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ON THE COMPOUND NORMAL DISTRIBUTIONS

By

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While the so-called compound Poisson's distributions are frequently spoken of [1], there is no such with the normal distributions to our poor knowledge. Namely, if a random variable x has a probability density $\varphi(x;\theta_1,\theta_2,\cdots)$, and the parameters θ_i being again random variables distribute with the frequency function $\psi(\theta_1,\theta_2,\cdots)$, the compound φ-distribution is defined by

$$f(x)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\varphi(x;\theta_1,\theta_2,\cdots)\psi(\theta_1,\theta_2,\cdots)d\theta_1\,d\theta_2\cdots, \tag{1}$$

with

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\psi(\theta_1,\theta_2,\cdots)d\theta_1\,d\theta_2\cdots=1.$$

In particular the compound normal distribution with mean a and variance σ^2 is

$$f(x)=\int_0^{\infty}\int_{-\infty}^{\infty}\varphi(x;a,\sigma)\psi(a,\sigma)\,da\,d\sigma, \tag{2}$$

where

$$\varphi(x;a,\sigma)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}\quad\text{and}\quad\int_0^{\infty}\int_{-\infty}^{\infty}\psi(a,\sigma)\,da\,d\sigma=1.$$

We shall discuss the latter somewhat in detail. When ψ is known f is obtainable merely by integration, while, if f is given, ψ should be found by solving (2) as an integral equation. Thereby theoretically Laplace transform and practically Gauss' method of numerical integration by selected ordinates might be efficiently utilized.

§ 1.

All integrands in (1) and (2) being assumed to be positive and integrable, the order of integrations can be changed, and

$$\left.\begin{aligned}f(x)&=\int_0^{\infty}\left[\int_{-\infty}^{\infty}\varphi(x,a,\sigma)\psi(a,\sigma)\,da\right]d\sigma\equiv\int_0^{\infty}\varphi_2(x,\sigma)d\sigma\\&=\int_{-\infty}^{\infty}\left[\int_0^{\infty}\varphi(x,a,\sigma)\psi(a,\sigma)\,d\sigma\right]da\equiv\int_{-\infty}^{\infty}\varphi_1(x,a)da.\end{aligned}\right\} \tag{3}$$

1) E. g. W. Feller, Probability Theory and its Applications, 1952, p. 221.

Or, if we set

$$\left.\begin{aligned} f_1(x,a)&=\frac{\varphi_1(x,a)}{\psi_1(a)}, & f_2(x,\sigma)&=\frac{\varphi_2(x,\sigma)}{\psi_2(\sigma)}, \\ \text{with}\quad \psi_1(a)&=\int_0^\infty \psi(a,\sigma)d\sigma, & \psi_2(\sigma)&=\int_{-\infty}^\infty \psi(a,\sigma)da, \end{aligned}\right\} \tag{4}$$

both f_1 and f_2 are also compound normal frequency functions, although they might get out of normality in form, and

$$f(x)=\int_0^\infty f_2(x,\sigma)d\Psi_2(\sigma)=\int_{-\infty}^\infty f_1(x,a)d\Psi_1(a), \tag{5}$$

where $\Psi_1(a)$ and $\Psi_2(\sigma)$ stand for cumulative distribution functions of a and σ respectively, and $d\Psi_1(a)=\psi_1(a)da$, $d\Psi_2(\sigma)=\psi_2(\sigma)d\sigma$.

In particular, if a and σ be independent of each other

$$\psi(a,\sigma)=\psi_1(a)\psi_2(\sigma), \tag{6}$$

and

$$f_1(x,a)=\int_0^\infty \varphi(x,a,\sigma)d\Psi_2(\sigma), \quad f_2(x,\sigma)=\int_{-\infty}^\infty \varphi(x,a,\sigma)d\Psi_1(a), \tag{7}$$

yet (5) still hold.

Theorem 1. *$f_2(x,\sigma)$ is normal in x when and only when $\psi(a,\sigma)$ is normal in a.*

For, let

$$\psi(a,\sigma)=\frac{\psi_2(\sigma)}{\sqrt{2\pi}\,\tau}\exp\left\{-\frac{(a-m)^2}{2\tau^2}\right\}, \tag{8}$$

where m and τ are constant if a and σ independent, otherwise both or one of them shall be variable as functions of σ[2]. On account of (4) and (3) we have

$$\begin{aligned} f_2(x,\sigma)&=\int_{-\infty}^\infty \varphi(x,a,\sigma)\psi(a,\sigma)da/\psi_2(\sigma) \\ &=\frac{1}{2\pi\sigma\tau}\int_{-\infty}^\infty \exp\left\{-\frac{(x-a)^2}{2\sigma^2}-\frac{(a-m)^2}{2\tau^2}\right\}da \\ &=\frac{1}{\sqrt{2\pi(\sigma^2+\tau^2)}}\exp\left\{-\frac{(x-m)^2}{2(\sigma^2+\tau^2)}\right\}, \end{aligned} \tag{9}$$

which shows that $f_2(x,\sigma)$ is $N(x, m, \sqrt{\sigma^2+\tau^2})$.

To prove the converse, we have to solve the integral equation

$$\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}\frac{\psi(a,\sigma)}{\psi_2(\sigma)}da=\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}, \tag{10}$$

2) We have assumed that $\psi(a,\sigma)$ is normal in a, which means that m and τ in (8) do not contain a.

in which the right hand side is written in the standardized form instead of the last expression in (9), because σ, τ and m can be temporarily as constants considered. As a matter of fact, in consequence of scattering of a in (10), the dispersion of the resultant distribution should be greater than before integration, and therefore $\sigma^2 < 1$.

Now putting $x = \sigma s$, $a = -\sigma t$ equation (10) reduces to

$$\int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}(s+t)^2\} \frac{\psi(-\sigma t, \sigma)}{\psi_2(\sigma)} dt = \exp\{-\tfrac{1}{2}\sigma^2 s^2\},$$

viz.

$$\int_{-\infty}^{\infty} e^{-st} g(t)\, dt = \mathfrak{f}(s), \tag{11}$$

where

$$g(t) = \exp\left(-\frac{t^2}{2}\right) \cdot \psi(-\sigma t, \sigma)/\psi_2(\sigma) \qquad \text{and} \qquad \mathfrak{f}(s) = \exp\{\tfrac{1}{2}(1-\sigma^2)s^2\}.$$

This integral equation presents a Laplace transform, and a known inversion formula is capable to be applied[3]. Assuming that (11) converges absolutely on the line $\Re\, s = c$ in the complex s-plane, the inversion formula enunciates

$$g(t) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \mathfrak{f}(s) e^{st} ds \qquad (s = c + i\eta)$$

On calculating this limiting value, we obtain

$$g(t) = \frac{1}{\pi} \exp\{ct + \tfrac{1}{2}(1-\sigma^2)c^2\} \int_0^{\infty} \exp\{-A\eta^2\} \cos B\eta\, d\eta,$$

where $A = \frac{1}{2}(1-\sigma^2) > 0$, $B = t + c(1-\sigma^2)$. By use of a known formula

$$\int_0^{\infty} e^{-\alpha\xi^2} \cos\beta\xi\, d\xi = \tfrac{1}{2}\sqrt{\frac{\pi}{\alpha}} \exp\left\{-\frac{\beta^2}{4\alpha}\right\}, \quad (\alpha > 0)$$

we have

$$g(t) = \frac{1}{\sqrt{2\pi(1-\sigma^2)}} \exp\left\{-\frac{t^2}{2(1-\sigma^2)}\right\} \quad (1 > \sigma^2).$$

Remembering that $t = -a/\sigma$ and $\exp\left(\frac{t^2}{2}\right) \cdot g(t) = \psi(a, \sigma)/\psi_2(\sigma)$, we attain finally

$$\psi(a, \sigma) = \frac{\psi_2(\sigma)}{\sqrt{2\pi(1-\sigma^2)}} \exp\left\{-\frac{a^2}{2(1-\sigma^2)}\right\}, \tag{12}$$

which completes the proof[4].

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3) Cf. D. V. Widder, The Laplace Transform, 1946, p. 241.

4) Since $\psi(a, \sigma)$ is to be real positive, $\psi_2(\sigma)$ in (12) shall be zero for $1 < \sigma < \infty$. Also for $\sigma = 1-0$, $\psi(a, \sigma)$ presents an indeterminate form, but then on interpretting the main factor as a singular normal distribution, $\psi(a, \sigma)\, da\, d\sigma$ tends to $\psi_2(\sigma) d\sigma$, which becomes in general an infinitesimal, unless $\psi_2(\sigma)$ has there a finite jump δ, so that $\int_{1-0}^{1} \psi(a, \sigma) da\, d\sigma = \delta$.

Corollary. In case a and σ are independent, $f_2(x, \sigma)$ is normal if and only if $\psi_1(a)$ is normal[5].

Theorem 2. *$\psi(a,\sigma)$ or $\psi_1(a)$ being normal in a, the final distribution $f(x)$ in general becomes non-normal, and rather specially it offers a single normal distribution.*

By theorem 1 the frequency function $f_2(x, \sigma)$ becomes normal, but

$$f(x)=\int_0^\infty f_2(x,\sigma)d\Psi_2(\sigma)=\int_0^\infty \frac{1}{\sqrt{2\pi(\sigma^2+\tau^2)}}\exp\left\{-\frac{(x-m)^2}{2(\sigma^2+\tau^2)}\right\}d\Psi_2(\sigma) \qquad (13)$$

is simply a superposition of normal distributions. Specially, if it happens that only on a discrete set S, $m=m_0$, $\sigma^2+\tau^2=\sigma_0^2$ and $\int_S d\Psi_2(\sigma)=1$ hold, so that $d\Psi_2(\sigma)=0$ on the complementary continuous set S', then $f(x)$ reduces to $\varphi(x, m_0, \sigma_0)$. Here, of course, the integral should be understood as Stieltjes' one. However, to discuss the case that $\psi_2(\sigma)$ is of continuous type, we should consult with the equation

$$\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}=\int_0^\infty \frac{1}{\sqrt{2\pi(\sigma^2+\tau^2)}}\exp\left\{-\frac{(x-m)^2}{2(\sigma^2+\tau^2)}\right\}d\Psi_2(\sigma), \qquad (14)$$

where m and τ are some functions of σ and $\Psi_2(\sigma)$ represents the cumlative distribution function. Or, expressing (14) in form of characteristics

$$\exp\{-t^2/2\}=\int_0^\infty \exp\{imt-(\sigma^2+\tau^2)t^2/2\}d\Psi_2(\sigma),$$

viz.

$$1=\int_0^\infty \exp\{imt-\tfrac{1}{2}(\sigma^2+\tau^2-1)t^2\}\, d\Psi_2(\sigma). \qquad (15)$$

Hence

$$\int_0^\infty \exp\{-\tfrac{1}{2}(\sigma^2+\tau^2-1)t^2\}\,(\cos mt+i\sin mt)\, d\Psi_2(\sigma)=1. \qquad (16)$$

The imaginary part's appearance being only superficial, we may write

$$\int_0^\infty \exp\{-\tfrac{1}{2}(\sigma^2+\tau^2-1)t^2\}\,(\cos mt+\sin mt)\, d\Psi_2(\sigma)=1.$$

By virtue of the first mean value theorem

$$(\cos m_\theta t+\sin m_\theta t)\int_0^\infty \exp\{-\tfrac{1}{2}(\sigma^2+\tau^2-1)t^2\}\, d\Psi_2(\sigma)=1$$

and

$$\cos m_\theta t\int_0^\infty \exp\{-\tfrac{1}{2}(\sigma^2+\tau^2-1)t^2\}\, d\Psi_2(\sigma)=1,$$

$$\sin m_\theta t\int_0^\infty \exp\{-\tfrac{1}{2}(\sigma^2+\tau^2-1)t^2\}\, d\Psi_2(\sigma)=0,$$

5) This does not mean that $f(x)$ becomes normal: Compare e.g. Ex. 4 in §3.

in view of (16). Hence we have $m_0 t = 2n\pi$ and

$$\int_0^\infty \exp\{-\tfrac{1}{2}(\sigma^2+\tau^2-1)t^2\}\, d\Psi_2(\sigma) = 1.$$

But, if $\sigma^2+\tau^2 \neq 1$ on a continuous subset S with non-zero measure, the order of magnitude of this integral could be altered from 1 by taking t^2 sufficiently large. Hence it must hold that $\sigma^2+\tau^2=1$ on the whole continuous integration interval. Further, now that $\sigma^2+\tau^2-1=0$, the real part of (16) becomes

$$1 = \int_0^\infty \cos mt\, d\Psi_2(\sigma) = \int_0^\infty \left(1 - \frac{m^2t^2}{2} + \cdots\right) d\Psi_2(\sigma)$$

for all values of t, so that

$$1 = \int_0^\infty d\Psi_2(\sigma), \qquad 0 = \int_0^\infty m^2 d\Psi_2(\sigma), \qquad \cdots\cdots$$

Hence $m=0$ throughout and $\tau^2 = 1-\sigma^2$. Thus we obtain from (8)

$$\left.\begin{aligned} \psi(a,\sigma) &= \frac{\psi_2(\sigma)}{\sqrt{2\pi(1-\sigma^2)}} \exp\left\{-\frac{a^2}{2(1-\sigma^2)}\right\} & 0<\sigma<1 \\ &= 0 & \sigma>1 \end{aligned}\right\} \tag{17}$$

where, it is no matter whatsoever $\psi_2(\sigma)$ may be, only if $\psi_2(\sigma) \geqq 0$ and $\int_0^1 \psi_2(\sigma)d\sigma = 1$ consists. Or, more specially if we assume the rectangular distribution $\psi_2(\sigma)=1$ in $0<\sigma<1$, we obtain

$$\psi(a,\sigma) = \frac{1}{\sqrt{2\pi(1-\sigma^2)}} \exp\left\{-\frac{a^2}{2(1-\sigma^2)}\right\} \qquad (1>\sigma^2) \tag{18}$$

as a typical solution of the integral equation

$$\varphi(x,0,1) = \int_0^\infty \int_{-\infty}^\infty \varphi(x,a,\sigma)\psi(a,\sigma)\, da d\sigma. \tag{19}$$

The above proof is little pleasing. A more rigorous proof is postponed for a future work together with the following problem: Starting from $\psi(a,\sigma)$ that is not normal in a (even the singular normal distribution being exclusive) so that $f_2(x,\sigma)$ is non-normal, can the final distribution $f(x)$ be normal after all? If our conjecture be permitted, we surmise that this shall be impossible.

§ 2.

We shall show that the normality of $\psi(a,\sigma)$ in a does not necessitate the final normal distribution.

Ex. 1. Let

$$\psi(a,\sigma)=k\sigma^{-n}\exp\left\{-\frac{(a-b)^2+c^2}{2\sigma^2}\right\}, \tag{1}$$

where $c>0$, $n>2$ and $k=\sqrt{\frac{2}{\pi}}\left(\frac{c}{\sqrt{2}}\right)^{n-2}\Big/\Gamma\left(\frac{n}{2}-1\right)$. Performing integrations, the compound normal distribution becomes

$$f(x)=\int_0^\infty\int_{-\infty}^\infty \varphi(x,a,\sigma)\psi(a,\sigma)\,da\,d\sigma=\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}-1\right)}\frac{1}{\sqrt{2\pi}\,c}\left[1+\frac{(x-b)^2}{2c^2}\right]^{-\frac{n-1}{2}} \tag{2}$$

a Student-like distribution. Really, on taking $\alpha>0$, and writing

$$c^2=\frac{n-2}{2}\alpha^2, \qquad x=\xi\alpha, \qquad b=\beta\alpha,$$

we get

$$f(x)dx=f(\alpha\xi)\,\alpha\,d\xi=\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}-1\right)}\frac{1}{\sqrt{(n-2)\pi}}\left[1+\frac{(\xi-\beta)^2}{n-2}\right]^{-\frac{n-1}{2}}d\xi=s_{n-2}(\xi)d\xi \tag{3}$$

which is Student's distribution with $n-2$ degrees of freedom.

In particular, if $n=3$, $b=0$, $c^2=\frac{1}{2}$, so that

$$\psi(a,\sigma)=\frac{1}{\sqrt{2\pi\sigma^3}}\exp\left\{-\frac{1+2a^2}{4\sigma^2}\right\}, \tag{4}$$

then

$$f(x)=\frac{1}{\pi(1+x^2)} \qquad \text{(Cauchy's distribution).} \tag{5}$$

Conversely, given $f(x)$, to find $\psi(a,\sigma)$, we ought to solve the integral equation

$$f(x)=\int_0^\infty\int_{-\infty}^\infty \varphi(x,a,\sigma)\psi(a,\sigma)\,da\,d\sigma \tag{6}$$

such that $\psi(a,\sigma)\geqq 0$, $\iint\psi(a,\sigma)\,da\,d\sigma=1$, the latter of which, however, follows naturally from the equation itself, as we integrate (6) in regard to x, assuming Fubini. But the above kind of integral equation with two parameters seems not yet to have been thoroughly treated and even the existence of the solution, its uniqueness and continuity &c. are not clear. For the present we shall assume all these affirmatively, except the uniqueness, for, evidently solution (1.17) shows that it contains a somewhat arbitrary function $\psi_2(\sigma)$. Hence to get a solution of (6) we are obliged to proceed after Gauss' method of numerical integration as follows:

At first transforming the variable as

$$a=\beta \tan\frac{\pi}{2}t,\quad \sigma=\gamma\tan\frac{\pi}{4}(1+u), \tag{7}$$

$$\psi(a,\sigma)=\psi\left(\beta \tan\frac{\pi}{2}t,\quad \gamma\tan\frac{\pi}{4}(1+u)\right)=Z_{\beta\gamma}(t,u)$$

with arbitrary β,γ, equation (6) can be written as

$$f(x)=\tfrac{1}{4}\int_{-1}^{1}\int_{-1}^{1}X_{\beta\gamma}(x,t,u)\ Y(t,u)\ Z_{\beta\gamma}(t,u)\ dtdu \tag{8}$$

where

$$X_{\beta\gamma}(x,t,u)=\frac{\beta}{\gamma}\exp\{-(x-\beta\tan\frac{\pi}{2}t)^2/2\gamma^2\tan^2\frac{\pi}{4}(1+u)\}, \tag{9}$$

$$Y(t,u)=\sqrt{\frac{\pi^3}{2}}\sec^2\frac{\pi}{2}t\sec^2\frac{\pi}{4}(1+u)/\tan\frac{\pi}{4}(1+u). \tag{10}$$

Then, by means of Gauss' method of selected ordinates we get

$$f(x)=\sum_{\mu=1}^{m}\sum_{\nu=1}^{n}R_\mu R_\nu y_{\mu\nu}, \tag{11}$$

where

$$y_{\mu\nu}=X_{\beta\gamma}(x,t_\mu,u_\nu)Y(t_\mu,u_\nu)Z_{\beta\gamma}(t_\mu,u_\nu), \tag{12}$$

where $X_{\beta\gamma}$ and Y are prescribed while $Z_{\beta\gamma}(t_\mu,u_\nu)$ to be found, the number of which being $mn=l$. Therefore, if we select $x=x_\lambda$ $(\lambda=1,2,\cdots\cdots,l)$ appropriately, we have the following l equations :

$$f(x_\lambda)=\sum_{\mu=1}^{m}\sum_{\nu=1}^{n}R_\mu R_\nu X_{\beta\gamma}(x_\lambda,t_\mu,u_\nu)\ Y(t_\mu,u_\nu)\ Z_{\beta\gamma}(t_\mu,u_\nu). \tag{13}$$

Solving these simultaneous linear equations, the values of l unknown $Z_{\beta\gamma}(t_\mu,u_\nu)$ could be determined.

Making $\beta=\gamma=1$, we get the values of $Z(t,u)$ at l points (t_μ,u_ν) and consequently the values of $z=\psi(a,\sigma)$ at (a_μ,σ_ν) and thus the outline of the surface $z=\psi(a,\sigma)$ would be manifested. To amplify the plotting points any more we may make $\beta,\gamma=1,2,\cdots,\frac{1}{2}$, &c., combine them in various ways and the shape of the distribution surface could be acculated.

After the above plan, I. Wajiki executed numerical computations of *Ex.* 1, i.e. equation (6) with (5) taking $m=n=5$, the result of which, however, was very unpleasing: the calculated values are much more multiplied with theoretical ones.

However, the adoption of Gauss' method of selected ordinates for the case of double integral is by no means of no promise. Really

Ex. 2. Letting e.g. $\psi(a,\sigma)=ka/\sigma^3$ in $0<a<\sqrt{1-(\sigma-1)^2}$ and $1<\sigma<2$, but $\psi(a,\sigma)=0$ everywhere else, we obtain $k=\dfrac{2}{1-\log_e 2}=6.5177831$, and

$$f(x)=\int_1^2\int_0^{\sqrt{1-(\sigma-1)^2}}\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}\psi(a,\sigma)\,da\,d\sigma,$$

so that by exact integrations

$$f(0)=\int_1^2\int_0^{\sqrt{1-(\sigma-1)^2}}\frac{ka}{\sqrt{2\pi}\,\sigma^4}\exp\left\{-\frac{a^2}{2\sigma^2}\right\}da\,d\sigma=0.2770032,$$

while, on transforming $\sigma=\frac{1}{2}(3+u)$ and $a=\frac{1}{2}\sqrt{1-(\sigma-1)^2}(1+t)=\frac{1}{4}\sqrt{(3+u)(1-u)}(1+t)$, we get

$$f(0)=\frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}k\sqrt{\frac{2}{\pi}}\frac{(1-u)(1+t)}{(3+u)^3}\exp\left\{-\frac{(1-u)(1+t)^2}{8(3+u)}\right\}dt\,du$$

and whence

$$f(0)=\sum_{\mu=1}^{5}\sum_{\nu=1}^{5}R_\mu R_\nu\sqrt{\frac{2}{\pi}}\,k\frac{(1-u_\nu)(1+t_\mu)}{(3+u_\nu)^3}\exp\left\{-\frac{(1-u_\nu)(1+t_\mu)^2}{8(3+u_\nu)}\right\}=0.2770029.$$

Thus the theoretical value obtained by exact double integral coincides pretty good with the value calculated by Gauss' approximation.

§ 3.

Specially we consider the case that a and σ are independent, so that $\psi(a,\sigma)=\psi_1(a)\psi_2(\sigma)$. The integral equation now becomes

$$f(x)=\int_0^\infty\int_{-\infty}^\infty\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}\psi_1(a)\psi_2(\sigma)\,da\,d\sigma. \qquad (1)$$

In this case, if one of unknown functions ψ_1, ψ_2 be presumed, the other could be therewith decided by solving the usual integral equation with one parameter.

1° If ψ_2 is presumed, and consequently

$$\int_0^\infty\exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}\frac{\psi_2(\sigma)}{\sqrt{2\pi}\sigma}\,d\sigma\equiv K(a,x), \qquad (2)$$

which forms a symmetrical kernel, is made known, to find ψ_1 we have to solve the integral equation

$$f(x)=\int_{-\infty}^\infty K(a,x)\psi_1(a)\,da. \qquad (3)$$

2° If ψ_1 is known and so also

$$\int_{-\infty}^\infty\exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}\frac{\psi_1(a)}{\sqrt{2\pi}\sigma}\,da\equiv H(\sigma,x), \qquad (4)$$

then ψ_2 should be determined from the integral equation

$$f(x)=\int_0^\infty H(\sigma,x)\psi_2(\sigma)d\sigma. \tag{5}$$

Both (3) and (5) belong to Fredholm's equations of the first kind. So they may be somewhat theoretically discussed, although below only numerical computations are stressed.

Ex. 3. Let the frequency function of σ be a truncated one, and

$$\left.\begin{aligned}\psi_2(\sigma)&=0, && (0<\sigma<1)\\ &=1/\sigma^2 && (1<\sigma<\infty).\end{aligned}\right\} \tag{6}$$

Then we have by (2)

$$K(a,x)=\frac{1}{\sqrt{2\pi}(x-a)^2}[1-\exp\{-\tfrac{1}{2}(x-a)^2\}]. \tag{7}$$

On the other hand, making

$$\left.\begin{aligned}\psi_1(a)&=\tfrac{1}{2}, && (-1<a<1)\\ &=0 && (|a|>1),\end{aligned}\right\} \tag{8}$$

we get

$$f(x)=\frac{1}{2\sqrt{2\pi}(x^2-1)}[2-(x+1)\exp\{-\tfrac{1}{2}(x-1)^2\}+(x-1)\exp\{-\tfrac{1}{2}(x+1)^2\}]$$
$$+\tfrac{1}{2}[\Phi(x+1)-\Phi(x-1)], \tag{9}$$

where $\quad \Phi(x)=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^x \exp\left(-\frac{t^2}{2}\right)dt.$

Conversely presumed (9) and (6), $\psi_1(a)$ is to be sought as the solution of equation (3), viz.

$$f(x)=\int_{-\infty}^\infty [1-\exp\{-\tfrac{1}{2}(x-a)^2\}]\frac{\psi_1(a)da}{\sqrt{2\pi}(x-a)^2}\equiv\int_{-\infty}^\infty K(a,x)\psi_1(a)da. \tag{10}$$

We assume that $\psi_1(a)$ is continuous, except at points $a=\pm1$, because (9) behaves at $x=\pm1$ singular though apparently. Gauss' method of numerical integration could be applied so far as the integrand is continuous throughout the considered interval, so that we should separate the whole integration interval as follows:

$$f(x)=\int_{-\infty}^\infty=\int_{-1}^1+\int_1^\infty+\int_{-\infty}^{-1}=\text{(i)}+\text{(ii)}+\text{(iii)}. \tag{11}$$

Firstly

$$\text{(i)}\ =\tfrac{1}{2}\int_{-1}^1 2K(a,x)\psi_1(a)da=\sum_{\kappa=1}^k R_\kappa 2K(a_\kappa,x)\psi_1(a_\kappa).$$

Secondly, putting $a=\sec\frac{\pi}{4}(1+t)$,

$$\text{(ii)} =\tfrac{1}{2}\int_{-1}^{1}\frac{\pi}{2}K\Big(\sec\frac{\pi}{4}(1+t),x\Big)\sec\frac{\pi}{4}(1+t)\tan\frac{\pi}{4}(1+t)\psi_1\Big(\sec\frac{\pi}{4}(1+t)\Big)dt$$

$$=\tfrac{1}{2}\int_{-1}^{1}L(t,x)\psi_1(t)dt=\sum_{\lambda=1}^{l}R_\lambda L(t_\lambda,x)\psi_1(t_\lambda).$$

Thirdly

$$\text{(iii)}=\tfrac{1}{2}\int_{-1}^{1}\frac{\pi}{2}K\Big(-\sec\frac{\pi}{4}(1+t),x\Big)\sec\frac{\pi}{4}(1+t)\tan\frac{\pi}{4}(1+t)\psi_1\Big(-\sec\frac{\pi}{4}(1+t)\Big)dt$$

$$=\tfrac{1}{2}\int_{-1}^{1}M(t,x)\psi_1(t)dt=\sum_{\mu=1}^{m}R_\mu M(t_\mu,x)\psi_1(t_\mu).$$

These three expressions being substituted in (11) and taking $x=x_\nu$ we obtain

$$f(x_\nu)=\sum_{\kappa=1}^{k}2R_\kappa K(t_\kappa,x_\nu)\psi_1(t_\kappa)+\sum_{\lambda=1}^{l}R_\lambda L(t_\lambda,x_\nu)\psi_1(t_\lambda)$$

$$+\sum_{\mu=1}^{m}R_\mu M(t_\mu,x_\nu)\psi_1(t_\mu),\quad \nu=1,2,\cdots,n(=k+l+m).$$

Solving these n linear equations simultaneously with respect to n unknowns ψ_1's, their roots yield the values of $U=\psi_1(a)$ at $a=a_\kappa$, a_λ, a_μ and throw light on its graph. After this scheme I. Wajiki taking $k=l=m=3$, obtained the following nine equations:

1) when $x=0$:

$$0.09574U_1+0.17731U_2+0.09574U_3+0.01236U_4+0.12450U_5$$
$$+0.17135U_6+0.01236U_7+0.12450U_8+0.17135U_9=0.18433;$$

2) when $x=0.1$:

$$0.09209U_1+0.17686U_2+0.09911U_3+0.01292U_4+0.13189U_5$$
$$+0.17755U_6+0.01177U_7+0.11720U_8+0.16547U_9=0.18398;$$

3) when $x=-0.1$:

$$0.09911U_1+0.1786U_2+0.09209U_3+0.01177U_4+0.11720U_5$$
$$+0.16547U_6+0.01292U_7+0.13189U_8+0.17755U_9=0.18398;$$

4) when $x=0.2$:

$$0.08822U_1+0.17555U_2+0.10215U_3+0.01346U_4+0.13933U_5$$
$$+0.18409U_6+0.001118U_7+0.11008U_8+0.15989U_9=0.18290;$$

5) when $x=-0.2$:

$$0.10215U_1+0.17555U_2+0.08822U_3+0.01118U_4+0.11008U_5$$
$$+0.15989U_6+0.01346U_7+0.13933U_8+0.18409U_9=0.18290;$$

6) when $x=0.3$:

$$0.08419U_1+0.17338U_2+0.10480U_3+0.01395U_4+0.14672U_5$$
$$+0.19100U_6+0.01058U_7+0.10320U_8+0.15459U_9=0.18115;$$

7) when $x=-0.3$:

$$0.10480U_1+0.17338U_2+0.08419U_3+0.01058U_4+0.10320U_5$$
$$+0.15459U_6+0.01395U_7+0.14672U_8+0.19100U_9=0.18115;$$

8) when $x=0.4$:

$$0.08006U_1+0.17040U_2+0.10702U_3+0.14409U_4+0.15397U_5$$
$$+0.19830U_6+0.00999U_7+0.9659U_8+0.14954U_9=0.17868;$$

9) when $x=-0.4$:

$$0.10702U_1+0.17040U_2+0.08006U_3+0.00999U_4+0.09659U_5$$
$$+0.14954U_6+0.14409U_7+0.15397U_8+0.19830U_9=0.17868.$$

Assuming

$$\psi_1(a_\kappa)=U_1=U_2=U_3=\tfrac{1}{2},\quad \psi_1(a_\lambda)=U_4=U_5=U_6=0,\quad \psi_1(a_\mu)=U_7=U_8=U_9=0,$$

and calculating the left handed sides, we obtain the following equations

1)	$x=0$:	$0.18439=0.18433$,	6)	$x=0.3$:	$0.18118=0.18115$,
2)	$x=0.1$:	$0.18403=0.18398$,	7)	$x=-03$:	$0.18118=0.18115$,
3)	$x=-0.1$:	$0.18403=0.18398$,	8)	$x=0.4$:	$0.17874=0.17868$,
4)	$x=0.2$:	$0.18296=0.18290$,	9)	$x=-0.4$:	$0.17874=0.17868$.
5)	$x=-0.2$:	$0.18296=0.18290$,			

Thus the figures on both the sides agree almost up to the fourth decimal place.

Ex. 4. If $\psi_1(a)=\dfrac{1}{\sqrt{2\pi}}\exp(-\tfrac{1}{2}a^2)$, then (4) becomes

$$H(\sigma,x)=\frac{1}{2\pi\sigma}\int_{-\infty}^{\infty}\exp\left\{-\frac{(x-a)^2}{2\sigma^2}-\frac{a^2}{2}\right\}da=\frac{1}{\sqrt{2\pi(1+\sigma^2)}}\exp\left\{-\frac{x^2}{2(1+\sigma^2)}\right\}. \tag{12}$$

Further, assuming

$$\psi_2(\sigma)=\sigma/(1+\sigma^2)^{\frac{3}{2}}, \tag{13}$$

we obtain

$$f(x)=\frac{1}{\sqrt{2\pi x^2}}\left\{1-\exp\left(-\frac{x^2}{2}\right)\right\}, \tag{14}$$

Thus equation (5) reduces to

$$\frac{1}{\sqrt{2\pi}x^2}\left\{1-\exp\left(-\frac{x^2}{2}\right)\right\}=\int_0^{\infty}\frac{1}{\sqrt{2\pi(1+\sigma^2)}}\exp\left\{-\frac{x^2}{2(1+\sigma^2)}\right\}\psi_2(\sigma)d\sigma, \tag{15}$$

the solution of which is nothing but expression (13).

The integral equation (15) can be solved theoretically on referring to Laplace transform[6]. Namely, on setting $2s=x^2$, $t=(1+\sigma^2)^{-1}$, the interval $0<\sigma<\infty$ is trans-

6) D. V. Widder, loc. cit., p. 66.

formed into $1>t>0$, and equation (15) to

$$\frac{1-e^{-s}}{s}=\int_0^1 e^{-st}\psi_2\left(\sqrt{\frac{1-t}{t}}\right)\frac{dt}{t\sqrt{1-t}}. \tag{16}$$

Hence, if we write

$$\mathfrak{f}(s)=(1-e^{-s})/s \tag{17}$$

and

$$\left.\begin{aligned} g(t)&=\psi_2\left(\sqrt{\frac{1-t}{t}}\right)\Big/t\sqrt{1-t} \quad &(0<t<1)\\ &=0, &(1<t<\infty)\end{aligned}\right\} \tag{18}$$

the above equation reduces to

$$\mathfrak{f}(s)=\int_0^\infty e^{-st}g(t)dt. \tag{19}$$

By a known inversion formula, we get for $c>0$

$$g(t)=\lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\mathfrak{f}(s)e^{st}ds=\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{1-e^{-c-i\eta}}{c+i\eta}e^{(c+i\eta)t}d\eta$$

$$=\frac{e^{ct}}{\pi}\int_0^\infty\frac{c\cos\eta t+\eta\sin\eta t-e^{-c}[c\cos\eta(t-1)+\eta\sin\eta(t-1)}{c^2+\eta^2}d\eta.$$

But

$$\int_0^\infty\frac{\gamma\cos\beta\eta}{\gamma^2+\eta^2}d\eta=\int_0^\infty\frac{\eta\sin\beta\eta}{\gamma^2+\eta^2}d\eta=\frac{\pi}{2}e^{-\beta\gamma} \qquad \text{for} \quad \beta>0,\ \gamma>0$$

hold, as easily shown by the theory of residues. These formulas being applied to the before standing integrals with caution about signs, we obtain the following result:

$$g(t)=1, \quad \text{if } 0<t<1, \quad \text{but otherwise} \quad g(t)=0.$$

Remembering that $g(t)=\psi_2\left(\sqrt{\frac{1-t}{t}}\right)\Big/t\sqrt{1-t}$ and $t=(1+\sigma^2)^{-1}$, we get

$$\psi_2\left(\sqrt{\frac{1-t}{t}}\right)=t\sqrt{1-t} \text{ in } 1>t>0, \quad \text{viz. } \psi_2(\sigma)=\sigma/(1+\sigma^2)^{\frac{3}{2}} \text{ in } 0<\sigma<\infty,$$

which agrees with (13).

In order to solve the same integral equation numerically, first transforming (15) by $\sigma=\tan\frac{\pi}{4}(1+t)$, we have

$$f(x)=\tfrac{1}{2}\int_{-1}^{1}\tfrac{1}{2}\sqrt{\frac{\pi}{2}}\exp\left\{-\frac{x^2}{2}\cos^2\frac{\pi}{4}(1+t)\right\}.\ \sec\frac{\pi}{4}(1+t).\ \psi_2\left(\tan\frac{\pi}{4}(1+t)\right)dt.$$

Setting further $\sec\frac{\pi}{4}(1+t)\psi_2(\tan\frac{\pi}{4}(1+t))=\chi(t)$, we have only to compute by aid of

Gauss' method of n ordinates $\frac{2}{\pi x^2}\left(1-e^{-x^2/2}\right)=\sum_{\nu=1}^{n} R_\nu \exp\{-\frac{x^2}{2}\cos\frac{\pi}{4}(1+t_\nu)\}\chi(t_\nu)$. Letting e.g. $n=5$, and $x=0$, 0.5, 1, 1.5, 2, we have five equations involving five unknowns $\chi(t_\nu)$. Solving these simultaneous linear equations with respect to $\chi(t_\nu)$, T. Kawashiro calculated the values of $\psi_2(\sigma)$ as in the following table, the true values being those obtained from (13):

$\sigma=\tan\frac{\pi}{4}(1+t_\nu)$	0.0733	0.3792	1	2.6368	13.5465
cal. $\psi_2(\sigma)$	0.0717	0.3087	0.3560	0.1180	0.0057
true $\psi_2(\sigma)$	0.0732	0.3100	0.3536	0.1176	0.0054