Journal of Gakugei, Tokushima University
Vol. VII. 1956

ON A SPECIAL SEMILATTICE WITH A MINIMAL CONDITION.

By

Takayuki TAMURA

(Received September 30, 1956)

By a semilattice we mean a commutative idempotent semigroup, namely, a partly ordered set which has a least upper bound of any two elements[1]. In the present paper we shall discuss the structure of a special semilattice, which will be called an unbounded dispersed semilattice with a certain minimal condition.

§1. Flowing Semilattice.

Let S be a semilattice and let $a, b, c, \cdots, x, y, \cdots$ be elements of S. At first we explain the notations $[b,c]$ $[a,*)$ $(*,a]$ $(*,a)$ as following.

For $b \leqq c$
$$[b,c]=\{x;\, b\leqq x\leqq c,\ x\in S\},$$
$$[a,*)=\{x;\, a\leqq x,\ x\in S\},$$
$$(*,a]=\{x;\, x\leqq a,\ x\in S\},$$
$$(*,a)=\{x;\, x< a,\ x\in S\}.$$

Lemma 1. *The following conditions are all equivalent.*

(1.1) *If $b<c$, then $[b,c]$ is a chain in S.*

(1.2) *For any $a\in S$, $[a,*)$ froms a chain in S.*

(1.3) *For any a, x, $y\in S$, either $ax\geqq ay$ or $ax\leqq ay$.*

(1.4) *There are no x,y,z such that $z<x$, $z<y$, and $x \gtreqless\!\!\!\!/\; y$ hold simultaneously. In other words, there is no lower bound of incomparable x and y.*

Proof. (1.1)→(1.2). If $[a,*)$ is not a chain for some a, there are incomparable $x,y\in[a,*)$. Let z be a least upper bound of x and y. Then both x and y belong to $[a,z]$. This contradicts with (1.1).

(1.2)→(1.3.) Suppose that ax and ay are incomparable. Considering $a\leqq ax$, $a\leqq ay$, that is, $[a,*)$ is not a chain. This conflicts with (1.2).

(1.3)→(1.4). If (1.4) is false, there are x, y, z such that $z\leqq x$, $z\leqq y$ and $x \gtreqless\!\!\!\!/\; y$, in other words, $x=zx'$, $y=zy'$, and $zx' \gtreqless\!\!\!\!/\; zy'$, contradicting with (1.3).

(1.4)→(1.1). Suppose that (1.1) is not valid. A certain set $[b,c]$ is not a chain.

1) See G. Birkhoff, Lattice theory. In the partly ordered set, $a\geqq b$ is defined as $ab=a$. Accordingly xy is a least upper bound of x and y. $a>b$ means $a\geqq b$ but $a\neq b$.

Then there are incomparable x and y such that $b<x<c$ and $b<y<c$. This conflicts with (1.4).

If a semilattice S satisfies the condition of Lemma 1, S is called a *flowing semilattice.*

Lemma 2. *In a flowing semilattice S, if $ab>ac$, then $ab=bc$. Conversely if $ab=bc$ and $bc\neq ac$, then $ab>ac$.*

Proof. According to (1.3) of Lemma 1, any two of the three elements ab, bc, and ca are comparable so that they form a chain. We consider the three cases in the present Lemma.

(i) $ab>ac>bc$, (ii) $ab\geqq bc\geqq ac$,

(iii) $bc>ab>ac$.

However we can show that (i) and (iii) are impossible in the following manner. If (i) holds, $ac\geqq a$, $ac>bc\geqq b$, hence $ac\geqq ab$, contradicting with the assumption $ab>ac$. In the case of (iii), $ab\geqq b$, $ab>ac\geqq c$, hence $ab\geqq bc$. This also conflicts with the inequality (iii) $bc>ab$. Thus we have proved possibility of (ii). Now, from $bc\geqq b$, $bc\geqq ac\geqq a$, it follows that $bc\geqq ab$ and so $ab=bc$. We shall prove the latter half of this lemma. Since $ab=bc\neq ac$, either $ab>ac$ or $ac>ab$ by (1.3) of Lemma 1. If $ac>ab$, we obtain $ac=bc$ by ther former half of of this lemma. This contradicts with the assumption. Hence we have only $ab>ac$.

Theorem 1. *Let S be a flowing semilattice and let a,b,c be any elements of S. Then two at least of the three elements ab,bc, and ca are equal. Conversely if a semilattice S satisfies this condition, it is a flowing semilattice.*

Proof. As far as the former half of the theorem is concerned, we may show that only one of the following four cases arises:

(1) $ab=bc=ca$, (2) $ca<ab=bc$,

(3) $ab<ca=bc$, (4) $bc<ab=ca$.

According to (1.3), ab and ca are comparable. By Lemma 2, $ab>ca$ implies $ab=bc$, $ab<ca$ implies $ac=bc$; if $ab=ca$ and $ac\neq bc$, then $ab>bc$. Therefore the above four cases are obtained.

Conversely suppose that S satisfies the above conditions, neverthless, that S is not flowing. By Lemma 1, there are incomparable x,y and their lower bound z: $z<x$, $z<y$, $x\neq y$. Of course $zx=x$, $zy=y$, and $xy\neq x$, $xy\neq y$, because x and y are incomparable. Therefore any two of the three elements zx, zy, and xy are not equal. This contradicts with the assumption. Thus the theorem has been proved.

§ 2. Dispersed Semilattice.

Lemma 3. *In a flowing semilattice S, the following conditions ase all equivalent.*

(2.1) *For any $b,c\in S$, $b<c$, $[b,c]$ is finite.*

(2.2) *For any $a\in S$, $[a,*)$ is mapped isomorphically into the chain composed of all positive* *integers.*

(2.3) *Any maximal chain of S is mapped isomorphically into the chain of all integers.*

Proof. (2.1)→(2.2) Let x be an element of $(a,*)$. Since $[a,x]$ is finite, a positive integer k is determined such that

$$a=x_0<x_1<\cdots<x_k=x.$$

It is clear that the correspondence $x\leftrightarrow k$ is one to one and preserves the ordering.

(2.2)→(2.3). Let C be a maximal chain of S, and a be any fixed element of C. The subset $[a,*)$ of C consists of

$$a=x_0<x_1<x_2<\cdots<x_n<\cdots.$$

If the subset $(*,a)$ of C is not empty, then, for any $z\in(*,a)$, $[z,*)$ is isomorphic into the set of all positive integers, and a is certainly contained in $[z,*)$,

$$z=z_0<z_1<\cdots<z_l=a,$$

that is, $[z,a]$ is finite. We rewrite them as follows:

$$z=z_0=x_{-l},\ z_1=x_{-l+1},\cdots,\ z_{l-1}=x_{-1},\ a=z_l=x_0,$$

and then $z=x_{-l}<x_{-l+1}<\cdots<x_{-1}<x_0=a$,

so $x\to -l$ preserves the ordering.

(2.3)→(2.1). This is obvious.

If a flowing semilattice S satisfies the conditions of Lemma 3, then S is called a *dispersed semilattice.*

Lemma 4. *If S is a dispersed semilattice, then S is a complete semilattice, in other words, S contains the least upper bound of any subset.*

Proof. Let b be an upper bound of any subset T, and a an element of T. Since $[a,b]$ is finite by Lemma 3, we can find the least p_0 of elements which are upper bounds of T and belong to $[a,b]$. In the following manner, it is proved that this p_0 is required one. Let u be any upper bound of T. According to Theorem 1, the two at least of the three elements ab, au, and bu are equal, that is, one of the following identities (1), (2), and (3) holds.

(1) $b=ab=au=u,$ (2) $bu=au=u,$

(3) $bu=ab=b.$

In all cases, it is concluded that either $u\in[a,b]$ or $b\leqq u$, consequently we have $p_0\leqq u$ i.e. p_0 is the least upper bound of T.

Now we shall define a terminology, the *length of an element.*

Since $[a,x]$, for $a<x$, is finite, a non-negative integer k is determined such that

$$a=x_0<x_1<\cdots<x_k=x.$$

This k is called the length of an element x to a, and k is denoted by $k=l_a(x)$. We make a promise $l_a(a)=0$.

Hereafter S denotes a dispersed semilattice.

Lemma 5. *Let* $a\leqq x$, $a\leqq y$.

(2.4) $x=y$ *if and only if* $l_a(x)=l_a(y)$.

(2.5) $x>y$ *if and only if* $l_a(x)>l_a(y)$.

(2.6) *If* $a\leqq b\leqq c$, *then* $l_a(b)+l_b(c)=l_a(c)$.

Proof. By Lemmas 1 and 3, $[a,*)$ is a chain and $[a, ax]$, $[a, ay]$ are finite. This lemma is obvious.

Lemma 6. $ab=ac$ *if and only if*

(2.5) $l_b(ab)+l_c(bc)=l_b(bc)+l_c(ca)$.

Proof. Necessity of (2.5). Let $ab=ac=p$.
$p\geqq b$, and $p\geqq c$ imply $p\geqq bc\geqq b$ and $p\geqq bc\geqq c$. By Lemma 5,

$$l_b(p)=l_b(bc)+l_{bc}(p), \qquad l_c(p)=l_c(bc)+l_{bc}(p).$$

From the two identities, we get (2.5) directly.

Sufficiency of (2.5). Suppose $ab>ac$ under (2.5). According to Lemmas 2 and 5, $ab=bc$, $l_b(ab)=l_b(bc)$. By (2.5), we have $l_c(bc)=l_c(ca)$, so $bc=ca$; consequently $ab=bc=ca$, contradicting with $ab>ac$. This leads to $ab\not>ac$, similarly $ab\not<ac$. Hence $ab=ac$.

Gathering Theorem 1 and Lemma 6 into together,

Lemma 7. *If S is a dispersed semilattice, then, for any* a,b, $c\in S$, *one of the following identities holds.*

(2.5) $\quad l_b(ab)+l_c(bc)=l_b(bc)+l_c(ca)$,

(2.6) $\quad l_c(bc)+l_a(ca)=l_c(ca)+l_a(ab)$

(2.7) $\quad l_a(ca)+l_b(ab)=l_a(ab)+l_b(bc)$.

In detail,

(2.5), (2.6), *and* (2.7) *hold at the same time if and only if* $ab=bc=ca$.

(2.5) *holds if and only if* $bc<ab=ac$.

(2.6) *holds if and only if* $ca<bc=ba$.

(2.7) *holds if and only if* $ab<ca=cb$.

Hereafter we shall provide a dispersed semilattice S with a minimal condition as follows.

For any $x\in S$, there is a minimal element a of S such that $a\leqq x$.

Let M be the set of all minimal elements of S. Naturally distinct minimal elements are incomparable. Let $M\times M$ be the set of all pairs (a,b) of elemements a,b of M.

Consider a mapping which associates $(a,b)\in M\times M$ with a pair $(l_a(ab), l_b(ab))$ of non-negative integers $l_a(ab)$, $l_b(ab)$, where we rewrite $f(a;(a,b))=l_a(ab)$, $f(b;(a,b))=l_b(ab)$. These satisfy the following conditions.

(2.8) $f(a;(a,b))\geqq 0$, and $f(a;(a,b))=0$ if and only if $a=b$.

(2.9) $f(a;(a,b))=f(a;(b,a))$.

(2.10) For any a,b,c, one least of the following three identities holds.

$$f(b;(a,b))+f(c;(b,c))=f(b;(b,c))+f(c;(c,a)),$$
$$f(c;(b,c))+f(a;(c,a))=f(c;(c,a))+f(a;(a,b)),$$
$$f(a;(c,a))+f(b;(a,b))=f(a;(a,b))+f(b;(b,c)).$$

On the other hand, we shall see that a dispersed semilattice with a minimal condition is characterized by a mapping

$$(a,b)\to(f(a;(a,b)),\ f(b;(a,b)))$$

If S is a dispersed semilattice with the minimal condition, any element x of S determines one minimal element a at least and a non-negative integer $l_a(x)$ such as above mentioned.

Lemma 8. *Let a and b be minimal element of S. $l_a(x)$ is bounded for fixed a and varying x, if and only if $l_b(x)$ is bounded for fiexed b and varying x.*

Proof. If $l_a(x)$ is unbounded, then $[a,*)$ is infinite. Since $[ab,*)\subset[a,*)$, $[ab,*)$ is infinite and also $[b,*)$ is so, which means that $l_b(x)$ is unbouded.

Lemma 9. *$l_a(x)$ is bounded for varying x if only if S has the greatest element.*

Proof. If S has the greatest element g, $[a,g]$ is finite and any element $x\geqq a$ is included in $[a,g]$ because S is a dispersed semilattice. Hence $l_a(x)$ is bounded. Conversely if S has not the greatest element, then, for any x, there is y such that $y\geqq x$. Accordingly we have an infinite chain

$$a=x_0<x_1<x_2<\cdots<x_n<\cdots\ ,$$

so $l_a(x)$ is unbounded. When S has not the greatest element, S is called *unbounded.*

In this paper we shall treat construction of unbounded dispersed semilattice with the minimal condition, in which $l_a(x)$ is unbounded, that is, $l_a(x)$ is valued throughout non-negative integers.

§ 3. Construction of an Unbounded Dispersed Semilattice with a Minimal Condition.

In this paragraph, consider M as an abstract set, and suppose that a mapping of $(a,b)\in M\times M$ to a pair of non-negative integers: $(f(a;(a,b)), f(b;(a,b)))$ where $f(a;(a,b))$ and $f(b;(a,b))$ satisfy the following conditions.

(3.1) $f(a;(a,b))\geqq 0$, and $f(a;(a,b))=0$ if and only if $a=b$.

(3.2) $f(a;(a,b))=f(a;(b,a))$ for any $a,b\in M$.

(3.3) For any $a,b,c\in M$, one at least of (3.3.1), (3.3.2), and (3.3.3.) holds.

$$(3.3.1)\quad \begin{cases} f(a;(a,b))=f(a;(a,c)) \\ f(c;(c,a))\geqq f(c;(c,b)) \\ f(b;(a,b))+f(c;(b,c))=f(b;(b,c))+f(c;(c,a)) \end{cases}$$

$$(3.3.2)\quad \begin{cases} f(b;(b,c))=f(b;(a,b)) \\ f(a;(a,b))\geqq f(a;(a,c)) \\ f(c;(b,c))+f(a;(c,a))=f(c;(c,a))+f(a;(a,b)) \end{cases}$$

$$(3.3.3)\quad \begin{cases} f(c;(c,a))=f(c;(b,c)) \\ f(b;(b,c))\geqq f(b;(b,a)) \\ f(a;(c,a))+f(b;(a,b))=f(a;(a,b))+f(b;(b,c)) \end{cases}$$

Now, let S be the set of all pairs of an element of M and an element of the set K of all non-negative integers.

$$S=K\times M=\{(\lambda,a);\ \lambda\in K,\ a\in M\}.$$

We introduce an ordering into S as follows.

$(\lambda,a)\geqq(\mu,b)$ means $\lambda\geqq f(a;(a,b))$ as well as $\lambda+f(b;(a,b))\geqq\mu+f(a;(a,b))$.

Lemma 10. *This ordering of S is a quasi-ordering.*

Proof. $(\lambda,a)\geqq(\lambda,a)$ is easily shown by the definition.
We shall show only a transitive law:

$(\lambda,a)\geqq(\mu,b)$ and $(\mu,b)\geqq(\nu,c)$ imply $(\lambda,a)\geqq(\nu,c)$.

The Proof of $\lambda\geqq f(a;(a,c))$. By $(\lambda,a)\geqq(\mu,b)$, we have $\lambda\geqq f(a;(a,b))$. In the case of (3.3.1) or (3.3.2), we get directly $\lambda\geqq f(a;(a,c))$. In the case (3.3.3),

$$\begin{aligned} \lambda &\geqq \mu+f((a;(a,b))-f(b;(a,b)) && (\text{by}(\lambda,a)\geqq(\mu,b)) \\ &\geqq f(b;(b,c))+f(a;(a,b))-f(b;(a,b)) && (\text{by } \mu\geqq f(b;(b,c)) \text{ because } (\mu,b)\geqq(\nu,c),) \\ &=f(a;(c,a)). && (\text{by the third identity of (3.3.3).}) \end{aligned}$$

The Proof of $\lambda+f(c;(a,c))\geqq\nu+f(a;(a,c))$.
By the definition of $(\lambda,a)\geqq(\mu,b)$ and $(\mu,b)\geqq(\nu,c)$, we have $\lambda-\mu\geqq f(a;(a,b))-f(b;(a,b))$, $\mu-\nu\geqq f(b;(b,c))-f(c;(b,c))$, so that $\lambda-\nu=\lambda-\mu+\mu-\nu$

$$\begin{aligned} &\geqq f(a;(a,b))-f(b;(a,b))+f(b;(b,c))-f(c;(b,c)) \\ &=f(a;(a,c))-f(c;(a,c)). \end{aligned}$$

The last identity is obtained from the first and third identities in all cases (3.3.1), (3.3.2), and (3.3.3). Now we define $(\lambda,a)=(\mu,b)$ as $(\lambda,a)\geqq(\mu,b)$ and $(\lambda,a)\leqq(\mu,b)$.

Lemma 11. $(\lambda,a)=(\mu,b)$ *if and only if*

$$\lambda\geqq f(a;(a,b)) \text{ and } \lambda+f(b;(a,b))=\mu+f(a;(a,b)).$$

Proof. By the definition, if $(\lambda,a)=(\mu,b)$ then

$$(3.4) \qquad \lambda \geqq f(a;(a,b)), \qquad (3.5) \qquad \lambda + f(b;(a,b)) \geqq \mu + f(a;(a,b)),$$
$$(3.6) \qquad \mu \geqq f(b;(b,a)), \qquad (3.7) \qquad \mu + f(a;(b,a)) \geqq \lambda + f(b;(b,a)).$$

hold simultaneously. From (3.5) and (3.7) we have

$$(3.8) \qquad \lambda + f(b;(a,b)) = \mu + f(a;(a,b)),$$

while (3,6) is implied by (3.4) and (3.8). The converse is clear.

Lemma 12. *$(\lambda, a) > (\mu, a)$ if and only if $\lambda > \mu$. $(\lambda, a) = (\mu, a)$ if and only if $\lambda = \mu$.*

Proof. Since $f(a;(a,a)) = 0$, this lemma is derived easily from the definition.

Lemma 13. *The set of all (ξ, x) such that $(\lambda, a) \leqq (\xi, x)$ is a chain.*

Proof. Let $(\lambda, a) \leqq (\xi, x)$. Since $\xi \geqq f(x;(a,x))$, we have

$$(\xi, x) = (\xi - f(x;(a,x)) + f(a;(a,x)),\ a)$$

where naturally $\xi - f(x;(a,x)) + f(a;(a,x)) \geqq \lambda$ by $(\lambda, a) \leqq (\xi, x)$. It is immediately shown that $(\lambda, a) \leqq (\xi_1, x_1)$and $(\lambda, a) \leqq (\xi_2, x_2)$ imply (ξ_1, x_1) and (ξ_2, x_2) are comparable.

By Lemmas 12 and 13, directly

Lemma 14. $(\lambda_0, a_0) \leqq (\xi, x) \leqq (\lambda_1, a_0)$, $\lambda_0 \leqq \lambda_1$ *if and only if* $(\xi, x) = (\lambda, a_0)$, $\lambda_0 \leqq \lambda \leqq \lambda_1$.

Lemma 15. *(λ, a) and (μ, b) are incomparable if and only if $\lambda < f(a;(a,b))$, $\mu < f(b;(a,b))$.*

Proof. It is clear that if (λ, a) and (μ, b) are comparable,

$$\lambda \geqq f(a;(a,b)) \text{ or } \mu \geqq f(b;(a,b)).$$

Conversely we show that if $\lambda \geqq f(a;(a,b))$ or $\mu \geqq f(b;(a,b))$, then (λ, a) and (μ, b) are comparable.

$$\lambda \geqq f(a;(a,b)) \text{ and } \mu < f(b;(a,b)) \text{ imply } (\lambda, a) \geqq (\mu, b),$$
$$\lambda < f(a;(a,b)) \text{ and } \mu \geqq f(b;(a,b)) \text{ imply } (\lambda, a) \leqq (\mu, b).$$

If $\lambda \geqq f(a;(a,b))$ and $\mu \geqq f(b;(a,b))$, then

$$\lambda - \mu \geqq f(a;(a,b)) - f(b;(a,b)) \text{ implies } (\lambda, a) \geqq (\mu, b),$$
$$\lambda - \mu \leqq f(a;(a,b)) - f(b;(a,b)) \text{ implies } (\lambda, a) \leqq (\mu, b).$$

This lemma has been proved.

Lemma 16. *For (λ, a) and (μ, b), we define (ν, c) in the following manner.*

$$\begin{aligned} (\nu, c) &= (\lambda, a) \quad \text{if} \quad (\lambda, a) \geqq (\mu, b), \\ (\nu, c) &= (\mu, b) \quad \text{if} \quad (\lambda, a) \leqq (\mu, b), \\ (\nu, c) &= (f(a;(a,b)), a) \text{ if } (\lambda, a) \not\gtreqless (\mu, b) \\ &= (f(b;(a,b)), b). \end{aligned}$$

Then (ν, c) is a least upper bound of (λ, a) and (μ, b).

Proof. We shall prove only a case where (λ, a) and (μ, b) are incomparable. First,

$(f(a;(a,b)),\ a)=(f(b;(a,b)),\ b)$ follows from the definition of equality. By Lemma 15,

$$\lambda< f(a;(a,b)), \qquad \mu< f(b;(a,b)),$$

hence

$$(\lambda,a)<(f(a;(a,b)),\ a)=(\nu,c),$$
$$(\mu,b)<(f(b;(a,b)),\ b)=(\nu,c).$$

Let us show that (ν,c) is the least upper bound of (λ,a) and (μ,b). Let (ς,d) be any upper bound of (λ,a) and (μ,b). Any (ζ,d) is comparable to (ν,c) because of Lemma 9. Suppose that there is an upper bound (ζ_0,d_0) of (λ,a) and (μ,b) such that $(\lambda,a)<(\zeta_0,d_0)<(\nu,c)$. By Lemma 14, there is η such that $(\zeta_0,d_0)=(\eta,a)$ where $\eta<f(a;(a,b))$. Remember $\mu<f(b;(a,b))$, (η,a) and (μ,b) are incomparable because of Lemma 15. This contradicts with the assumption that $(\zeta_0,d_0)\geqq(\mu,b)$. Therefore $(\zeta,d)\geqq(\nu,c)$ for any upper bound (ζ,d) of (λ,a) and (μ,b). Thus the proof of the lemma has been completed.

Thus it has been proved that S^* is a semilattice.

Lemma 17. *A minimal element of S^* is $(0,a)$, for each $a\in M$.*

Proof. Suppose that $(\lambda,b)\leqq(0,a.)$ By the definition of inequality, $f(a;(a,b))=0$ which implies $a=b$. By Lemma 12, we have $\lambda=0$. Thus the $(0,a)$ has been proved to be minimal. Conversely, (λ,b), $\lambda\neq0$, is not a minimal element because $(\lambda,b)>(0,b)$.

Lemma 18. *S^* satisfies the minimal condition.*

Proof. For any $(\lambda,a)\in S^*$, $(0,a)\leqq(\lambda,a)$ where $(0,a)$ is minnimal.

Thus we have seen that S^* is a dispersed semilattice with the minimal condition by means of Lemmas 13, 14, and 18. Furthermore it follows from Lemma 12 that S^* is unbounded.

Theorem 2. *An unbounded dispersed semilattice with the minimal condition is characterized by a set M and the mapping which associates $(a,b)\in M\times M$ with a pair of non-negative integers*

$$(f(a;(a,b)),\ f(b;(a,b)))$$

where f satisfies the conditions (3.1), (3.2), *and* (3.3).

Thus the problem of construction of such a semilattice is reduced to the study of f, which remain unsolved here.

§ 4. Remarks.

"The minimal condion" in the present paper is weaker than the so-called "descending chain condition", which means that a chain $x_1>x_2>\cdots$ ceases in a finite number.

Example. The following example satisfies the minimal condition, but does not the descending chain condition.

Let I be the set of all integers i, and let I' be the set of all i' where $i\leftrightarrow i'$ is one-to-one. Now S is defined as the set union of I and I': $S=I\cup I'$; and we define the multiplication in the following manner.

$$ij = i'j = ij' = \max(i, j),$$
$$i'j' = \max(i, j), \text{ if } i' \neq j',$$
$$i'i' = i'.$$

Finally we give a necessary and suffiicient condition for S to satisfy the descending chain condition.

Theorem 3. *S^* does not satisfy the descending chain condion if and only if M contains an infinite sequence $\{a_i\}$, where $a_i \neq a_j$, $i \neq j$, $a_i \in M$, such that*

$$f(a_i; (a_{i-1}, a_i)) > f(a_i; (a_i, a_{i+1})), \ i = 2, 3, \cdots.$$

Proof. Suppose that S^* satisfies the descending chain condition, there is an in finite sequence

$$(\lambda_1, b_1) > (\lambda_2, b_2) > \cdots$$

of elements of S^*. Since the element $(0, b_1)$ is minimal, $(0, b_1) \neq (\lambda_j, b_j)$ for all j. Then all the inequalities $(\lambda_j, b_j) > (0, b_1)$, $j \geqq 1$, do not hold. Because, otherwise

$$(\lambda_1, b_1) \geqq (\lambda_j, b_j) > (0, b_1) \text{ for all } j \geqq 1,$$

while $[(0, b_1), (\lambda_1, b_1)]$ is finite since S^* is dispersed, arriving at the contradiction. Now, there is a positive integer k such that

$$(\lambda_j, b_j) < (0, b_1), \ \ 1 \leqq j \leqq k, \text{ and } (\lambda_{k+1}, b_{k+1}) \not> (0, b_1)$$

where naturally $(\lambda_j, b_j) \not> (0, b_1)$, $j \geqq k+1$.
Rewrite $a_1 = b_1$, $a_2 = b_{k+1}$. If $a_1, a_2, \cdots, a_{i-1}$ are obtained, then we let $a_i = b_{k_i}$ where

$$(\lambda_{k_i-1}, b_{k_i-1}) > (0, a_{i-1}) \text{ and } (\lambda_{k_i}, b_{k_i}) \not> (0, a_{i-1}), \ k_{i-1} > k_i.$$

Thus the sequence $\{a_i\}$ in M is determined. As easily seen, $(\lambda_{k_i-1}, b_{k_i-1})$ is an upper bound of $(0, a_{i-1})$ and $(0, a_i)$ so that $(\lambda_{k_i-1}, b_{k_i-1}) \geqq (0, a_{i-1})(0, a_i) < (0, a_i)$. On the other hand, as S^* is dispersed, $[(0, a_i), (\lambda_{k_i-1}, b_{k_i-1})]$ is a finite chain which contains (λ_{k_i}, b_{k_i}). Since $(\lambda_{k_i}, b_{k_i}) \geqq (0, a_{i-1})(0, a_i)$ is impossible, we have $(\lambda_{k_i-1}, b_{k_i-1}) \geqq (0, a_{i-1})(0, a_i) > (\lambda_{k_i}, b_{k_i})$, and similarly $(\lambda_{k_{i+1}-1}, b_{k_{i+1}-1}) \geqq (0, a_i)(0, a_{i+1}) > (\lambda_{k_{i+1}}, b_{k_{i+1}})$. Hence we have seen that $\{a_i\}$ satisfies

$$(0, a_{i-1})(0, a_i) > (0, a_i)(0, a_{i+1}),$$

and utilizing Lemma 16,

$$(f(a_i; (a_{i-1}, a_i)), a_i) > (f(a_i; (a_i, a_{i+1})), a_i).$$

By Lemma 12, $f(a_i; (a_{i-1}, a_i)) > f(a_i; (a_i, a_{i+1}))$.

Conversely if an infinite sequence $\{a_i\}$ fulfils the above condition, we can easily show that there is a sequence $\{b_i\}$ where $b_i = (0, a_{i-1})(0, a_i)$, satisfying $b_i > b_{i+1}$ $i = 2, 3, \cdots$. Hence the descending chain condition is not valid in S^*. Thus the proof of this theorem has been completed.