Journal of Gakugei, Tokushima University
Vol. VII. 1956.

ON THE SPACE WITH DOMINANT AFFINE CONNECTION.

By

Yoshihiro ICHIJŌ

(Received september 30, 1956)

In this paper we shall consider an n-dimensional space V_n with dominant affine connection, in which the quantities $\Gamma^{\alpha}_{\beta i}$(1) determing the relation between tangent spaces attaching to every point are given. However in this case the tangent spaces of V_n are $m(>n)$-dimensional affine spaces. We call $\Gamma^{\alpha}_{\beta i}$ the coefficients of dominant affine connection.

§ 1. Consider an n-dimensional space V_n with dominant affine connection where a current point $\mathfrak{x}$ is given by a system of coordinates $(x^1, x^2, \cdots, x^n)$ and linearly independent m vectors $\mathfrak{x}_\lambda$ which compose a frame of a tangent space attaching to this point $\mathfrak{x}$ are given.

Then these vectors satisfy the equations

(1.1) $$d\mathfrak{x}_\alpha = \Gamma^{\lambda}_{\alpha i}\, \mathfrak{x}_\lambda\, dx^i.$$

Let $\mathfrak{x}_i$ be the linearly independent n vectors satisfying the equations

(1.2) $$d\mathfrak{x} = \mathfrak{x}_i\, dx_i.$$

Being the vectors on m-dimensional affine space A_m, $\mathfrak{x}_i$ must satisfy

(1.3) $$\mathfrak{x}_i = B_i^{\cdot\alpha}\, \mathfrak{x}_\alpha,$$

and contravariant vectors v^i on A_n can be written

(1.4) $$V^\lambda = B_i^{\cdot\lambda}\, v^i,$$

from which we see the quantities $B_i^{\cdot\lambda}$ are n contravariant vectors.

Let $\mathfrak{x}_P$ be $m-n$ linearly independent vectors of $\mathfrak{x}_i$, and A_{m-n} be an $(m-n)$-dimensional subspace of A_m, then we define $B_P^{\cdot\alpha}$ by the equations

(1.5) $$\mathfrak{x}_P = B_P^{\cdot\alpha}\, \mathfrak{x}_\alpha.$$

We find similarly

(1.4)′ $$V^\lambda = B_P^{\cdot\lambda}\, v^P$$

where v^P is a vector on A_{m-n}, and $B_P^{\cdot\lambda}$ are p contravariant vectors.

The rank of the matrix

(1) In this paper we shall denote by $\alpha, \beta, \gamma, \lambda, \mu, \nu, \cdots$ the suffices which take the value $1, 2, \cdots, m$; by $a, b, c, \cdots, i, j, \cdots, r$, those which take the yalue $\dot{1}, \dot{2}, \cdots, \dot{n}$, and P, Q, R, S, those which take the value $\dot{n}+\dot{1}, \dot{n}+\dot{2}, \cdots, \dot{m}$,

$$\begin{pmatrix} \vdots \\ B_i^{\cdot\lambda} \\ \vdots \\ B_P^{\cdot\lambda} \\ \vdots \end{pmatrix}$$

is m, then we have the inverse matrix

$$\begin{pmatrix} \vdots \\ B^i_{\cdot\lambda} \\ \vdots \\ B^P_{\cdot\lambda} \\ \vdots \end{pmatrix},$$

from which we see the relations

$$B^i_{\cdot\lambda} B_k^{\cdot\lambda} = \delta^i_k, \quad B^i_{\cdot\lambda} B_P^{\cdot\lambda} = 0, \quad B^P_{\cdot\lambda} B_k^{\cdot\lambda} = 0, \quad B^P_{\cdot\lambda} B_Q^{\cdot\lambda} = \delta^P_Q \tag{1.6}$$

$$B_i^{\cdot\lambda} B^i_{\cdot\mu} + B_P^{\cdot\lambda} B^P_{\cdot\mu} = \delta^\lambda_\mu.$$

For the displacement on V_n we put

$$d\mathfrak{x}_i = \gamma^k_{ij} dx^j \mathfrak{x}_k + H^P_{i\,j} dx^j \mathfrak{x}_P, \tag{1.7}$$

on the other hand we see

$$d\mathfrak{x}_i = [B_i^{\cdot\alpha}{}_{,j} + B_i^{\cdot\beta} \Gamma^\alpha_{\beta j}] \mathfrak{x}_\alpha dx^j,$$

where comma means partial derivative, and comparing with (1.7) we obtain

$$\frac{\partial B_i^{\cdot\alpha}}{\partial x^j} = -\Gamma^\alpha_{\beta j} B_i^{\cdot\beta} + \gamma^k_{ij} B_k^{\cdot\alpha} + H^Q_{i\,j} B_Q^{\cdot\alpha}. \tag{1.8}$$

Similarly, putting

$$d\mathfrak{x}_P = H^k_{P\,j} dx^j \mathfrak{x}_k + H^Q_{P\,j} dx^j \mathfrak{x}_Q, \tag{1.9}$$

we obtain

$$\frac{\partial B_P^{\cdot\alpha}}{\partial \mathfrak{x}^j} = -\Gamma^\alpha_{\beta j} B_P^{\cdot\beta} + H^P_{k\,j} B_k^{\cdot\alpha} + H^Q_{P\,j} B_Q^{\cdot\alpha}. \tag{1.10}$$

Now consider the transformation of coordinate

$$x'^i = x'^i(x^1, x^2, \cdots, x^n)$$

and change of the frame

$$\mathfrak{x}_\lambda = A^{\lambda'}_\lambda \mathfrak{x}_{\lambda'}$$

where the rank of the matrix $(A^{\lambda'}_\lambda)$ is m and $(A^\lambda_{\lambda'})$ is the inverse matrix of $(A^{\lambda'}_\lambda)$. Then from (1.3), (1.5), $B_i^{\cdot\lambda}$ and $B_P^{\cdot\lambda}$ are transformed by the laws

$$B_{i'}^{\cdot\lambda'} = A^{\lambda'}_\lambda \frac{\partial x^i}{\partial x^{i'}} B_i^{\cdot\lambda}, \qquad B_P^{\cdot\lambda'} = A^{\lambda'}_\lambda B_P^{\cdot\lambda}. \tag{1.11}$$

Moreover, the transformation law of a composite tensor, that is to say, the tensor which may involve Latin and Greak indices, is[2]

(1.12) $$T^{\cdots\alpha'\cdots i'\cdots}_{\cdots\beta'\cdots j'\cdots}=T^{\cdots\alpha\cdots i\cdots}_{\cdots\beta\cdots j\cdots}\cdots A^{\alpha'}_{\alpha}\cdots A^{\beta}_{\beta'}\cdots\frac{\partial x^{i'}}{\partial x^{i}}\cdots\frac{\partial x^{j}}{\partial x^{j'}}\cdots,$$

and hence $B_i^{\cdot\lambda}$ is a composite tensor.

Comparing

$$d\mathfrak{x}_{\alpha'}=\Gamma^{\beta'}_{\alpha' i'}\mathfrak{x}_{\beta'}dx,$$

with

$$d\mathfrak{x}_{\alpha'}=d(A^{\beta}_{\alpha'}\mathfrak{x}_{\beta}),$$

we obtain

(1.13) $$\Gamma^{\beta'}_{\alpha' i'}=A^{\beta}_{\beta'}(A^{\alpha}_{\alpha'}\Gamma^{\beta}_{\alpha i}+A^{\beta}_{\alpha', i})\frac{\partial x^{i}}{\partial x^{i'}}.$$

On the other hand, considering the case where we do not change the vector frame, we see

$$d\mathfrak{x}_{i'}=[\gamma^{k'}_{i'j'}B_k^{\cdot\alpha}\frac{\partial x^{k}}{\partial x^{k'}}+H^{P}_{i'j'}B_P^{\cdot\alpha}]\mathfrak{x}_{\alpha}\frac{\partial x^{j'}}{\partial x^{j}}dx^{j}.$$

Here, differentiating the relation $\mathfrak{x}_{i'}=\frac{\partial x^{i}}{\partial x^{i'}}\mathfrak{x}_{i}$, and comparing with the above equation give

(1.14) $$\gamma^{k'}_{i'j'}\frac{\partial x^{k}}{\partial x^{k'}}=\frac{\partial^2 x^{k}}{\partial x^{i'}\partial x^{j'}}+\gamma^{k}_{ij}\frac{\partial x^{i}}{\partial x^{i'}}\frac{\partial x^{j}}{\partial x^{j'}},$$

$$H^{P}_{i'j'}=H^{P}_{ij}\frac{\partial x^{i}}{\partial x^{i'}}\frac{\partial x^{j}}{\partial x^{j'}}.$$

Also from (1.10) we obtain

(1.15) $$H^{k'}_{P j'}=H^{k}_{Pj}\frac{\partial x^{k'}}{\partial x^{k}}\frac{\partial x^{j'}}{\partial x^{j}},\quad H^{h'}_{P j'}=H^{k}_{Pj}\frac{\partial x^{k'}}{\partial x^{k}}.$$

§ **2.** From the integrability conditions of the equations (1.8) we obtain

(2.1) $$R^{\alpha}_{\beta jk}B_i^{\cdot\beta}=R^{h}_{ijk}B_h^{\cdot\alpha}+[H^{h}_{Pk}H^{P}_{ij}-H^{h}_{Pj}H^{P}_{ik}]B_h^{\cdot\alpha}$$
$$+[(\gamma^{l}_{ij}H^{P}_{lk}-\gamma^{l}_{ik}H^{P}_{lj})+(H^{P}_{ij,k}-H^{P}_{ik,j}+H^{Q}_{ij}H^{P}_{Qk}-H^{Q}_{ik}H^{P}_{Qj})]B_P^{\cdot\alpha},$$

where the quantity $R^{\alpha}_{\beta jk}$ is a curvature tensor of V_n, i.e.

(2.2) $$R^{\alpha}_{\beta jk}=\Gamma^{\alpha}_{\beta j,k}-\Gamma^{\alpha}_{\beta k,j}+\Gamma^{\alpha}_{\sigma j}\Gamma^{\sigma}_{\beta k}-\Gamma^{\alpha}_{\sigma k}\Gamma^{\sigma}_{\beta j},$$

and R^{h}_{ijk} is a curvature tensor for γ^{i}_{jk}, i.e.

(2) A. D. Michal and J. L. Botsfold; Geometries involving affine connections and general linear connections. An extension of the recent Einstein-Mayer geometry. Annali di mat. 12 (1934) p. p. 13~32.

(2.3) $$R^h_{ijk}=\gamma^h_{ij,k}-\gamma^h_{ik,j}+\gamma^l_{ij}\gamma^h_{lk}-\gamma^l_{ik}\gamma^h_{lj}.$$

In the same manner, from (1.10) we obtain

(2.4) $$R^\alpha_{\beta ij}B_P^{\cdot\beta}=[H^Q_{Pi,j}-H^Q_{Pj,i}+H^R_{Pi}H^Q_{Rj}-H^R_{Pj}H^Q_{Ri}+H^l_{Pi}H^Q_{lj}-H^k_{Pj}H^Q_{ki}]B_Q^{\cdot\alpha}$$
$$+[H^l_{Pi,j}-H^l_{Pj,i}+H^k_{Pi}\gamma^l_{kj}-H^k_{Pj}\gamma^l_{ki}+H^Q_{Pi}H^l_{Qj}-H^Q_{Pj}H^l_{Qi}]B_l^{\cdot\alpha}.$$

Covariant derivative of the composite tensor $T^{\cdot\cdot\alpha\cdots i\cdots}_{\cdots\beta\cdots j\cdots}$ is given from (1.12) and (1.14) by

(2.5) $$T^{\cdot\cdot\alpha\cdots i\cdots}_{\cdots\beta\cdots j\cdots;k}=\frac{\partial T^{\cdot\cdot\alpha\cdots i\cdots}_{\cdots\beta\cdots j\cdots}}{\partial x^k}+\cdots+\Gamma^\alpha_{\lambda k}T^{\cdot\cdot\lambda\cdots i\cdots}_{\cdots\beta\cdots j\cdots}+\cdots+\gamma^i_{ln}T^{\cdot\cdot\alpha\cdots l\cdots}_{\cdots\beta\cdots j\cdots}+\cdots$$
$$\cdots-\Gamma^\lambda_{\beta k}T^{\cdot\cdot\alpha\cdots i\cdots}_{\cdots\lambda\cdots j\cdots}-\cdots-\gamma^l_{jk}T^{\cdot\cdot\alpha\cdots i}_{\cdots\beta\cdots l}-\cdots.$$

Especially

(2.6) $$B_{j;k}^{\cdot\lambda}=B_{j,k}^{\cdot\lambda}+B_j^{\cdot\mu}\Gamma^\lambda_{\mu k}-B_i^{\cdot\lambda}\gamma^i_{jk}=H^P_{jk}B_P^{\cdot\lambda},$$

and

(2.7) $$V^\lambda_{;k}=B_i^{\cdot\lambda}v^i_{;k}+H^P_{jk}B_P^{\cdot\lambda}v^j,$$

where we put $V^\lambda=B_i^{\cdot}v^i$(3).

For the extension of Ricci equation we obtain

(2.8) $$T^{\cdot\cdot\alpha\cdots a\cdots}_{\cdots\beta\cdots b\cdots;i;j}-T^{\cdot\cdot\alpha\cdots a\cdots}_{\cdots\beta\cdots b\cdots;j;i}$$
$$=\cdots+R^\alpha_{\lambda ij}T^{\cdot\cdot\lambda\cdots a\cdots}_{\cdots\beta\cdots b\cdots}+\cdots+R^a_{lij}T^{\cdot\cdot\alpha\cdots l\cdots}_{\cdots\beta\cdots b\cdots}+\cdots$$
$$\cdots-R^\lambda_{\beta ij}T^{\cdot\cdot\alpha\cdots a\cdots}_{\cdots\lambda\cdots l\cdots}-\cdots-R^l_{bij}T^{\cdot\cdot\alpha\cdots a\cdots}_{\cdots\beta\cdots l\cdots}-\cdots.$$

§ 3. In the space connected dominantly with the given functions $\Gamma^\lambda_{\mu i}$ whose trasformation laws are given by the relations (1.13), when we determine the quantities $B_i^{\cdot\lambda}$ and $B_P^{\cdot\lambda}$ from the relations (1.13) and (1.15), we may determine four kinds of the quantities γ^k_{ij}, H^P_{ij}, H^P_{Qj} and H^i_{Pj} from the relations (1.8) and (1.10)

(3.1) $$\gamma^k_{ij}=B^k_{\cdot\alpha}\frac{\partial B_i^{\cdot\alpha}}{\partial x^j}+\Gamma^\alpha_{\beta j}B^k_{\cdot\alpha}B_i^{\cdot\beta},$$

(3.2) $$H^P_{ij}=B^P_{\cdot\alpha}\frac{\partial B_i^{\cdot\alpha}}{\partial x^j}+\Gamma^\alpha_{\beta j}B_i^{\cdot\beta}B^P_{\cdot\alpha},$$

(3.3) $$H^P_{Qj}=B^P_{\cdot\alpha}\frac{\partial B_Q^{\cdot\alpha}}{\partial x^j}+\Gamma^\alpha_{\beta j}B^P_{\cdot\alpha}B_Q^{\cdot\beta},$$

(3.4) $$H^i_{Qj}=B^i_{\cdot\alpha}\frac{\partial B_Q^{\cdot\alpha}}{\partial x^j}+\Gamma^\alpha_{\beta j}B^i_{\cdot\alpha}B_\theta^{\cdot\alpha}.$$

Conversely, from (1.8) and (1.10) we obtain

(3) K. Yano: Sur la theorie der espace à hyperconnection euclidienn. I. et II; Proc. Jap. Acad (21) (1945) p. p. 156∼163 et p. p.164∼170.

$$\Gamma^{\alpha}_{\beta j}=[H^{Q}_{Pj}\,B^{\cdot\alpha}_{Q}+H^{i}_{Pj}\,B^{\cdot\alpha}_{i}-\frac{\partial B^{\cdot\alpha}_{P}}{\partial x^{j}}]B^{P}_{\cdot\beta}$$
$$+[\gamma^{k}_{ij}\,B^{\cdot\alpha}_{k}+H^{P}_{i\,j}\,B^{\cdot\alpha}_{P}-\frac{\partial B^{\cdot\alpha}_{i}}{\partial x^{j}}]B^{i}_{\cdot\beta}. \tag{3.5}$$

Therefore, when we take arbitralily γ^{k}_{ij}, $H^{P}_{i\,j}$ and H^{Q}_{Pj}, H^{i}_{Pj} that may satisfy the transformation laws (1.14) and (1.15) respectively, and determine the quantities $\Gamma^{\alpha}_{\beta j}$ from the relations (3.5), we see the quantities satisfy the transformation law (1,13) from (1,11), (1,14) and (1.15), that is,

$$\Gamma^{\alpha'}_{\beta' j'}=B^{i'}_{\cdot\beta'}(\gamma^{k'}_{i'j'}B^{\cdot\alpha'}_{k'}+H^{P}_{i'j'}B^{\cdot\alpha}_{P}-\frac{\partial B^{\cdot\alpha}_{i'}}{\partial x^{j'}})+B^{P}_{\cdot\beta'}(H^{Q}_{Pj'}B^{\cdot\alpha'}_{Q}+H^{i'}_{Pj'}B^{\cdot\alpha'}_{i'}-\frac{\partial B^{\cdot\alpha'}_{P}}{\partial x^{j'}})$$

$$=B^{i}_{\cdot\beta}\,A^{\beta}_{\beta'}\frac{\partial x^{i'}}{\partial x^{i}}\left[\left(\frac{\partial^2 x^k}{\partial x^{i'}\partial x^{j'}}+\gamma^{k}_{ij}\frac{\partial x^{i}}{\partial x^{i'}}\frac{\partial x^{j}}{\partial x^{j'}}\right)B^{\cdot\alpha}_{k}\,A^{\alpha'}_{\alpha}-\frac{\partial(B^{\cdot\alpha}_{k}\frac{\partial x^{k}}{\partial x^{i'}}A^{\alpha'}_{\alpha})}{\partial x^{j}}\frac{\partial x^{j}}{\partial x^{j'}}\right]$$

$$+B^{P}_{\cdot\beta}\,A^{\beta}_{\beta'}\left[H^{Q}_{Pj}\frac{\partial x^{j}}{\partial x^{j'}}B^{\cdot\alpha}_{Q}\,A^{\alpha'}_{\alpha}+H^{k}_{Pj}\frac{\partial x^{j}}{\partial x^{j'}}\frac{\partial x^{i'}}{\partial x^{k}}B^{\cdot\alpha}_{i}\frac{\partial x^{i}}{\partial x^{i'}}A^{\alpha'}_{\alpha}-\frac{\partial(B^{\cdot\alpha}_{P}\,A^{\alpha'}_{\alpha})}{\partial x^{j}}\frac{\partial x^{j}}{\partial x^{j'}}\right]$$

$$=A^{\alpha'}_{\alpha}\frac{\partial x^{j}}{\partial x^{j'}}[A^{\beta}_{\beta'}\,\Gamma^{\alpha}_{\beta j}+A^{\alpha}_{\beta',j}].$$

Hence, *when on each point of an n-dimensional space V_n we give the vector field $B^{\cdot\lambda}_{i}$, $B^{\cdot\lambda}_{P}$ and the quantities γ^{k}_{ij}, $H^{P}_{i\,j}$ and H^{Q}_{Pj}, H^{i}_{Pj} which satisfy respectively the transformation laws* (1.14) *and* (1.15), *then we may determine coefficients of dominant affine connection which satisfy the relations* (3.5).

Moreover for the curvature tensor, similarly from (2.1), (2.4) and (1.16), we obtain

$$R^{\alpha}_{\beta jk}=B^{i}_{\cdot\beta}\,B^{\cdot\alpha}_{h}\,[R^{h}_{ijk}+H^{h}_{Pk}\,H^{P}_{i\,j}-H^{h}_{Pj}\,H^{P}_{P\,ik}]$$
$$+B^{i}_{\cdot\beta}\,B^{\cdot\alpha}_{Q}\,[H^{Q}_{i\,j,k}-H^{Q}_{i\,k,j}+H^{P}_{i\,j}H^{Q}_{Pk}-H^{P}_{i\,k}H^{Q}_{Pj}+\gamma^{l}_{ij}\,H^{Q}_{l\,k}-\gamma^{l}_{ik}H^{P}_{l\,j}]$$
$$+B^{P}_{\cdot\beta}\,B^{\cdot\alpha}_{h}\,[H^{Q}_{Pj,k}-H^{h}_{Pk,j}+H^{l}_{Pj}\,\gamma^{h}_{lk}-H^{l}_{Pk}\,\gamma^{h}_{lj}+H^{Q}_{Pj}H^{l}_{Qk}-H^{Q}_{Pk}H^{h}_{Qj}]$$
$$+B^{P}_{\cdot\beta}B^{\cdot\alpha}_{Q}\,[H^{Q}_{Pj,k}-H^{Q}_{Pk,j}+H^{R}_{Pj}\,H^{Q}_{Rj}-H^{R}_{Pk}H^{Q}_{Rj}+H^{l}_{Pj}H^{Q}_{l\,k}-H^{l}_{Pk}H^{Q}_{l\,j}], \tag{3.6}$$

and this is equivalent to (2.1) and (2.4).

§ 4. In this paragraph we shall consider a necessary and sufficient condition that a dominantly affinly connected space be a sub-variety of an affinely connected space.

Consider an m-dimensional space V_m with affine connection where a current point A is given by a system of coordinates $(y^1, y^2, \cdots, y^m)$, and the connection is given by the following equations;

$$dA=A_{\alpha}dy^{\alpha}, \qquad dA_{\alpha}=\Gamma^{\beta}_{\lambda\alpha}\,A_{\beta}\,dy^{\alpha}. \tag{4.1}$$

In V_m we consider an n-dimensional variety V'_n defined by the equations

$$y^{\alpha}=y^{\alpha}(x^{1}, x^{2},\cdots, x^{n}) \tag{4.2}$$

when current point A displaces on V'_n, we have

$$dA = A_\alpha B_i^{\cdot\alpha} dx^i \tag{4.3}$$

where

$$B_i^{\cdot\alpha} = \frac{\partial y^\alpha}{\partial x^i} \tag{4.4}$$

If we define

$$A_i = A_\alpha B_i^{\cdot\alpha}, \qquad A_P = A_\alpha B_P^{\cdot\alpha} \tag{4.5}$$

where $B_P^{\cdot\alpha}$ are $m-n$ contravariant vectors and the determinant $|B_i^{\cdot\alpha}, B_P^{\cdot\alpha}|$ is not equal to zero, we see A_i, A_P are m-linearly independent vectors of V_m. For the displacement on V_n we see

$$dA = A_i dx^i \tag{4.1$'$}$$

and we put

$$dA_i = (\gamma_{ij}^k + H_{i\,j}^P A_P) dx^j, \tag{4.6}$$

$$dA_P = (H_{Pj}^k A_k + H_{Pj}^Q A_Q) dx^j. \tag{4.7}$$

Differentiating (4.5) and comparing with (4.6), (4.7), we obtain

$$\frac{\partial B_i^{\cdot\alpha}}{\partial x^j} = -\Gamma_{P\mu}^\alpha B_i^{\cdot\lambda} B_j^{\cdot\mu} + \gamma_{ij}^k B_k^{\cdot\alpha} + H_{ij}^P B_P^{\cdot\alpha}, \tag{4.8}$$

$$\frac{\partial B_P^{\cdot\alpha}}{\partial x^j} = -\Gamma_{\lambda\mu}^\alpha B_P^{\cdot\lambda} B_j^{\cdot\mu} + H_{Pj}^k B_k^{\cdot\alpha} + H_{Pj}^Q B_Q^{\cdot\alpha}. \tag{4.9}$$

As the quanty $B_i^{\cdot\alpha}$ defined by the equations (4.4) must satisfy the integrability conditions of the equations $\dfrac{\partial^2 y^\alpha}{\partial x^i \partial x^j} = \dfrac{\partial^2 y^\alpha}{\partial x^j \partial x^i}$ we obtain

$$B_i^{\cdot\lambda} B_j^{\cdot\mu} (\Gamma_{\lambda\mu}^\alpha - \Gamma_{\mu\lambda}^\alpha) = B_k^{\cdot\alpha} (\gamma_{ij}^k - \gamma_{ji}^k) + B_P^{\cdot\alpha} (H_{i\,j}^P - H_{j\,i}^P). \tag{4.10}$$

Moreover from the integrability conditions of the system of epuations (4.8) and (4.9), we obtain[4)]

$$\begin{aligned} B_i^{\cdot\lambda} B_j^{\cdot\mu} B_k^{\cdot\nu} R_{\lambda\mu\nu}^\alpha = {} & B_l^{\cdot\alpha} (R_{ijk}^l + H_{i\,j}^P H_{Pj}^l - H_{i\,k}^P H_{P\,j}^l) \\ & + B_P^{\cdot\alpha} (H_{i\,j,k}^P - H_{i\,k,j}^P + H_{i\,j}^Q H_{Qk}^P - H_{i\,k}^Q H_{Qj}^P + \gamma_{ij}^l H_{Pk}^l - \gamma_{ik}^l H_{l\,j}^P), \end{aligned} \tag{4.11}$$

$$\begin{aligned} B_P^{\cdot\lambda} B_j^{\cdot\mu} B_k^{\cdot\nu} R_{\lambda\mu\nu}^\alpha = {} & B_l^{\cdot\alpha} (H_{Pj,k}^l - H_{Pk,j}^l + H_{Pj}^Q H_{Qk}^l - H_{Pk}^Q H_{Pj}^l + H_{Pj}^h \gamma_{hk}^l - H_{Pk}^h \gamma_{hj}^l) \\ & + B_Q^{\cdot\alpha} (H_{Pj,k}^Q - H_{Pk,j}^Q + H_{Pj}^R H_{Rk}^Q - H_{Pk}^R H_{Rj}^Q + H_{Pj}^l H_{l\,k}^Q - H_{Pk}^l H_{l\,j}^Q). \end{aligned} \tag{4.12}$$

In the present case the quantities $\Gamma_{\beta j}^\alpha$ are given (i.e. the quantities $B_i^{\cdot\alpha}$, $B_P^{\cdot\alpha}$, $H_{i\,j}^P$, γ_{ij}^k,

(4) M. Matsumoto; Affinely connected spaces of class one.
Mem. of Colleg. of Science. Univ. of Kyoto. Vol. 26. (1951) p.p. 235～249.

H^P_{Qj} and H^i_{Pj} are given), that is to say, considering space V_n is a space with dominant affine connections, and we should like to determine the quantities $\Gamma^\alpha_{\beta\lambda}$ which are the coefficients of affine connection of enveloping space V_m. Obviously y^α (the coordinate system of V_m) must satisfy the relations

(4.13) $$B_i^{\cdot\alpha}=\frac{\partial y^\alpha}{\partial x^i},$$

and on V_n

(4.14) $$dy^\alpha=B_i^{\cdot\alpha}\,dx^i,$$

then we obtain

(4.15) $$\Gamma^\beta_{\lambda i}=\Gamma^\beta_{\lambda\alpha}\,B_i^{\cdot\alpha}.$$

Hence,these functions $\Gamma^\beta_{\lambda\alpha}$ must satisfy firstly the transformation laws of coefficients of connection, secondly the equations (4.14), i.e., the integrability conditions of system of equations (4.13), and finaly (4.11) and (4.12), i.e., the integrability conditions of (4.8) and (4.9).

When these three sorts of conditions are satisfied, we may defin $(y^1, y^2,\cdots,y^m)$ which are the solutions of the system of equations

$$B_i^{\cdot\alpha}=\frac{\partial y^\alpha}{\partial x^i},$$

and find the m-dimensional space V_m with coefficients ofaffine connection $\Gamma^\beta_{\lambda\alpha}$ which has a system of coordinate y^α defined above and the considering space V_n as an n-dimensional sub-variety.

Consequently three sorts of conditions are the necessary and sufficient conditions that the space V_n can be embedded in an m-dimensional affinely connected space as an n-dimensional variety.

From (4.15) we must put

(4.16) $$\Gamma^\alpha_{\lambda\mu}=B^{\cdot i}_{\cdot\mu}\,\Gamma^\alpha_{\lambda i}+B^{\cdot P}_{\cdot\mu}\,\Gamma^\alpha_{\lambda P},$$

where the quantities $\Gamma^\alpha_{\lambda P}$ are arbitraty functions of x^i.

Now we consider the transformation of coordinates y^α and x^i, then $\Gamma^\alpha_{\beta i}$ are transformed by the relations

$$\Gamma^{\beta'}_{\alpha' i'}=A^{\beta'}_\beta(A^\alpha_{\alpha'}\Gamma^\beta_{\alpha i}+A^\beta_{\alpha',i})\frac{\partial x^i}{\partial x^{i'}},$$

where we must put

(4.17) $$\frac{\partial y^{\alpha'}}{\partial y^\alpha}=A^{\alpha'}_\alpha,$$

So we see $$B_{i'}^{\cdot\mu'}\,\Gamma^{\beta'}_{\alpha'\mu'}=A^{\beta'}_\beta(A^\alpha_{\alpha'}\,B_i^{\cdot\mu}\Gamma^\beta_{\alpha\mu}+A^\beta_{\alpha',i})\,\frac{\partial x^i}{\partial x^{i'}}.$$

and from (4.13) $$B_{i'}^{\cdot\,\mu'}=\frac{\partial y^{\mu'}}{\partial x^{i'}}=A_{\mu}^{\mu'}\,B_{i}^{\cdot\,\mu}\frac{\partial x^{i}}{\partial x^{i'}}.$$

From these equations we obtain

$$\Gamma^{\beta'}_{\alpha'\mu'}=\frac{\partial y^{\beta'}}{\partial y^{\beta}}\left[\frac{\partial^2 y^{\beta}}{\partial y^{\alpha'}\partial y^{\mu'}}+\Gamma^{\beta}_{\alpha\mu}\frac{\partial y^{\mu}}{\partial y^{\mu'}}\frac{\partial y^{\alpha}}{\partial y^{\alpha'}}\right], \tag{4.18}$$

that is to say, the quantities $\Gamma^{\alpha}_{\lambda\mu}$ defined by (4.16) satisfy the transformation laws of the coefficients of affine connection. Hence we see that the first conditions are satisfied by the equations (4.13), (4.14) and (4.15).

Next, we consider the second conditions, while these conditions are the relations (4.10).

On the other hand from the relation

$$\Gamma^{\alpha}_{\lambda\mu}\,B_{j}^{\cdot\,\mu}=\Gamma^{\alpha}_{\lambda j},$$

we obtain

$$B^{k}_{\cdot\,\alpha}(B_{i}^{\cdot\,\lambda}\,\Gamma^{\alpha}_{\lambda j}-B_{j}^{\cdot\,\lambda}\,\Gamma^{\alpha}_{\lambda i})=\gamma^{k}_{ij}-\gamma^{k}_{ji}, \tag{4.19}$$

$$B^{P}_{\cdot\,\alpha}(B_{i}^{\cdot\,\lambda}\,\Gamma^{\alpha}_{\lambda j}-B_{j}^{\cdot\,\lambda}\,\Gamma^{\alpha}_{\lambda i})=H^{P}_{i\,j}-H^{P}_{j\,i}. \tag{4.20}$$

Finally the last conditions are the equations (4.11) and (4.12). While the relations (2.1) and (2.4) must be satisfied in V_n, the equations (4.11) and (4.12) are identically satisfied when the relations (4.13) are satisfied since we can obtain

$$R^{\alpha}_{\beta jk}=R^{\alpha}_{\beta\mu\lambda}\,B_{j}^{\cdot\,\mu}\,B_{k}^{\cdot\,\lambda}. \tag{4.21}$$

For (4,21) from (4,13) we see $B_{j}^{\cdot\,\mu}{}_{,k}=B_{k}^{\cdot\,\mu}{}_{,j}$,

then
$$\begin{aligned}R^{\alpha}_{\beta jk}&=\Gamma^{\alpha}_{\beta j,k}-\Gamma^{\alpha}_{\beta k,j}+\Gamma^{\alpha}_{\sigma j}\,\Gamma^{\sigma}_{\beta j}-\Gamma^{\alpha}_{\sigma k}\Gamma^{\sigma}_{\beta j}\\&=\Gamma^{\alpha}_{\beta\mu,\lambda}\,B_{k}^{\cdot\,\lambda}\,B_{j}^{\cdot\,\mu}-\Gamma^{\alpha}_{\beta\lambda,\mu}\,B_{\mu}^{\cdot\,\lambda}\,B_{j}^{\cdot\,\mu}+\Gamma^{\alpha}_{\beta\mu}(B_{j}^{\cdot\,\mu}{}_{,k}-B_{k}^{\cdot\,\mu}{}_{,j})\\&\quad+B_{j}^{\cdot\,\mu}\,B_{k}^{\cdot\,\lambda}(\Gamma^{\alpha}_{\sigma\mu}\,\Gamma^{\sigma}_{\beta\lambda}-\Gamma^{\alpha}_{\sigma\lambda}\,\Gamma^{\sigma}_{\beta\mu})\\&=R^{\alpha}_{\beta\mu\lambda}B_{j}^{\cdot\,\mu}B_{k}^{\cdot\,\lambda}.\end{aligned}$$

Consequently the relations (4.11) and (4.12) are equivalent in consequences of (2.1) and (2.4) respectively, that is to say, in consequences of (3.6).

A necessary and sufficient condition that an n-dimensional space with dominant affine connections be an n-dimensional variety of an m-dimensional space with affine connection is that the relations (4.19), (4.20) and (3.6) *be satisfied.*

§ 5. In this paragraph we consider the case where γ^{i}_{jk} are equal to γ^{i}_{kj}. While on a space with symmetric affine connection the equations of geodesic lines are

$$\frac{\delta}{ds}\left(\frac{dx^{i}}{ds}\right)=0$$

that is to say, $$\frac{d^2x^i}{ds^2}+\gamma^{\,i}_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0,$$

where s is affine parameter.

Now in the space V_n with dominant affine connection when we put

$$V^\lambda=\frac{dx^i}{ds}B_i^{\cdot\lambda}, \tag{5.1}$$

the condition that a curve $x^i=x^i(s)$ is developed into a straight line when we develop V_n in an m-dimensional affine space along this curve is written

$$\frac{\delta}{ds}(V^\lambda)=0 \tag{5.2}$$

we have $$\Big(H^P_{j\,k}\frac{dx^j}{ds}\frac{dx^k}{ds}\Big)B_P^{\cdot\lambda}+\Big(\frac{d^2x^i}{ds^2}+\gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}\Big)B_i^{\cdot\lambda}=0$$

Consequently we obtoin

$$\frac{d^2x^i}{ds^2}+\gamma^{\,i}_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0, \tag{5.3}$$

$$H^P_{j\,k}\frac{dx^j}{ds}\frac{dx^k}{ds}=0 \tag{5.4}$$

Then we call the curves which are solutions of the system of equations (5.3) and (5.4) geodesic line in V_n and asymptotic line respectively.

If a curve in V_n can be developed into a straight line in A_m this curve must be geodesic and asymptotic line in V_n. Especially the necessary and sufficient condition that all geodesic lines in V_n can be developed into straight lines is

$$H^P_{i\,j}+H^P_{j\,i}=0. \tag{5.5}$$

While geodesic lines and asymptotic lines are respectively same for two connections whose coefficients are in the relations

$$\bar{\gamma}^i_{jk}=\gamma^i_{jk}+\delta^i_j\,\psi_k+\delta^i_k\,\psi_j, \tag{5.6}$$

$$\bar{H}^P_{j\,k}=\rho H^P_{j\,k}+\Omega^P_{j\,k}. \tag{5.7}$$

where ψ_i is an arbitrary covariant vector, ρ is an arbitrary scalar and $\Omega^P_{j\,k}$ are p arbitrary skew symmetric tensors. In the equations (3.5) for $\bar{\Gamma}^\alpha_{\beta j}$, substituting for $\bar{\gamma}^i_{jk}$ and $\bar{H}^P_{j\,k}$ their expressions (5.6) and (5.7) respectively, we obtain

$$\bar{\Gamma}^\alpha_{\beta j}=\Gamma^\alpha_{\beta j}+B^k_{\cdot\beta}\,B_k^{\cdot\alpha}\,\psi_j+B^i_{\cdot\beta}\,B_j^{\cdot\alpha}\,\psi_i+(\rho-1)B^i_{\cdot\beta}\,B_P^{\cdot\alpha}\,H^P_{i\,j}+\Omega^P_{i\,j}B^i_{\cdot\beta}\,B_P^{\cdot\alpha}. \tag{5.8}$$

Conversly in the case where the quantities $\Gamma^\alpha_{\beta j}$ and $\bar{\Gamma}^\alpha_{\beta j}$ are connected with the relation (5.8) we obtain (5.6), (5.7) and

$$\bar{H}^P_{Qj}=H^P_{Qj},\qquad \bar{H}^i_{Qj}=H^i_{Qj}$$

Consequently, *the geodesic lines and asymptotic lines are same respectively for two connections whose coefficients are in the relations* (5.8).

We say that the dominant affine connection of coefficients $\bar{\Gamma}^{\alpha}_{\beta j}$ is obtaind from that with the cefficients $\Gamma^{\alpha}_{\beta j}$ by a projective change of the dominnat affine connection.

We see from (1.13)

$$\Gamma^{\lambda}_{\mu i}=A^{\lambda}_{\mu'}\frac{\partial x^{i'}}{\partial x^{i}}[A^{\nu'}_{\mu}\ \Gamma^{\mu'}_{\nu' i'}+A^{\mu'}_{\mu\ ,i'}]. \tag{5.9}$$

Contracting for λ and μ we obtain

$$\Gamma^{\lambda}_{\lambda i}=\frac{\partial x^{i'}}{\partial x^{i}}\Gamma^{\mu'}_{\mu' i'}+\frac{\partial \log \Delta}{\partial x^{i}}, \tag{5.10}$$

where

$$\Delta=|A^{\mu'}_{\lambda}| \tag{5.11}$$

On the other hand from the relation (5.8) contracting for α and β we have

$$\psi_j=\frac{1}{n+1}(\bar{\Gamma}^{\alpha}_{\alpha j}-\Gamma^{\alpha}_{\alpha j}) \tag{5.12}$$

from which and (5.8) we find that *the quantities*

$$\Pi^{\alpha}_{\beta j}=\Gamma^{\alpha}_{\beta j}-\frac{1}{n+1}\{B^{k}_{\cdot\beta}\,B^{\cdot\alpha}_{k}\ \Gamma^{\lambda}_{\lambda j}+B^{\cdot\alpha}_{j}\ B^{k}_{\cdot\beta}\ \Gamma^{\lambda}_{\lambda k}\}-B^{i}_{\cdot\beta}\,B^{\cdot\alpha}_{P}H^{P}_{i\ j} \tag{5.13}$$

are independent of a projective change of dominant affine connection.

For the relations between the function Π^{i}_{jk} and the analogous function in a coordinate system $x^{i'}$ we find from the relations (5.13), (5.19), (1.11) and (1.14)

$$\Pi^{\alpha}_{\beta j}A^{\lambda'}_{\beta}=A^{\lambda'}_{\beta}\frac{\partial x^{i'}}{\partial x^{j}}\Pi^{\beta'}_{\lambda' j'}+A^{\beta'}_{\beta\ ,j}-\frac{1}{n+1}B^{k'}_{\cdot\alpha'}\,B^{\cdot\beta'}_{k'}\ A^{\alpha'}_{\beta}\frac{\partial \log \Delta}{\partial x^{\alpha}}-\frac{1}{n+1}B^{k}_{\cdot\beta}B^{\cdot\beta'}_{j'}\frac{\partial x^{j'}}{\partial x^{j}}\frac{\partial \log \Delta}{\partial x^{k}}. \tag{5.14}$$

Moreover we find that *the quantities*

$$W^{\alpha}_{\beta jk}=B^{\cdot\alpha}_{h}\ B^{i}_{\cdot\beta}\ W^{h}_{i jk} \tag{5.15}$$

and

$$\begin{aligned} W'^{\alpha}_{\beta jk}=&B^{\cdot\alpha}_{h}B^{i}_{\cdot\beta}\ W^{h}_{ijk}+B^{\cdot\alpha}_{Q}\ B^{P}_{\cdot\beta}(H^{Q}_{Pj,k}-H^{Q}_{Pk,j}+H^{R}_{Pj}\ H^{Q}_{Rk}-H^{R}_{Pk}H^{Q}_{Rj})\\ &+B^{P}_{\cdot\beta}\ B^{\cdot\alpha}_{h}(H^{h}_{Pj,k}-H^{h}_{Pk,j}+H^{l}_{Pj}\ \gamma^{h}_{lk}-H^{l}_{pk}\ \gamma^{h}_{lj}+H^{Q}_{Pj}\ H^{h}_{Qk}-H^{Q}_{Pk}\ H^{h}_{Qj})\\ &-\frac{1}{n+1}(H^{h}_{Pj}\ \Gamma^{\sigma}_{\sigma k}+H^{h}_{Pk}\ \Gamma^{\sigma}_{\sigma j}+\delta^{h}_{k}\ H^{l}_{Pj}\ \Gamma^{\sigma}_{\sigma l}+\delta^{h}_{j}H^{l}_{P}\ \Gamma^{\sigma}_{\sigma k}) \end{aligned} \tag{5.16}$$

are independent of a projective change of dominant affine connection, where W^{h}_{ijk} *is so-called Weyl projective curvature tensor for* γ^{i}_{jk}.

Journal of Gakugei, Tokushima Uoiversity
Vol. VII. 1956.

A NOTE ON SUBORDINATION

Hitoshi ABE

(Received September 30, 1956)

1. Introduction The following result is well known as Hurwitz-Bochner's theorem [1][2].(1)

If $w=f(z)$ *is regular in* $|z|<1$, $f(0)=0$, $f'(0)=1$, *and* $f(z)\neq 0$

except $z=0$, *the conformal image of* $f(z)$ *assumes every value in* $|w|<1/16$. In the present paper we generalize this theorem and deal with the related problems with it by the principle of subordination.

As a preliminary remark we shall give a notion of $Q(z)$ whose properties are as follows [3].

$$Q(z)=16z\prod_{n=1}^{\infty}\left(\frac{1+z^{2n}}{1+z^{2n-1}}\right)^{8},\qquad (|z|<1).$$

$$Q(z)=J\left(\frac{1}{\pi i}\log z\right),\text{ where } J(z) \text{ is the elliptic modular function.}$$

Let the surface M be the conformal image of $|z|<1$ by $Q(z)$. M has no branch point. M covers every point of w-plane except $w=0,1,\infty$. M does not cover $w=1,\infty$, but the $w=0$ is covered by one sheet of M only.

2. Lemma 1. $\qquad |Q(z)|\leqq Q(-r),\qquad (|z|=r<1).$

Proof. Each factor of infinite products which constitutes $Q(z)$ has its greatest absolute value on $|z|=r$ only when $z=-r$, and therefore we have the above estimate clearly.

Lemma 2. *Let* $w=f(z)=a_1z+\cdots\cdots$, *be regular and* $f(z)\neq 0$ *except* $z=0$ *in* $|z|<1$. *If* $f(z)$ *omits a value* α *in* $|z|<1$, *the following estimates are got.*

(i) $|a_1|\leqq 16|\alpha|$, *that is, the conformal image of* $f(z)$ *assumes every value in* $|w|<|a_1|/16$.

(ii) $|f(z)|\leqq|\alpha|\cdot Q(-r),\qquad (|z|=r<1).$

Proof. Let us consider

$$P(z)=Q^{-1}\left(\frac{f(z)}{\alpha}\right),$$

where $Q^{-1}(w)$ is the inverse function of $Q(z)$.

$f(z)$ leaves out $w=0$ except $z=0$ in $|z|<1$, and therefore $P(z)$ is analytic in $0<|z|<1$. On the other hand the regularity of $P(z)$ hold good at $z=0$. Hence $f(z)/\alpha$ is subordinate to $Q(z)$ and by the principle of subordination we get

(1) The bracket denotes the number of the references.

$$|\alpha^{-1}f'(0)|\leqq Q'(0)$$

$$|\alpha^{-1}f(z)|\leqq|Q(z)|\leqq Q(-r) \quad (|z|=r<1).$$

These complete the proof.

Remark. The above estimates are sharp as is shown by $f_0(z)=\alpha Q(z)$. $|\alpha|$ in the latter result of this lemma is can not be made smaller than $|a_1|/16$. The former result was got by the same method by Bochner [1][3].

Theorem 1. *Let* $w=f(z)=a_p z^p+\cdots\cdots$, *be regular and* $f(z)\neq 0$ except $z=0$ *in* $|z|<1$.

(i) *The values taken by* $w=f(z)$ *cover the circle* $|w|<|a_p|/16^p$ p times or more times.

(ii) *The conformal image of the unit circle by* $w=f(z)$ *completely covers the interior of a circle about the origin whose rudius is* $|a_p|/16$. *We can state this result in detail, namely,*
If $w=f(z)$ omit a value α,

(ii′) $$|a_p|\leqq 16|\alpha|$$

(iii) $$|f(z)|\leqq|\alpha|\cdot Q(-r^p) \qquad (|z|=r<1).$$

These bounds are best possible.

Proof.

First in order to prove the former result we consider $g(z)=(f(z))^{1/p}$. Since $f(z)\neq 0$ except $z=0$ in $|z|<1$, $g(z)$ is regular and $g(z)\neq 0$ except $z=0$ in $|z|<1$. Hence when lemma 2 is used with respect to $g(z)$, the proof of (i) will be given.

Secondly we prove the latter results (ii) and (iii). Let us consider

$$F(z)=Q(z^p)=16z^p+\cdots\cdots.$$

The Riemann surface onto which the unit circle is mapped by $F(z)$ has no branch point, and covers every point in w-plane infinite times except $w=0,1,\infty$. It does not cover $w=1$, ∞, but $w=0$ is covered by its p sheets only.

Now we put

$$R(z)=F^{-1}\left(\frac{f(z)}{\alpha}\right).$$

Under the same conditions in the proof of lemma 2, $R(z)$ is regular in $0<|z|<1$, and $R(z)$ is regular at $z=0$ more. We can get easy

$$|R'(0)|=\left|\frac{a_p}{16\alpha}\right|^{\frac{1}{p}}$$

Furthermore $$R(0)=0, \qquad |R(z)|<1.$$

Hence $$|R'(0)|\leqq 1, \text{ that is, } |a_p|\leqq 16|\alpha|.$$

On the other hand $f(z)$ is subordinate to $\alpha F(z)$ and therefore

$$|f(z)|\leqq\max_{|z|=r}|\alpha F(z)|\leqq|\alpha|\cdot Q(-r^p)(|z|=r<1).$$

Theorem 2. *Let $w=f(z)=a_1z+\cdots\cdots$, be regular and $f(z)\neq 0$ except $z=0$ in $|z|<1$. If $f(z)$ leaves out two values α and $-\alpha$, then*

$$|a_1|\leq 4|\alpha|, \qquad |f(z)|\leq|\alpha|\cdot Q^{\frac{1}{2}}(-r^2).$$

Namely, if $w=f(z)=z+\cdots\cdots$, is an odd regular function in $|z|<1$ and $f(z)\neq 0$ except $z=0$, the values taken by $w=f(z)$ cover fully $|w|<1/4$. These bounds are best possible.(2)

Proof. As the superordinate function to $f(z)/\alpha$ we consider

$$Q_1(z)=\sqrt{Q(z^2)}=4z+\cdots\cdots\cdots.$$

And then $Q_1(z)$ is an odd regular function in $|z|<1$ and the other properties of $Q_1(z)$ are quite similar to ones of $Q(z)$ except the fact that it does not assume both 1, and -1[2]. Namely $Q_1^{-1}\left(\frac{f(z)}{\alpha}\right)$ is analytici n $|z|<1$ like the case of lemma 1 and therefore we have the forme result of this theorem by the principle of subordination. The latter result is evident.

We can moreover generalize this theorem, that is,

Theorem. 2′. *Let $w=f(z)=a_pz^p+\cdots\cdots$, be regular in $|z|<1$, and $f(z)\neq 0$ except $z=0$. If $f(z)$ leaves out the values α, and $-\alpha$, then*

$$|a_p|\leq 4|\alpha|, \quad |f(z)|\leq|\alpha|\cdot Q^{\frac{1}{2}}(-r^{2p}), \quad (|z|=r<1).$$

Proof. If we consider the superordinate function

$$Q^{\frac{1}{2}}(z^{2p})=4z^p+\cdots\cdots\cdots\cdots,$$

the results are evident.

Theorem 3. *Let $w=f(z)=a_p\, z^p+\cdots\cdots$, be regular and not zero except $z=0$ in $|z|<1$. If $f(z)$ leaves out the values $-\infty\leq w\leq\frac{-1}{4}$, $f(z)$ is subject to the following inequalities.*

$$|a_p|\leq 1, \quad |f(z)|\leq\frac{r^p}{(1-r^p)^2}, \ (|z|=r<1)$$

These bounds are sharp as is shown by

$$F(z)=\frac{z^p}{(1-z^p)^2}$$

Proof. As the superordinate function to $f(z)$ we consider $F(z)$. The Riemann surface onto whieh the unit circle is mapped by $F(z)$ has no branch point except $z=0$. Hence

$$S(z)=F^{-1}(f(z))$$

is analytic in $0<|z|<1$. On the other hand $S(z)$ has regularity at $z=0$ whose derivertive has $(a_p)^{\frac{1}{p}}$ there. Moreover $S(0)=0$ and $|S(z)|<1$. Hence we have the above results.

---

(2) Ifthe condition that $f(z)\neq 0$ except $z=0$, is omitted, we have the following result, which is well known [3].

$$|a_1|\leq k|\alpha|, \qquad k=\Gamma^4(\tfrac{1}{4})/4\pi^2$$

We can get the following estimates by means of the same method also.

Theorem 4. *If* $w=f(z)=a_p z^p+\cdots\cdots$, *is regular in* $|z|<1$, *leaves out the values* $-\infty\leqq w\leqq-\alpha$, *and* $\alpha\leqq w\leqq\infty$, $(\alpha>0)$, *and vanishes at* $z=0$ *only,*

$$|a_p|\leqq 2\alpha, \qquad |f(z)|\leqq\frac{2\alpha r^p}{1-r^{2p}}. \qquad (|z|=r).$$

These bounds are best possible as is shown by

$$F(z)=\frac{2\alpha z^p}{(1+z^{2p})}$$

3. We consider the case where $f(z)$ is meromorphic in $|z|<1$.

Theorem 5. *Let* $w=f(z)=a_1 z+a_3 z^3+\cdots\cdots$, *be an odd meromorphic function in* $|z|<1$ *and* $f(z)\neq 0$ *except* $z=0$, *then the image by this function covers fully the cirle*

$$|w|<\frac{|a_1|}{8}.$$

The result is best possible as is shown by

$$f_0(z)=\frac{2Q_1(z)}{1+Q_1^2(z)}$$

Proof. It is clear that $f_0(z)$ is an odd meromorphic function and does not take 1 and -1 b because of the property of $Q_1(z)$.
Let us consider

$$F(z)=\frac{1}{1-f_0(z)}=\frac{1+Q_1^2(z)}{(1-Q_1(z))^2}$$

Then

$$F'(z)=\frac{2(1+Q_1(z))}{(1-Q_1(z))^3}Q_1'(z)\neq 0 \quad (|z|<1),$$

because $Q_1'(z)\neq 0$ in $|z|<1$.

If $f(z)$ leaves out α and $-\alpha$, we consider $g(z)=f(z)/\alpha$. The function $(1-g(z))^{-1}$ is subordinate to $F(z)$ because of the proerty of $F(z)$. And therefore

$$\left|\frac{a_1}{\alpha}\right|\leqq F'(0)=8$$

This completes the proof.

Remark. If $f(z)=a_1 z\cdots\cdots$, is meromorphic and vanishes at $z=0$ only in $|z|<1$, $f(z)$ takes at least one value of each coupl $\pm w$ belonging to the circl

$$|w|<\frac{|a_1|}{8}$$

The proof is quite similar if $Q(z)(2-Q(z))^{-1}$ is considered as the superordinate function.

Here we state the related result with this theorem.

Theorem 6. *Let* $f(z)=a_1z+\cdots\cdots$, *be an odd meromorphic function in* $|z|<1$, *then the image by* $f(z)$ *covers fully the circle*

$$|w|<\frac{2|a_1|\pi^2}{\Gamma^4(\frac{1}{4})}$$

This result is best possible.

Proof. Like the case in theorem 5 we consider

$$f_0(z)=\frac{2F(z)}{1+F^2(z)},\quad F(z)=2J\left[i\left(\frac{1+z}{1-z}\right)\right]-1.$$

$F(z)$ leaves out 1 and -1, but every value except these values is taken infinite times. Furthermore $F'(z)\neq 0$ in $|z|<1$, because $J(z)$ has no branch point [3]. Hereafter we may do like the proof of theorem 5.

References

[1] S. Bochner, Remarks on the theorems of Picad-Landau and Picard-Schottky, J. London Math. Soc, (1926), pp.100-103.

[2] J. N. Littlewood, Lectures on the theory of functions, Oxford (1944), pp. 196–197.

[3] Z. Nehari, Conformal Mapping, Mcgraw-Hill, (1952), pp. 226-236, pp. 318-330.