Journal of Gakugei, Tokushima University
Vol. VII. 1956.

NOTES ON POWER SERIES IN ABSTRACT SPACES

By

Takeshi WATANABE, Masaaki TERAI and Isae SHIMODA

(Received September 30, 1956)

The analytic function in the abstract space developed remarkably in the recent time. Many theorems of analytic functions has been extended to the complex Banach spaces by Angus E. Taylor and others. We'll proceed here to a investigation on the some characteristics of a power series in the complex Banach spaces. Let E,E' be complex-B-spaces.

Theorem 1 *In an E'-valued function $f(x)=\sum_{n=0}^{\infty}h_n(x)$ defined on the complex-Banach-space E, if $\lim_{n\to\infty}\frac{\sup_{\|x\|=1}\|h_n(x)\|}{\sup_{\|x\|=1}\|h_{n-1}(x)\|}=\frac{1}{\lambda}\cdots(1)$ exists, λ is the radius of bound of $f(x)$.*

Proof We prove the case in $0<\frac{1}{\lambda}(=l<\infty)$. The case in $l=0,\infty$ is proved as well. We give

$$0<l-\varepsilon<\frac{\sup_{\|x\|=1}\|h_{n+1}(x)\|}{\sup_{\|x\|=1}\|h_n(x)\|}<l+\varepsilon,$$

for an arbitrary positive number ε and $n_0(\varepsilon)\leqq n$. Therefore

$$l-\varepsilon<\frac{\sup\|h_{n_0+1}(x)\|}{\sup\|h_{n_0}(x)\|}<l+\varepsilon$$

$$l-\varepsilon<\frac{\sup\|h_{n_0+2}(x)\|}{\sup\|h_{n_0+1}(x)\|}<l+\varepsilon$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$l-\varepsilon<\frac{\sup\|h_n(x)\|}{\sup\|h_{n-1}(x)\|}<l+\varepsilon$$

If we multiplicate respectively these inequality, we have

$$\sup_{\|x\|=1}\|h_{n_0}(x)\|(l-\varepsilon)^{n-n_0}<\sup_{\|x\|=1}\|h_n(x)\|<\sup_{\|x\|=1}\|h_{n_0}(x)\|(l+\varepsilon)^{n-n_0}$$

Since ε is arbitrary,

$$\overline{\lim_{n\to\infty}}\sqrt[n]{\sup_{\|x\|=1}\|h_n(x)\|}=l=\frac{1}{\lambda}\cdots\cdots(2)$$

Hence λ in (1) is the radius of bound. This completes the proof.

Theorem 2. *Put* $\overline{\lim_{n\to\infty}} \dfrac{\sup\limits_{\|x\|=1}\|h_n(x)\|}{\sup\limits_{\|x\|=1}\|h_{n-1}(x)\|}=\dfrac{1}{\sigma}$, *then* $f(x)$ *is bounded (in the large) in* $\|x\|<\sigma$.

Proof For an arbitrary positive number ε and $n_0(\varepsilon)\leqq n$ we have

$$\frac{\sup\limits_{\|x\|=1}\|h_n(x)\|}{\sup\limits_{\|x\|=1}\|h_{n-1}(x)\|}\leqq\frac{1}{\sigma-\varepsilon},$$

Put $y=\dfrac{x}{\|x\|}$ for any element x in $0<\|x\|<\sigma-2\varepsilon$, then the set of y is a set in the $\|x\|=1$. Thus it follows that

$$\frac{\sup\limits_{0<\|x\|<\sigma-2\varepsilon}\|h_n(x)\|}{\sup\limits_{0<\|x\|<\sigma-2\varepsilon}\|h_{n-1}(x)\|}=\frac{\sup\limits_{0<\|x\|<\sigma-2\varepsilon}\|h_n\left(\frac{x}{\|x\|}\right)\|}{\sup\limits_{0<\|x\|<\sigma-2\varepsilon}\|h_{n-1}\left(\frac{x}{\|x\|}\right)\|}\cdot\|x\|$$

$$\leqq\frac{\sup\limits_{\|x\|=1}\|h_n(x)\|}{\sup\limits_{\|x\|=1}\|h_{n-1}(x)\|}(\sigma-2\varepsilon)\leqq\frac{\sigma-2\varepsilon}{\sigma-\varepsilon}=\sigma'<1$$

Therefore, since $\dfrac{\sup\limits_{0<\|x\|<\sigma-2\varepsilon}\|h_n(x)\|}{\sup\limits_{0<\|x\|<\sigma-2\varepsilon}\|h_{n-1}(x)\|}\leqq\sigma'$ in the sphere $\|x\|<\sigma-2\varepsilon$, consequently,

$$\sup_{0<\|x\|<\sigma-2\varepsilon}\|h_{n_0+1}(x)\|\leqq\sigma'\sup_{0<\|x\|<\sigma-2\varepsilon}\|h_{n_0}(x)\|\leqq\sigma'\sup_{0<\|x\|<\sigma-2\varepsilon}M\|x\|^{n_0},\text{ for a suitable } M.$$

$$\leqq\sigma\, M(\sigma-2\varepsilon)^{n_0}=\sigma' N,\text{ where } N=M(\sigma-2\varepsilon)^{n_0}.$$

Since $\|h_{n_0+1}(x)\|$ does not take its maximum at an inner point, $\sup\limits_{\|x\|<\sigma-2\varepsilon}\|h_{n_0+1}(x)\|\leqq\sigma' N$.

Similarly,

$$\sup_{\|x\|<\sigma-2\varepsilon}\|h_{n_0+2}(x)\|\leqq\sigma'\sup_{0<\|x\|\sigma-2\varepsilon}\|h_{n_0+1}(x)\|\leqq\sigma'^2 N$$

....

$$\sup_{\|x\|<\sigma-2\varepsilon}\|h_{n_0+n}(x)\|\leqq\sigma'\sup_{0<\|x\|<\sigma-2\varepsilon}\|h_{n_0+n-1}(x)\|\leqq\sigma'^n N$$

Hence,

$$\|f(x)\|=\|\sum_{n=0}^{\infty}h_n(x)\|\leqq\sum_{n=0}^{\infty}\|h_n(x)\|\leqq\sum_{n=0}^{n_0-1}\sup_{\|x\|<\sigma-2\varepsilon}\|h_n(x)\|$$

$$+\sup_{\|x\|<\sigma-2\varepsilon}\sum^{\infty}\|h_n(x)\|<\sum_{n=0}^{n_0-1}M(\sigma-2\varepsilon)^n+N\frac{1}{1-\sigma'}<\infty$$

Since ε is arbitrary positive number, we see that $f(x)$ is bounded (in the large) in $\|x\|<\sigma$.

In this theorem, σ is not necessarily a radius of bound of $f(x)$, because the following example shows.

Now, let $p(>1)$ be an any finite posive integer, m be a positive integer and

$|a_1|<|a_2|<\cdots<|a_p|$ be complex numbers such that $|<|a_n|<k<\infty$, where k is a constant.

In the power series $f(x)=\sum_{n=0}^{\infty}h_n(x)(=\sum_{n=0}^{\infty}a_n x_n^n)$ which have the coefficients such that

$$a_n=\frac{k}{a_1^n a_2^n\cdots a_p^n} \qquad \textit{when } n=pm,$$

$$a_n=\frac{k}{a_1^n a_2^n\cdots a_p^{n+1}} \qquad \textit{when } n=pm+1,$$

$$a_n=\frac{k}{a_1^n a_2^n\cdots a_{p-2}^n a_{p-1}^{n+1} a_p^{n+1}} \quad \textit{when } n=pm+2,$$

....

$$a_n=\frac{k}{a_1^n\cdots a_{p-i}^n a_{p-i+1}^{n+1}\cdots a_p^{n+1}}, \quad \textit{when } n=pm+i \ (0\leqq i\leqq p-1),$$

$$a_n=\frac{k}{a_1^n a_2^{n+1}\cdots\cdots a_p^{p+1}}, \qquad \textit{when } n=p(m+1)-1$$

it follows that,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\left\|\frac{a_1^{pm}\, a_2^{pm}\cdots a_p^{pm}}{a_1^{pm+1}a_2^{pm+1}\cdots a_p^{pm+2}}\right\|=\frac{1}{|a_1a_2\cdots a_p^2|}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \textit{when } n=pm,$$

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{a_1^{pm+1}\cdots\cdots a_{p-1}^{pm+1}\, a_p^{pm+2}}{a_1^{pm+2}\cdots\cdots a_{p-1}^{pm+3}\, a_p^{pm+3}}\right|=\frac{1}{|a_1a_2\cdots a_{p-1}^2 a_p|} \quad \textit{when } n=pm+1,$$

....

$$\left|\frac{a_{n+1}}{a_n}\right|=\cdots\cdots=\frac{1}{|a_1\cdots a_{p-i-1}a_{p-i}^2 a_{p-i+1}\cdots a_p|} \quad \frac{\textit{when } n=pm+i}{(0\leqq i\leqq p-i)}$$

....

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{a^{pm+p-1}a_2^{pm+p}\cdots a_p^{pm+p}}{a_1^{pm+p}a_2^{pm+p}\cdots a_p^{pm+p}}\right|=\frac{1}{|a_1|}=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \textit{when } p=p(m+1)-1$$

Hence, if we put $|a_n|=\sup_{\|x\|=1}\|h_n(x)\|$, then it follows that

$$\overline{\lim_{n\to\infty}}\frac{\sup_{\|x\|=1}\|h_n(x)\|}{\sup_{\|x\|=1}\|h_{n-1}(x)\|}=\frac{1}{|a_1|},\cdots\cdots(1),$$

where $|a_1|$ implies σ by the assumption.

On the other hand, since

$$\lim_{n\to\infty}\sqrt[n]{\sup_{\|x\|=1}\|h_n(x)\|}=\lim_{n\to\infty}\sqrt[pm]{k/|a_1^{pm}a_2^{pm}\cdots a_p^{pm}|}=\cdots\cdots\cdots\cdots$$

$$=\lim_{n\to\infty}\sqrt[pm+i]{k/|a_p^{pm+i}\cdots a_{p-i}^{pm+i}\, a_{pm-i+1}^{pm+i+1}\cdots a_p^{pm+i+1}|}=\cdots\cdots\cdots\cdots$$

$$\lim\sqrt[pm+p-1]{k/a_1{}^{pm+p-1}a_2^{pm+p}\cdots a_p^{p2n+p}|}=\frac{1}{|a_1a_2\cdots a_p|},$$

then $\lim_{n\to\infty} \sqrt[n]{\sup_{\|x\|=1}\|h_n(x)\|} = \frac{1}{|a_1 a_2 \cdots a_p|}$ ......(2), where $|a_1 a_2 \cdots\cdots a_p|$ implies s of theorem 4. This shows to the fact that $\sigma \neq s$.

Theorem 3. *Put* $\overline{\lim_{n\to\infty}} \sup_{\|x\|=1} \frac{\|h_n(x)\|}{\|h_{n-1}(x)\|} = \frac{1}{\lambda'}$, *then* $f(x)$ *is bounded (in the large) in* $\|x\| < \lambda'$.

Proof. It is clear from $\sup_{\|x\|=1} \frac{\|h_n(x)\|}{\|h_{n-1}(x)\|} \geq \frac{\sup_{\|x\|=1}\|h_n(x)\|}{\sup_{\|x\|=1}\|h_{n-1}(x)\|}$.

λ' of Theorem 3 is not necessarily a radius of bound of $f(x)$ as the following example shows.

Let us consider the function $f(x) = \sum_{n=1}^{\infty} n x_n^n$ in the complex l_2-space æ such that æ $\ni {}^{\forall}x = (x_1, x_2, \cdots, x_n, \cdots)$ where $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. Then we can express the homogeneous polynomial of degree n such that $h_n(x) = n x_n^n$. Therefore, the redius of bound λ of a power series $f(x)$ is given by

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{\|x\|=1}\|h_n(x)\|} = \overline{\lim_{n\to\infty}} \sqrt[n]{\sup_{\|x\|=1} n|x_n|^n} = \overline{\lim_{n\to\infty}} \sqrt[n]{n} = 1,$$

since $\|x\| = 1$ if $x = (0,0,\cdots,0,1,0,\cdots)$, where only n-th coordinate is 1 and others are zero.

However, $\sup_{\|x\|=1} \frac{\|h_n(x)\|}{\|h_{n-1}(x)\|} = \sup_{\|x\|=1} \frac{n|x_n|^n}{(n-1)|x_{n-1}|^{n-1}} = \infty$

Further $\overline{\lim_{n\to\infty}} (\sup_{\|x\|=1} \frac{\|h_n(x)\|}{\|h_{n-1}(x)\|}) = \infty$. Hence $\lambda' = 0$. This shows the fact that $\lambda \neq \lambda'$.

Corollary. *If* $\lim_{n\to\infty} \sup_{\|x\|=1} \frac{\|h_n(x)\|}{\|h_{n-1}(x)\|} = \frac{1}{\lambda''}$ *exists, then* $f(x)$ *is bounded (in the large) in* $\|x\| < \lambda''$.

Of course, λ'' is not necessarily a radius of bound of $f(x)$.

Theorem 4 *If* $\sup_{\|x\|=1} \lim_{n\to\infty} \frac{\|h_n(x)\|}{\|h_{n-1}(x)\|} = \frac{1}{s}$ *exists in a power series* $f(x) = \sum_{n=0}^{\infty} h_n(x)$, *then* s *is a radius of analyticity of* $f(x)$.

Proof By the assumption we have $\lim_{n\to\infty} \frac{\|h_n(x)\|}{\|h_{n-1}(x)\|} \leq \frac{1}{s}$ for an arbitrary point x on $\|x\| = 1$, and also $\frac{\|h_n(x)\|}{\|h_{n-1}(x)\|} \leq \frac{1}{s-\varepsilon}$ for ${}^{\forall}\varepsilon > 0$, $n_0(\varepsilon, x) \leq n$.

Consequently,

$$\frac{\|h_{n_0+1}(x)\|}{\|h_{n_0}(x)\|} \leq \frac{1}{s-\varepsilon}, \quad \frac{\|h_{n_0+2}(x)\|}{\|h_{n_0+1}(x)\|} \leq \frac{1}{s-\varepsilon}, \quad \cdots\cdots \quad \frac{\|h_n(x)}{\|h_{n-1}(x)\|} \leq \frac{1}{s-\varepsilon},$$

If we multiplicate respectively these inequality, then we have

$$\|h_n(x)\| \leq \left(\frac{1}{s-\varepsilon}\right)^{n-n_0} \|h_{n_0}(x)\|.$$

Hence $\overline{\lim_{n\to\infty}} \sqrt[n]{\|h_n(x)\|} \leq \lim_{n\to\infty}\left(\frac{1}{s-\varepsilon}^{\frac{n-n_0}{n}} \sqrt[n]{\|h_{n_0}(x)\|}\right)=\frac{1}{s-\varepsilon}$, and since ε is arbitrary on $\|x\|=1$,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\|h_n(x)\|} \leq \frac{1}{s} \cdots\cdots(1)$$

If we take a suitable x such that $\frac{1}{s+\frac{\varepsilon}{2}} \leq \lim_{n\to\infty} \frac{\|h_n(x)\|}{\|h_{n-1}(x)\|}$ is satisfied for a given positive number ε, then

$$\frac{1}{s+\varepsilon} \leq \frac{\|h_n(x)\|}{\|h_{n-1}(x)\|} \quad \text{for } n \geqq n_0(\varepsilon, x).$$

Consequently, we have the following result as well, that is,

$$\left(\frac{1}{s+\varepsilon}\right)^{n-n_0} \|h_{n_0}(x)\| \leq \|h_n(x)\|,$$

and also $\lim_{n\to\infty}\left(\frac{1}{s+\varepsilon}\right)^{\frac{n-n_0}{n}} \sqrt[n]{\|h_{n_0}(x)\|} \leq \overline{\lim_{n\to\infty}} \sqrt[n]{\|h_n(x)\|}$. Hence,

$$\frac{1}{s+\varepsilon} \leq \overline{\lim_{n\to\infty}} \sqrt[n]{\|h_n(x)\|} \leq \sup_{\|x\|=1} \overline{\lim_{n\to\infty}} \sqrt[n]{\|h_n(x)\|}.$$

Since ε is arbitrary $\frac{1}{s} \leq \sup_{\|x\|=1} \overline{\lim_{n\to\infty}} \sqrt[n]{\|h_n(x)\|} \cdots\cdots(2)$.

Therefore, s is a radius of analyticity of $f(x)$ from (1) and (2).

This completes the proof.

Theorem 5 *Put* $\sup_{\|x\|=1} \overline{\lim_{n\to\infty}} \frac{\|h_n(x)\|}{\|h_{n-1}(x)\|} = \frac{1}{\mu'}$, *then $f(x)$ is analytic in $\|x\|<\mu'$, which is not necessarily a radius of analyticity.*

Proof For an arbitrary element x in the set $\|x\|=1$ we have

$$\overline{\lim_{n\to\infty}} \frac{\|h_n(x)\|}{\|h_{n-1}(x)\|} \leq \frac{1}{\mu'}$$

Then $\frac{\|h_n(x)\|}{\|h_{n-1}(x)\|} \leq \frac{1}{\mu'-\varepsilon}$ for an arbitrary positive number $\varepsilon>0$ and $n_0(\varepsilon,x) \leq n$.

Therefore,

$$\|h_{n_0+1}(x)\| \leq \frac{1}{\mu'-\varepsilon} \|h_{n_0}(x)\|,$$

$$\|h_{n_0+2}(x)\| \leq \frac{1}{\mu'-\varepsilon} \|h_{n_0+1}(x)\|,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\|h_n(x)\| \leq \frac{1}{\mu'-\varepsilon} \|h_{n-1}(x)\|.$$

If we multiplicate respectively these inequality, then we have

$\|h_n(x)\| \leq \left(\frac{1}{\mu'-\varepsilon}\right)^{h-n_0} \|h_{n_0}(x)\|$, and also $\sqrt[n]{\|h_n(x)\|} \leq \frac{1}{\mu'-\varepsilon}^{\frac{n-n_0}{n}} \sqrt[n]{\|h_{n_0}(x)\|}$.

Thus if we take the limit, then we have $\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(x)\|}\leq\lim_{n\to\infty}\left(\frac{1}{\mu'-\varepsilon}\right)^{\frac{n-n_0}{n}}\sqrt[n]{\|h_{n_0}(x)\|}$,

namely, $\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(x)\|}\leq\frac{1}{\mu'-\varepsilon}$.

Therefore, $\sup_{\|x\|=1}\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(x)\|}\leq\frac{1}{\mu'-\varepsilon}$. Since ε is arbitrary, $\sup_{\|x\|=1}\overline{\lim_{n\to\infty}}\sqrt[n]{\|h_n(x)\|}\leq\frac{1}{\mu'}$.
Hence, $f(x)$ is analytic in $\|x\|<\mu'$, which is not necessarily a radius of analytictiy.

Theorem 6 (*The extension of Tauber's theorem*)

Let the radius of analyticity of $f(x)=\sum_{n=0}^{\infty}h_n(x)$ *be s, x be a point on the boundary of the sphere of convergence, and O be center of the sphere of convergence. When* α *converges to* 1 *along the radius which join o and x,* $\lim_{\alpha\to 1}f(\alpha x)=A$ *exists, and also* $\sum_{n=0}^{\infty}h_n(x)$ *converges as* $n\|h_n(x)\|\to 0$. *Then, the sum* $\sum_{n=0}^{\infty}h_n(x)$ *equals to A.*

References

Isae Shimoda : On power series in abstract spaces, Mathematica Japonicae Vol. 1, No. 2.