end{cases}$$

$$(3.3.3)\quad \begin{cases} f(c;(c,a))=f(c;(b,c)) \\ f(b;(b,c))\geqq f(b;(b,a)) \\ f(a;(c,a))+f(b;(a,b))=f(a;(a,b))+f(b;(b,c)) \end{cases}$$

Now, let S be the set of all pairs of an element of M and an element of the set K of all non-negative integers.

$$S=K\times M=\{(\lambda,a);\ \lambda\in K,\ a\in M\}.$$

We introduce an ordering into S as follows.

$(\lambda,a)\geqq(\mu,b)$ means $\lambda\geqq f(a;(a,b))$ as well as $\lambda+f(b;(a,b))\geqq\mu+f(a;(a,b))$.

Lemma 10. *This ordering of S is a quasi-ordering.*

Proof. $(\lambda,a)\geqq(\lambda,a)$ is easily shown by the definition.
We shall show only a transitive law:

$(\lambda,a)\geqq(\mu,b)$ and $(\mu,b)\geqq(\nu,c)$ imply $(\lambda,a)\geqq(\nu,c)$.

The Proof of $\lambda\geqq f(a;(a,c))$. By $(\lambda,a)\geqq(\mu,b)$, we have $\lambda\geqq f(a;(a,b))$. In the case of (3.3.1) or (3.3.2), we get directly $\lambda\geqq f(a;(a,c))$. In the case (3.3.3),

$$\lambda\geqq\mu+f((a;(a,b))-f(b;(a,b)) \qquad (\text{by}(\lambda,a)\geqq(\mu,b))$$

$$\geqq f(b;(b,c))+f(a;(a,b))-f(b;(a,b)) \qquad (\text{by } \mu\geqq f(b;(b,c)) \text{ because } (\mu,b)\geqq(\nu,c),)$$

$$=f(a;(c,a)). \qquad (\text{by the third identity of (3.3.3).})$$

The Proof of $\lambda+f(c;(a,c))\geqq\nu+f(a;(a,c))$.
By the definition of $(\lambda,a)\geqq(\mu,b)$ and $(\mu,b)\geqq(\nu,c)$, we have $\lambda-\mu\geqq f(a;(a,b))-f(b;(a,b))$, $\mu-\nu\geqq f(b;(b,c))-f(c;(b,c))$, so that $\lambda-\nu=\lambda-\mu+\mu-\nu$

$$\geqq f(a;(a,b))-f(b;(a,b))+f(b;(b,c))-f(c;(b,c))$$
$$=f(a;(a,c))-f(c;(a,c)).$$

The last identity is obtained from the first and third identities in all cases (3.3.1), (3.3.2), and (3.3.3). Now we define $(\lambda,a)=(\mu,b)$ as $(\lambda,a)\geqq(\mu,b)$ and $(\lambda,a)\leqq(\mu,b)$.

Lemma 11. $(\lambda,a)=(\mu,b)$ *if and only if*

$$\lambda\geqq f(a;(a,b)) \text{ and } \lambda+f(b;(a,b))=\mu+f(a;(a,b)).$$

Proof. By the definition, if $(\lambda,a)=(\mu,b)$ then

(3.4) $\lambda \geqq f(a;(a,b))$, (3.5) $\lambda + f(b;(a,b)) \geqq \mu + f(a;(a,b))$,

(3.6) $\mu \geqq f(b;(b,a))$, (3.7) $\mu + f(a;(b,a)) \geqq \lambda + f(b;(b,a))$.

hold simultaneously. From (3.5) and (3.7) we have

$$\lambda + f(b;(a,b)) = \mu + f(a;(a,b)), \tag{3.8}$$

while (3,6) is implied by (3.4) and (3.8). The converse is clear.

Lemma 12. *$(\lambda, a) > (\mu, a)$ if and only if $\lambda > \mu$. $(\lambda, a) = (\mu, a)$ if and only if $\lambda = \mu$.*

Proof. Since $f(a;(a,a)) = 0$, this lemma is derived easily from the definition.

Lemma 13. *The set of all (ξ, x) such that $(\lambda, a) \leqq (\xi, x)$ is a chain.*

Proof. Let $(\lambda, a) \leqq (\xi, x)$. Since $\xi \geqq f(x;(a,x))$, we have

$$(\xi, x) = (\xi - f(x;(a,x)) + f(a;(a,x)),\ a)$$

where naturally $\xi - f(x;(a,x)) + f(a;(a,x)) \geqq \lambda$ by $(\lambda, a) \leqq (\xi, x)$. It is immediately shown that $(\lambda, a) \leqq (\xi_1, x_1)$and $(\lambda, a) \leqq (\xi_2, x_2)$ imply (ξ_1, x_1) and (ξ_2, x_2) are comparable.

By Lemmas 12 and 13, directly

Lemma 14. *$(\lambda_0, a_0) \leqq (\xi, x) \leqq (\lambda_1, a_0)$, $\lambda_0 \leqq \lambda_1$ if and only if $(\xi, x) = (\lambda, a_0)$, $\lambda_0 \leqq \lambda \leqq \lambda_1$.*

Lemma 15. *(λ, a) and (μ, b) are incomparable if and only if $\lambda < f(a;(a,b))$, $\mu < f(b;(a,b))$.*

Proof. It is clear that if (λ, a) and (μ, b) are comparable,

$$\lambda \geqq f(a;(a,b)) \text{ or } \mu \geqq f(b;(a,b)).$$

Conversely we show that if $\lambda \geqq f(a;(a,b))$ or $\mu \geqq f(b;(a,b))$, then (λ, a) and (μ, b) are comparable.

$$\lambda \geqq f(a;(a,b)) \text{ and } \mu < f(b;(a,b)) \text{ imply } (\lambda, a) \geqq (\mu, b),$$
$$\lambda < f(a;(a,b)) \text{ and } \mu \geqq f(b;(a,b)) \text{ imply } (\lambda, a) \leqq (\mu, b).$$

If $\lambda \geqq f(a;(a,b))$ and $\mu \geqq f(b;(a,b))$, then

$$\lambda - \mu \geqq f(a;(a,b)) - f(b;(a,b)) \text{ implies } (\lambda, a) \geqq (\mu, b),$$
$$\lambda - \mu \leqq f(a;(a,b)) - f(b;(a,b)) \text{ implies } (\lambda, a) \leqq (\mu, b).$$

This lemma has been proved.

Lemma 16. *For (λ, a) and (μ, b), we define (ν, c) in the following manner.*

$$\begin{aligned} (\nu, c) &= (\lambda, a) \quad \text{if} \quad (\lambda, a) \geqq (\mu, b), \\ (\nu, c) &= (\mu, b) \quad \text{if} \quad (\lambda, a) \leqq (\mu, b), \\ (\nu, c) &= (f(a;(a,b)), a) \text{ if } (\lambda, a) \not\geqq\not\leqq (\mu, b) \\ &= (f(b;(a,b)), b). \end{aligned}$$

Then (ν, c) is a least upper bound of (λ, a) and (μ, b).

Proof. We shall prove only a case where (λ, a) and (μ, b) are incomparable. First,

$(f(a;(a,b)),\ a)=(f(b;(a,b)),\ b)$ follows from the definition of equality. By Lemma 15,

$$\lambda < f(a;(a,b)), \qquad \mu < f(b;(a,b)),$$

hence

$$(\lambda, a) < (f(a;(a,b)),\ a) = (\nu, c),$$
$$(\mu, b) < (f(b;(a,b)),\ b) = (\nu, c).$$

Let us show that (ν, c) is the least upper bound of (λ, a) and (μ, b). Let (ς, d) be any upper bound of (λ, a) and (μ, b). Any (ζ, d) is comparable to (ν, c) because of Lemma 9. Suppose that there is an upper bound (ζ_0, d_0) of (λ, a) and (μ, b) such that $(\lambda, a) < (\zeta_0, d_0) < (\nu, c)$. By Lemma 14, there is η such that $(\zeta_0, d_0) = (\eta, a)$ where $\eta < f(a;(a,b))$. Remember $\mu < f(b;(a,b))$, (η, a) and (μ, b) are incomparable because of Lemma 15. This contradicts with the assumption that $(\zeta_0, d_0) \geqq (\mu, b)$. Therefore $(\zeta, d) \geqq (\nu, c)$ for any upper bound (ζ, d) of (λ, a) and (μ, b). Thus the proof of the lemma has been completed.

Thus it has been proved that S^* is a semilattice.

Lemma 17. *A minimal element of S^* is $(0, a)$, for each $a \in M$.*

Proof. Suppose that $(\lambda, b) \leqq (0, a.)$ By the definition of inequality, $f(a;(a,b)) = 0$ which implies $a = b$. By Lemma 12, we have $\lambda = 0$. Thus the $(0, a)$ has been proved to be minimal. Conversely, (λ, b), $\lambda \neq 0$, is not a minimal element because $(\lambda, b) > (0, b)$.

Lemma 18. *S^* satisfies the minimal condition.*

Proof. For any $(\lambda, a) \in S^*$, $(0, a) \leqq (\lambda, a)$ where $(0, a)$ is minnimal.

Thus we have seen that S^* is a dispersed semilattice with the minimal condition by means of Lemmas 13, 14, and 18. Furthermore it follows from Lemma 12 that S^* is unbounded.

Theorem 2. *An unbounded dispersed semilattice with the minimal condition is characterized by a set M and the mapping which associates $(a,b) \in M \times M$ with a pair of non-negative integers*

$$(f(a;(a,b)),\ f(b;(a,b)))$$

where f satisfies the conditions (3.1), (3.2), *and* (3.3).

Thus the problem of construction of such a semilattice is reduced to the study of f, which remain unsolved here.

§ 4. Remarks.

"The minimal condion" in the present paper is weaker than the so-called "descending chain condition", which means that a chain $x_1 > x_2 > \cdots$ ceases in a finite number.

Example. The following example satisfies the minimal condition, but does not the descending chain condition.

Let I be the set of all integers i, and let I' be the set of all i' where $i \leftrightarrow i'$ is one-to-one. Now S is defined as the set union of I and I': $S = I \cup I'$; and we define the multiplication in the following manner.

$$ij = i'j = ij' = \max(i, j),$$
$$i'j' = \max(i, j), \text{ if } i' \neq j',$$
$$i'i' = i'.$$

Finally we give a necessary and suffiicient condition for S to satisfy the descending chain condition.

Theorem 3. *S^* does not satisfy the descending chain condion if and only if M contains an infinite sequence $\{a_i\}$, where $a_i \neq a_j$, $i \neq j$, $a_i \in M$, such that*

$$f(a_i; (a_{i-1}, a_i)) > f(a_i; (a_i, a_{i+1})), \ i = 2, 3, \cdots.$$

Proof. Suppose that S^* satisfies the descending chain condition, there is an in finite sequence

$$(\lambda_1, b_1) > (\lambda_2, b_2) > \cdots$$

of elements of S^*. Since the element $(0, b_1)$ is minimal, $(0, b_1) \neq (\lambda_j, b_j)$ for all j. Then all the inequalities $(\lambda_j, b_j) > (0, b_1)$, $j \geqq 1$, do not hold. Because, otherwise

$$(\lambda_1, b_1) \geqq (\lambda_j, b_j) > (0, b_1) \text{ for all } j \geqq 1,$$

while $[(0, b_1), (\lambda_1, b_1)]$ is finite since S^* is dispersed, arriving at the contradiction. Now, there is a positive integer k such that

$$(\lambda_j, b_j) < (0, b_1), \ 1 \leqq j \leqq k, \text{ and } (\lambda_{k+1}, b_{k+1}) \not> (0, b_1)$$

where naturally $(\lambda_j, b_j) \not> (0, b_1)$, $j \geqq k+1$.
Rewrite $a_1 = b_1$, $a_2 = b_{k+1}$. If $a_1, a_2, \cdots, a_{i-1}$ are obtained, then we let $a_i = b_{k_i}$ where

$$(\lambda_{k_i - 1}, b_{k_i - 1}) > (0, a_{i-1}) \text{ and } (\lambda_{k_i}, b_{k_i}) \not> (0, a_{i-1}), \ k_{i-1} > k_i.$$

Thus the sequence $\{a_i\}$ in M is determined. As easily seen, $(\lambda_{k_i-1}, b_{k_i-1})$ is an upper bound of $(0, a_{i-1})$ and $(0, a_i)$ so that $(\lambda_{k_i-1}, b_{k_i-1}) \geqq (0, a_{i-1})(0, a_i) < (0, a_i)$. On the other hand, as S^* is dispersed, $[(0, a_i), (\lambda_{k_i-1}, b_{k_i-1})]$ is a finite chain which contains (λ_{k_i}, b_{k_i}). Since $(\lambda_{k_i}, b_{k_i}) \geqq (0, a_{i-1})(0, a_i)$ is impossible, we have $(\lambda_{k_i-1}, b_{k_i-1}) \geqq (0, a_{i-1})(0, a_i) > (\lambda_{k_i}, b_{k_i})$, and similarly $(\lambda_{k_{i+1}-1}, b_{k_{i+1}-1}) \geqq (0, a_i)(0, a_{i+1}) > (\lambda_{k_{i+1}}, b_{k_{i+1}})$. Hence we have seen that $\{a_i\}$ satisfies

$$(0, a_{i-1})(0, a_i) > (0, a_i)(0, a_{i+1}),$$

and utilizing Lemma 16,

$$(f(a_i; (a_{i-1}, a_i)), a_i) > (f(a_i; (a_i, a_{i+1})), a_i).$$

By Lemma 12, $f(a_i; (a_{i-1}, a_i)) > f(a_i; (a_i, a_{i+1}))$.

Conversely if an infinite sequence $\{a_i\}$ fulfils the above condition, we can easily show that there is a sequence $\{b_i\}$ where $b_i = (0, a_{i-1})(0, a_i)$, satisfying $b_i > b_{i+1}$ $i = 2, 3, \cdots$. Hence the descending chain condition is not valid in S^*. Thus the proof of this theorem has been completed.

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EINE VERALLGEMEINERUNG DER LAPLACE-TRANSFORMATION

Von

Yoshikatsu WATANABE

(Eingegangen am 30 September, 1956)

Während die gewöhnliche Laplace-Transformation

$$f(s)=\int_0^\infty e^{-su}F(u)du$$

auf der Funktion einer Vairiable bezieht, erwäge ich diejenige in bezug auf der Funktion von n Variablen

$$f(s_1,\cdots\cdots,s_n)=\int_0^\infty\cdots\cdots\int_0^\infty \exp\left\{-\sum_{\nu=1}^{n} s_\nu u_\nu\right\} F(u_1,\cdots\cdots,u_n)du_1\cdots\cdots du_n,$$

und ins besondere den Fall $n=2$

$$f(s,t)=\int_0^\infty\int_0^\infty e^{-su-tv}F(u,v)\,dudv.$$

Bei noch besondereren Fall $s=t$ wird es

$$f(s,s)=\int_0^\infty\int_0^\infty e^{-s(u+v)}F(u,v)dudv,$$

was, gesetzt $u+v=\xi$, $u-v=\eta$,

$$f(s,s)=\tfrac{1}{2}\int_0^\infty e^{-s\xi}d\xi\int_{-\xi}^{\xi}F\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)d\eta=\int_0^\infty e^{-s\xi}G(\xi)d\xi$$

also auf den Fall $n=1$ sich reduziert. Jedoch die verschiedenen Ergebnisse zu übersehen, scheint es mir vielmehr besser die Sache in paralleler Weise zu dem Falle einer Variable zu betrachten. Dazu hat man nur einiges darüber maßgebendes Buch[1)] fast wörterlich umzuschreiben. Ich habe es mit meinem privaten Wünsch zu den Text umständlich verstehen können getan.

§ 1.

Es sei das Grundbereich, in dem die vorkommenden Funktionen definiert sind, immer $0\leqq u<\infty$, $0\leqq v<\infty$. Die Funktion $F(u,v)$ sei in jedem endlichen Teilbereich $0\leqq u\leqq U$, $0\leqq v\leqq V$ summierbar im Lebesqueschen Sinn, was zur Folge hat, daß neben

1) Z. B., G. Doetsch, Handbuch der Laplace-Transformaiton, 1950.

$\int_0^U\int_0^V F(u,v)dvdu$ immer auch $\int_0^U\int_0^V |F(u,v)|dvdu$ existiert. Dann ist für jedes komplexe s,t auch $e^{-su-tv}F(u,v)$ noch eine derartige, da e^{-su-tv} in jedem endlichen Bereich beschränkt bleibt. Es existiert also für jedes endliche U und V

$$\int_0^U\int_0^V e^{-su-tv}F(u,v)dvdu \quad \text{und} \quad \int_0^U\int_0^V |e^{-su-tv}F(u,v)|dvdu.$$

Gibt es ein reelles oder komplexes Wertepaar s_0, t_0 derart, daß

(1) $$\lim_{\omega,\omega'\to\infty}\int_0^\omega\int_0^{\omega'} e^{-s_0u-t_0v}F(u,v)dvdu=f(s_0,t_0)$$

existiert, so heiße $f(s_0,t_0)$ ein **zwei-parametriges Laplace-Integral.**

Lemma 1. Zur Konvergenz des Limes (1) ist notwendig und hinreichend, daß man zu jedem $\varepsilon>0$ ein $\Omega=\Omega(\varepsilon)>0$ so wählen kann, daß

(2) $$\left|\int_0^{\omega_2}\int_0^{\omega_2'}-\int_0^{\omega_1}\int_0^{\omega_1'}\right|<\varepsilon \quad \text{mit ausgelassenen } e^{-s_0u-t_0v}F(u,v)dvdu$$

für jene Wertepaare ω_1,ω_1' und ω_1,ω_2' mit $\Omega\leqq\omega_1\leqq\omega_2$ und $\Omega\leqq\omega_1'\leqq\omega_2'$ gilt. Oder, noch ausführlich bedeutet (2), daß alle drei Ungleichungen

(3) $$(\text{i})\equiv\left|\int_0^{\omega_1}\int_{\omega'_1}^{\omega'_2}\right|<\frac{\varepsilon}{3}, \qquad (\text{ii})\equiv\left|\int_{\omega_1}^{\omega_2}\int_0^{\omega'_1}\right|<\frac{\varepsilon}{3}, \qquad (\text{iii})\equiv\left|\int_{\omega_1}^{\omega_2}\int_{\omega'_1}^{\omega'_2}\right|<\frac{\varepsilon}{3}$$

erfüllt sind. Zwar ist es klar, daß (3) hinreichend für (2) ist. Aber für $\omega_2=\omega_1$ oder $\omega_2'=\omega_1'$ stimmt (2) mit (i) oder (ii) überein, und da

$$(\text{iii})\leqq\left|\int_0^{\omega_2}\int_0^{\omega_2'}-\int_0^{\omega_1}\int_0^{\omega_1'}\right|+(\text{i})+(\text{ii})$$

ist, so muß (iii) stets bestehen, wenn (2) und folglich (i) und (ii) auch gelten, also ist (3) für Bestehen von (2) notwendig.

Lemma 2. Ein besonderer Fall ist der, daß Laplace-Integral absolut konvergiert, d. h. daß

$$\lim_{\omega,\omega'\to\infty}\int_0^\omega\int_0^{\omega'}|e^{-s_0u-t_0v}F(u,v)|dvdu=\lim_{\omega,\omega'\to\infty}\int_0^\omega\int_0^{\omega'}e^{-\sigma_0u-\tau_0v}|F(u,v)|dvdu.$$

mit $\sigma_0=\Re s$ und $\tau_0=\Re t_0$ existiert. Dazu ist notwendig und hinreichend, daß man zu jedem $\varepsilon>0$ ein $\Omega=\Omega(\varepsilon)>0$ so bestimmen kann, daß nach (3) die folgenden drei Ungleichungen mit ausgelassenen $\exp\{-\sigma_0u-\tau_0v\}|F(u,v)|dvdu$

(4) $$\left|\int_0^{\omega_2}\int_{\omega_1'}^{\omega_2'}\right|<\varepsilon, \qquad \left|\int_{\omega_1}^{\omega_2}\int_0^{\omega_2'}\right|<\varepsilon \quad \text{und} \quad \left|\int_{\omega_1}^{\omega_2}\int_{\omega_1'}^{\omega_2'}\right|<\varepsilon$$

für $\Omega\leqq\omega_1\leqq\omega_2$, $\Omega\leqq\omega_1'\leqq\omega_2'$ bestehen.

Satz. 1. *Ist $F(u,v)$ außer einem passenden Rechteck beschränkt,*

(5) $$|F(u,v)| \leqq C$$

für $u \geqq U$ *mit* $v \geqq 0$, *sowie für* $v \geqq V$ *mit* $u \geqq 0$, *so ist das Laplace-Integral für jedes Wertepaar* s_0, t_0 *mit* $\Re s_0 > 0$, $\Re t_0 > 0$ *abrolut konvergent.*

Beweis. Es sei σ_0, τ_0 ein feste Zahlenpaar mit $\Re s_0 = \sigma_0 > 0$, $\Re t_0 = \tau_0 > 0$, so ist für $U \leqq \omega_1 \leqq \omega_2$ sowie $V \leqq \omega_1' \leqq \omega_2'$ mit ausgelassenen $\exp\{-\sigma_0 u - \tau_0 v\}|F(u.v)|dvdu$

$$\left|\int_0^{\omega_1}\int_{\omega_1'}^{\omega_2'}\right| < C\int_0^{\omega_1} e^{-\sigma_0 u}du \int_{\omega_1'}^{\omega_2'} e^{-\tau_0 v}dv < \frac{C}{\sigma_0\tau_0}e^{-\tau_0\omega_1'} < \varepsilon$$

für $\omega_1' > \Omega(\varepsilon)$, und ebenso sind für $\omega_1 > \Omega(\varepsilon)$

$$\left|\int_{\omega_1}^{\omega_2}\int_0^{\omega_1'}\right| < \varepsilon, \qquad \left|\int_{\omega_1}^{\omega_2}\int_{\omega_1'}^{\omega_2'}\right| < \frac{C}{\sigma_0\tau_0}e^{-\sigma_0\omega_1 - \tau_0\omega_1'} < \varepsilon.$$

Also bestehen Bedingungen (4), w.z.b.w.

Deutet man die Parameter s und t als komplexe Variablen in jeden s– und t–Ebene, so konvergiert das Lapalce-Integral

$$\int_0^\infty\int_0^\infty e^{-su-tv}F(u,v)dv\,du$$

für eine beschränkte und in jedem endlichen Bereich integrierbare Funktion mindestesn in der Mannigfaltigkeit der offenen Halbebenen $\Re s > 0$, $\Re t > 0$, und zwar sogar absolut. Es kann aber in einem noch umfangreicheren Gebiet absolut konvergieren; so ist z. B. für $F(u,v) = e^{-u-v}$ offenkundig für $\Re s > -1$, $\Re t > -1$ absolut konvergent

$$\int_0^\infty\int_0^\infty e^{-su-tv}e^{-u-v}dudv = \frac{1}{(s+1)(t+1)} \qquad \text{für} \quad \Re(s+1) > 0, \quad \Re(t+1) > 0.$$

Es ist leicht einzusehen, daß jedes in einem Punkt (s_0, t_0) absolut konvergente Laplace-Integral sofort in der Mannigfaltigkeit zweier ganzer Halbebenen konvergiert. Es gilt nämlich

Satz 2. *Ist ein Laplace-Integral in einem Punkt* (s_0, t_0) *absolut konvergent, so ist es in der abgeschlossenen Mannigfaltigkeit* $\Re s \geqq \Re s_0$, $\Re t \geqq \Re t_0$ *absolut konvergent.*

Beweis. Für $\Re s = \sigma \geqq \sigma_0 = \Re s_0$, $\Re t = \tau \geqq \tau_0 = \Re t_0$ gilt

$$\int_0^{\omega_1}\int_{\omega_1'}^{\omega_2'} |e^{-su-tv}F(u,v)|\,dv\,du = \int_0^{\omega_1}\int_{\omega_1'}^{\omega_2'} \left|e^{-(s-s_0)u-(t-t_0)v}e^{-s_0u-t_0v}\right| F(u,v)dv\,du$$

$$= \int_0^{\omega_1}\int_{\omega_1'}^{\omega_2'} e^{-(\sigma-\sigma_0)u-(\tau-\tau_0)v}|e^{-s_0u-t_0v}F(u,v)|\,dv\,du \leqq \int_0^{\omega_1}\int_{\omega_1'}^{\omega_2'} |e^{-s_0u-t_0v}F(u_1v)|\,dv\,du.$$

Ganz ebenso

$$\int_{\omega_1}^{\omega_2}\int_0^{\omega_1'} |e^{-su-tv}F(u,v)|\,dv\,du \leqq \int_{\omega_1}^{\omega_2}\int_0^{\omega_1'} |e^{-s_0u-t_0v}F(u,v)|\,dv\,du$$

und

$$\int_{\omega_1}^{\omega_2}\int_{\omega_1'}^{\omega_2'} |e^{-su-tv}F(u,v)|\,dv\,du \leqq \int_{\omega_1}^{\omega_2}\int_{\omega_1'}^{\omega_2'} |e^{-s_0u-t_0v}\;F(u,v)|\,dv\,du.$$

Wenn die Bedingung (4) für s_0,t_0 erfüllt ist, so erst recht für s,t.

Satz 3. *Das genaue Gebiet absoluter Konvergenz des Laplace-Integrals ist die Mannigfaltigkeit offener oder abgeschlossener Halbebenen* $\mathfrak{R}s>\alpha$ *(oder* $\geqq\alpha$*) und* $\mathfrak{R}t>\alpha'$ *(oder* $\geqq\alpha'$*), wobei* α,α' *auch gleich* $-\infty$ *oder* $+\infty$ *sein können.*

Jeder Rand $\mathfrak{R}s=\alpha$, $\mathfrak{R}t=\alpha'$ kann nur entweder ganz (wie bei der Funktion $F(u,v)=[1+(u^2+v^2)^2]^{-1}; \alpha=\alpha'=0$), oder gar nicht (wie bei der Funktion $F(u,v)=[1+u^2+v^2]^{-1}$; $\alpha=\alpha'=0$) zum Gebiet absoluter Konvergenz gehören.

Das Wertepaar (α,α') heißt die **Koordinaten absoluter Konvergenz,** das Gebiet $\{\mathfrak{R}s>\alpha,\ \mathfrak{R}t>\alpha'\}$, oder $\{\mathfrak{R}s\geqq\alpha,\ \mathfrak{R}t\geqq\alpha'\}$ falls nochmalige Konvergenzen auf ganzen Ränder stattfinden, die **Halbebenen-Mannigfaltigkeit absoluter Konvergenz** des Laplace-Integrals.

§ 2.

Wir fragen nun, wie das Gebiet von s- und t-Werten aussieht, wo das Laplace-Integral nicht notwendig absolut, sondern einfach konvergiert. Dazu schicken wir einen Satz voraus, den wir unten etwas allgemeiner formulieren, als es hier nötig wäre.

Satz 4. *Wenn* $G(u,v)=0(\rho^k)$ *mit* $k\geqq 0$ *für* $\rho=\sqrt{u^2+v^2}\to\infty$ *ist, so ist das Integral*

$$(6)\qquad \int_0^\infty\int_0^\infty (st)^{\frac{k}{2}+1}e^{-su-tv}G(u,v)\,du\,dv$$

$$=\int_0^{\frac{\pi}{2}}\int_0^\infty (st)^{\frac{k}{2}+1}\exp\{-\rho(s\cos\theta+\ t\sin\theta)\}\,G\,(\rho\cos\theta,\ \rho\sin\theta)\,\rho\,d\rho\,d\theta$$

in der Mannigfaltigkeit jedes ziemlich abgerundeten Winkelraumes $\mathfrak{W}$ ($s=0$ *als Spitze und* $|\arg s|\leqq\psi<\frac{\pi}{2}$ *aber mit Ausnahme eines Kreises K von Mittelpunkt* $s=0$ *mit beliebig kleinem Radius) und desjeniges* $\mathfrak{W}'$ ($t=0$, $|\arg t|\leqq\psi<\frac{\pi}{2}$, *ausgenommen* K') *zwar gleichmaßig konvergent.*

Beweis. Zu jedem $\varepsilon>0$ gibt es ein $\omega>0$, so daß $|G(u,v)|<\varepsilon\rho^k$ für $\rho\geqq\omega$ ist. Dann ist der Rest für $\omega\leqq\rho_1<\rho_2$ und jedes s in $\mathfrak{W}$ und t in $\mathfrak{W}'$

$$|R|=\left|\int_0^{\frac{\pi}{2}}\int_{\rho_1}^{\rho_2}(st)^{\frac{k}{2}+1}\exp\{-\rho(s\cos\theta+t\sin\theta)\}\,G\,(\rho\cos\theta,\ \rho\sin\theta)\rho\,d\rho\,d\theta\right|$$

$$\leqq\varepsilon|st|^{\frac{k}{2}+1}\int_0^{\frac{\pi}{2}}\int_{\rho_1}^{\rho_2}\exp\{-\rho(\sigma\cos\theta+\tau\sin\theta)\}\rho^{k+1}d\rho\,d\theta,$$

wobei $0<\sigma=\mathfrak{R}s\leqq|s|$ und $0<\tau=\mathfrak{R}t\leqq|t|$ sind. Durch Einsetzung von $\rho(\sigma\cos\theta+\tau\sin\theta)=w$ wird

$$|R|<\varepsilon\left|\frac{st}{\sigma\tau}\right|^{\frac{k}{2}+1}\int_0^{\frac{\pi}{2}}\left(\frac{\sqrt{\sigma\tau}}{\sigma\cos\theta+\tau\sin\theta}\right)^{k+2}d\theta\int_0^\infty e^{-w}w^{k+1}dw$$

$$< \frac{\varepsilon\Gamma(k+2)}{(\cos\psi)^{k+2}} \int_0^{\frac{\pi}{2}} \left[\frac{\sqrt{\cos\gamma}\ \sin\gamma}{\cos(\theta-\gamma)}\right]^{k+2} d\theta,$$

wo $\gamma = \tan^{-1}\tau/\sigma$ ist und wegen angenommener Abrundungen der Winkelspitzen $0 < \varepsilon_1 \leqq \sigma, \tau$ demzufolge $0 < \varepsilon_2 < \gamma < \frac{\pi}{2} - \varepsilon_2$ und das letzte Integral $< \pi/(\sqrt{2}\sin\varepsilon_2)^{k+2} = M$ wird. Daher gilt unsere Abschätzung

$$|R| < \varepsilon\Gamma(k+2)M/(\cos\psi)^{k+2} \qquad \text{unabhängig von } s \text{ und t.}$$

Es ist aber möglich die Abrunderungen wie viel klein auch immer zu machen. Anderseits für $s = t = 0$ ist die Abschätzung trivialerweise erfüllt.

Zusatz 4. Bei $s=t$ ergibt es sich, daß, falls $G(u,v) = 0(\rho^k)$ mit $k \geqq 0$ ist,

$$\int_0^\infty\int_0^\infty s^{k+2} e^{-s(u+v)}\, G(u,v)\,du\,dv \tag{7}$$

in $\mathfrak{W}\left(0, |\arg s| \leqq \psi < \frac{\pi}{2}\right)$ gleichmäßig konvergiert. Ferner, wenn $G(u,v)$ eine Funktion der Variable u allein $= g(u)$ und $g(u) = 0(u^k)$ ist, so reduziert sich der Ausdruck (7) auf

$$\int_0^\infty s^{k+1} e^{-su} g(u)\,du, \tag{8}$$

das in $\mathfrak{W}$ gleichmäßig konvergiert.

Wir vorbereiten noch ein

Lemma 3. Die partielle Integration zweier Variablen kann folgendermaßen geleistet werden

$$\int_a^b\int_c^d f(x,y)g_{xy}(x,y)\,dy\,dx = \Big[\!\!\Big[f(x,y)g(x,y) \Big]\!\!\Big]_{a,c}^{b,d} - \int_a^b \Big[f_x(x,y)g(x,y)\Big]_c^d dx \tag{9}$$

$$- \int_c^d \Big[f_y(x,y)g(x,y)\Big]_a^b dy + \int_a^b\int_c^d f_{xy}(x,y)g(x,y)\,dy\,dx,$$

wobei

$$\Big[\!\!\Big[f(x,y)g(x,y) \Big]\!\!\Big]_{a,c}^{b,d} = f(b,d)g(b,d) - f(b,c)g(b,c) - f(a,d)g(a,d) + f(a,c)g(a,c). \tag{10}$$

Satz 5. *Konvergiert ein Laplace-Integral für* s_0, t_0

$$\int_0^\infty\int_0^\infty e^{-s_0u-t_0v}\, F(u,v)\,du\,dv = f_0, \tag{11}$$

so kanvergiert das Laplace-Integral

$$\int_0^\infty\int_0^\infty e^{-su-tv} F(u,v)\,du\,dv \tag{12}$$

gleichmäßig in der mannigfaltigkeit $\{\mathfrak{W}, \mathfrak{W}'\}$, *wo* $\mathfrak{W}$ *einen Winkelraum in s-Ebene mit* s_0 *als Spitze aber etwas abgerundet und* $|\arg(s-s_0)| \leqq \psi < \frac{\pi}{2}$ *sowie* $\mathfrak{W}'$ *denjenigen in t-Ebene bedeuten. Ins-*

besondere konvergiert es also für jeden Punkt (s,t) *der offenen Mannigfaltigkeit* $(\Re s=\sigma>\sigma_0=\Re s_0, \Re t=\tau>\tau_0=\Re t_0)$. *Wird ferner*

$$\int_0^u\int_0^v e^{-s_0u-t_0v}F(u,v)dv\,du=\Phi(u,v), \qquad also \qquad e^{-s_0u-t_0v}F(u,v)=\Phi_{uv}(u,v) \tag{13}$$

gesetzt, so läßt sich das Laplace-Integral (11) *für jedes* s,t *mit* $\sigma>\sigma_0, \tau>\tau_0$ *durch folgendes andere*

$$(s-s_0)(t-t_0)\int_0^\infty\int_0^\infty e^{-(s-s_0)u-(t-t_0)v}\Phi(u,v)du\,dv, \tag{14}$$

und für $\sigma>\sigma_0, \tau>\tau_0$ *und* $s=s_0$, $t=t_0$ *durch*

$$f_0+\int_0^\infty\int_0^\infty (s-s_0)(t-t_0)e^{-(s-s_0)u-(t-t_0)v}[\Phi(u,v)-f_0]dv\,du \tag{15}$$

darstellen. Das letztere Integral konvergiert gleichmäßig in jeder Mannigfaltigkeit $\{\mathfrak{W}, \mathfrak{W}'\}$.

Beweis. Aus Lemma 3 folgt bei jetztigem Fall für beliebige komplexe s,t

$$\int_0^\omega\int_0^\omega e^{-su-tv}F(u,v)\,dv\,du=\int_0^\omega\int_0^{\omega'} e^{-(s-s_0)u-(t-t_0)v}e^{-s_0u-t_0v}F(u,v)\,dv\,du$$

$$=e^{-(s-s_0)\omega-(t-t_0)\omega'}\Phi(\omega,\omega')+\int_0^\omega(s-s_0)e^{-(s-s_0)u-(t-t_0)\omega'}\Phi(u,\omega')\,du$$

$$+\int_0^{\omega'}(t-t_0)e^{-(s-s_0)\omega-(t-t_0)v}\Phi(\omega,v)dv+\int_0^\omega\int_0^{\omega'}(s-s_0)(t-t_0)e^{-(s-s_0)u-(t-t_0)v}\Phi(u,v)dv\,du,$$

da $\Phi(u,v)$ verschwindet, wenn eines oder beide von u,v veschwindet. Addiert man hierzu die Gleichung

$$0=f_0\{1-e^{-(s-s_0)\omega}\}\{1-e^{-(t-t_0)\omega'}\}-\int_0^\omega\int_0^{\omega'}(s-s_0)(t-t_0)e^{-(s-s_0)u-(t-t_0)v}f_0\,dv\,du,$$

so ergibt sich

$$\int_0^\omega\int_0^{\omega'}e^{-su-tv}F(u,v)\,dv\,du=f_0\{1-e^{-(s-s_0)\omega}\}\{1-e^{-(t-t_0)\omega'}\}$$

$$+e^{-(s-s_0)\omega-(t-t_0)\omega'}\Phi(\omega,\omega')+e^{-(t-t_0)\omega'}\int_0^\omega(s-s_0)e^{-(s-s_0)u}\Phi(u,\omega')\,du$$

$$+e^{-(s-s_0)\omega}\int_0^{\omega'}(t-t_0)e^{-(t-t_0)v}\Phi(\omega,v)\,dv+\int_0^\omega\int_0^{\omega'}(s-s_0)(t-t_0)e^{-(s-s_0)u-(t-t_0)v}[\Phi(u,v)-f_0]\,dv\,du$$

$$=f_0+e^{-(s-s_0)\omega-(t-t_0)\omega'}[\Phi(\omega,\omega')-f_0]+e^{-(t-t_0)\omega'}\int_0^\omega(s-s_0)e^{-(s-s_0)u}[\Phi(u,\omega')-f_0]du$$

$$+e^{-(s-s_0)\omega}\int_0^{\omega'}(t-t_0)e^{-(t-t_0)v}[\Phi(\omega,v)-f_0]dv+\int_0^\omega\int_0^{\omega'}(s-s_0)(t-t_0)e^{-(s-s_0)u-(t-t_0)v}[\Phi(u,v)-f_0]\,dv\,du$$

$$=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}.$$

Wegen $\Phi(\omega,\omega')\to f_0$ für $\omega,\omega'\to\infty$ konvergiert II auf rechten Seite für $\omega,\omega'\to\infty$ in $\{\Re s \geqq \Re s_0,\ \Re t\geqq \Re t_0\}$ (also $|e^{-(s-s_0)u-(t-t_0)v}|\leqq 1$) gleichmäßig und zwar gegn 0. Bezüglich III ist es gleich 0 falls $s=s_0,\ t=t_0$, sonst aber konvergiert das Integral nach Zusatz (8) (mit $s-s_0$ an Stelle von s und $k=1$, da $\Phi(u,\omega')-f_0=O(1)=0(v)$ ist[2)]) gleichmaßig in $\mathfrak{W}$, und der Faktor gegen 0 in $\mathfrak{W}'$, dieselben für IV, und schließlich V nach Satz 4 (mit $k=0$ und $s-s_0$, $t-t_0$ an Stelle von s, t) gleichmaßig in $\{\mathfrak{W},\ \mathfrak{W}'\}$, also die linke Seite gleichmäßig in $\{\mathfrak{W},\ \mathfrak{W}'\}$ und zwar gegen

$$\int_0^\infty\int_0^\infty e^{-su-tv}F(u,v)\,dv\,du=f_0+\int_0^\infty\int_0^\infty(s-s_0)(t-t_0)e^{-(s-s_0)u-(t-t_0)v}[\Phi(u,v)-f_0]\,dv\,du.$$

Das ist der Ausdruck (15), der für $s=s_0$, $t=t_0$ und für jedes s, t mit $\Re s>\Re s_0$ bzw. $\Re t>\Re t_0$ gilt, da man jedes solche s, t in $\mathfrak{W}$ bzw. $\mathfrak{W}'$ einfassen kann. Läßt man den Punkt (s_0, t_0) weg, und betrachtet nur $\Re s>\Re s_0$ bzw. $\Re t>\Re t_0$, so ist

$$(s-s_0)(t-t_0)\int_0^\infty\int_0^\infty e^{-(s-s_0)u-(t-t_0)v}f_0\,du\,dv$$

konvergent und gleich f_0, so daß sich (15) auf (14) reduziert.

Satz 6. *Bleiben sämtliche Limites von*

$$\Phi(U,V)=\int_0^V\int_0^U e^{-s_0u-t_0v}F(u,v)\,dv\,du \qquad (U,V\geqq 0) \tag{16}$$

für $U+V\to\infty$ *alle und jedes endlich und ins besondere* (11) *gültig, so sind die in* (14) *und* (15) *vorkommenden Integral für* $\sigma>\sigma_0$, $\tau>\tau_0$ *absolut konvergent.*

Beweis. Die Funktionen $\Phi(u,v)$ und $\Phi(u,v)-f_0$ sind für $u,v\geqq 0$ stetig und haben endliche Grenzwerte für $u+v\to\infty$, sind also beschränkt[3)], so daß die Integrale (14) und (15) nach Satz 1 für $\Re(s-s_0)>0$, $\Re(t-t_0>0$ absolut konvergieren.

Aus der Tatsache, daß die Konvergenz des Laplace-Integrals in einem Punkt (s_0, t_0) die Konvergenz in der Mannigfaltigkeit der offenen Halbebenen $\Re s>\Re s_0$, $\Re t>\Re t_0$ nach sich zieht, ergibt sich nun wörtlich bei der absoluten Konvergenz.

Satz 7. *Das genaue Gebiet der (einfachen) Konvergenz des Laplace-Integrals ist eine Mannigfaltigkeit* $\Re s>\beta$, $\Re t>\beta'$, *deren Ränder* $\Re s=\beta$ *bzw.* $\Re t=\beta'$ *ganz, teilweise oder gar nicht zum Konvergenzgebiet gehören können. Eines oder beide von* β,β' *können auch je* $-\infty$ *oder* $+\infty$ *sein.*

Das Wertepaar (β,β') heißt die **Konvergenz-Koordinaten**, das Gebiet $\Re s>\beta$, $\Re t>\beta'$ mit Einschluß der eventuellen Konvergenzpunkte auf $\Re s=\beta$, $\Re t=\beta'$ die

2) Dabei braucht man als $s-s_0=(s-s_0)^2/(s-s_0)$ sich denken, und dies ist plausible wegen $|s-s_0|\neq 0$ infolge der Abrundung der Spitze von $\mathfrak{W}$.

3) Es seien z. B. $s_0=t_0=0$ und $F(v,v)\equiv F(u)=1,-1,0$ jenachdem $0\leqq u<1$, $1\leqq u\leqq 2$ oder $2<u<\infty$ ist. Dann wird $\Phi(U,V)=\int_0^U\int_0^V F(u,v)\,dv\,du=UV$, $(2-U)V$ oder 0. Daher gelten $\Phi(0,\infty)=0$, $\Phi(0<U<2,\infty)=\infty$, $\Phi(2\leqq U<\infty,\infty)=0$ und $\Phi(\infty,0\leqq V\leqq\infty)=0$, also $f_0=0$, jedoch ist dieses $\Phi(U,V)$ keinerlei beschränkt, und zu unserer Kategorie nicht gehört, obgleich das Laplace-Integral $\int_0^\infty\int_0^\infty e^{-su-tv}F(u,v)\,du\,dv$ für $\Re t>0$, $\Re s>0$ absolut konvergiert. Es möchte also möglich sein, meine Annahme (16) etwas zu erschwächen können.

Konvergenz-Mannigfaltigkeit der Halbebenen, die Geraden $\Re s=\beta$ bzw. $\Re t=\beta'$ die **Konvergenzgeraden**[4].

Daß tatsächlich auf den Ränder $\Re s=\beta$, $\Re t=\beta'$ alle Möglichkeiten des Konvergenzverhaltens vorliegen können, ziehen, folgende Beispiele:

a) $$F(u,v)=[1+(u^2+v^2)^2]^{-1}, \qquad \beta=\beta'=0;$$

in allen Punkten der Geraden $\Re s=0$, $\Re t=0$ konvergiert das Laplace Integral, sogar absolut.

b) $$F(u,v)=(1+u^2+v^2)^{-1}, \qquad \beta=\beta'=0;$$

in Punkte $s=0, t=0$ divergiert das Laplace-Integral, in allen anderen Punkten mit $\Re s=0$, $\Re t=0$ (also $s=iy$, $t=iy'$, wo y, y' reell und beide zwei $\neq 0$, obgleich aber nur eines $=0$ sein möge) konvergiert es, aber nicht absolut, denn zwar

$$\int_0^\infty e^{-iyu}du\int_0^\infty \frac{dv}{1+u^2+v^2}=\frac{\pi}{2}\int_0^\infty \frac{\cos yu}{\sqrt{1+u^2}}du-\frac{\pi}{2}i\int_0^\infty \frac{\sin yu}{\sqrt{1+u^2}}du$$

und diese Integrale sind für $y\neq 0$ konvergent, wie ihre Darstellung durch unendliche Reihen erkennen läßt.

c) $$F(u,v)=1, \qquad \beta=\beta'=0;$$

in allen Punkten mit $\Re s=0$, $\Re t=0$ divergiert das Laplace-Integral.

§ 3.

Um di Koordinaten einfacher und absoluter Konvergenz festzustellen, kann man sich offenbar auf reelle s, t beschränken. Natürlich ist

$$\beta\leqq\alpha, \qquad \beta'\leqq\alpha'.$$

Zuerst wollen wir uns zum Falle $s=t$ beschränken, viz.

$$\int_0^\infty\int_0^\infty e^{-s(u+v)}F(u,v)\,du\,dv,$$

und seine gemeinsamen Konvergenzabszissen β bestimmen.

Satz 8. *Wird mit $U+V=W$*

$$\overline{\lim_{W\to\infty}}\frac{1}{W}\log\left|\int_0^U\int_0^V F(u,v)\,dv\,du\right|=\lambda \tag{17}$$

gesetzt, so ist

4) Oder, da jene Ordinaten von ihre Punkte zusammengefassen eine Ebene bilden, so möge die Gesamtheit $\{\Re s=\beta, \Re t=\beta'\}$ als Konvergenzränderebene genannt werden —— freilich ist dies ganz andere als die Konvergenz-Halbebene des gewöhnilichen einparametrigen Laplace-Integrals.

a) $\beta \leqq \lambda$, *b*) $\lambda \leqq \mathrm{Max}(0, \beta)$, *c*) $\beta = \lambda$ *für* $\lambda > 0$.

Beweis. *a*) λ sei endlich. Wir zeigen daß

$$\int_0^\infty \int_0^\infty e^{-(\lambda+\delta)(u+v)} F(u,v)\,du\,dv$$

für jedes $\delta > 0$ konvergiert. Setzen wir zur Abkürzung

$$\int_0^u \int_0^v F(u,v)\,dv\,du = \varphi(u,v) \text{ und daher } F(u,v) = \varphi_{uv}(u,v),$$

so folgt durch partielle Integration (9)

$$\int_0^\omega \int_0^{\omega'} e^{-(\lambda+\delta)(u+v)} F(u,v)\,dv\,du = e^{-(\lambda+\delta)(\omega+\omega')}\varphi(\omega,\omega') + (\lambda+\delta)\int_0^\omega e^{-(\lambda+\delta)(u+\omega')}\varphi(u,\omega')\,du$$

$$+ (\lambda+\delta)\int_0^{\omega'} e^{-(\lambda+\delta)(\omega+v)}\varphi(\omega,v)\,dv + (\lambda+\delta)^2 \int_0^\omega \int_0^{\omega'} e^{-(\lambda+\delta)(u+v)}\varphi(u,v)\,dv\,du$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.$$

Für $u+v=w>W$ gilt nach (17)

$$\frac{1}{w}\log|\varphi(u,v)| < \lambda + \delta/2, \quad \text{also } |\varphi(u,v)| < e^{(\lambda+\delta/2)w},$$

so daß

$$e^{-(\lambda+\delta)w}|\varphi(u,v)| < e^{-\frac{\delta}{2}w}.$$

Hieraus erstens folgt, daß bei $\omega+\omega'=\Omega>W$

$$|\mathrm{I}| \leqq e^{-(\lambda+\delta)\Omega}|\varphi(\omega,\omega')| < e^{-\frac{\delta}{2}\Omega} \to 0 \qquad \text{für} \quad \Omega\to\infty.$$

Zweitens hat II für $\Omega\to\infty$ einen Grenzwert. Denn, falls ω mit Ω gegen ∞ strebt,

$$\mathrm{II} = (\lambda+\delta)\int_0^W e^{-(\lambda+\delta)(u+\omega')}\varphi(u,\omega')\,du + R,$$

wo

$$|R| \leqq |\lambda+\delta| \int_W^\omega e^{-(\lambda+\delta)(u+\omega')}|\varphi(u,\omega')|\,du \leqq |\lambda+\delta| \int_W^\omega e^{-\frac{\delta}{2}(u+\omega')}\,du$$

$$\leqq |\lambda+\delta| \int_W^\infty e^{-\frac{\delta}{2}u}du = \frac{2|\lambda+\delta|}{\delta}e^{-\delta W/2} < \varepsilon.$$

Oder, falls ω endlich bleibt, dann für $W>\omega$, $R=0$ und

$$|\mathrm{II}| \leqq |\lambda+\delta| \int_0^\omega e^{-(\lambda+\delta)(u+\omega')}|\varphi(u,\omega')|\,du < |\lambda+\delta|\,\omega e^{-\frac{\delta}{2}\Omega} \to 0 \quad \text{für } \Omega\to\infty.$$

Ganz ebenso sich verhält III. Schließlich kann man mit Berücksichtigung des Lemmas 1 beweisen, daß IV für $\Omega\to\infty$ gegen einen Grenzwert strebt, so daß dasgleiche für die linke Seite gilt. Also Also $\beta \leqq \lambda$. Ist $\lambda = -\infty$, so gilt derselbe Beweis mit jeder beliebigen negativen Zahl A an statt von λ, und es folgt $\beta \leqq A$, also $\beta = -\infty$. Ist $\lambda = +\infty$, so ist die Aussage $\beta \leqq \lambda$ trivial.

b) Es sei $s_0 > 0$ und $\Phi(U, V, s_0) = \int_0^U \int_0^V e^{-s_0(u+v)} F(u,v)\,dv\,du$ mit $U, V \geqq 0$

für jedes $W = U + V \to \infty$ konvergent[5], und demgemäß $|\Phi(u, v, s_0)| \leqq C$, da $\Phi(u, v, s_0)$ als Integral in jedem endlichen Gebiet stetig ist, während in jedem unendlichen Gebiet es noch beschränkt bleibt. Durch partielle Integration erhält man

$$\Phi(\omega, \omega', 0) \equiv \varphi(\omega, \omega') = \int_0^\omega \int_0^{\omega'} e^{s_0(u+v)} e^{-s_0(u+v)} F(u,v)\,dv\,du$$

$$= e^{s_0(\omega+\omega')} \Phi(\omega, \omega', s_0) - s_0 \int_0^\omega e^{s_0(u+\omega')} \Phi(u, \omega', s_0)\,du$$

$$- s_0 \int_0^{\omega'} e^{s_0(\omega+v)} \Phi(\omega, v, s_0)\,dv + s_0^2 \int_0^\omega \int_0^{\omega'} e^{s_0(u+v)} \Phi(u, v, s_0)\,dv\,du.$$

Also gilt

$$|\varphi(\omega, \omega')| \leqq Ce^{s_0\Omega} + C(e^{s_0\Omega} - e^{s_0\omega'}) + C(e^{s_0\Omega} - e^{s_0\omega}) + C(e^{s_0\omega} - 1)(e^{s_0\omega'} - 1)$$

$$\leqq 4Ce^{s_0\Omega} \leqq e^{(s_0+\delta)\Omega}, \qquad (\delta > 0)$$

wenn $\omega + \omega' = \Omega$ etwas groß $\geqq W$ wird, und $4C \leqq e^{\delta\Omega}$ gilt. Folglich ist außer dem rechtwinkligem Dreieck mit Nullpunkt als Spitze und gleichen Seiten W, $|\varphi(\omega, \omega')| < e^{(s_0+\delta)\Omega}$. Hieraus aber ergibt sich

$$\frac{\log |\varphi(\omega, \omega')|}{\Omega} < s_0 + \delta, \quad \text{also } \lambda \leqq s_0 + \delta \text{ für jedes } \delta > 0.$$

Mithin $\lambda \leqq s_0$. Ist die Konvergenzabszisse $\beta \geqq 0$ und endlich, so kann s_0 jede Zahl $> \beta$ bedeuten, und man erhält $\lambda \leqq \beta$. Für $\beta = \infty$ ist die Aussage trivial. Ist dagegen $\beta < 0$, so kann s_0 jede Zahl > 0 bedeuten, so daß $\lambda \leqq 0$ gilt. Es ist also allgemein $\lambda \leqq$ Max $(0, \beta)$.

c) Aus *a)* und *b)* ergeben sich die Behauptung *c)* folgendermaßen: Ist $\lambda > 0$, so liefert *b)*: $0 < \lambda \leqq$ Max $(0, \beta)$. Also kann nicht Max $(0, \beta) = 0$, sondern nur $= \beta$ sein. Aus $\beta \leqq \lambda$ und $\lambda \leqq \beta$ folgt $\beta = \lambda$.

Zusatz 8. Es sei $\lambda > 0$ in (17) und $F(u,v) = F_1(u)$ eine Funktion von u allein. Dann ergeben siech

$$\int_0^\infty \int_0^\infty e^{-\lambda(u+v)} F(u,v)\,dv\,du = \frac{1}{\lambda} \int_0^\infty e^{-\lambda u} F_1(u)\,du,$$

---

5) Oder, mindestens mögen alle Limites schwingend sein. Also falls beide $U, V \to \infty$ streben, so liegt $|\Phi(U,V)|$ zwissen gewiße $|P_1 + iQ_1| \leqq C$ und $|P_2 + iQ_2| \leqq C$ für das Gebiet $U > \Omega$, $V > \Omega$. Sonst, dagegen, liegt eines von U, V z. B. U an endlichen Wert U_0 nahe, so gilt das gleiche für das Gebiet $U_0 - \varepsilon < U < U_0 + \varepsilon$, $\Omega < V \to \infty$.

$$\int_0^U\int_0^V F(u,v)\,dv\,du = V\int_0^U F_1(v)du \quad \text{oder} \quad 0 \ (V=0).$$

Demzufolge reduziert sich (17) auf

$$\overline{\lim_{W\to\infty}} \frac{1}{W} \log \left|\int_0^U\int_0^V F(u,v)\,dv\,du\right| = \overline{\lim_{U\to\infty}} \frac{1}{U} \log \left|\int_0^U F_1(u)du\right|,$$

da $U/W \leqq 1$ ist. Also ist die positive Konvergenzabszisse des Laplace-Integral $\int_0^\infty e^{-\lambda u}F_1(u)du$ durch

$$\lambda = \overline{\lim_{U\to\infty}} \frac{1}{U} \log \left|\int_0^U F_1(u)du\right| \tag{18}$$

gegeben.

Satz 9. *Unter der Voraussetzung, daß* $\int_0^U\int_0^V F(u,v)\,dv\,du$ *für jedes* $U+V\to\infty$ *existiert, werde mit* $U+V=W$

$$\overline{\lim_{W\to\infty}} \frac{1}{W} \log \left|\int_U^\infty\int_V^\infty F(u,v)\,dv\,du\right| = \mu \tag{19}$$

gesetzt. Ist $\beta<0$, *so ist* $\beta=\mu$.

Beweis. *a)* Wenn $\beta<0$ ist, so konvergiert das Laplace-Integral für $s=0$ schon $\int_0^\infty\int_0^\infty F(u,v)\,du\,dv$ und damit $\int_\omega^\infty\int_{\omega'}^\infty F(u,v)\,dv\,du = \psi(\omega,\omega')$ für $\omega,\omega'\geqq 0$ hat also einen Sinn. Überdies ist $\psi(u,v)\to 0$ für jedes $u+v=w\to\infty$, und folglich $\log|\psi(u,v)|\to-\infty$, so daß gewiß $\mu\leqq 0$ sein soll. Ferner folgt es durch partielle Integration

$$\int_0^\omega\int_0^{\omega'} e^{-(\mu+\delta)(u+v)}F(u,v)\,dv\,du$$

$$= e^{-(\mu+\delta)(\omega+\omega')}\psi(\omega,\omega') - e^{-(\mu+\delta)\omega}\psi(\omega,0) - e^{-(\mu+\delta)\omega'}\psi(0,\omega') + \psi(0,0)$$

$$+ \int_0^\omega (\mu+\delta)e^{-(\mu+\delta)u}\left[e^{-(\mu+\delta)\omega'}\psi(u,\omega') - \psi(u,0)\right]du$$

$$+ \int_0^{\omega'} (\mu+\delta)e^{-(\mu+\delta)v}\left[e^{-(\mu+\delta)\omega}\psi(\omega,v) - \psi(0,v)\right]dv$$

$$+ \int_0^\omega\int_0^{\omega'} (\mu+\delta)^2 e^{-(\mu+\delta)(u+v)}\psi(u,v)\,dv\,du$$

$$= \mathrm{I} - \mathrm{II} - \mathrm{III} + \mathrm{IV} + \mathrm{V} + \mathrm{VI} + \mathrm{VII}.$$

Außer dem rechtwinkligem Dreieck mit gleichem Schenkel W (passend groß) glit

$$\frac{1}{w}\log|\psi(u,v)| < \mu + \frac{\delta}{2} \quad \text{für} \quad u+v=w>W,$$

also

$$|\psi(u,v)| < e^{(\mu+\frac{1}{2}\delta)w}, \quad \text{so daß} \quad e^{-(\mu+\delta)w}|\psi(u,v)| < e^{-\frac{1}{2}\delta w}.$$

Hieraus aber folgt, daß

$$|\mathrm{I}| \leqq e^{-\frac{1}{2}\delta\Omega} \to 0 \qquad \text{bei} \qquad \Omega = \omega + \omega' \to \infty.$$

Für II dasgleichen falls ω mit $\Omega \to \infty$ strebt, oder sonst bleibt als konstant. Ganz ebenso ist für III. Das letzte Integral VII für $\omega + \omega' \to \infty$, also für $\rho = \sqrt{\omega^2 + \omega'^2} \to \infty$ wegen Satzes 4 hat einen Grenzwert, während für V sowie VI noch dieselben wegen Zusatzes (8) gelten. Daher gilt dasgleiche für ihrer Gesamtheit und das Integral

$$\int_0^\infty \int_0^\infty e^{-(\mu+\delta)(u+v)} F(u,v)\,du\,dv$$

existiert. Also ist $\beta \leqq \mu$.

b) Es sei $s_1 < 0$ und $\Phi(u,v) = \int_0^u \int_0^v e^{-s_1(u+v)} F(u,v)\,dv\,du$ bei $u+v \to \infty$

jedesmal limitierbar. Folglich ist das Integral

$$\psi(\omega, \omega') = \int_\omega^\infty \int_{\omega'}^\infty F(u,v)\,dv\,du = \int_\omega^\infty \int_{\omega'}^\infty e^{s_1(u+v)} \Phi_{uv}(u,v)\,dv\,du$$

sinnlich und durch partielle Integration wie folgt gegeben:

$$\psi(\omega,\omega') = \Big[\!\Big[e^{s_1(u+v)} \Phi(u,v) \Big]\!\Big]_{\omega,\omega'}^{\infty,\infty} - s_1 \int_\omega^\infty \Big[e^{s_1(u+v)} \Phi(u,v) \Big]_{\omega'}^\infty du$$

$$- s_1 \int_{\omega'}^\infty \Big[e^{s_1(u+v)} \Phi(u,v) \Big]_\omega^\infty dv + s_1^2 \int_\omega^\infty \int_{\omega'}^\infty e^{s_1(u+v)} \Phi(v,v)\,dv\,du.$$

Da $\Phi(\omega,\omega')$ für $u, v \geqq 0$ stetig ist und für jedes $u + v \to \infty$ limitierbar ist, so ist $|\Phi(u,v)| \leqq C$ für alle $u, v \geqq 0$. Also zunächst wegen $s_1 < 0$ strebt $e^{s_1(u+v)}\Phi(u,v) \to 0$ für $u+v \to \infty$ und demzufolge

$$\psi(\omega,\omega') = e^{s_1(\omega+\omega')} \Phi(\omega,\omega') + s_1 \int_\omega^\infty e^{s_1(u+\omega')} \Phi(u,\omega')\,du$$

$$+ s_1 \int_{\omega'}^\infty e^{s_1(u+v)} \Phi(\omega,v)\,dv + s_1^2 \int_\omega^\infty \int_{\omega'}^\infty e^{s_1(u+v)} \Phi(u,v)\,dv\,du,$$

und weiter

$$|\psi(\omega,\omega')| \leqq Ce^{s_1(\omega+\omega')} - s_1 C \int_\omega^\infty e^{s_1(u+w')} du - s_1 C \int_{\omega'}^\infty e^{s_1(\omega+v)}\,dv + s_1^2 C \int_\omega^\infty \int_{\omega'}^\infty e^{s_1(u+v)}\,dv\,du$$

$$= 4Ce^{s_1(\omega+\omega')}.$$

Deshalb ist für $\omega + \omega' = \Omega > W$ bei $\delta > 0$, zwar $|\psi(\omega,\omega')| < e^{(s_1+\delta)\Omega}$, und hieraus ergibt sich

$$\frac{1}{\Omega} \log |\psi(\omega,\omega')| < s_1 + \delta, \quad \text{also } \mu \leqq s_1 + \delta \text{ für jedes } \delta > 0.$$

Mithin $\mu \leqq s_1$. Ist die Konvergenzabszisse $\beta < 0$, so kann s_1 jede negative Zahl $> \beta$

bedeuten, und man erhält $\mu \leqq \beta$. Mit dem Ergebnis unter *a*) zusammengenommen, liefert das $\mu = \beta$.

Zusatz 9. Es sei $\beta = \mu < 0$ in (19) und $F(u,v) = e^{(\mu-1)v} F_1(u)$. Dann gelten

$$\int_0^\infty \int_0^\infty e^{-\mu(u+v)} F(u,v)\,du\,dv = \int_0^\infty e^{-\mu u} F_1(u)\,du,$$

$$\int_\omega^\infty \int_{\omega'}^\infty F(u,v)\,dv\,du = \frac{e^{(\mu-1)\omega'}}{1-\mu} \int_\omega^\infty F_1(u)\,du.$$

Folglich reduziert sich (19) auf

$$\overline{\lim_{\omega+\omega'\to\infty}} \frac{1}{\omega+\omega'} \left\{ (\mu-1)\omega' - \log(1-\mu) + \log \left| \int_\omega^\infty F_1(u)du \right| \right\},$$

in welche, um den Limes tatsächlich superior zu machen, es wie $\frac{\omega'}{\omega+\omega'} \to 0$, so daß $\frac{\omega}{\omega+\omega'} \to 1$ genommen werden soll. Daher ist die negative Konvergenzabszisse vom Laplace-Integral $\int_0^\infty e^{-\mu u} F_1(u)\,du$ durch

$$\mu = \overline{\lim_{\omega\to\infty}} \frac{1}{\omega} \log \left| \int_\omega^\infty F_1(u)du \right| (<0) \tag{20}$$

gegeben.

Satz 10. *Die Konvergenzabszisse β des Laplace-Integrals*

$$\int_0^\infty \int_0^\infty e^{-s(u+v)} F(u,v)\,du\,dv$$

bestimmt sich allgemein aus den Zahlen λ und μ folgendermaßen: Für $\lambda > 0$ ist $\beta = \lambda$, für $\lambda < 0$ ist $\beta = \mu$, für $\lambda = 0$ ist $\beta = 0$ oder μ. Deswegen soll es $\beta = 0$ sein, falls $\lambda = \mu = 0$.

Beweis. Stellt sich $\lambda > 0$ heraus, so ist $\beta = \lambda$ nach Satz 8. Ist $\lambda < 0$, so ist $\beta \leqq \lambda < 0$ nach Satz 8, also $\beta = \mu$ nach Satz 9. Ist $\lambda = 0$, so ist $\beta \leqq 0$ nach Satz 8, also entweder $\beta = 0$, oder aber $\beta < 0$, also $\beta = \mu$ nach Satz 9. Die Entscheidung kann man durch die Probe, ob $s = \frac{\mu}{2}$ Konvergenz-oder Divergenz-punkt ist, herbeiführen.

Ersetzt man $F(u,v)$ durch $|F(u,v)|$, so erhält man entsprechende Sätze über die Abszisse α der absoluten Konvergenz.

§ 4.

Da die aus Satz 10 bestimmte Abszisse β_0 lediglich $\beta_0 = \text{Max}\,(\beta, \beta')$ gibt, so braucht man noch etwaige kleineren Abszisse zu finden. Dazu bildet man mit $\delta > 0$

$$\int_0^\infty e^{-(\beta_0+\delta)v} F(u,v)\,dv = F_1(u), \qquad \int_0^\infty e^{-(\beta_0+\delta)u} F(u,v)\,du = F_2(v)$$

und sucht nach (18) oder (20) die Konvergenz-Abszisse γ_i des Laplace-Integrals

$$\int_0^\infty e^{-\gamma w} F_i(w)\,dw \qquad (i=1,2)$$

aus. Falls $\gamma_i < \beta_0$ ist, so sind die gesuchte Konvergenz-Koordinaten (γ_1, β_0) oder (β_0, γ_2). Dabei keinerwegs können beide γ_1, $\gamma_2 < \beta_0$ sein. Denn, falls beide $\gamma_1, \gamma_2 < \beta_0$ sind, so sind $\beta_0 - \gamma_i = \varepsilon_i > 0$ $(i=1,2)$. Es sei z. B. $\varepsilon_1 \leqq \varepsilon_2$ und setze man $\varepsilon_1 - \varepsilon_1' = \varepsilon_2 - \varepsilon_2' = \varepsilon > 0$ mit $\varepsilon_i' > 0$. Dazu hat man nur sie zu währen, so daß $0 < \varepsilon_1' < \varepsilon_1$ und $0 < \varepsilon_2' = \varepsilon_2 - \varepsilon_1 + \varepsilon_1' = \varepsilon_2 - (\varepsilon_1 - \varepsilon_1') < \varepsilon_2$. Dann gilt $\beta_0 - \varepsilon = \gamma_i + \varepsilon_i - \varepsilon = \gamma_i + \varepsilon_i' > \gamma_i$, und folglich soll

$$\int_0^\infty\int_0^\infty e^{-(\beta_0-\varepsilon)(u+v)} F(u,v)\,du\,dv \qquad (\varepsilon > 0)$$

wegen $\varepsilon_i' > 0$ nach Satz 5 konvergieren, was aber Sätzes 8, 9 unmöglich sein soll.

Wir wollen schließlich einige Beispiele hinzufügen.

1. Für $F(u,v) = e^{u+v^2 i}$ ist es ersichtlich $\beta = 1$, $\beta' = 0$. Sind nämlich $\Re s > 1$, $t = 0$, so ergibt sich

$$f(s,0) = \int_0^\infty\int_0^\infty e^{-su} e^{u+v^2 i}\,du\,dv = \int_0^\infty e^{-(s-1)u}\,du \int_0^\infty (\cos v^2 + i \sin v^2)\,dv.$$

Setzt man ferner $v = \sqrt{\theta}$, so erhält man

$$f(s,0) = \frac{1}{2(s-1)} \int_0^\infty \frac{\cos\theta + i\sin\theta}{\sqrt{\theta}}\,d\theta = \text{konst.}$$

Um dasselbe Resultat aus (17) formal zu finden, hat man zu sehen, daß

$$\int_0^U\int_0^V e^{u+v^2 i}\,dv\,du = (e^U - 1)\int_0^V (\cos v^2 + i \sin v^2)\,dv = O(e^U),$$

und damit
$$\overline{\lim_{U+V\to\infty}} \frac{1}{U+V} \log e^U = 1 = \beta.$$

Läßt es sich $\Re s > 1$ aber $0 < \Re t \leqq 1$ gelten, so wird

$$\int_0^\infty\int_0^\infty e^{-su-tv} e^{u+v^2 i}\,du\,dv = \frac{1}{2(s-1)} \int_0^\infty e^{-t\sqrt{\theta}} \frac{(\cos\theta + i\sin\theta)}{\sqrt{\theta}}\,d\theta$$

konvergent. Da aber das Integral

$$\int_0^\Theta \frac{\cos\theta + i\sin\theta}{\sqrt{\theta}}\,d\theta = g(\Theta)$$

für $\Theta \to \infty$ endlich und $\neq 0$ ist, so ist nach (18)

$$\overline{\lim_{\Theta\to\infty}} \frac{1}{\Theta} \log |g(\Theta)| = 0 = \beta'.$$

Also sind die Konvergenz-Koordinaten $\beta = 1$, $\beta' = 0$.

2. Für $F(u,v) = (1 + ui) e^{2u+v+uvi}$ ergibt sich

$$\Phi(u,v)=\int_0^V\int_0^U F(u,v)\,dv\,du=e^{2U+V}\left\{\frac{e^{2UVi}-e^{-2U}}{2+iV}-\frac{e^{-V}(1-e^{-2U})}{4}\right\}$$

und nach (17)

$$\overline{\lim_{U+V\to\infty}}\frac{1}{U+V}\log|\Phi(U,V)|=\overline{\lim_{U+V\to\infty}}\frac{2U+V+0(1)}{U+V}=2,$$

also ist $\beta_0=\mathrm{Max}(\beta,\beta')=2$. Um die etwaige kleinere Abszisse zu finden, setzt man mit $s=2+\delta$ $(\delta>0)$

$$f(s,t)=\int_0^\infty\int_0^\infty e^{-su-tv}(1+ui)e^{2u+v+uvi}\,dv\,du=\int_0^\infty e^{-tv}G(v)\,dv,$$

so folgt, daß

$$\int_0^V G(v)\,dv=\int_0^V e^v\left[\frac{1}{\delta-iv}+\frac{i}{(\delta-iv)^2}\right]dv,$$

und mit benutzung des zweiten Mittelwertsatzes das letztere Integral gleich $e^V O(1)$ und $\neq 0$ wird. Daher wird nach (18)

$$\beta'=\overline{\lim_{V\to\infty}}\frac{1}{V}\log\left|\int_0^V G(v)\,dv\right|=\overline{\lim_{V\to\infty}}\frac{1}{V}\log\left|e^V O(1)\right|=1.$$

Also sind die Konvergenz-Koordinaten $\beta=2$, $\beta'=1$.

3. Wir betrachten eine Erweiterung eines von Widder[6] angegebenen merkwürdigen interessanten Beispiels:

$$f(s,t)=\int_0^\infty\int_0^\infty e^{-su-tu}e^{u+v}\sin e^{u+v}\,du\,dv$$

und suchen die Konvergenz-Koordinaten zu finden.

Durch die Koordinaten-Transformation (Rotation): $u=\dfrac{\xi-\eta}{\sqrt{2}}$, $\eta=\dfrac{\xi+\eta}{\sqrt{2}}$ erhält man

$$f(s,s)\equiv f(s)=\int_0^\infty\int_{-\xi}^{\xi} e^{-(s-1)\sqrt{2}\xi}\sin e^{\sqrt{2}\xi}\,d\eta\,d\xi$$

$$=2\int_0^\infty \xi e^{-(s-1)\sqrt{2}\xi}\sin e^{\sqrt{2}\xi}\,d\xi=\int_1^\infty\frac{\log x\sin x}{x^s}\,dx,\qquad (x=e^{\sqrt{2}\xi}).$$

Also wird der Integrand periodisch, aber doch die Amplitude-Funktion $x^{-s}\log x$ (>0) an $x=e^{1/s}$ einen Maximumwert $1/se$ annimmt, und nachdem monoton wie $x^{-s+\delta}$ abnimmt, sofern $s>\delta>0$ ist. Daher konvergiert das letzte Integral für $\Re s>0$, wie man aus ihre Darstellung durch unendliche Reihe erkennen kann. Also erhält man $\beta=\beta'=0$, was wegen Symmetrie keine weitere Erniedrigung erlaubt. Jedoch für $\Re s>0$ hat man durch partielle Integration

$$f(s)=(1-s)\int_1^\infty\frac{\cos x}{x^{s+1}}\,dx,$$

6) D.V. Widder, The Laplace Transform, 1946, p. 58.

das für $\Re s > -1$ noch konvergiert. Daher dient der letztere Ausdruck als die analytische Fortsetzung über den Rand $\Re s = 0$ hinaus bis auf $\Re s = -1$. Weiter erhält man durch nochmalige partielle Integration

$$f(s) = -(1-s)\sin 1 + (1-s^2)\int_1^\infty \frac{\sin x}{x^{s+2}}\,dx,$$

das für $\Re s > -2$ konvergiert, und $f(s)$ ist noch bis auf $\Re s = -2$ fortsetzbar, u.s.w. Also lautet, daß es in der ganzer Ebene keinen singulären Puukt der Funktion gibt und $f(s)$ zwar eine Ganzefunktion ist.

4. Schließlich beschäftige ich mich mit ein Beispiel negativer Konvergenzabszissen. Es sei z.B.

$$F(u,v) = u^{m-1}v^{n-1}e^{-pu-qv-uv} \qquad \text{mit} \quad m > n > 0, \quad p > q > 0.$$

Schon ist seiner Integral vorhanden, da

$$0 < \int_0^\infty\int_0^\infty F(u,v)\,du\,dv < \int_0^\infty u^{m-n-1}e^{-pu}\,du\int_0^\infty e^{-uv}(uv)^{n-1}d(uv) = \frac{\Gamma(m-n)\Gamma(n)}{p^{m-n}}$$

gilt, und demgemäß ist $\beta, \beta' \leqq 0$. Da aber die Berechnung aus Formeln bei negativer Abszisse etwas verwickelt wird, seien es Einfachheit wegen beispielsweise $m=2$, $n=1$, $p=2$, $q=1$, also wird $F(u,v) = ue^{-uv-2u-v}$ und

$$\psi(U,V) = \int_U^\infty\int_V^\infty ue^{-2u}e^{-uv-v}\,dv\,du = e^{-V}\int_U^\infty ue^{-(2+V)u}\frac{du}{u+1}$$

$$= \frac{e^{-V}}{2+V}e^{-(2+V)U} - e^{-V}\int_U^\infty e^{-(2+V)u}\frac{du}{u+1}.$$

Mit benutzung des zweiten Mittelwertsatzes erhält man

$$\psi(U,V) = e^{-2U-UV}\left[e^{-V} - \log\frac{U_1+1}{U+1}\right] \qquad (0 \leqq U < U_1 < \infty),$$

und daraus wegen (19)

$$\mu = \overline{\lim_{U+V\to\infty}}\frac{1}{U+V}\log|\psi(U,V)| = \overline{\lim_{U+V\to\infty}}\frac{-2U-V-UV}{U+V}.$$

Falls beide $U, V \to \infty$ streben, so wird $\lim = -\infty$, während bei festes $U = u_0$ oder $V = v_0$, $\lim = -1 - u_0$ oder $-2 - v_0$, wo $0 \leqq u_0, v_0 < \infty$ sind. Daher der Limes superior $= -1$ bei $u_0 = 0$, also $\mu = -1$ sein soll. Ferner nimmt man $t = -1$ an, so folgt

$$\int_0^\infty\int_0^\infty e^{-su-tv}ue^{-2u-v-uv}\,dv\,du = \int_0^\infty e^{-(s+2)u}\,du = \frac{1}{s+2},$$

und daraus offentlich $\beta = -2$, $\beta' = -1$.

Vielmehr mögen wir ohne Benutzung von Formeln folgendermaßen heuristisch fortfahren. Da das Integral $\int_0^\infty e^{-sx}e^{-bx}x^{a-1}\,dx$ mit $a>0,\ b>0$ für $\Re s>-b$ konvergiert aber für $\Re s<-b$ divergiert, so ist seiner Konvergenzabszisse $\beta=-b$. In unseren Falle zwar konvergiert das Laplace-Integral für $s=t=-q$, weil

$$\int_0^\infty\int_0^\infty e^{q(u+v)}\,e^{-pu-qv-uv}u^{m-1}v^{n-1}\,du\,dv$$

$$=\int_0^\infty u^{m-n-1}e^{-(p-q)u}du\int_0^\infty e^{-uv}(uv)^{n-1}d(uv)=\frac{\Gamma(m-n)\Gamma(n)}{(p-q)^{m-n}}$$

ist, aber das für $s=t<-q-r\ (r>0)$ divergiert, also $\beta_0=\mathrm{Max}(\beta,\beta')=-q$.

Setzt man für $t=-q+\delta\,(\delta>0)$

$$f(s,t)=\int_0^\infty\int_0^\infty e^{-su-tv}F(u,v)\,du\,dv=\int_0^\infty e^{-su}G(u)\,du$$

mit

$$G(u)=\int_0^\infty e^{-tv}F(u,v)\,dv,$$

so gilt

$$G(u)=\int_0^\infty e^{(q-\delta)v}F(u,v)\,dv=\int_0^\infty u^{m-1}v^{n-1}e^{-pu-\delta v-uv}dv=\frac{\Gamma(n)u^{m-1}e^{-pu}}{(u+\delta)^n},$$

und daraus folgt

$$f(s,-q+\delta)=\Gamma(n)\int_0^\infty e^{-(s+p)u}\frac{u^{m-1}}{(u+\delta)^n}du,$$

dessen Konvergenzabszisse gefunden zu werden braucht. Es ist nun evident, daß das letztere Integral konvergiert oder divergiert, jenachdem $\Re s>$ oder $<-p$ ist. Daher $\beta=-p,\ \beta'=-q$, was eben zu finden war.

Berichtigung zu meiner früheren Note "Aufgaben betreffend das Irrfahrtproblem", dieses Journ. Vol. VI, S. 45. **Von Y. Watanabe.**

Tatsächlich bedeutet Formel (2.1) die Wahrscheinlichkeit im Falle, daß durch Elevator der Punkt nur aufwärts aber nicht abwärts gehen kann. Unter dort gemachter Voraussetzung, daß Auf- bezw. Abgehen möglich sind, soll die Wahrscheinlichkeit berichtigt werden, wie folgt:

$$P_m(x,y,z)=\frac{1}{(2\pi)^3}\iiint\left(\frac{2\cos\varphi+2\cos\psi+e^{i\theta}}{5}\right)^m \exp(-ix\varphi-iy\psi)\sum_\nu \exp(-iz_\nu\theta)\,d\varphi d\psi d\theta,$$

wobei $z=0$ oder 1 ist, während jedes $z_\nu \equiv z$ (mod 2) und die Summe mit z beginnt und durch $z_l=m-|x|-|y|$ vollendet, jedennoch für die Fälle, daß entweder $m<|x|+|y|+z$ oder $m\not\equiv x+y+z$ (mod 2) ist, es in $\sum_\nu$ einziges Glied z allein zu einnehmen ist, und dann $P_m(x,y,z)$ auf Null reduziert. Daraus aber folgt, daß $P_{2n}(0,0,0)\cong\frac{5}{8n\pi}$ für $n\to\infty$ gilt, deswegen $\sum P_{2n}(0,0,0)$ noch divergiert und lim $\Omega_n=1$ besteht.

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ON THE SPACE WITH DOMINANT AFFINE CONNECTION.

By

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In this paper we shall consider an n-dimensional space V_n with dominant affine connection, in which the quantities $\Gamma^{\alpha}_{\beta i}$ (1) determing the relation between tangent spaces attaching to every point are given. However in this case the tangent spaces of V_n are $m(>n)$-dimensional affine spaces. We call $\Gamma^{\alpha}_{\beta i}$ the coefficients of dominant affine connection.

§ **1.** Consider an n-dimensional space V_n with dominant affine connection where a current point $\mathfrak{x}$ is given by a system of coordinates $(x^1, x^2, \cdots, x^n)$ and linearly independent m vectors $\mathfrak{x}_\lambda$ which compose a frame of a tangent space attaching to this point $\mathfrak{x}$ are given.

Then these vectors satisfy the equations

$$d\mathfrak{x}_\alpha = \Gamma^{\lambda}_{\alpha i}\, \mathfrak{x}_\lambda\, dx^i. \tag{1.1}$$

Let $\mathfrak{x}_i$ be the linearly independent n vectors satisfying the equations

$$d\mathfrak{x} = \mathfrak{x}_i\, dx_i. \tag{1.2}$$

Being the vectors on m-dimensional affine space A_m, $\mathfrak{x}_i$ must satisfy

$$\mathfrak{x}_i = B_{i}^{\cdot\alpha}\, \mathfrak{x}_\alpha, \tag{1.3}$$

and contravariant vectors v^i on A_n can be written

$$V^\lambda = B_{i}^{\cdot\lambda}\, v^i, \tag{1.4}$$

from which we see the quantities $B_{i}^{\cdot\lambda}$ are n contravariant vectors.

Let $\mathfrak{x}_P$ be $m-n$ linearly independent vectors of $\mathfrak{x}_i$, and A_{m-n} be an $(m-n)$-dimensional subspace of A_m, then we define $B_{P}^{\cdot\alpha}$ by the equations

$$\mathfrak{x}_P = B_{P}^{\cdot\alpha}\, \mathfrak{x}_\alpha. \tag{1.5}$$

We find similarly

$$V^\lambda = B_{P}^{\cdot\lambda}\, v^P \tag{1.4$'$}$$

where v^P is a vector on A_{m-n}, and $B_{P}^{\cdot\lambda}$ are p contravariant vectors.

The rank of the matrix

(1) In this paper we shall denote by $\alpha, \beta, \gamma, \lambda, \mu, \nu, \cdots$ the suffices which take the value $1, 2, \cdots, m$; by $a, b, c, \cdots, i, j, \cdots, r$, those which take the value $\dot{1}, \dot{2}, \cdots, \dot{n}$, and P, Q, R, S, those which take the value $\dot{n}+\dot{1}, \dot{n}+\dot{2}, \cdots, \dot{m}$,

$$\begin{pmatrix} \vdots \\ B_{i}^{\cdot\lambda} \\ \vdots \\ B_{P}^{\cdot\lambda} \\ \vdots \end{pmatrix}$$

is m, then we have the inverse matrix

$$\begin{pmatrix} \vdots \\ B^{i}_{\cdot\lambda} \\ \vdots \\ B^{P}_{\cdot\lambda} \\ \vdots \end{pmatrix},$$

from which we see the relations

$$(1.6) \qquad B^{i}_{\cdot\lambda} B_{k}^{\cdot\lambda}=\delta^{i}_{k}, \quad B^{i}_{\cdot\lambda}B_{P}^{\cdot\lambda}=0, \quad B^{P}_{\cdot\lambda}B_{k}^{\cdot\lambda}=0, \quad B^{P}_{\cdot\lambda}B_{Q}^{\cdot\lambda}=\delta^{P}_{Q}$$

$$B_{i}^{\cdot\lambda}B^{i}_{\cdot\mu}+B_{P}^{\cdot\lambda}B^{P}_{\cdot\mu}=\delta^{\lambda}_{\mu}.$$

For the displacement on V_n we put

$$(1.7) \qquad d\mathfrak{x}_i=\gamma^{k}_{ij}dx^{j}\mathfrak{x}_k+H^{P}_{i\,j}dx^{j}\mathfrak{x}_P,$$

on the other hand we see

$$d\mathfrak{x}_i=[B_{i}^{\cdot\alpha}{}_{,j}+B_{i}^{\cdot\beta}\Gamma^{\alpha}_{\beta j}]\mathfrak{x}_\alpha dx^{j},$$

where comma means partial derivative, and comparing with (1.7) we obtain

$$(1.8) \qquad \frac{\partial B_{i}^{\cdot\alpha}}{\partial x^{j}}=-\Gamma^{\alpha}_{\beta j}B_{i}^{\cdot\beta}+\gamma^{k}_{ij}B_{k}^{\cdot\alpha}+H^{Q}_{i\,j}B_{Q}^{\cdot\alpha}.$$

Similarly, putting

$$(1.9) \qquad d\mathfrak{x}_P=H^{k}_{P\,j}dx^{j}\mathfrak{x}_k+H^{Q}_{P\,j}dx^{j}\mathfrak{x}_Q,$$

we obtain

$$(1.10) \qquad \frac{\partial B_{P}^{\cdot\alpha}}{\partial \mathfrak{x}^{j}}=-\Gamma^{\alpha}_{\beta j}B_{P}^{\cdot\beta}+H^{P}_{k\,j}B_{k}^{\cdot\alpha}+H^{Q}_{P\,j}B_{Q}^{\cdot\alpha}.$$

Now consider the transformation of coordinate

$$x'^{i}=x'^{i}(x^{1},x^{2},\cdots,x^{n})$$

and change of the frame

$$\mathfrak{x}_\lambda=A^{\lambda'}_{\lambda}\mathfrak{x}_{\lambda'}$$

where the rank of the matrix $(A^{\lambda'}_{\lambda})$ is m and $(A^{\lambda}_{\lambda'})$ is the inverse matrix of $(A^{\lambda'}_{\lambda})$. Then from (1.3), (1.5), $B_{i}^{\cdot\lambda}$ and $B_{P}^{\cdot\lambda}$ are transformed by the laws

$$(1.11) \qquad B_{i'}^{\cdot\lambda'}=A^{\lambda'}_{\lambda}\frac{\partial x^{i}}{\partial x^{i'}}B_{i}^{\cdot\lambda}, \qquad B_{P}^{\cdot\lambda'}=A^{\lambda'}_{\lambda}B_{P}^{\cdot\lambda}.$$

Moreover, the transformation law of a composite tensor, that is to say, the tensor which may involve Latin and Greak indices, is[2]

$$T^{\cdot\cdot\alpha'\cdots i'\cdots}_{\cdots\beta'\cdots j'\cdots}=T^{\cdot\cdot\alpha\cdots i\cdots}_{\cdots\beta\cdots j\cdots}\cdots A^{\alpha'}_{\alpha}\cdots A^{\beta}_{\beta'}\cdots\frac{\partial x^{i'}}{\partial x^{i}}\cdots\frac{\partial x^{j}}{\partial x^{j'}}\cdots, \tag{1.12}$$

and hence $B_i^{\cdot\lambda}$ is a composite tensor.

Comparing

$$d\mathfrak{x}_{\alpha'}=\Gamma^{\beta'}_{\alpha' i'}\mathfrak{x}_{\beta'}dx,$$

with

$$d\mathfrak{x}_{\alpha'}=d(A^{\beta}_{\alpha'}\mathfrak{x}_{\beta}),$$

we obtain

$$\Gamma^{\beta'}_{\alpha' i'}=A^{\beta}_{\beta'}(A^{\alpha}_{\alpha'}\Gamma^{\beta}_{\alpha i}+A^{\beta}_{\alpha',i})\frac{\partial x^{i}}{\partial x^{i'}}. \tag{1.13}$$

On the other hand, considering the case where we do not change the vector frame, we see

$$d\mathfrak{x}_{i'}=[\gamma^{k'}_{i'j'}B_k^{\cdot\alpha}\frac{\partial x^{k}}{\partial x^{k'}}+H^{P}_{i'j'}B_P^{\cdot\alpha}]\mathfrak{x}_{\alpha}\frac{\partial x^{j'}}{\partial x^{j}}dx^{j}.$$

Here, differentiating the relation $\mathfrak{x}_{i'}=\frac{\partial x^{i}}{\partial x^{i'}}\mathfrak{x}_{i}$, and comparing with the above equation give

$$\gamma^{k'}_{i'j'}\frac{\partial x^{k}}{\partial x^{k'}}=\frac{\partial^2 x^{k}}{\partial x^{i'}\partial x^{j'}}+\gamma^{k}_{ij}\frac{\partial x^{i}}{\partial x^{i'}}\frac{\partial x^{j}}{\partial x^{j'}}, \tag{1.14}$$

$$H^{P}_{i'j'}=H^{P}_{ij}\frac{\partial x^{i}}{\partial x^{i'}}\frac{\partial x^{j}}{\partial x^{j'}}.$$

Also from (1.10) we obtain

$$H^{k'}_{P j'}=H^{k}_{Pj}\frac{\partial x^{k'}}{\partial x^{k}}\frac{\partial x^{j'}}{\partial x^{j}},\ H^{k'}_{P j'}=H^{k}_{Pj}\frac{\partial x^{k'}}{\partial x^{k}}. \tag{1.15}$$

§ 2. From the integrability conditions of the equations (1.8) we obtain

$$\begin{aligned}R^{\alpha}_{\beta jk}B_i^{\cdot\beta}=R^{h}_{ijk}B_h^{\cdot\alpha}+[H^{h}_{Pk}H^{P}_{ij}-H^{h}_{Pj}H^{P}_{ik}]B_h^{\cdot\alpha}\\+[(\gamma^{l}_{ij}H^{P}_{lk}-\gamma^{l}_{ik}H^{P}_{lj})+(H^{P}_{ij,k}-H^{P}_{ik,j}+H^{Q}_{ij}H^{P}_{Qk}-H^{Q}_{ik}H^{P}_{Qj})]B_P^{\cdot\alpha},\end{aligned} \tag{2.1}$$

where the quantity $R^{\alpha}_{\beta jk}$ is a curvature tensor of V_n, i.e.

$$R^{\alpha}_{\beta jk}=\Gamma^{\alpha}_{\beta j,k}-\Gamma^{\alpha}_{\beta k,j}+\Gamma^{\alpha}_{\sigma j}\Gamma^{\sigma}_{\beta k}-\Gamma^{\alpha}_{\sigma k}\Gamma^{\sigma}_{\beta j}, \tag{2.2}$$

and R^{h}_{ijk} is a curvature tensor for γ^{i}_{jk}, i.e.

(2) A. D. Michal and J. L. Botsfold; Geometries involving affine connections and general linear connections. An extension of the recent Einstein-Mayer geometry. Annali di mat. 12 (1934) p. p. 13~32.

$$(2.3)\qquad R^h_{ijk}=\gamma^h_{ij,k}-\gamma^h_{ik,j}+\gamma^l_{ij}\gamma^h_{lk}-\gamma^l_{ik}\gamma^h_{lj}.$$

In the same manner, from (1.10) we obtain

$$(2.4)\qquad R^\alpha_{\beta ij}B_P^{\cdot\beta}=[H^Q_{Pi,j}-H^Q_{Pj,i}+H^R_{Pi}H^Q_{Rj}-H^R_{Pj}H^Q_{Ri}+H^l_{Pi}H^Q_{lj}-H^k_{Pj}H^Q_{ki}]B_Q^{\cdot\alpha}$$
$$+[H^l_{Pi,j}-H^l_{Pj,i}+H^k_{Pi}\gamma^l_{kj}-H^k_{Pj}\gamma^l_{ki}+H^Q_{Pi}H^l_{Qj}-H^Q_{Pj}H^l_{Qi}]B_l^{\cdot\alpha}.$$

Covariant derivative of the composite tensor $T^{\cdot\cdot\alpha\cdots i\cdots}_{\cdots\beta\cdots j\cdots}$ is given from (1.12) and (1.14) by

$$(2.5)\qquad T^{\cdot\cdot\alpha\cdots i\cdots}_{\cdots\beta\cdots j\cdots;k}=\frac{\partial T^{\cdot\cdot\alpha\cdots i\cdots}_{\cdots\beta\cdots j\cdots}}{\partial x^k}+\cdots+\Gamma^\alpha_{\lambda k}T^{\cdot\cdot\lambda\cdots i\cdots}_{\cdots\beta\cdots j\cdots}+\cdots+\gamma^i_{ln}T^{\cdot\cdot\alpha\cdots l\cdots}_{\cdots\beta\cdots j\cdots}+\cdots$$
$$\cdots-\Gamma^\lambda_{\beta k}T^{\cdot\cdot\alpha\cdots i\cdots}_{\cdots\lambda\cdots j\cdots}-\cdots-\gamma^l_{jk}T^{\cdot\cdot\alpha\cdots i}_{\cdots\beta\cdots l}-\cdots.$$

Especially

$$(2.6)\qquad B_{j;k}^{\cdot\lambda}=B_{j,k}^{\cdot\lambda}+B_j^{\cdot\mu}\Gamma^\lambda_{\mu k}-B_i^{\cdot\lambda}\gamma^i_{jk}=H^P_{jk}B_P^{\cdot\lambda},$$

and

$$(2.7)\qquad V^\lambda_{;k}=B_i^{\cdot\lambda}v^i_{;k}+H^P_{jk}B_P^{\cdot\lambda}v^j,$$

where we put $V^\lambda=B_i^{\cdot}v^i$[3].

For the extension of Ricci equation we obtain

$$(2.8)\qquad T^{\cdot\cdot\alpha\cdots a\cdots}_{\cdots\beta\cdots b\cdots;i;j}-T^{\cdot\cdot\alpha\cdots a\cdots}_{\cdots\beta\cdots b\cdots;j;i}$$
$$=\cdots+R^\alpha_{\lambda ij}T^{\cdot\cdot\lambda\cdots a\cdots}_{\cdots\beta\cdots b\cdots}+\cdots+R^a_{lij}T^{\cdot\cdot\alpha\cdots l\cdots}_{\cdots\beta\cdots b\cdots}+\cdots$$
$$\cdots-R^\lambda_{\beta ij}T^{\cdot\cdot\alpha\cdots a\cdots}_{\cdots\lambda\cdots l\cdots}-\cdots-R^l_{bij}T^{\cdot\cdot\alpha\cdots a\cdots}_{\cdots\beta\cdots l\cdots}-\cdots.$$

§ 3. In the space connected dominantly with the given functions $\Gamma^\lambda_{\mu i}$ whose trasformation laws are given by the relations (1.13), when we determine the quantities $B_i^{\cdot\lambda}$ and $B_P^{\cdot\lambda}$ from the relations (1.13) and (1.15), we may determine four kinds of the quantities γ^k_{ij}, H^P_{ij}, H^P_{Qj} and H^i_{Pj} from the relations (1.8) and (1.10)

$$(3.1)\qquad \gamma^k_{ij}=B^k_{\cdot\alpha}\frac{\partial B_i^{\cdot\alpha}}{\partial x^j}+\Gamma^\alpha_{\beta j}B^k_{\cdot\alpha}B_i^{\cdot\beta},$$

$$(3.2)\qquad H^P_{ij}=B^P_{\cdot\alpha}\frac{\partial B_i^{\cdot\alpha}}{\partial x^j}+\Gamma^\alpha_{\beta j}B_i^{\cdot\beta}B^P_{\cdot\alpha},$$

$$(3.3)\qquad H^P_{Qj}=B^P_{\cdot\alpha}\frac{\partial B_Q^{\cdot\alpha}}{\partial x^j}+\Gamma^\alpha_{\beta j}B^P_{\cdot\alpha}B_Q^{\cdot\beta},$$

$$(3.4)\qquad H^i_{Qj}=B^i_{\cdot\alpha}\frac{\partial B_Q^{\cdot\alpha}}{\partial x^j}+\Gamma^\alpha_{\beta j}B^i_{\cdot\alpha}B_\theta^{\cdot\alpha}.$$

Conversely, from (1.8) and (1.10) we obtain

(3) K. Yano: Sur la theorie der espace à hyperconnection euclidienn. I. et II; Proc. Jap. Acad (21) (1945) p.p. 156~163 et p.p.164~170.

$$(3.5)\qquad \Gamma^{\alpha}_{\beta j}=[H^{Q}_{Pj}\,B^{\cdot\alpha}_{Q}+H^{i}_{Pj}\,B^{\cdot\alpha}_{i}-\frac{\partial B^{\cdot\alpha}_{P}}{\partial x^{j}}]B^{P}_{\cdot\beta}$$

$$+[\gamma^{k}_{ij}\,B^{\cdot\alpha}_{k}+H^{P}_{i\,j}\,B^{\cdot\alpha}_{P}-\frac{\partial B^{\cdot\alpha}_{i}}{\partial x^{j}}]B^{i}_{\cdot\beta}.$$

Therefore, when we take arbitralily γ^{k}_{ij}, $H^{P}_{i\,j}$ and H^{Q}_{Pj}, H^{i}_{Pj} that may satisfy the transformation laws (1.14) and (1.15) respectively, and determine the quantities $\Gamma^{\alpha}_{\beta j}$ from the relations (3.5), we see the quantities satisfy the transformation law (1,13) from (1,11), (1,14) and (1.15), that is,

$$\Gamma^{\alpha'}_{\beta' j'}=B^{i'}_{\cdot\beta'}(\gamma^{k'}_{i'j'}B^{\cdot\alpha'}_{k'}+H^{P}_{i'j'}B^{\cdot\alpha}_{P}-\frac{\partial B^{\cdot\alpha}_{i'}}{\partial x^{j'}})+B^{P}_{\cdot\beta'}(H^{Q}_{Pj'}B^{\cdot\alpha'}_{Q}+H^{i'}_{Pj'}B^{\cdot\alpha'}_{i'}-\frac{\partial B^{\cdot\alpha'}_{P}}{\partial x^{j'}})$$

$$=B^{i}_{\cdot\beta}\,A^{\beta}_{\beta'}\frac{\partial x^{i'}}{\partial x^{i}}\left[\left(\frac{\partial^{2}x^{k}}{\partial x^{i'}\partial x^{j'}}+\gamma^{k}_{ij}\frac{\partial x^{i}}{\partial x^{i'}}\frac{\partial x^{j}}{\partial x^{j'}}\right)B^{\cdot\alpha}_{k}\,A^{\alpha'}_{\alpha}-\frac{\partial(B^{\cdot\alpha}_{k}\frac{\partial x^{k}}{\partial x^{i'}}A^{\alpha'}_{\alpha})}{\partial x^{j}}\frac{\partial x^{j}}{\partial x^{j'}}\right]$$

$$+B^{P}_{\cdot\beta}\,A^{\beta}_{\beta'}\left[H^{Q}_{Pj}\frac{\partial x^{j}}{\partial x^{j'}}B^{\cdot\alpha}_{Q}\,A^{\alpha'}_{\alpha}+H^{k}_{Pj}\frac{\partial x^{j}}{\partial x^{j'}}\frac{\partial x^{i'}}{\partial x^{k}}B^{\cdot\alpha}_{i}\frac{\partial x^{i}}{\partial x^{i'}}A^{\alpha'}_{\alpha}-\frac{\partial(B^{\cdot\alpha}_{P}\,A^{\alpha'}_{\alpha})}{\partial x^{j}}\frac{\partial x^{j}}{\partial x^{j'}}\right]$$

$$=A^{\alpha'}_{\alpha}\frac{\partial x^{j}}{\partial x^{j'}}[A^{\beta}_{\beta'}\,\Gamma^{\alpha}_{\beta j}+A^{\alpha}_{\beta',j}].$$

Hence, *when on each point of an n-dimensional space V_n we give the vector field $B^{\cdot\lambda}_{i}$, $B^{\cdot\lambda}_{P}$ and the quantities γ^{k}_{ij}, $H^{P}_{i\,j}$ and H^{Q}_{Pj}, H^{i}_{Pj} which satisfy respectively the transformation laws* (1.14) *and* (1.15), *then we may determine coefficients of dominant affine connection which satisfy the relations* (3.5).

Moreover for the curvature tensor, similarly from (2.1), (2.4) and (1.16), we obtain

$$(3.6)\qquad R^{\alpha}_{\beta jk}=B^{i}_{\cdot\beta}\,B^{\cdot\alpha}_{h}\,[R^{h}_{ijk}+H^{h}_{Pk}\,H^{P}_{i\,j}-H^{h}_{Pj}\,H^{P}_{P\,ik}]$$

$$+B^{i}_{\cdot\beta}\,B^{\cdot\alpha}_{Q}\,[H^{Q}_{i\,j,k}-H^{Q}_{i\,k,j}+H^{P}_{i\,j}H^{Q}_{Pk}-H^{P}_{i\,k}H^{Q}_{Pj}+\gamma^{l}_{ij}\,H^{Q}_{l\,k}-\gamma^{l}_{ik}H^{P}_{l\,j}]$$

$$+B^{P}_{\cdot\beta}\,B^{\cdot\alpha}_{h}\,[H^{Q}_{Pj,k}-H^{h}_{Pk,j}+H^{l}_{Pj}\,\gamma^{h}_{lk}-H^{l}_{Pk}\,\gamma^{h}_{lj}+H^{Q}_{Pj}H^{l}_{Qk}-H^{Q}_{Pk}H^{h}_{Qj}]$$

$$+B^{P}_{\cdot\beta}B^{\cdot\alpha}_{Q}\,[H^{Q}_{Pj,k}-H^{Q}_{Pk,j}+H^{R}_{Pj}\,H^{Q}_{Rj}-H^{R}_{Pk}H^{Q}_{Rj}+H^{l}_{Pj}H^{Q}_{l\,k}-H^{l}_{Pk}H^{Q}_{l\,j}],$$

and this is equivalent to (2.1) and (2.4).

§ **4.** In this paragraph we shall consider a necessary and sufficient condition that a dominantly affinly connected space be a sub-variety of an affinely connected space.

Consider an m-dimensional space V_m with affine connection where a current point A is given by a system of coordinates $(y^1, y^2, \cdots, y^m)$, and the connection is given by the following equations;

$$(4.1)\qquad dA=A_{\alpha}dy^{\alpha},\qquad dA_{\alpha}=\Gamma^{\beta}_{\lambda\alpha}\,A_{\beta}\,dy^{\alpha}.$$

In V_m we consider an n-dimensional variety V'_n defined by the equations

$$(4.2)\qquad y^{\alpha}=y^{\alpha}(x^{1}, x^{2},\cdots, x^{n})$$

when current point A displaces on V'_n, we have

$$dA = A_\alpha B_i^{\cdot\alpha} dx^i \tag{4.3}$$

where

$$B_i^{\cdot\alpha} = \frac{\partial y^\alpha}{\partial x^i} \tag{4.4}$$

If we define

$$A_i = A_\alpha B_i^{\cdot\alpha}, \qquad A_P = A_\alpha B_P^{\cdot\alpha} \tag{4.5}$$

where $B_P^{\cdot\alpha}$ are $m-n$ contravariant vectors and the determinant $|B_i^{\cdot\alpha}, B_P^{\cdot\alpha}|$ is not equal to zero, we see A_i, A_P are m-linearly independent vectors of V_m. For the displacement on V_n we see

$$dA = A_i dx^i \tag{4.1$'$}$$

and we put

$$dA_i = (\gamma_{ij}^k + H_{i\ j}^P A_P) dx^j, \tag{4.6}$$

$$dA_P = (H_{Pj}^k A_k + H_{Pj}^Q A_Q) dx^j. \tag{4.7}$$

Differentiating (4.5) and comparing with (4.6), (4.7), we obtain

$$\frac{\partial B_i^{\cdot\alpha}}{\partial x^j} = -\Gamma_{P\mu}^\alpha B_i^{\cdot\lambda} B_j^{\cdot\mu} + \gamma_{ij}^k B_k^{\cdot\alpha} + H_{ij}^P B_P^{\cdot\alpha}, \tag{4.8}$$

$$\frac{\partial B_P^{\cdot\alpha}}{\partial x^j} = -\Gamma_{\lambda\mu}^\alpha B_P^{\cdot\lambda} B_j^{\cdot\mu} + H_{Pj}^k B_k^{\cdot\alpha} + H_{Pj}^Q B_Q^{\cdot\alpha}. \tag{4.9}$$

As the quanty $B_i^{\cdot\alpha}$ defined by the equations (4.4) must satisfy the integrability conditions of the equations $\dfrac{\partial^2 y^\alpha}{\partial x^i \partial x^j} = \dfrac{\partial^2 y^\alpha}{\partial x^j \partial x^i}$ we obtain

$$B_i^{\cdot\lambda} B_j^{\cdot\mu} (\Gamma_{\lambda\mu}^\alpha - \Gamma_{\mu\lambda}^\alpha) = B_k^{\cdot\alpha} (\gamma_{ij}^k - \gamma_{ji}^k) + B_P^{\cdot\alpha} (H_{i\ j}^P - H_{j\ i}^P). \tag{4.10}$$

Moreover from the integrability conditions of the system of epuations (4.8) and (4.9), we obtain[4)]

$$B_i^{\cdot\lambda} B_j^{\cdot\mu} B_k^{\cdot\nu} R_{\lambda\mu\nu}^\alpha = B_l^{\cdot\alpha} (R_{ijk}^l + H_{i\ j}^P H_{Pj}^l - H_{i\ k}^P H_{Pj}^l) \tag{4.11}$$
$$+ B_P^{\cdot\alpha} (H_{i\ j,k}^P - H_{i\ k,j}^P + H_{i\ j}^Q H_{Qk}^P - H_{i\ k}^Q H_{Qj}^P + \gamma_{ij}^l H_{Pk}^l - \gamma_{ik}^l H_{l\ j}^P),$$

$$B_P^{\cdot\lambda} B_j^{\cdot\mu} B_k^{\cdot\nu} R_{\lambda\mu\nu}^\alpha = B_l^{\cdot\alpha} (H_{Pj,k}^l - H_{Pk,j}^l + H_{Pj}^Q H_{Qk}^l - H_{Pk}^Q H_{Pj}^l + H_{Pj}^h \gamma_{hk}^l - H_{Pk}^h \gamma_{hj}^l) \tag{4.12}$$
$$+ B_Q^{\cdot\alpha} (H_{Pj,k}^Q - H_{Pk,j}^Q + H_{Pj}^R H_{Rk}^Q - H_{Pk}^R H_{Rj}^Q + H_{Pj}^l H_{l\ k}^Q - H_{Pk}^l H_{l\ j}^Q).$$

In the present case the quantities $\Gamma_{\beta j}^\alpha$ are given (i.e. the quantities $B_i^{\cdot\alpha}$, $B_P^{\cdot\alpha}$, $H_{i\ j}^P$, γ_{ij}^k,

(4) M. Matsumoto; Affinely connected spaces of class one.
Mem. of Colleg. of Science. Univ. of Kyoto. Vol. 26. (1951) p.p. 235~249.

H^P_{Qj} and H^i_{Pj} are given), that is to say, considering space V_n is a space with dominant affine connections, and we should like to determine the quantities $\Gamma^\alpha_{\beta\lambda}$ which are the coefficients of affine connection of enveloping space V_m. Obviously y^α (the coordinate system of V_m) must satisfy the relations

(4.13)
$$B_i^{\cdot\alpha} = \frac{\partial y^\alpha}{\partial x^i},$$

and on V_n

(4.14)
$$dy^\alpha = B_i^{\cdot\alpha}\, dx^i,$$

then we obtain

(4.15)
$$\Gamma^\beta_{\lambda i} = \Gamma^\beta_{\lambda\alpha} B_i^{\cdot\alpha}.$$

Hence,these functions $\Gamma^\beta_{\lambda\alpha}$ must satisfy firstly the transformation laws of coefficients of connection, secondly the equations (4.14), i.e., the integrability conditions of system of equations (4.13), and finaly (4.11) and (4.12), i.e., the integrability conditions of (4.8) and (4.9).

When these three sorts of conditions are satisfied, we may defin $(y^1, y^2, \cdots, y^m)$ which are the solutions of the system of equations

$$B_i^{\cdot\alpha} = \frac{\partial y^\alpha}{\partial x^i},$$

and find the m-dimensional space V_m with coefficients of affine connection $\Gamma^\beta_{\lambda\alpha}$ which has a system of coordinate y^α defined above and the considering space V_n as an n-dimensional sub-variety.

Consequently three sorts of conditions are the necessary and sufficient conditions that the space V_n can be embedded in an m-dimensional affinely connected space as an n-dimensional variety.

From (4.15) we must put

(4.16)
$$\Gamma^\alpha_{\lambda\mu} = B^i_{\cdot\mu}\, \Gamma^\alpha_{\lambda i} + B^P_{\cdot\mu}\, \Gamma^\alpha_{\lambda P},$$

where the quantities $\Gamma^\alpha_{\lambda P}$ are arbitraty functions of x^i.

Now we consider the transformation of coordinates y^α and x^i, then $\Gamma^\alpha_{\beta i}$ are transformed by the relations

$$\Gamma^{\beta'}_{\alpha' i'} = A^{\beta'}_\beta (A^\alpha_{\alpha'} \Gamma^\beta_{\alpha i} + A^\beta_{\alpha', i}) \frac{\partial x^i}{\partial x^{i'}},$$

where we must put

(4.17)
$$\frac{\partial y^{\alpha'}}{\partial y^\alpha} = A^{\alpha'}_\alpha,$$

So we see
$$B_{i'}^{\cdot\mu'}\, \Gamma^{\beta'}_{\alpha'\mu'} = A^{\beta'}_\beta (A^\alpha_{\alpha'}\, B_i^{\cdot\mu} \Gamma^\beta_{\alpha\mu} + A^\beta_{\alpha', i}) \frac{\partial x^i}{\partial x^{i'}}.$$

and from (4.13) $$B_{i'}^{\cdot\,\mu'}=\frac{\partial y^{\mu'}}{\partial x^{i'}}=A_{\mu}^{\mu'}\,B_{i}^{\cdot\,\mu}\frac{\partial x^{i}}{\partial x^{i'}}.$$

From these equations we obtain

$$\Gamma^{\beta'}_{\alpha'\mu'}=\frac{\partial y^{\beta'}}{\partial y^{\beta}}\left[\frac{\partial^2 y^{\beta}}{\partial y^{\alpha'}\partial y^{\mu'}}+\Gamma^{\beta}_{\alpha\mu}\frac{\partial y^{\mu}}{\partial y^{\mu'}}\frac{\partial y^{\alpha}}{\partial y^{\alpha'}}\right], \tag{4.18}$$

that is to say, the quantities $\Gamma^{\alpha}_{\lambda\mu}$ defined by (4.16) satisfy the transformation laws of the coefficients of affine connection. Hence we see that the first conditions are satisfied by the equations (4.13), (4.14) and (4.15).

Next, we consider the second conditions, while these conditions are the relations (4.10).

On the other hand from the relation

$$\Gamma^{\alpha}_{\lambda\mu}\,B_{j}^{\cdot\,\mu}=\Gamma^{\alpha}_{\lambda j},$$

we obtain

$$B^{k}_{\cdot\,\alpha}(B_{i}^{\cdot\,\lambda}\,\Gamma^{\alpha}_{\lambda j}-B_{j}^{\cdot\,\lambda}\,\Gamma^{\alpha}_{\lambda i})=\gamma^{k}_{ij}-\gamma^{k}_{ji}, \tag{4.19}$$

$$B^{P}_{\cdot\,\alpha}(B_{i}^{\cdot\,\lambda}\,\Gamma^{\alpha}_{\lambda j}-B_{j}^{\cdot\,\lambda}\,\Gamma^{\alpha}_{\lambda i})=H^{P}_{i\,j}-H^{P}_{j\,i}. \tag{4.20}$$

Finally the last conditions are the equations (4.11) and (4.12). While the relations (2.1) and (2.4) must be satisfied in V_n, the equations (4.11) and (4.12) are identically satisfied when the relations (4.13) are satisfied since we can obtain

$$R^{\alpha}_{\beta jk}=R^{\alpha}_{\beta\mu\lambda}\,B_{j}^{\cdot\,\mu}\,B_{k}^{\cdot\,\lambda}. \tag{4.21}$$

For (4,21) from (4,13) we see $B_{j}^{\cdot\,\mu}{}_{,k}=B_{k}^{\cdot\,\mu}{}_{,j}$,

then
$$\begin{aligned}R^{\alpha}_{\beta jk}&=\Gamma^{\alpha}_{\beta j,k}-\Gamma^{\alpha}_{\beta k,j}+\Gamma^{\alpha}_{\sigma j}\,\Gamma^{\sigma}_{\beta j}-\Gamma^{\alpha}_{\sigma k}\Gamma^{\sigma}_{\beta j}\\&=\Gamma^{\alpha}_{\beta\mu,\lambda}\,B_{k}^{\cdot\,\lambda}\,B_{j}^{\cdot\,\mu}-\Gamma^{\alpha}_{\beta\lambda,\mu}\,B_{\mu}^{\cdot\,\lambda}\,B_{j}^{\cdot\,\mu}+\Gamma^{\alpha}_{\beta\mu}(B_{j}^{\cdot\,\mu}{}_{,k}-B_{k}^{\cdot\,\mu}{}_{,j})\\&\quad+B_{j}^{\cdot\,\mu}\,B_{k}^{\cdot\,\lambda}(\Gamma^{\alpha}_{\sigma\mu}\,\Gamma^{\sigma}_{\beta\lambda}-\Gamma^{\alpha}_{\sigma\lambda}\,\Gamma^{\sigma}_{\beta\mu})\\&=R^{\alpha}_{\beta\mu\lambda}B_{j}^{\cdot\,\mu}B_{k}^{\cdot\,\lambda}.\end{aligned}$$

Consequently the relations (4.11) and (4.12) are equivalent in consequences of (2.1) and (2.4) respectively, that is to say, in consequences of (3.6).

A necessary and sufficient condition that an n-dimensional space with dominant affine connections be an n-dimensional variety of an m-dimensional space with affine connection is that the relations (4.19), (4.20) and (3.6) *be satisfied.*

§ 5. In this paragraph we consider the case where γ^{i}_{jk} are equal to γ^{i}_{kj}. While on a space with symmetric affine connection the equations of geodesic lines are

$$\frac{\delta}{ds}\left(\frac{dx^{i}}{ds}\right)=0$$

that is to say,
$$\frac{d^2x^i}{ds^2}+\gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0,$$

where s is affine parameter.

Now in the space V_n with dominant affine connection when we put

$$V^\lambda=\frac{dx^i}{ds}B_i^{\cdot\lambda}, \tag{5.1}$$

the condition that a curve $x^i=x^i(s)$ is developed into a straight line when we develop V_n in an m-dimensional affine space along this curve is written

$$\frac{\delta}{ds}(V^\lambda)=0 \tag{5.2}$$

we have
$$\left(H^P_{j\,k}\frac{dx^j}{ds}\frac{dx^k}{ds}\right)B_P^{\cdot\lambda}+\left(\frac{d^2x^i}{ds^2}+\gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}\right)B_i^{\cdot\lambda}=0$$

Consequently we obtoin

$$\frac{d^2x^i}{ds^2}+\gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0, \tag{5.3}$$

$$H^P_{j\,k}\frac{dx^j}{ds}\frac{dx^k}{ds}=0 \tag{5.4}$$

Then we call the curves which are solutions of the system of equations (5.3) and (5.4) geodesic line in V_n and asymptotic line respectively.

If a curve in V_n can be developed into a straight line in A_m this curve must be geodesic and asymptotic line in V_n. Especially the necessary and sufficient condition that all geodesic lines in V_n can be developed into straight lines is

$$H^P_{i\,j}+H^P_{j\,i}=0. \tag{5.5}$$

While geodesic lines and asymptotic lines are respectively same for two connections whose coefficients are in the relations

$$\bar{\gamma}^i_{jk}=\gamma^i_{jk}+\delta^i_j\,\psi_k+\delta^i_k\,\psi_j, \tag{5.6}$$

$$\bar{H}^P_{j\,k}=\rho H^P_{j\,k}+\Omega^P_{j\,k}. \tag{5.7}$$

where ψ_i is an arbitrary covariant vector, ρ is an arbitrary scalar and $\Omega^P_{j\,k}$ are p arbitrary skew symmetric tensors. In the equations (3.5) for $\bar{\Gamma}^\alpha_{\beta j}$, substituting for $\bar{\gamma}^i_{jk}$ and $\bar{H}^P_{j\,k}$ their expressions (5.6) and (5.7) respectively, we obtain

$$\bar{\Gamma}^\alpha_{\beta j}=\Gamma^\alpha_{\beta j}+B^k_{\cdot\beta}B_k^{\cdot\alpha}\,\psi_j+B^i_{\cdot\beta}B_j^{\cdot\alpha}\,\psi_i+(\rho-1)B^i_{\cdot\beta}B_P^{\cdot\alpha}H^P_{i\,j}+\Omega^P_{i\,j}B^i_{\cdot\beta}B_P^{\cdot\alpha}. \tag{5.8}$$

Conversly in the case where the quantities $\Gamma^\alpha_{\beta j}$ and $\bar{\Gamma}^\alpha_{\beta j}$ are connected with the relation (5.8) we obtain (5.6), (5.7) and

$$\bar{H}^P_{Qj}=H^P_{Qj},\qquad \bar{H}^i_{Qj}=H^i_{Qj}$$

Consequently, *the geodesic lines and asymptotic lines are same respectively for two connections whose coefficients are in the relations* (5.8).

We say that the dominant affine connection of coefficients $\bar{\Gamma}^{\alpha}_{\beta j}$ is obtaind from that with the cefficients $\Gamma^{\alpha}_{\beta j}$ by a projective change of the dominnat affine connection.

We see from (1.13)

$$\Gamma^{\lambda}_{\mu i}=A^{\lambda}_{\mu'}\frac{\partial x^{i'}}{\partial x^{i}}[A^{\nu'}_{\mu}\ \Gamma^{\mu'}_{\nu' i'}+A^{\mu'}_{\mu}{}_{,i'}]. \tag{5.9}$$

Contracting for λ and μ we obtain

$$\Gamma^{\lambda}_{\lambda i}=\frac{\partial x^{i'}}{\partial x^{i}}\Gamma^{\mu'}_{\mu' i'}+\frac{\partial \log \Delta}{\partial x^{i}}, \tag{5.10}$$

where

$$\Delta=|A^{\mu'}_{\lambda}| \tag{5.11}$$

On the other hand from the relation (5.8) contracting for α and β we have

$$\psi_{j}=\frac{1}{n+1}(\bar{\Gamma}^{\alpha}_{\alpha j}-\Gamma^{\alpha}_{\alpha j}) \tag{5.12}$$

from which and (5.8) we find that *the quantities*

$$\Pi^{\alpha}_{\beta j}=\Gamma^{\alpha}_{\beta j}-\frac{1}{n+1}\{B^{k}_{\cdot\beta}\,B^{\cdot\alpha}_{k}\ \Gamma^{\lambda}_{\lambda j}+B^{\cdot\alpha}_{j}\ B^{k}_{\cdot\beta}\ \Gamma^{\lambda}_{\lambda k}\}-B^{i}_{\cdot\beta}\,B^{\cdot\alpha}_{P}H^{P}_{i\ j} \tag{5.13}$$

are independent of a projective change of dominant affine connection.

For the relations between the function Π^{i}_{jk} and the analogous function in a coordinate system $x^{i'}$ we find from the relations (5.13), (5.19), (1.11) and (1.14)

$$\Pi^{\alpha}_{\beta j}A^{\lambda'}_{\beta}=A^{\lambda'}_{\beta}\frac{\partial x^{i'}}{\partial x^{j}}\Pi^{\beta'}_{\lambda' j'}+A^{\beta'}_{\beta}{}_{,j}-\frac{1}{n+1}B^{k'}_{\cdot\alpha'}\,B^{\cdot\beta'}_{k'}\ A^{\alpha'}_{\beta}\frac{\partial \log \Delta}{\partial x^{\alpha}}-\frac{1}{n+1}B^{k}_{\cdot\beta}B^{\cdot\beta'}_{j'}\frac{\partial x^{j'}}{\partial x^{j}}\frac{\partial \log \Delta}{\partial x^{k}}. \tag{5.14}$$

Moreover we find that *the quantities*

$$W^{\alpha}_{\beta jk}=B^{\cdot\alpha}_{h}\ B^{i}_{\cdot\beta}\ W^{h}_{i jk} \tag{5.15}$$

and

$$\begin{aligned}W'^{\alpha}_{\beta jk}=&B^{\cdot\alpha}_{h}B^{i}_{\cdot\beta}\ W^{h}_{ijk}+B^{\cdot\alpha}_{Q}\ B^{P}_{\cdot\beta}(H^{Q}_{Pj,k}-H^{Q}_{Pk,j}+H^{R}_{Pj}\ H^{Q}_{Rk}-H^{R}_{Pk}H^{Q}_{Rj})\\&+B^{P}_{\cdot\beta}\ B^{\cdot\alpha}_{h}(H^{h}_{Pj,k}-H^{h}_{Pk,j}+H^{l}_{Pj}\ \gamma^{h}_{lk}-H^{l}_{pk}\ \gamma^{h}_{lj}+H^{Q}_{Pj}\ H^{h}_{Qk}-H^{Q}_{Pk}\ H^{h}_{Qj})\\&-\frac{1}{n+1}(H^{h}_{Pj}\ \Gamma^{\sigma}_{\sigma k}+H^{h}_{Pk}\ \Gamma^{\sigma}_{\sigma j}+\delta^{h}_{k}\ H^{l}_{Pj}\ \Gamma^{\sigma}_{\sigma l}+\delta^{h}_{j}H^{l}_{P}\ \Gamma^{\sigma}_{\sigma k})\end{aligned} \tag{5.16}$$

are independent of a projective change of dominant affine connection, where W^{h}_{ijk} *is so-called Weyl projective curvature tensor for* γ^{i}_{jk}.

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A NOTE ON SUBORDINATION

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1. Introduction The following result is well known as Hurwitz-Bochner's theorem [1][2].(1)

If $w=f(z)$ *is regular in* $|z|<1$, $f(0)=0$, $f'(0)=1$, *and* $f(z)\neq 0$

except $z=0$, *the conformal image of* $f(z)$ *assumes every value in* $|w|<1/16$. In the present paper we generalize this theorem and deal with the related problems with it by the principle of subordination.

As a preliminary remark we shall give a notion of $Q(z)$ whose properties are as follows [3].

$$Q(z)=16z\prod_{n=1}^{\infty}\left(\frac{1+z^{2n}}{1+z^{2n-1}}\right)^8, \qquad (|z|<1).$$

$$Q(z)=J\left(\frac{1}{\pi i}\log z\right), \text{ where } J(z) \text{ is the elliptic modular function.}$$

Let the surface M be the conformal image of $|z|<1$ by $Q(z)$. M has no branch point. M covers every point of w-plane except $w=0,1,\infty$. M does not cover $w=1,\infty$, but the $w=0$ is covered by one sheet of M only.

2. Lemma 1. $\qquad |Q(z)|\leq Q(-r), \qquad (|z|=r<1).$

Proof. Each factor of infinite products which constitutes $Q(z)$ has its greatest absolute value on $|z|=r$ only when $z=-r$, and therefore we have the above estimate clearly.

Lemma 2. *Let* $w=f(z)=a_1z+\cdots\cdots$, *be regular and* $f(z)\neq 0$ *except* $z=0$ *in* $|z|<1$. *If* $f(z)$ *omits a value* α *in* $|z|<1$, *the following estimates are got.*

(i) $|a_1|\leq 16|\alpha|$, *that is, the conformal image of* $f(z)$ *assumes every value in* $|w|<|a_1|/16$.

(ii) $|f(z)|\leq|\alpha|\cdot Q(-r), \qquad (|z|=r<1).$

Proof. Let us consider

$$P(z)=Q^{-1}\left(\frac{f(z)}{\alpha}\right),$$

where $Q^{-1}(w)$ is the inverse function of $Q(z)$.
$f(z)$ leaves out $w=0$ except $z=0$ in $|z|<1$, and therefore $P(z)$ is analytic in $0<|z|<1$. On the other hand the regularity of $P(z)$ hold good at $z=0$. Hence $f(z)/\alpha$ is subordinate to $Q(z)$ and by the principle of subordination we get

(1) The bracket denotes the number of the references.

$$|\alpha^{-1}f'(0)|\leqq Q'(0)$$

$$|\alpha^{-1}f(z)|\leqq|Q(z)|\leqq Q(-r)\quad(|z|=r<1).$$

These complete the proof.

Remark. The above estimates are sharp as is shown by $f_0(z)=\alpha Q(z)$. $|\alpha|$ in the latter result of this lemma is can not be made smaller than $|a_1|/16$. The former result was got by the same method by Bochner [1][3].

Theorem 1. *Let* $w=f(z)=a_pz^p+\cdots\cdots$, *be regular and* $f(z)\neq0$ except $z=0$ *in* $|z|<1$.

(i) *The values taken by* $w=f(z)$ *cover the circle* $|w|<|a_p|/16^p$ p times or more times.

(ii) *The conformal image of the unit circle by* $w=f(z)$ *completely covers the interior of a circle about the origin whose rudius is* $|a_p|/16$. *We can state this result in detail, namely,*
If $w=f(z)$ omit a value α,

(ii') $$|a_p|\leqq16|\alpha|$$

(iii) $$|f(z)|\leqq|\alpha|\cdot Q(-r^p)\qquad(|z|=r<1).$$

These bounds are best possible.

Proof.

First in order to prove the former result we consider $g(z)=(f(z))^{1/p}$. Since $f(z)\neq0$ except $z=0$ in $|z|<1$, $g(z)$ is regular and $g(z)\neq0$ except $z=0$ in $|z|<1$. Hence when lemma 2 is used with respect to $g(z)$, the proof of (i) will be given.

Secondly we prove the latter results (ii) and (iii). Let us consider

$$F(z)=Q(z^p)=16z^p+\cdots\cdots.$$

The Riemann surface onto which the unit circle is mapped by $F(z)$ has no branch point, and covers every point in w-plane infinite times except $w=0,1,\infty$. It does not cover $w=1$, ∞, but $w=0$ is covered by its p sheets only.
Now we put

$$R(z)=F^{-1}\left(\frac{f(z)}{\alpha}\right).$$

Under the same conditions in the proof of lemma 2, $R(z)$ is regular in $0<|z|<1$, and $R(z)$ is regular at $z=0$ more. We can get easy

$$|R'(0)|=\left|\frac{a_p}{16\alpha}\right|^{\frac{1}{p}}$$

Furthermore $$R(0)=0,\qquad|R(z)|<1.$$

Hence $$|R'(0)|\leqq1,\ \text{that is,}\ |a_p|\leqq16|\alpha|.$$

On the other hand $f(z)$ is subordinate to $\alpha F(z)$ and therefore

$$|f(z)|\leqq\max_{|z|=r}|\alpha F(z)|\leqq|\alpha|\cdot Q(-r^p)(|z|=r<1).$$

Theorem 2. *Let $w=f(z)=a_1z+\cdots\cdots$, be regular and $f(z)\neq 0$ except $z=0$ in $|z|<1$. If $f(z)$ leaves out two values α and $-\alpha$, then*

$$|a_1|\leqq 4|\alpha|, \qquad |f(z)|\leqq|\alpha|\cdot Q^{\frac{1}{2}}(-r^2).$$

Namely, if $w=f(z)=z+\cdots\cdots$, is an odd regular function in $|z|<1$ and $f(z)\neq 0$ except $z=0$, the values taken by $w=f(z)$ cover fully $|w|<1/4$. These bounds are best possible.(2)

Proof. As the superordinate function to $f(z)/\alpha$ we consider

$$Q_1(z)=\sqrt{Q(z^2)}=4z+\cdots\cdots\cdots.$$

And then $Q_1(z)$ is an odd regular function in $|z|<1$ and the other properties of $Q_1(z)$ are quite similar to ones of $Q(z)$ except the fact that it does not assume both 1, and -1[2]. Namely $Q_1^{-1}\left(\frac{f(z)}{\alpha}\right)$ is analytici n $|z|<1$ like the case of lemma 1 and therefore we have the forme result of this theorem by the principle of subordination. The latter result is evident.

We can moreover generalize this theorem, that is,

Theorem. 2′. *Let $w=f(z)=a_pz^p+\cdots\cdots$, be regular in $|z|<1$, and $f(z)\neq 0$ except $z=0$. If $f(z)$ leaves out the values α, and $-\alpha$, then*

$$|a_p|\leqq 4|\alpha|, \quad |f(z)|\leqq|\alpha|\cdot Q^{\frac{1}{2}}(-r^{2p}), \quad (|z|=r<1).$$

Proof. If we consider the superordinate function

$$Q^{\frac{1}{2}}(z^{2p})=4z^p+\cdots\cdots\cdots\cdots,$$

the results are evident.

Theorem 3. *Let $w=f(z)=a_p\, z^p+\cdots\cdots$, be regular and not zero except $z=0$ in $|z|<1$. If $f(z)$ leaves out the values $-\infty\leqq w\leqq\frac{-1}{4}$, $f(z)$ is subject to the following inequalities.*

$$|a_p|\leqq 1, \quad |f(z)|\leqq\frac{r^p}{(1-r^p)^2}, \ (|z|=r<1)$$

These bounds are sharp as is shown by

$$F(z)=\frac{z^p}{(1-z^p)^2}$$

Proof. As the superordinate function to $f(z)$ we consider $F(z)$. The Riemann surface onto whieh the unit circle is mapped by $F(z)$ has no branch point except $z=0$. Hence

$$S(z)=F^{-1}(f(z))$$

is analytic in $0<|z|<1$. On the other hand $S(z)$ has regularity at $z=0$ whose deriver-tive has $(a_p)^{\frac{1}{p}}$ there. Moreover $S(0)=0$ and $|S(z)|<1$. Hence we have the above results.

(2) Ifthe condition that $f(z)\neq 0$ except $z=0$, is omitted, we have the following result, which is well known [3].

$$|a_1|\leqq k|\alpha|, \qquad k=\Gamma^4(\tfrac{1}{4})/4\pi^2$$

We can get the following estimates by means of the same method also.

Theorem 4. *If* $w=f(z)=a_p z^p+\cdots\cdots$, *is regular in* $|z|<1$, *leaves out the values* $-\infty\leqq w\leqq-\alpha$, *and* $\alpha\leqq w\leqq\infty$, $(\alpha>0)$, *and vanishes at* $z=0$ *only*,

$$|a_p|\leqq 2\alpha, \qquad |f(z)|\leqq\frac{2\alpha r^p}{1-r^{2p}}. \qquad (|z|=r).$$

These bounds are best possible as is shown by

$$F(z)=\frac{2\alpha z^p}{(1+z^{2p})}$$

3. We consider the case where $f(z)$ is meromorphic in $|z|<1$.

Theorem 5. *Let* $w=f(z)=a_1 z+a_3 z^3+\cdots\cdots$, *be an odd meromorphic function in* $|z|<1$ *and* $f(z)\neq 0$ *except* $z=0$, *then the image by this function covers fully the cirle*

$$|w|<\frac{|a_1|}{8}.$$

The result is best possible as is shown by

$$f_0(z)=\frac{2Q_1(z)}{1+Q_1^2(z)}$$

Proof. It is clear that $f_0(z)$ is an odd meromorphic function and does not take 1 and -1 b because of the property of $Q_1(z)$.

Let us consider

$$F(z)=\frac{1}{1-f_0(z)}=\frac{1+Q_1^2(z)}{(1-Q_1(z))^2}$$

Then

$$F'(z)=\frac{2(1+Q_1(z))}{(1-Q_1(z))^3}Q_1'(z)\neq 0 \quad (|z|<1),$$

because $Q_1'(z)\neq 0$ in $|z|<1$.

If $f(z)$ leaves out α and $-\alpha$, we consider $g(z)=f(z)/\alpha$. The function $(1-g(z))^{-1}$ is subordinate to $F(z)$ because of the proerty of $F(z)$. And therefore

$$\left|\frac{a_1}{\alpha}\right|\leqq F'(0)=8$$

This completes the proof.

Remark. If $f(z)=a_1 z\cdots\cdots$, is meromorphic and vanishes at $z=0$ only in $|z|<1$, $f(z)$ takes at least one value of each coupl $\pm w$ belonging to the circl

$$|w|<\frac{|a_1|}{8}$$

The proof is quite similar if $Q(z)(2-Q(z))^{-1}$ is considered as the superordinate function.

Here we state the related result with this theorem.

Theorem 6. *Let $f(z)=a_1z+\cdots\cdots$, be an odd meromorphic function in $|z|<1$, then the image by $f(z)$ covers fully the circle*

$$|w|<\frac{2|a_1|\pi^2}{\Gamma^4(\frac{1}{4})}$$

This result is best possible.

Proof. Like the case in theorem 5 we consider

$$f_0(z)=\frac{2F(z)}{1+F^2(z)},\ F(z)=2J\left[i\left(\frac{1+z}{1-z}\right)\right]-1.$$

$F(z)$ leaves out 1 and -1, but every value except these values is taken infinite times. Furthermore $F'(z)\neq 0$ in $|z|<1$, because $J(z)$ has no branch point [3]. Hereafter we may do like the proof of theorem 5.

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ON THE COMPOUND NORMAL DISTRIBUTIONS

By

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While the so-called compound Poisson's distributions are frequently spoken of [1], there is no such with the normal distributions to our poor knowledge. Namely, if a random variable x has a probability density $\varphi(x;\theta_1,\theta_2,\cdots)$, and the parameters θ_i being again random variables distribute with the frequency function $\psi(\theta_1,\theta_2,\cdots)$, the compound φ-distribution is defined by

$$f(x)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\varphi(x;\theta_1,\theta_2,\cdots)\psi(\theta_1,\theta_2,\cdots)d\theta_1\,d\theta_2\cdots, \tag{1}$$

with
$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\psi(\theta_1,\theta_2,\cdots)d\theta_1\,d\theta_2\cdots=1.$$

In particular the compound normal distribution with mean a and variance σ^2 is

$$f(x)=\int_0^{\infty}\int_{-\infty}^{\infty}\varphi(x;a,\sigma)\psi(a,\sigma)\,da\,d\sigma, \tag{2}$$

where

$$\varphi(x;a,\sigma)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}\quad\text{and}\quad\int_0^{\infty}\int_{-\infty}^{\infty}\psi(a,\sigma)\,da\,d\sigma=1.$$

We shall discuss the latter somewhat in detail. When ψ is known f is obtainable merely by integration, while, if f is given, ψ should be found by solving (2) as an integral equation. Thereby theoretically Laplace transform and practically Gauss' method of numerical integration by selected ordinates might be efficiently utilized.

§ 1.

All integrands in (1) and (2) being assumed to be positive and integrable, the order of integrations can be changed, and

$$\left.\begin{aligned}f(x)&=\int_0^{\infty}\left[\int_{-\infty}^{\infty}\varphi(x,a,\sigma)\psi(a,\sigma)\,da\right]d\sigma\equiv\int_0^{\infty}\varphi_2(x,\sigma)d\sigma\\&=\int_{-\infty}^{\infty}\left[\int_0^{\infty}\varphi(x,a,\sigma)\psi(a,\sigma)\,d\sigma\right]da\equiv\int_{-\infty}^{\infty}\varphi_1(x,a)da.\end{aligned}\right\} \tag{3}$$

1) E. g. W. Feller, Probability Theory and its Applications, 1952, p. 221.

Or, if we set

$$\left.\begin{array}{ll} f_1(x,a)=\dfrac{\varphi_1(x,a)}{\psi_1(a)}, & f_2(x,\sigma)=\dfrac{\varphi_2(x,\sigma)}{\psi_2(\sigma)}, \\ \psi_1(a)=\displaystyle\int_0^\infty \psi(a,\sigma)d\sigma, & \psi_2(\sigma)=\displaystyle\int_{-\infty}^\infty \psi(a,\sigma)da, \end{array}\right\} \tag{4}$$

with

both f_1 and f_2 are also compound normal frequency functions, although they might get out of normality in form, and

$$f(x)=\int_0^\infty f_2(x,\sigma)d\Psi_2(\sigma)=\int_{-\infty}^\infty f_1(x,a)d\Psi_1(a), \tag{5}$$

where $\Psi_1(a)$ and $\Psi_2(\sigma)$ stand for cumulative distribution functions of a and σ respectively, and $d\Psi_1(a)=\psi_1(a)da$, $d\Psi_2(\sigma)=\psi_2(\sigma)d\sigma$.

In particular, if a and σ be independent of each other

$$\psi(a,\sigma)=\psi_1(a)\psi_2(\sigma), \tag{6}$$

and

$$f_1(x,a)=\int_0^\infty \varphi(x,a,\sigma)d\Psi_2(\sigma), \quad f_2(x,\sigma)=\int_{-\infty}^\infty \varphi(x,a,\sigma)d\Psi_1(a), \tag{7}$$

yet (5) still hold.

Theorem 1. *$f_2(x,\sigma)$ is normal in x when and only when $\psi(a,\sigma)$ is normal in a.*

For, let

$$\psi(a,\sigma)=\frac{\psi_2(\sigma)}{\sqrt{2\pi}\,\tau}\exp\left\{-\frac{(a-m)^2}{2\tau^2}\right\}, \tag{8}$$

where m and τ are constant if a and σ independent, otherwise both or one of them shall be variable as functions of σ[2]. On account of (4) and (3) we have

$$\begin{aligned} f_2(x,\sigma)&=\int_{-\infty}^\infty \varphi(x,a,\sigma)\psi(a,\sigma)da/\psi_2(\sigma) \\ &=\frac{1}{2\pi\sigma\tau}\int_{-\infty}^\infty \exp\left\{-\frac{(x-a)^2}{2\sigma^2}-\frac{(a-m)^2}{2\tau^2}\right\}da \\ &=\frac{1}{\sqrt{2\pi(\sigma^2+\tau^2)}}\exp\left\{-\frac{(x-m)^2}{2(\sigma^2+\tau^2)}\right\}, \end{aligned} \tag{9}$$

which shows that $f_2(x,\sigma)$ is $N(x, m, \sqrt{\sigma^2+\tau^2})$.

To prove the converse, we have to solve the integral equation

$$\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}\frac{\psi(a,\sigma)}{\psi_2(\sigma)}da=\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}, \tag{10}$$

2) We have assumed that $\psi(a,\sigma)$ is normal in a, which means that m and τ in (8) do not contain a.

in which the right hand side is written in the standardized form instead of the last expression in (9), because σ, τ and m can be temporarily as constants considered. As a matter of fact, in consequence of scattering of a in (10), the dispersion of the resultant distribution should be greater than before integration, and therefore $\sigma^2<1$.

Now putting $x=\sigma s$, $a=-\sigma t$ equation (10) reduces to

$$\int_{-\infty}^{\infty}\exp\{-\tfrac{1}{2}(s+t)^2\}\frac{\psi(-\sigma t,\sigma)}{\psi_2(\sigma)}dt=\exp\{-\tfrac{1}{2}\sigma^2 s^2\},$$

viz.

$$\int_{-\infty}^{\infty}e^{-st}g(t)\,dt=\mathfrak{f}(s), \tag{11}$$

where

$$g(t)=\exp\left(-\frac{t^2}{2}\right)\cdot\psi(-\sigma t,\sigma)/\psi_2(\sigma) \qquad \text{and} \qquad \mathfrak{f}(s)=\exp\{\tfrac{1}{2}(1-\sigma^2)s^2\}.$$

This integral equation presents a Laplace transform, and a known inversion formula is capable to be applied[3]. Assuming that (11) converges absolutely on the line $\Re\ s=c$ in the complex s-plane, the inversion formula enunciates

$$g(t)=\lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\mathfrak{f}(s)e^{st}ds \qquad (s=c+i\eta)$$

On calculating this limiting value, we obtain

$$g(t)=\frac{1}{\pi}\exp\{ct+\tfrac{1}{2}(1-\sigma^2)c^2\}\int_0^{\infty}\exp\{-A\eta^2\}\cos B\eta\,d\eta,$$

where $A=\frac{1}{2}(1-\sigma^2)>0$, $B=t+c(1-\sigma^2)$. By use of a known formula

$$\int_0^{\infty}e^{-\alpha\xi^2}\cos\beta\xi\,d\xi=\tfrac{1}{2}\sqrt{\frac{\pi}{\alpha}}\exp\left\{-\frac{\beta^2}{4\alpha}\right\}, \quad (\alpha>0)$$

we have

$$g(t)=\frac{1}{\sqrt{2\pi(1-\sigma^2)}}\exp\left\{-\frac{t^2}{2(1-\sigma^2)}\right\} \quad (1>\sigma^2).$$

Remembering that $t=-a/\sigma$ and $\exp\left(\frac{t^2}{2}\right)\cdot g(t)=\psi(a,\sigma)/\psi_2(\sigma)$, we attain finally

$$\psi(a,\sigma)=\frac{\psi_2(\sigma)}{\sqrt{2\pi(1-\sigma^2)}}\exp\left\{-\frac{a^2}{2(1-\sigma^2)}\right\}, \tag{12}$$

which completes the proof[4].

3) Cf. D. V. Widder, The Laplace Transform, 1946, p. 241.

4) Since $\psi(a,\sigma)$ is to be real positive, $\psi_2(\sigma)$ in (12) shall be zero for $1<\sigma<\infty$. Also for $\sigma=1-0$, $\psi(a,\sigma)$ presents an indeterminate form, but then on interpretting the main factor as a singular normal distribution, $\psi(a,\sigma)\,da\,d\sigma$ tends to $\psi_2(\sigma)d\sigma$, which becomes in general an infinitesimal, unless $\psi_2(\sigma)$ has there a finite jump δ, so that $\int_{1-0}^{1}\psi(a,\sigma)da\,d\sigma=\delta$.

Corollary. In case a and σ are independent, $f_2(x, \sigma)$ is normal if and only if $\psi_1(a)$ is normal[5].

Theorem 2. *$\psi(a,\sigma)$ or $\psi_1(a)$ being normal in a, the final distribution $f(x)$ in general becomes non-normal, and rather specially it offers a single normal distribution.*

By theorem 1 the frequency function $f_2(x, \sigma)$ becomes normal, but

$$f(x)=\int_0^\infty f_2(x,\sigma)d\Psi_2(\sigma)=\int_0^\infty \frac{1}{\sqrt{2\pi(\sigma^2+\tau^2)}}\exp\left\{-\frac{(x-m)^2}{2(\sigma^2+\tau^2)}\right\}d\Psi_2(\sigma) \qquad (13)$$

is simply a superposition of normal distributions. Specially, if it happens that only on a discrete set S, $m=m_0$, $\sigma^2+\tau^2=\sigma_0^2$ and $\int_S d\Psi_2(\sigma)=1$ hold, so that $d\Psi_2(\sigma)=0$ on the complementary continuous set S', then $f(x)$ reduces to $\varphi(x, m_0, \sigma_0)$. Here, of course, the integral should be understood as Stieltjes' one. However, to discuss the case that $\psi_2(\sigma)$ is of continuous type, we should consult with the equation

$$\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}=\int_0^\infty \frac{1}{\sqrt{2\pi(\sigma^2+\tau^2)}}\exp\left\{-\frac{(x-m)^2}{2(\sigma^2+\tau^2)}\right\}d\Psi_2(\sigma), \qquad (14)$$

where m and τ are some functions of σ and $\Psi_2(\sigma)$ represents the cumlative distribution function. Or, expressing (14) in form of characteristics

$$\exp\{-t^2/2\}=\int_0^\infty \exp\{imt-(\sigma^2+\tau^2)t^2/2\}d\Psi_2(\sigma),$$

viz.

$$1=\int_0^\infty \exp\{imt-\tfrac{1}{2}(\sigma^2+\tau^2-1)t^2\}\, d\Psi_2(\sigma). \qquad (15)$$

Hence

$$\int_0^\infty \exp\{-\tfrac{1}{2}(\sigma^2+\tau^2-1)t^2\}\,(\cos mt+i\sin mt)\, d\Psi_2(\sigma)=1. \qquad (16)$$

The imaginary part's appearance being only superficial, we may write

$$\int_0^\infty \exp\{-\tfrac{1}{2}(\sigma^2+\tau^2-1)t^2\}\,(\cos mt+\sin mt)\, d\Psi_2(\sigma)=1.$$

By virtue of the first mean value theorem

$$(\cos m_\theta t+\sin m_\theta t)\int_0^\infty \exp\{-\tfrac{1}{2}(\sigma^2+\tau^2-1)t^2\}\, d\Psi_2(\sigma)=1$$

and

$$\cos m_\theta t\int_0^\infty \exp\{-\tfrac{1}{2}(\sigma^2+\tau^2-1)t^2\}\, d\Psi_2(\sigma)=1,$$

$$\sin m_\theta t\int_0^\infty \exp\{-\tfrac{1}{2}(\sigma^2+\tau^2-1)t^2\}\, d\Psi_2(\sigma)=0,$$

5) This does not mean that $f(x)$ becomes normal: Compare e.g. Ex. 4 in §3.

in view of (16). Hence we have $m_\theta t = 2n\pi$ and

$$\int_0^\infty \exp\{-\tfrac{1}{2}(\sigma^2+\tau^2-1)t^2\}\, d\Psi_2(\sigma) = 1.$$

But, if $\sigma^2+\tau^2 \neq 1$ on a continuous subset S with non-zero measure, the order of magnitude of this integral could be altered from 1 by taking t^2 sufficiently large. Hence it must hold that $\sigma^2+\tau^2=1$ on the whole continuous integration interval. Further, now that $\sigma^2+\tau^2-1=0$, the real part of (16) becomes

$$1 = \int_0^\infty \cos mt\, d\Psi_2(\sigma) = \int_0^\infty \left(1 - \frac{m^2t^2}{2} + \cdots\right) d\Psi_2(\sigma)$$

for all values of t, so that

$$1 = \int_0^\infty d\Psi_2(\sigma), \qquad 0 = \int_0^\infty m^2 d\Psi_2(\sigma), \qquad \cdots\cdots$$

Hence $m=0$ throughout and $\tau^2 = 1-\sigma^2$. Thus we obtain from (8)

$$\left.\begin{aligned} \psi(a,\sigma) &= \frac{\psi_2(\sigma)}{\sqrt{2\pi(1-\sigma^2)}} \exp\left\{-\frac{a^2}{2(1-\sigma^2)}\right\} \qquad & 0<\sigma<1 \\ &= 0 & \sigma>1 \end{aligned}\right\} \tag{17}$$

where, it is no matter whatsoever $\psi_2(\sigma)$ may be, only if $\psi_2(\sigma) \geqq 0$ and $\int_0^1 \psi_2(\sigma)d\sigma = 1$ consists. Or, more specially if we assume the rectangular distribution $\psi_2(\sigma)=1$ in $0<\sigma<1$, we obtain

$$\psi(a,\sigma) = \frac{1}{\sqrt{2\pi(1-\sigma^2)}} \exp\left\{-\frac{a^2}{2(1-\sigma^2)}\right\} \qquad (1>\sigma^2) \tag{18}$$

as a typical solution of the integral equation

$$\varphi(x,0,1) = \int_0^\infty \int_{-\infty}^\infty \varphi(x,a,\sigma)\psi(a,\sigma)\, da d\sigma. \tag{19}$$

The above proof is little pleasing. A more rigorous proof is postponed for a future work together with the following problem: Starting from $\psi(a,\sigma)$ that is not normal in a (even the singular normal distribution being exclusive) so that $f_2(x,\sigma)$ is non-normal, can the final distribution $f(x)$ be normal after all? If our conjecture be permitted, we surmise that this shall be impossible.

§ 2.

We shall show that the normality of $\psi(a,\sigma)$ in a does not necessitate the final normal distribution.

Ex. 1. Let

$$\psi(a,\sigma)=k\sigma^{-n}\exp\left\{-\frac{(a-b)^2+c^2}{2\sigma^2}\right\}, \tag{1}$$

where $c>0$, $n>2$ and $k=\sqrt{\frac{2}{\pi}}\left(\frac{c}{\sqrt{2}}\right)^{n-2}\Big/\Gamma\left(\frac{n}{2}-1\right)$. Performing integrations, the compound normal distribution becomes

$$f(x)=\int_0^\infty\int_{-\infty}^\infty \varphi(x,a,\sigma)\psi(a,\sigma)\,da\,d\sigma=\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}-1\right)}\frac{1}{\sqrt{2\pi}\,c}\left[1+\frac{(x-b)^2}{2c^2}\right]^{-\frac{n-1}{2}} \tag{2}$$

a Student-like distribution. Really, on taking $\alpha>0$, and writing

$$c^2=\frac{n-2}{2}\alpha^2, \qquad x=\xi\alpha, \qquad b=\beta\alpha,$$

we get

$$f(x)dx=f(\alpha\xi)\,\alpha\,d\xi=\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}-1\right)}\frac{1}{\sqrt{(n-2)\pi}}\left[1+\frac{(\xi-\beta)^2}{n-2}\right]^{-\frac{n-1}{2}}d\xi=s_{n-2}(\xi)d\xi \tag{3}$$

which is Student's distribution with $n-2$ degrees of freedom.

In particular, if $n=3$, $b=0$, $c^2=\frac{1}{2}$, so that

$$\psi(a,\sigma)=\frac{1}{\sqrt{2\pi\sigma^3}}\exp\left\{-\frac{1+2a^2}{4\sigma^2}\right\}, \tag{4}$$

then

$$f(x)=\frac{1}{\pi(1+x^2)} \qquad \text{(Cauchy's distribution)}. \tag{5}$$

Conversely, given $f(x)$, to find $\psi(a,\sigma)$, we ought to solve the integral equation

$$f(x)=\int_0^\infty\int_{-\infty}^\infty \varphi(x,a,\sigma)\psi(a,\sigma)\,da\,d\sigma \tag{6}$$

such that $\psi(a,\sigma)\geqq 0$, $\iint\psi(a,\sigma)\,da\,d\sigma=1$, the latter of which, however, follows naturally from the equation itself, as we integrate (6) in regard to x, assuming Fubini. But the above kind of integral equation with two parameters seems not yet to have been thoroughly treated and even the existence of the solution, its uniqueness and continuity &c. are not clear. For the present we shall assume all these affirmatively, except the uniqueness, for, evidently solution (1.17) shows that it contains a somewhat arbitrary function $\psi_2(\sigma)$. Hence to get a solution of (6) we are obliged to proceed after Gauss' method of numerical integration as follows:

At first transforming the variable as

$$a=\beta \tan\frac{\pi}{2}t,\quad \sigma=\gamma\tan\frac{\pi}{4}(1+u), \tag{7}$$

$$\psi(a,\sigma)=\psi\left(\beta \tan\frac{\pi}{2}t,\quad \gamma\tan\frac{\pi}{4}(1+u)\right)=Z_{\beta\gamma}(t,u)$$

with arbitrary β, γ, equation (6) can be written as

$$f(x)=\tfrac{1}{4}\int_{-1}^{1}\int_{-1}^{1}X_{\beta\gamma}(x,t,u)\ Y(t,u)\ Z_{\beta\gamma}(t,u)\ dtdu \tag{8}$$

where

$$X_{\beta\gamma}(x,t,u)=\frac{\beta}{\gamma}\exp\{-(x-\beta\tan\frac{\pi}{2}t)^2/2\gamma^2\tan^2\frac{\pi}{4}(1+u)\}, \tag{9}$$

$$Y(t,u)=\sqrt{\frac{\pi^3}{2}}\sec^2\frac{\pi}{2}t\sec^2\frac{\pi}{4}(1+u)/\tan\frac{\pi}{4}(1+u). \tag{10}$$

Then, by means of Gauss' method of selected ordinates we get

$$f(x)=\sum_{\mu=1}^{m}\sum_{\nu=1}^{n}R_\mu R_\nu y_{\mu\nu}, \tag{11}$$

where

$$y_{\mu\nu}=X_{\beta\gamma}(x,t_\mu,u_\nu)Y(t_\mu,u_\nu)Z_{\beta\gamma}(t_\mu,u_\nu), \tag{12}$$

where $X_{\beta\gamma}$ and Y are prescribed while $Z_{\beta\gamma}(t_\mu,u_\nu)$ to be found, the number of which being $mn=l$. Therefore, if we select $x=x_\lambda$ ($\lambda=1,2,\cdots\cdots,l$) appropriately, we have the following l equations :

$$f(x_\lambda)=\sum_{\mu=1}^{m}\sum_{\nu=1}^{n}R_\mu R_\nu X_{\beta\gamma}(x_\lambda,t_\mu,u_\nu)\ Y(t_\mu,u_\nu)\ Z_{\beta\gamma}(t_\mu,u_\nu). \tag{13}$$

Solving these simultaneous linear equations, the values of l unknown $Z_{\beta\gamma}(t_\mu,u_\nu)$ could be determined.

Making $\beta=\gamma=1$, we get the values of $Z(t,u)$ at l points (t_μ,u_ν) and consequently the values of $z=\psi(a,\sigma)$ at (a_μ,σ_ν) and thus the outline of the surface $z=\psi(a,\sigma)$ would be manifested. To amplify the plotting points any more we may make $\beta,\gamma=1,2,\cdots,\frac{1}{2}$, &c., combine them in various ways and the shape of the distribution surface could be acculated.

After the above plan, I. Wajiki executed numerical computations of *Ex.* 1, i.e. equation (6) with (5) taking $m=n=5$, the result of which, however, was very unpleasing: the calculated values are much more multiplied with theoretical ones.

However, the adoption of Gauss' method of selected ordinates for the case of double integral is by no means of no promise. Really

Ex. 2. Letting e.g. $\psi(a,\sigma)=ka/\sigma^3$ in $0<a<\sqrt{1-(\sigma-1)^2}$ and $1<\sigma<2$, but $\psi(a,\sigma)=0$ everywhere else, we obtain $k=\dfrac{2}{1-\log_e 2}=6.5177831$, and

$$f(x)=\int_1^2\int_0^{\sqrt{1-(\sigma-1)^2}}\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}\psi(a,\sigma)\,da\,d\sigma,$$

so that by exact integrations

$$f(0)=\int_1^2\int_0^{\sqrt{1-(\sigma-1)^2}}\frac{ka}{\sqrt{2\pi}\,\sigma^4}\exp\left\{-\frac{a^2}{2\sigma^2}\right\}da\,d\sigma=0.2770032,$$

while, on transforming $\sigma=\frac{1}{2}(3+u)$ and $a=\frac{1}{2}\sqrt{1-(\sigma-1)^2}(1+t)=\frac{1}{4}\sqrt{(3+u)(1-u)}\,(1+t)$, we get

$$f(0)=\frac{1}{4}\int_{-1}^1\int_{-1}^1 k\sqrt{\frac{2}{\pi}}\frac{(1-u)(1+t)}{(3+u)^3}\exp\left\{-\frac{(1-u)(1+t)^2}{8(3+u)}\right\}dt\,du$$

and whence

$$f(0)=\sum_{\mu=1}^{5}\sum_{\nu=1}^{5}R_\mu R_\nu\sqrt{\frac{2}{\pi}}\,k\frac{(1-u_\nu)(1+t_\mu)}{(3+u_\nu)^3}\exp\left\{-\frac{(1-u_\nu)(1+t_\mu)^2}{8(3+u_\nu)}\right\}=0.2770029.$$

Thus the theoretical value obtained by exact double integral coincides pretty good with the value calculated by Gauss' approximation.

§ 3.

Specially we consider the case that a and σ are independent, so that $\psi(a,\sigma)=\psi_1(a)\psi_2(\sigma)$. The integral equation now becomes

$$f(x)=\int_0^\infty\int_{-\infty}^\infty\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}\psi_1(a)\psi_2(\sigma)\,da\,d\sigma. \tag{1}$$

In this case, if one of unknown functions ψ_1, ψ_2 be presumed, the other could be therewith decided by solving the usual integral equation with one parameter.

1° If ψ_2 is presumed, and consequently

$$\int_0^\infty\exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}\frac{\psi_2(\sigma)}{\sqrt{2\pi}\sigma}\,d\sigma\equiv K(a,x), \tag{2}$$

which forms a symmetrical kernel, is made known, to find ψ_1 we have to solve the integral equation

$$f(x)=\int_{-\infty}^\infty K(a,x)\psi_1(a)\,da. \tag{3}$$

2° If ψ_1 is known and so also

$$\int_{-\infty}^\infty\exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}\frac{\psi_1(a)}{\sqrt{2\pi}\sigma}\,da\equiv H(\sigma,x), \tag{4}$$

then ψ_2 should be determined from the integral equation

$$f(x)=\int_0^\infty H(\sigma,x)\psi_2(\sigma)d\sigma. \tag{5}$$

Both (3) and (5) belong to Fredholm's equations of the first kind. So they may be somewhat theoretically discussed, although below only numerical computations are stressed.

Ex. 3. Let the frequency function of σ be a truncated one, and

$$\left.\begin{aligned}\psi_2(\sigma)&=0, &&(0<\sigma<1)\\ &=1/\sigma^2 &&(1<\sigma<\infty).\end{aligned}\right\} \tag{6}$$

Then we have by (2)

$$K(a,x)=\frac{1}{\sqrt{2\pi}(x-a)^2}[1-\exp\{-\tfrac{1}{2}(x-a)^2\}]. \tag{7}$$

On the other hand, making

$$\left.\begin{aligned}\psi_1(a)&=\tfrac{1}{2}, &&(-1<a<1)\\ &=0 &&(|a|>1),\end{aligned}\right\} \tag{8}$$

we get

$$f(x)=\frac{1}{2\sqrt{2\pi}(x^2-1)}[2-(x+1)\exp\{-\tfrac{1}{2}(x-1)^2\}+(x-1)\exp\{-\tfrac{1}{2}(x+1)^2\}]$$
$$+\tfrac{1}{2}[\Phi(x+1)-\Phi(x-1)], \tag{9}$$

where $\qquad \Phi(x)=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^x \exp\left(-\frac{t^2}{2}\right)dt.$

Conversely presumed (9) and (6), $\psi_1(a)$ is to be sought as the solution of equation (3), viz.

$$f(x)=\int_{-\infty}^\infty [1-\exp\{-\tfrac{1}{2}(x-a)^2\}]\frac{\psi_1(a)da}{\sqrt{2\pi}(x-a)^2}\equiv\int_{-\infty}^\infty K(a,x)\psi_1(a)da. \tag{10}$$

We assume that $\psi_1(a)$ is continuous, except at points $a=\pm 1$, because (9) behaves at $x=\pm 1$ singular though apparently. Gauss' method of numerical integration could be applied so far as the integrand is continuous throughout the considered interval, so that we should separate the whole integration interval as follows:

$$f(x)=\int_{-\infty}^\infty=\int_{-1}^1+\int_1^\infty+\int_{-\infty}^{-1}=\text{(i)}+\text{(ii)}+\text{(iii)}. \tag{11}$$

Firstly

$$\text{(i)}\ =\tfrac{1}{2}\int_{-1}^1 2K(a,x)\psi_1(a)da=\sum_{\kappa=1}^k R_\kappa 2K(a_\kappa,x)\psi_1(a_\kappa).$$

Secondly, putting $a=\sec\frac{\pi}{4}(1+t)$,

$$\text{(ii)} =\tfrac{1}{2}\int_{-1}^{1}\frac{\pi}{2}K\Big(\sec\frac{\pi}{4}(1+t),x\Big)\sec\frac{\pi}{4}(1+t)\tan\frac{\pi}{4}(1+t)\psi_1\Big(\sec\frac{\pi}{4}(1+t)\Big)dt$$

$$=\tfrac{1}{2}\int_{-1}^{1}L(t,x)\psi_1(t)dt=\sum_{\lambda=1}^{l}R_\lambda L(t_\lambda,x)\psi_1(t_\lambda).$$

Thirdly

$$\text{(iii)}=\tfrac{1}{2}\int_{-1}^{1}\frac{\pi}{2}K\Big(-\sec\frac{\pi}{4}(1+t),x\Big)\sec\frac{\pi}{4}(1+t)\tan\frac{\pi}{4}(1+t)\psi_1\Big(-\sec\frac{\pi}{4}(1+t)\Big)dt$$

$$=\tfrac{1}{2}\int_{-1}^{1}M(t,x)\psi_1(t)dt=\sum_{\mu=1}^{m}R_\mu M(t_\mu,x)\psi_1(t_\mu).$$

These three expressions being substituted in (11) and taking $x=x_\nu$ we obtain

$$f(x_\nu)=\sum_{\kappa=1}^{k}2R_\kappa K(t_\kappa,x_\nu)\psi_1(t_\kappa)+\sum_{\lambda=1}^{l}R_\lambda L(t_\lambda,x_\nu)\psi_1(t_\lambda)$$

$$+\sum_{\mu=1}^{m}R_\mu M(t_\mu,x_\nu)\psi_1(t_\mu),\quad \nu=1,2,\cdots,n(=k+l+m).$$

Solving these n linear equations simultaneously with respect to n unknowns ψ_1's, their roots yield the values of $U=\psi_1(a)$ at $a=a_\kappa$, a_λ, a_μ and throw light on its graph. After this scheme I. Wajiki taking $k=l=m=3$, obtained the following nine equations:

1) when $x=0$:

$$0.09574U_1+0.17731U_2+0.09574U_3+0.01236U_4+0.12450U_5$$
$$+0.17135U_6+0.01236U_7+0.12450U_8+0.17135U_9=0.18433;$$

2) when $x=0.1$:

$$0.09209U_1+0.17686U_2+0.09911U_3+0.01292U_4+0.13189U_5$$
$$+0.17755U_6+0.01177U_7+0.11720U_8+0.16547U_9=0.18398;$$

3) when $x=-0.1$:

$$0.09911U_1+0.1786U_2+0.09209U_3+0.01177U_4+0.11720U_5$$
$$+0.16547U_6+0.01292U_7+0.13189U_8+0.17755U_9=0.18398;$$

4) when $x=0.2$:

$$0.08822U_1+0.17555U_2+0.10215U_3+0.01346U_4+0.13933U_5$$
$$+0.18409U_6+0.001118U_7+0.11008U_8+0.15989U_9=0.18290;$$

5) when $x=-0.2$:

$$0.10215U_1+0.17555U_2+0.08822U_3+0.01118U_4+0.11008U_5$$
$$+0.15989U_6+0.01346U_7+0.13933U_8+0.18409U_9=0.18290;$$

6) when $x=0.3$:

$$0.08419U_1+0.17338U_2+0.10480U_3+0.01395U_4+0.14672U_5$$
$$+0.19100U_6+0.01058U_7+0.10320U_8+0.15459U_9=0.18115;$$

7) when $x=-0.3$:

$$0.10480U_1+0.17338U_2+0.08419U_3+0.01058U_4+0.10320U_5 +0.15459U_6+0.01395U_7+0.14672U_8+0.19100U_9=0.18115;$$

8) when $x=0.4$:

$$0.08006U_1+0.17040U_2+0.10702U_3+0.14409U_4+0.15397U_5 +0.19830U_6+0.00999U_7+0.9659U_8+0.14954U_9=0.17868;$$

9) when $x=-0.4$:

$$0.10702U_1+0.17040U_2+0.08006U_3+0.00999U_4+0.09659U_5 +0.14954U_6+0.14409U_7+0.15397U_8+0.19830U_9=0.17868.$$

Assuming

$$\psi_1(a_\kappa)=U_1=U_2=U_3=\tfrac{1}{2},\ \ \psi_1(a_\lambda)=U_4=U_5=U_6=0,\ \ \psi_1(a_\mu)=U_7=U_8=U_9=0,$$

and calculating the left handed sides, we obtain the following equations

1)	$x=0$:	$0.18439=0.18433$,	6)	$x=0.3$:	$0.18118=0.18115$,
2)	$x=0.1$:	$0.18403=0.18398$,	7)	$x=-03$:	$0.18118=0.18115$,
3)	$x=-0.1$:	$0.18403=0.18398$,	8)	$x=0.4$:	$0.17874=0.17868$,
4)	$x=0.2$:	$0.18296=0.18290$,	9)	$x=-0.4$:	$0.17874=0.17868$.
5)	$x=-0.2$:	$0.18296=0.18290$,			

Thus the figures on both the sides agree almost up to the fourth decimal place.

Ex. 4. If $\psi_1(a)=\dfrac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}a^2)$, then (4) becomes

$$H(\sigma,x)=\frac{1}{2\pi\sigma}\int_{-\infty}^{\infty}\exp\left\{-\frac{(x-a)^2}{2\sigma^2}-\frac{a^2}{2}\right\}da=\frac{1}{\sqrt{2\pi(1+\sigma^2)}}\exp\left\{-\frac{x^2}{2(1+\sigma^2)}\right\}. \tag{12}$$

Further, assuming

$$\psi_2(\sigma)=\sigma/(1+\sigma^2)^{\frac{3}{2}}, \tag{13}$$

we obtain

$$f(x)=\frac{1}{\sqrt{2\pi x^2}}\left\{1-\exp\left(-\frac{x^2}{2}\right)\right\}, \tag{14}$$

Thus equation (5) reduces to

$$\frac{1}{\sqrt{2\pi}\,x^2}\left\{1-\exp\left(-\frac{x^2}{2}\right)\right\}=\int_0^{\infty}\frac{1}{\sqrt{2\pi(1+\sigma^2)}}\exp\left\{-\frac{x^2}{2(1+\sigma^2)}\right\}\psi_2(\sigma)d\sigma, \tag{15}$$

the solution of which is nothing but expression (13).

The integral equation (15) can be solved theoretically on referring to Laplace transform[6]. Namely, on setting $2s=x^2$, $t=(1+\sigma^2)^{-1}$, the interval $0<\sigma<\infty$ is trans-

6) D. V. Widder, loc. cit., p. 66.

formed into $1>t>0$, and equation (15) to

$$\frac{1-e^{-s}}{s}=\int_0^1 e^{-st}\psi_2\left(\sqrt{\frac{1-t}{t}}\right)\frac{dt}{t\sqrt{1-t}}. \tag{16}$$

Hence, if we write

$$\mathfrak{f}(s)=(1-e^{-s})/s \tag{17}$$

and

$$\left.\begin{aligned} g(t)&=\psi_2\left(\sqrt{\frac{1-t}{t}}\right)\Big/t\sqrt{1-t} \quad &(0<t<1)\\ &=0, &(1<t<\infty)\end{aligned}\right\} \tag{18}$$

the above equation reduces to

$$\mathfrak{f}(s)=\int_0^\infty e^{-st}g(t)dt. \tag{19}$$

By a known inversion formula, we get for $c>0$

$$g(t)=\lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\mathfrak{f}(s)e^{st}ds=\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{1-e^{-c-i\eta}}{c+i\eta}e^{(c+i\eta)t}d\eta$$

$$=\frac{e^{ct}}{\pi}\int_0^\infty\frac{c\cos\eta t+\eta\sin\eta t-e^{-c}[c\cos\eta(t-1)+\eta\sin\eta(t-1)}{c^2+\eta^2}d\eta.$$

But

$$\int_0^\infty\frac{\gamma\cos\beta\eta}{\gamma^2+\eta^2}d\eta=\int_0^\infty\frac{\eta\sin\beta\eta}{\gamma^2+\eta^2}d\eta=\frac{\pi}{2}e^{-\beta\gamma} \qquad \text{for} \quad \beta>0,\ \gamma>0$$

hold, as easily shown by the theory of residues. These formulas being applied to the before standing integrals with caution about signs, we obtain the following result:

$$g(t)=1, \quad \text{if } 0<t<1, \quad \text{but otherwise} \quad g(t)=0.$$

Remembering that $g(t)=\psi_2\left(\sqrt{\frac{1-t}{t}}\right)\Big/t\sqrt{1-t}$ and $t=(1+\sigma^2)^{-1}$, we get

$$\psi_2\left(\sqrt{\frac{1-t}{t}}\right)=t\sqrt{1-t} \text{ in } 1>t>0, \quad \text{viz. } \psi_2(\sigma)=\sigma/(1+\sigma^2)^{\frac{3}{2}} \text{ in } 0<\sigma<\infty,$$

which agrees with (13).

In order to solve the same integral equation numerically, first transforming (15) by $\sigma=\tan\frac{\pi}{4}(1+t)$, we have

$$f(x)=\tfrac{1}{2}\int_{-1}^{1}\tfrac{1}{2}\sqrt{\frac{\pi}{2}}\exp\left\{-\frac{x^2}{2}\cos^2\frac{\pi}{4}(1+t)\right\}.\ \sec\frac{\pi}{4}(1+t).\ \psi_2\left(\tan\frac{\pi}{4}(1+t)\right)dt.$$

Setting further $\sec\frac{\pi}{4}(1+t)\psi_2(\tan\frac{\pi}{4}(1+t))=\chi(t)$, we have only to compute by aid of

Gauss' method of n ordinates $\frac{2}{\pi x^2}\left(1-e^{-x^2/2}\right)=\sum_{\nu=1}^{n} R_\nu \exp\{-\frac{x^2}{2}\cos\frac{\pi}{4}(1+t_\nu)\}\chi(t_\nu)$.
Letting e.g. $n=5$, and $x=0$, 0.5, 1, 1.5, 2, we have five equations involving five unknowns $\chi(t_\nu)$. Solving these simultaneous linear equations with respect to $\chi(t_\nu)$, T. Kawashiro calculated the values of $\psi_2(\sigma)$ as in the following table, the true values being those obtained from (13):

$\sigma=\tan\frac{\pi}{4}(1+t_\nu)$	0.0733	0.3792	1	2.6368	13.5465
cal. $\psi_2(\sigma)$	0.0717	0.3087	0.3560	0.1180	0.0057
true $\psi_2(\sigma)$	0.0732	0.3100	0.3536	0.1176	0.0054

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NOTES ON POWER SERIES IN ABSTRACT SPACES

By

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The analytic function in the abstract space developed remarkably in the recent time. Many theorems of analytic functions has been extended to the complex Banach spaces by Angus E. Taylor and others. We'll proceed here to a investigation on the some characteristics of a power series in the complex Banach spaces. Let E,E' be complex-B-spaces.

Theorem 1 *In an E'-valued function $f(x)=\sum_{n=0}^{\infty}h_n(x)$ defined on the complex-Banach-space E, if* $\lim_{n\to\infty}\frac{\sup_{\|x\|=1}\|h_n(x)\|}{\sup_{\|x\|=1}\|h_{n-1}(x)\|}=\frac{1}{\lambda}\cdots(1)$ *exists, λ is the radius of bound of $f(x)$.*

Proof We prove the case in $0<\frac{1}{\lambda}(=l<\infty)$. The case in $l=0,\infty$ is proved as well. We give

$$0<l-\varepsilon<\frac{\sup_{\|x\|=1}\|h_{n+1}(x)\|}{\sup_{\|x\|=1}\|h_n(x)\|}<l+\varepsilon,$$

for an arbitrary positive number ε and $n_0(\varepsilon)\leqq n$. Therefore

$$l-\varepsilon<\frac{\sup\|h_{n_0+1}(x)\|}{\sup\|h_{n_0}(x)\|}<l+\varepsilon$$

$$l-\varepsilon<\frac{\sup\|h_{n_0+2}(x)\|}{\sup\|h_{n_0+1}(x)\|}<l+\varepsilon$$

....

$$l-\varepsilon<\frac{\sup\|h_n(x)\|}{\sup\|h_{n-1}(x)\|}<l+\varepsilon$$

If we multiplicate respectively these inequality, we have

$$\sup_{\|x\|=1}\|h_{n_0}(x)\|(l-\varepsilon)^{n-n_0}<\sup_{\|x\|=1}\|h_n(x)\|<\sup_{\|x\|=1}\|h_{n_0}(x)\|(l+\varepsilon)^{n-n_0}$$

Since ε is arbitrary,

$$\overline{\lim_{n\to\infty}}\sqrt[n]{\operatorname{sub}_{\|x\|=1}\|h_n(x)\|}=l=\frac{1}{\lambda}\cdots\cdots(2)$$

Hence λ in (1) is the radius of bound. This completes the proof.

Theorem 2. *Put* $\overline{\lim_{n\to\infty}} \dfrac{\sup\limits_{\|x\|=1}\|h_n(x)\|}{\sup\limits_{\|x\|=1}\|h_{n-1}(x)\|} = \dfrac{1}{\sigma}$, *then* $f(x)$ *is bounded (in the large) in* $\|x\|<\sigma$.

Proof For an arbitrary positive number ε and $n_0(\varepsilon)\leqq n$ we have

$$\frac{\sup\limits_{\|x\|=1}\|h_n(x)\|}{\sup\limits_{\|x\|=1}\|h_{n-1}(x)\|}\leqq\frac{1}{\sigma-\varepsilon},$$

Put $y=\dfrac{x}{\|x\|}$ for any element x in $0<\|x\|<\sigma-2\varepsilon$, then the set of y is a set in the $\|x\|=1$. Thus it follows that

$$\frac{\sup\limits_{0<\|x\|<\sigma-2\varepsilon}\|h_n(x)\|}{\sup\limits_{0<\|x\|<\sigma-2\varepsilon}\|h_{n-1}(x)\|}=\frac{\sup\limits_{0<\|x\|<\sigma-2\varepsilon}\|h_n\ (\frac{x}{\|x\|})\|}{\sup\limits_{0<\|x\|<\sigma-2\varepsilon}\|h_{n-1}(\frac{x}{\|x\|})\|}\cdot\|x\|$$

$$\leqq\frac{\sup\limits_{\|x\|=1}\|h_n(x)\|}{\sup\limits_{\|x\|=1}\|h_{n-1}(x)\|}(\sigma-2\varepsilon)\leqq\frac{\sigma-2\varepsilon}{\sigma-\varepsilon}=\sigma'<1$$

Therefore, since $\dfrac{\sup\limits_{0<\|x\|<\sigma-2\varepsilon}\|h_n(x)\|}{\sup\limits_{0<\|x\|<\sigma-2\varepsilon}\|h_{n-1}(x)\|}\leqq\sigma'$ in the sphere $\|x\|<\sigma-2\varepsilon$, consequently,

$$\sup_{0<\|x\|<\sigma-2\varepsilon}\|h_{n_0+1}(x)\|\leqq\sigma'\sup_{0<\|x\|<\sigma-2\varepsilon}\|h_{n_0}(x)\|\leqq\sigma'\sup_{0<\|x\|<\sigma-2\varepsilon}M\|x\|^{n_0}, \text{ for a suitable } M.$$

$$\leqq\sigma\ M(\sigma-2\varepsilon)^{n_0}=\sigma' N, \text{ where } N=M(\sigma-2\varepsilon)^{n_0}.$$

Since $\|h_{n_0+1}(x)\|$ does not take its maximum at an inner point, $\sup\limits_{\|x\|<\sigma-2\varepsilon}\|h_{n_0+1}(x)\|\leqq\sigma' N$.

Similarly,

$$\sup_{\|x\|<\sigma-2\varepsilon}\|h_{n_0+2}(x)\|\leqq\sigma'\sup_{0<\|x\|\sigma-2\varepsilon}\|h_{n_0+1}(x)\|\leqq\sigma'^2N$$

....

$$\sup_{\|x\|<\sigma-2\varepsilon}\|h_{n_0+n}(x)\|\leqq\sigma'\sup_{0<\|x\|<\sigma-2\varepsilon}\|h_{n_0+n-1}(x)\|\leqq\sigma'^nN$$

Hence,

$$\|f(x)\|=\|\sum_{n=0}^{\infty}h_n(x)\|\leqq\sum_{n=0}^{\infty}\|h_n(x)\|\leqq\sum_{n=0}^{n_0-1}\sup_{\|x\|<\sigma-2\varepsilon}\|h_n(x)\|$$

$$+\sup_{\|x\|<\sigma-2\varepsilon}\sum^{\infty}\|h_n(x)\|<\sum_{n=0}^{n_0-1}M(\sigma-2\varepsilon)^n+N\frac{1}{1-\sigma'}<\infty$$

Since ε is arbitrary positive number, we see that $f(x)$ is bounded (in the large) in $\|x\|<\sigma$.

In this theorem, σ is not necessarily a radius of bound of $f(x)$, because the following example shows.

Now, let $p(>1)$ be an any finite posive integer, m be a positive integer and

$|a_1|<|a_2|<\cdots<|a_p|$ be complex numbers such that $|<|a_n|<k<\infty$, where k is a constant.

In the power series $f(x)=\sum_{n=0}^{\infty}h_n(x)(=\sum_{n=0}^{\infty}a_n x_n^n)$ which have the coefficients such that

$$a_n=\frac{k}{a_1^n a_2^n\cdots a_p^n} \qquad \textit{when } n=pm,$$

$$a_n=\frac{k}{a_1^n a_2^n\cdots a_p^{n+1}} \qquad \textit{when } n=pm+1,$$

$$a_n=\frac{k}{a_1^n a_2^n\cdots a_{p-2}^n a_{p-1}^{n+1} a_p^{n+1}} \quad \textit{when } n=pm+2,$$

....

$$a_n=\frac{k}{a_1^n\cdots a_{p-i}^n a_{p-i+1}^{n+1}\cdots a_p^{n+1}}, \quad \textit{when } n=pm+i\ (0\leqq i\leqq p-1),$$

$$a_n=\frac{k}{a_1^n a_2^{n+1}\cdots\cdots a_p^{p+1}}, \qquad \textit{when } n=p(m+1)-1$$

it follows that,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\left\|\frac{a_1^{pm}\, a_2^{pm}\cdots a_p^{pm}}{a_1^{pm+1} a_2^{pm+1}\cdots a_p^{pm+2}}\right\|=\frac{1}{|a_1 a_2\cdots a_p^2|}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \textit{when } n=pm,$$

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{a_1^{pm+1}\cdots\cdots a_{p-1}^{pm+1} a_p^{pm+2}}{a_1^{pm+2}\cdots\cdots a_{p-1}^{pm+3} a_p^{pm+3}}\right|=\frac{1}{|a_1 a_2\cdots a_{p-1}^2 a_p|} \quad \textit{when } n=pm+1,$$

....

$$\left|\frac{a_{n+1}}{a_n}\right|=\cdots\cdots=\frac{1}{|a_1\cdots a_{p-i-1} a_{p-i}^2 a_{p-i+1}\cdots a_p|} \quad \frac{\textit{when } n=pm+i}{(0\leqq i\leqq p-i)}$$

....

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{a^{pm+p-1} a_2^{pm+p}\cdots a_p^{pm+p}}{a_1^{pm+p} a_2^{pm+p}\cdots a_p^{pm+p}}\right|=\frac{1}{|a_1|}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \textit{when } p=p(m+1)-1$$

Hence, if we put $|a_n|=\sup_{\|x\|=1}\|h_n(x)\|$, then it follows that

$$\overline{\lim_{n\to\infty}}\frac{\sup_{\|x\|=1}\|h_n(x)\|}{\sup_{\|x\|=1}\|h_{n-1}(x)\|}=\frac{1}{|a_1|},\cdots\cdots(1),$$

where $|a_1|$ implies σ by the assumption.

On the other hand, since

$$\lim_{n\to\infty}\sqrt[n]{\sup_{\|x\|=1}\|h_n(x)\|}=\lim_{n\to\infty}\sqrt[pm]{k/|a_1^{pm}a_2^{pm}\cdots a_p^{pm}|}=\cdots\cdots\cdots\cdots$$

$$=\lim_{n\to\infty}\sqrt[pm+i]{k/|a_p^{pm+i}\cdots a_{p-i}^{pm+i} a_{pm-i+1}^{pm+i+1}\cdots a_p^{pm+i+1}|}=\cdots\cdots\cdots\cdots$$

$$\lim\sqrt[pm+p-1]{k/a_1{}^{pm+p-1} a_2^{pm+p}\cdots a_p^{p2n+p}|}=\frac{1}{|a_1 a_2\cdots a_p|},$$

then $\lim_{n\to\infty}\sqrt[n]{\sup_{\|x\|=1}\|h_n(x)\|}=\frac{1}{|a_1a_2\cdots a_p|}$ ......(2), where $|a_1a_2\cdots\cdots a_p|$ implies s of theorem 4. This shows to the fact that $\sigma\neq s$.

Theorem 3. *Put* $\overline{\lim_{n\to\infty}}\sup_{\|x\|=1}\frac{\|h_n(x)\|}{\|h_{n-1}(x)\|}=\frac{1}{\lambda'}$, *then $f(x)$ is bounded (in the large) in $\|x\|<\lambda'$.*

Proof. It is clear from $\sup_{\|x\|=1}\frac{\|h_n(x)\|}{\|h_{n-1}(x)\|}\geq\frac{\sup_{\|x\|=1}\|h_n(x)\|}{\sup_{\|x\|=1}\|h_{n-1}(x)\|}$.

λ' of Theorem 3 is not necessarily a radius of bound of $f(x)$ as the following example shows.

Let us consider the function $f(x)=\sum_{n=1}^{\infty}nx_n^n$ in the complex l_2-space æ such that æ $\ni{}^{\forall}x=(x_1,x_2,\cdots,x_n,\cdots)$ where $\sum_{n=1}^{\infty}|x_n|^2<\infty$. Then we can express the homogeneous polynomial of degree n such that $h_n(x)=nx_n^n$. Therefore, the redius of bound λ of a power series $f(x)$ is given by

$$\overline{\lim_{n\to\infty}}\sqrt[n]{\sup_{\|x\|=1}\|h_n(x)\|}=\overline{\lim_{n\to\infty}}\sqrt[n]{\sup_{\|x\|=1}n|x_n|^n}=\overline{\lim_{n\to\infty}}\sqrt[n]{n}=1,$$

since $\|x\|=1$ if $x=(0,0,\cdots,0,1,0,\cdots)$, where only n-th coordinate is 1 and others are zero.

However, $\sup_{\|x\|=1}\frac{\|h_n(x)\|}{\|h_{n-1}(x)\|}=\sup_{\|x\|=1}\frac{n|x_n|^n}{(n-1)|x_{n-1}|^{n-1}}=\infty$

Further $\overline{\lim_{n\to\infty}}(\sup_{\|x\|=1}\frac{\|h_n(x)\|}{\|h_{n-1}(x)\|})=\infty$. Hence $\lambda'=0$. This shows the fact that $\lambda\neq\lambda'$.

Corollary. *If* $\lim_{n\to\infty}\sup_{\|x\|=1}\frac{\|h_n(x)\|}{\|h_{n-1}(x)\|}=\frac{1}{\lambda''}$ *exists, then $f(x)$ is bounded (in the large) in* $\|x\|<\lambda''$.

Of course, λ'' is not necessarily a radius of bound of $f(x)$.

Theorem 4 *If* $\sup_{\|x\|=1}\lim_{n\to\infty}\frac{\|h_n(x)\|}{\|h_{n-1}(x)\|}=\frac{1}{s}$ *exists in a power series* $f(x)=\sum_{n=0}^{\infty}h_n(x)$, *then s is a radius of analyticity of $f(x)$.*

Proof By the assumption we have $\lim_{n\to\infty}\frac{\|h_n(x)\|}{\|h_{n-1}(x)\|}\leq\frac{1}{s}$ for an arbitrary point x on $\|x\|=1$, and also $\frac{\|h_n(x)\|}{\|h_{n-1}(x)\|}\leq\frac{1}{s-\varepsilon}$ for ${}^{\forall}\varepsilon>0$, $n_0(\varepsilon,x)\leq n$.

Consequently,

$$\frac{\|h_{n_0+1}(x)\|}{\|h_{n_0}(x)\|}\leq\frac{1}{s-\varepsilon},\ \frac{\|h_{n_0+2}(x)\|}{\|h_{n_0+1}(x)\|}\leq\frac{1}{s-\varepsilon},\ \cdots\cdots\ \frac{\|h_n(x)}{\|h_{n-1}(x)\|}\leq\frac{1}{s-\varepsilon},$$

If we multiplicate respectively these inequality, then we have

$$\|h_n(x)\|\leq\left(\frac{1}{s-\varepsilon}\right)^{n-n_0}\|h_{n_0}(x)\|.$$

Hence $\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(x)\|}\leqq\lim_{n\to\infty}\left(\frac{1}{s-\varepsilon}^{\frac{n-n_0}{n}}\sqrt[n]{\|h_{n_0}(x)\|}\right)=\frac{1}{s-\varepsilon}$, and since ε is arbitrary on $\|x\|=1$,

$$\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(x)\|}\leqq\frac{1}{s}\cdots\cdots(1)$$

If we take a suitable x such that $\frac{1}{s+\frac{\varepsilon}{2}}\leqq\lim_{n\to\infty}\frac{\|h_n(x)\|}{\|h_{n-1}(x)\|}$ is satisfied for a given positive number ε, then

$$\frac{1}{s+\varepsilon}\leqq\frac{\|h_n(x)\|}{\|h_{n-1}(x)\|}\quad\text{for } n\geqq n_0(\varepsilon,x).$$

Consequently, we have the following result as well, that is,

$$\left(\frac{1}{s+\varepsilon}\right)^{n-n_0}\|h_{n_0}(x)\|\leqq\|h_n(x)\|,$$

and also $\lim_{n\to\infty}\left(\frac{1}{s+\varepsilon}\right)^{\frac{n-n_0}{n}}\sqrt[n]{\|h_{n_0}(x)\|}\leqq\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(x)\|}$. Hence,

$$\frac{1}{s+\varepsilon}\leqq\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(x)\|}\leqq\sup_{\|x\|=1}\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(x)\|}.$$

Since ε is arbitrary $\frac{1}{s}\leqq\sup_{\|x\|=1}\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(x)\|}\cdots\cdots(2)$.

Therefore, s is a radius of analyticity of $f(x)$ from (1) and (2).

This completes the proof.

Theorem 5 *Put* $\sup_{\|x\|=1}\overline{\lim_{n\to\infty}}\frac{\|h_n(x)\|}{\|h_{n-1}(x)\|}=\frac{1}{\mu'}$, *then $f(x)$ is analytic in $\|x\|<\mu'$, which is not necessarily a radius of analyticity.*

Proof For an arbitrary element x in the set $\|x\|=1$ we have

$$\overline{\lim_{n\to\infty}}\frac{\|h_n(x)\|}{\|h_{n-1}(x)\|}\leqq\frac{1}{\mu'}$$

Then $\frac{\|h_n(x)\|}{\|h_{n-1}(x)\|}\leqq\frac{1}{\mu'-\varepsilon}$ for an arbitrary positive number $\varepsilon>0$ and $n_0(\varepsilon,x)\leqq n$.

Therefore,

$$\|h_{n_0+1}(x)\|\leqq\frac{1}{\mu'-\varepsilon}\|h_{n_0}(x)\|,$$

$$\|h_{n_0+2}(x)\|\leqq\frac{1}{\mu'-\varepsilon}\|h_{n_0+1}(x)\|,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\|h_n(x)\|\leqq\frac{1}{\mu'-\varepsilon}\|h_{n-1}(x)\|.$$

If we multiplicate respectively these inequality, then we have

$\|h_n(x)\|\leqq\left(\frac{1}{\mu'-\varepsilon}\right)^{h-n_0}\|h_{n_0}(x)\|$, and also $\sqrt[n]{\|h_n(x)\|}\leqq\frac{1}{\mu'-\varepsilon}^{\frac{n-n_0}{n}}\sqrt[n]{\|h_{n_0}(x)\|}$.

Thus if we take the limit, then we have $\overline{\lim_{n\to\infty}} \sqrt[n]{\|h_n(x)\|} \leq \lim_{n\to\infty}\left(\frac{1}{\mu'-\varepsilon}\right)^{\frac{n-n_0}{n}} \sqrt[n]{\|h_{n_0}(x)\|}$,

namely, $\overline{\lim_{n\to\infty}} \sqrt[n]{\|h_n(x)\|} \leq \frac{1}{\mu'-\varepsilon}$.

Therefore, $\sup_{\|x\|=1} \overline{\lim_{n\to\infty}} \sqrt[n]{\|h_n(x)\|} \leq \frac{1}{\mu'-\varepsilon}$. Since ε is arbitrary, $\sup_{\|x\|=1} \overline{\lim_{n\to\infty}} \sqrt[n]{\|h_n(x)\|} \leq \frac{1}{\mu'}$.

Hence, $f(x)$ is analytic in $\|x\|<\mu'$, which is not necessarily a radius of analytictiy.

Theorem 6 (*The extension of Tauber's theorem*)

Let the radius of analyticity of $f(x)=\sum_{n=0}^{\infty} h_n(x)$ *be s, x be a point on the boundary of the sphere of convergence, and O be center of the sphere of convergence. When* α *converges to* 1 *along the radius which join o and x,* $\lim_{\alpha\to 1} f(\alpha x)=A$ *exists, and also* $\sum_{n=0}^{\infty} h_n(x)$ *converges as* $n\|h_n(x)\|\to 0$. *Then, the sum* $\sum_{n=0}^{\infty} h_n(x)$ *equals to A.*

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