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A NOTE ON SUBORDINATION

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1. Introduction The following result is well known as Hurwitz-Bochner's theorem [1][2].(1)

If $w=f(z)$ is regular in $|z|<1$, $f(0)=0$, $f'(0)=1$, and $f(z)\neq 0$

except $z=0$, the conformal image of $f(z)$ assumes every value in $|w|<1/16$. In the present paper we generalize this theorem and deal with the related problems with it by the principle of subordination.

As a preliminary remark we shall give a notion of $Q(z)$ whose properties are as follows [3].

$$Q(z)=16z\prod_{n=1}^{\infty}\left(\frac{1+z^{2n}}{1+z^{2n-1}}\right)^8, \qquad (|z|<1).$$

$$Q(z)=J\left(\frac{1}{\pi i}\log z\right), \text{ where } J(z) \text{ is the elliptic modular function.}$$

Let the surface M be the conformal image of $|z|<1$ by $Q(z)$. M has no branch point. M covers every point of w-plane except $w=0,1,\infty$. M does not cover $w=1,\infty$, but the $w=0$ is covered by one sheet of M only.

2. Lemma 1. $\qquad |Q(z)|\leqq Q(-r), \qquad (|z|=r<1).$

Proof. Each factor of infinite products which constitutes $Q(z)$ has its greatest absolute value on $|z|=r$ only when $z=-r$, and therefore we have the above estimate clearly.

Lemma 2. *Let $w=f(z)=a_1z+\cdots\cdots$, be regular and $f(z)\neq 0$ except $z=0$ in $|z|<1$. If $f(z)$ omits a value α in $|z|<1$, the following estimates are got.*

(i) $\quad |a_1|\leqq 16|\alpha|$, *that is, the conformal image of $f(z)$ assumes every value in $|w|<|a_1|/16$.*

(ii) $\quad |f(z)|\leqq|\alpha|\cdot Q(-r), \qquad (|z|=r<1).$

Proof. Let us consider

$$P(z)=Q^{-1}\left(\frac{f(z)}{\alpha}\right),$$

where $Q^{-1}(w)$ is the inverse function of $Q(z)$.

$f(z)$ leaves out $w=0$ except $z=0$ in $|z|<1$, and therefore $P(z)$ is analytic in $0<|z|<1$. On the other hand the regularity of $P(z)$ hold good at $z=0$. Hence $f(z)/\alpha$ is subordinate to $Q(z)$ and by the principle of subordination we get

(1) The bracket denotes the number of the references.

$$|\alpha^{-1}f'(0)|\leqq Q'(0)$$

$$|\alpha^{-1}f(z)|\leqq|Q(z)|\leqq Q(-r)\quad(|z|=r<1).$$

These complete the proof.

Remark. The above estimates are sharp as is shown by $f_0(z)=\alpha Q(z)$. $|\alpha|$ in the latter result of this lemma is can not be made smaller than $|a_1|/16$. The former result was got by the same method by Bochner [1][3].

Theorem 1. *Let* $w=f(z)=a_p z^p+\cdots\cdots$, *be regular and* $f(z)\neq 0$ except $z=0$ *in* $|z|<1$.

(i) *The values taken by* $w=f(z)$ *cover the circle* $|w|<|a_p|/16^p$ p times or more times.

(ii) *The conformal image of the unit circle by* $w=f(z)$ *completely covers the interior of a circle about the origin whose rudius is* $|a_p|/16$. *We can state this result in detail, namely,*
If $w=f(z)$ omit a value α,

(ii') $$|a_p|\leqq 16|\alpha|$$

(iii) $$|f(z)|\leqq|\alpha|\cdot Q(-r^p)\qquad(|z|=r<1).$$

These bounds are best possible.

Proof.

First in order to prove the former result we consider $g(z)=(f(z))^{1/p}$. Since $f(z)\neq 0$ except $z=0$ in $|z|<1$, $g(z)$ is regular and $g(z)\neq 0$ except $z=0$ in $|z|<1$. Hence when lemma 2 is used with respect to $g(z)$, the proof of (i) will be given.

Secondly we prove the latter results (ii) and (iii). Let us consider

$$F(z)=Q(z^p)=16z^p+\cdots\cdots.$$

The Riemann surface onto which the unit circle is mapped by $F(z)$ has no branch point, and covers every point in w-plane infinite times except $w=0,1,\infty$. It does not cover $w=1$, ∞, but $w=0$ is covered by its p sheets only.
Now we put

$$R(z)=F^{-1}\left(\frac{f(z)}{\alpha}\right).$$

Under the same conditions in the proof of lemma 2, $R(z)$ is regular in $0<|z|<1$, and $R(z)$ is regular at $z=0$ more. We can get easy

$$|R'(0)|=\left|\frac{a_p}{16\alpha}\right|^{\frac{1}{p}}$$

Furthermore $$R(0)=0,\qquad |R(z)|<1.$$

Hence $$|R'(0)|\leqq 1,\ \text{that is,}\ |a_p|\leqq 16|\alpha|.$$

On the other hand $f(z)$ is subordinate to $\alpha F(z)$ and therefore

$$|f(z)|\leqq\max_{|z|=r}|\alpha F(z)|\leqq|\alpha|\cdot Q(-r^p)(|z|=r<1).$$

Theorem 2. *Let $w=f(z)=a_1z+\cdots\cdots$, be regular and $f(z)\neq 0$ except $z=0$ in $|z|<1$. If $f(z)$ leaves out two values α and $-\alpha$, then*

$$|a_1|\leqq 4|\alpha|, \qquad |f(z)|\leqq|\alpha|\cdot Q^{\frac{1}{2}}(-r^2).$$

Namely, if $w=f(z)=z+\cdots\cdots$, is an odd regular function in $|z|<1$ and $f(z)\neq 0$ except $z=0$, the values taken by $w=f(z)$ cover fully $|w|<1/4$. These bounds are best possible.[2]

Proof. As the superordinate function to $f(z)/\alpha$ we consider

$$Q_1(z)=\sqrt{Q(z^2)}=4z+\cdots\cdots\cdots.$$

And then $Q_1(z)$ is an odd regular function in $|z|<1$ and the other properties of $Q_1(z)$ are quite similar to ones of $Q(z)$ except the fact that it does not assume both 1, and -1[2]. Namely $Q_1^{-1}\left(\frac{f(z)}{\alpha}\right)$ is analytici n $|z|<1$ like the case of lemma 1 and therefore we have the forme result of this theorem by the principle of subordination. The latter result is evident.

We can moreover generalize this theorem, that is,

Theorem. 2′. *Let $w=f(z)=a_pz^p+\cdots\cdots$, be regular in $|z|<1$, and $f(z)\neq 0$ except $z=0$. If $f(z)$ leaves out the values α, and $-\alpha$, then*

$$|a_p|\leqq 4|\alpha|, \quad |f(z)|\leqq|\alpha|\cdot Q^{\frac{1}{2}}(-r^{2p}), \quad (|z|=r<1).$$

Proof. If we consider the superordinate function

$$Q^{\frac{1}{2}}(z^{2p})=4z^p+\cdots\cdots\cdots\cdots,$$

the results are evident.

Theorem 3. *Let $w=f(z)=a_p z^p+\cdots\cdots$, be regular and not zero except $z=0$ in $|z|<1$. If $f(z)$ leaves out the values $-\infty\leqq w\leqq\frac{-1}{4}$, $f(z)$ is subject to the following inequalities.*

$$|a_p|\leqq 1, \quad |f(z)|\leqq\frac{r^p}{(1-r^p)^2}, \ (|z|=r<1)$$

These bounds are sharp as is shown by

$$F(z)=\frac{z^p}{(1-z^p)^2}$$

Proof. As the superordinate function to $f(z)$ we consider $F(z)$. The Riemann surface onto whieh the unit circle is mapped by $F(z)$ has no branch point except $z=0$. Hence

$$S(z)=F^{-1}(f(z))$$

is analytic in $0<|z|<1$. On the other hand $S(z)$ has regularity at $z=0$ whose derivertive has $(a_p)^{\frac{1}{p}}$ there. Moreover $S(0)=0$ and $|S(z)|<1$. Hence we have the above results.

(2) Ifthe condition that $f(z)\neq 0$ except $z=0$, is omitted, we have the following result, which is well known [3].

$$|a_1|\leqq k|\alpha|, \qquad k=\Gamma^4(\tfrac{1}{4})/4\pi^2$$

We can get the following estimates by means of the same method also.

Theorem 4. *If* $w=f(z)=a_pz^p+\cdots\cdots$, *is regular in* $|z|<1$, *leaves out the values* $-\infty\leqq w\leqq-\alpha$, *and* $\alpha\leqq w\leqq\infty$, $(\alpha>0)$, *and vanishes at* $z=0$ *only*,

$$|a_p|\leqq 2\alpha, \qquad |f(z)|\leqq\frac{2\alpha r^p}{1-r^{2p}}. \qquad (|z|=r).$$

These bounds are best possible as is shown by

$$F(z)=\frac{2\alpha z^p}{(1+z^{2p})}$$

3. We consider the case where $f(z)$ is meromorphic in $|z|<1$.

Theorem 5. *Let* $w=f(z)=a_1z+a_3z^3+\cdots\cdots$, *be an odd meromorphic function in* $|z|<1$ *and* $f(z)\neq 0$ *except* $z=0$, *then the image by this function covers fully the cirle*

$$|w|<\frac{|a_1|}{8}.$$

The result is best possible as is shown by

$$f_0(z)=\frac{2Q_1(z)}{1+Q_1^2(z)}$$

Proof. It is clear that $f_0(z)$ is an odd meromorphic function and does not take 1 and -1 b because of the property of $Q_1(z)$.
Let us consider

$$F(z)=\frac{1}{1-f_0(z)}=\frac{1+Q_1^2(z)}{(1-Q_1(z))^2}$$

Then

$$F'(z)=\frac{2(1+Q_1(z))}{(1-Q_1(z))^3}Q_1'(z)\neq 0 \quad (|z|<1),$$

because $Q_1'(z)\neq 0$ in $|z|<1$.

If $f(z)$ leaves out α and $-\alpha$, we consider $g(z)=f(z)/\alpha$. The function $(1-g(z))^{-1}$ is subordinate to $F(z)$ because of the proerty of $F(z)$. And therefore

$$\left|\frac{a_1}{\alpha}\right|\leqq F'(0)=8$$

This completes the proof.

Remark. If $f(z)=a_1z\cdots\cdots$, is meromorphic and vanishes at $z=0$ only in $|z|<1$, $f(z)$ takes at least one value of each coupl $\pm w$ belonging to the circl

$$|w|<\frac{|a_1|}{8}$$

The proof is quite similar if $Q(z)(2-Q(z))^{-1}$ is considered as the superordinate function.

Here we state the related result with this theorem.

Theorem 6. *Let $f(z)=a_1 z+\cdots\cdots$, be an odd meromorphic function in $|z|<1$, then the image by $f(z)$ covers fully the circle*

$$|w|<\frac{2|a_1|\pi^2}{\Gamma^4(\frac{1}{4})}$$

This result is best possible.

Proof. Like the case in theorem 5 we consider

$$f_0(z)=\frac{2F(z)}{1+F^2(z)},\ F(z)=2J\left[i\left(\frac{1+z}{1-z}\right)\right]-1.$$

$F(z)$ leaves out 1 and -1, but every value except these values is taken infinite times. Furthermore $F'(z)\neq 0$ in $|z|<1$, because $J(z)$ has no branch point [3]. Hereafter we may do like the proof of theorem 5.

References

[1] S. Bochner, Remarks on the theorems of Picad-Landau and Picard-Schottky, J. London Math. Soc, (1926), pp. 100-103.

[2] J. N. Littlewood, Lectures on the theory of functions, Oxford (1944), pp. 196–197.

[3] Z. Nehari, Conformal Mapping, Mcgraw-Hill, (1952), pp. 226-236, pp. 318-330.