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NOTES ON GENERAL ANALYSIS (VI)

Singular set

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In this note, the set of singular points of analytic functions in complex Banach spaces is composed of a number of singular subspaces, which are, of course, closed linear subspaces and functions are not analytic there. In the preceding paper[1], we investigated the singular subspace. If x_0 and y_0 do not belong to a singular subspace L_0, and $y_0 \neq \alpha x_0 + \beta y$ for any complex number α, β and any y in L_0, then x_0 and y_0 are called "*independent mutually of* L_0." If there exist two vectors at least which are indepedent mutually of L_0, and an E_2-valued function $f(x)$ is analytic on the outside of L_0 in E_1, then $f(x)$ is analytic on whole space E_1, where E_1, E_2 are complex Banach spaces. That is, the singular subspace L_0 is removable.
Generally, the singular set of an analytic function in complex Banach spaces is not necessarily a singular subspace.

In the first chapter of this paper, we discuss the case that a sigular set of an analytic function in complex Banach spaces is composed of many singular subspaces. For each singular subspace, there exist at least two vectors which are independent mutually of it. In this case, the singular set is removable. In the second of this paper, it is described that the singular subspace L_1 is removable under some conditions. Of course, for this singular subspace L_1 there exists only one vector which is independent mutually of it. In the third of this paper, the singular set is composed of two singular subspaces, For each singular subspace, there exists only one vector being independent mutually of it. The function with this singular set is not simple as the function with one singular subspace.

We shall state theorems which we shall need in the following discussions:

Theorem A.[1] *If there exist two vectors at least which are independent mutually of* L_0, *a homogeneous function* $f_n(x)$ *of degree n is a homogeneous polynomial of degree n, where* L_0 *is a singular subspace of* $f_n(x)$.

Theorem B. *Let* $h(x)$ *be a homogeneous function of degree n whose singular subspace is* L_1. *The necessary and sufficient condition that* $h(x)$ *should be a homogeneous polynomial is that*

$$\|h(x+\alpha y)\| \leqq K(x,y),$$

for a sufficiently small $|\alpha|$, *in which x is an arbitrary point in* L_1 *and y is an arbitrary outside point* *of* L_1 *and* $K(x,y)$ *is a positive constant with respect to* α .

§ 1. Removable singular set.

Let S be a sum of singular subspaces. For each singular subspace, there exist at least two vectors independent of the singular subspace. Suppose that non of the sequence of singular subspaces derived from S converges to any one of S. Then we have the next theorem.

Theorem 1. *If an* E_2*-valued function* $f(x)$ *defined on* E_1 *is analytic on the outside of* S, *then* $f(x)$ *is also analytic on* S. *That is,* S *is removable.*

Proof. Let x be a point of S being contained only one singular subspace L_0 of S. Since L_0 is not a limiting subspace of any sequence $\{L_n\}$ derived from S, we can find a neighbourhood $V(x)$ such that $f(x)$ is analytic on $V(x)$ excepting points of L_0. Let y be an arbitrary outside point of L_0 and β is a complex number satisfying $x+\beta y \in V(x)\cdot CL_0$, where CL_0 is a compliment of L_0. Then we have

$$f(x+\beta y)=\sum_{-\infty}^{\infty} h_n(x,y)\,\beta^n,$$

where

$$h_n(x,y)=\frac{1}{2\pi i}\int_C \frac{f(x+\alpha y)}{\alpha^{n+1}}d\alpha, \quad \text{for } n=0, \pm 1, \pm 2,\cdots$$

A circle C is defined by $|\alpha|=\rho$ such that $x+\alpha y$ lies in $V(x)$, if α lies on C.

We see that $h_n(x,y)$ is analytic as to y on the outside of L_0 and satisfies $h_n(x,\beta y)=\beta^n h_n(x,y)$. Then we see that $h_n(x,y)$ is analytic as to y, because L_0 is removable by Theorem A. This shows that $h_n(x,y)\equiv 0$ for $n<0$, and $h_n(x,y)$ is a homogeneous polynomial of degree n, if $n>0$. Then we have

$$f(x+y)=\sum_{n=0}^{\infty} h_n(x,y),$$

where $h_n(x,y)$ is a homogeneous polynomial of degree n.

Since $x+e^{i\theta}\rho y$ lies in $V(x)$ excepting L_0, there exists a neighbourhood $V(\theta)$ for each point $x+e^{i\theta}\rho y$ such that

$$\|f(z)-f(x+e^{i\theta}\rho y)\|<\varepsilon,$$

for an arbitrary positive number ε, if $z\in V(\theta)$, where $V(\theta)$ lies in $V(x)$ excepting L_0 and $V(\theta)$ is a set of points which satisfy $\|x+e^{i\theta}\rho y-z\|<\delta_\theta$ for a suitable positive number δ_θ determined by θ. Appealing to the covering theorem of Borel, we have $\theta_1,\theta_2,\cdots,\theta_k$, such that the set $\sum_{j=1}^{k} V(\theta_j, \frac{1}{2})$ covers the set $x+e^{i\theta}\rho y$ $(0\leqq\theta\leqq 2\pi)$, where $V(\theta_j,\frac{1}{2})$ is a neighbourhood of $x+e^{i\theta_j}\rho y$ such that $\|x+e^{i\theta_j}\rho y-z\|<\dfrac{\delta_{\theta_j}}{2}$.

Put $M=\underset{1\leq j\leq k}{Max.}\{\|f(x+e^{i\theta_j}\rho y)\|+\varepsilon\}$. If z lies in $\sum_{1}^{k} V(\theta_j)$, we have $\|f(z)\|\leqq M$. When

δ_0 is a positive number such that $0<\delta_0 \leqq \underset{1\leq j\leq k}{Max.} \frac{\delta_{\theta j}}{2}$, we have $x+e^{i\theta}V(y,\delta_0)\subset \sum_1^k U(\theta_j)$, for $0\leqq\theta\leqq 2\pi$, where $V(y, \delta_0)$ is a set of points which satisfy $\|y-z\|\leqq\delta_0$. Then

$$\|h_n(x,z)\| = \|\frac{1}{2\pi i}\int_C \frac{f(x+\alpha z)}{\alpha^{n+1}}d\alpha\|$$

$$\leqq \|\frac{1}{2\pi}\int_0^{2\pi}\frac{f(x+e^{i\theta}z)}{e^{in\theta}}d\theta\|$$

$$\leqq M,$$

where C is a circle whose radius is 1, for z lying in $V(y,\delta_0)$ and $n=0,1,2,\cdots$. Appealing to the lemma of Zorn[2] we see that $\|h_n(x,y)\|\leqq M$, when $\|y\|<\delta_0$, for $n=0,1,2,\cdots$. Thus we have

$$\sup_{\|y\|=1}\overline{\lim_{m\to\infty}}\sqrt[m]{\|h_m(x,y)\|}\,^{3)} = \sup_{\|y\|=1}\overline{\lim_{m\to\infty}}\sqrt[m]{\|h_m(x,\frac{\delta y}{\delta})\|}, \text{ for } 0<\delta<\delta_0,$$

$$= \frac{1}{\delta}\sup_{\|y\|=1}\overline{\lim_{m\to\infty}}\sqrt[m]{\|h_m(x,\delta y)\|}$$

$$= \frac{1}{\delta}\sup_{\|y\|=1}\overline{\lim_{m\to\infty}}\sqrt[m]{M}$$

$$= \frac{1}{\delta}.$$

This shows that the radius of analyticity[3] of $f(x+y)=\sum_{m=0}^{\infty}h_m(x,y)$ is not smaller than δ and we see that $f(x)$ is analytic at x lying only on L_0. Now, let L_0 and L_0' be arbitrary two singular subspaces in S. Then $L_0\cap L_0'$ is a singular subspace. We can see that $f(x)$ is analytic on a point lying only on $L_0\cap L_0'$ as well as L_0. And so on, we see that $f(x)$ is analytic on S by the transcendental mathematical induction.

Corollary. *If $h(x)$ is analytic on the outside of S and satisfies $h(\alpha x)=\alpha^n h(x)$ there for an arbitrary complex number α, $h(x)$ is a homogeneous polynomial of degree n.*

Proof. We see that $h(x)$ is analytic on whole spaces by Theorem 1. Since $h(x)$ is continuous, the equality $h(\alpha x)=\alpha^n h(x)$ is held also for a point x on S. This shows that $h(x)$ is a homogeneous polynomial of degree n. This completes the proof.

Thus we see that a singular set S is removable if S is composed of singular subspaces which are at least lower two dimensions than the space. But, when there do not exist two vectors which are independent mutually of an each subspace L_1 of S, S is not generally removable.

From now on, let L_1, L_2 be singular subspaces such that there exists at least one vector being independent mutually of each singular subspace L_i but do not exist two vectors being independent mutually of L_i, where $i=1,2$.

§ 2. Removable singular subspaces.

The singular subspace L_1 is removable under some conditions.

Theorem 2. *Let an E_2-valued function $f(x)$ defined on E_1 has a singular subspace L_1. If, for an arbitrary point x on L_1, there exists a neighbourhood $V(x)$ of x and a constant $K(x)$ such that $\|f(y)\| \leqq K(x)$ for y in $V(x)$,[4] then L_1 is removable.*

Proof. For an arbitrary point x in L_1 and an arbitrary outside point y of L_1, $f(x+\alpha y)$ is an analytic function of α, when $0<|\alpha|<\infty$. For a suitable positive number δ, $x+\alpha y \in V(x)$, when $|\alpha|<\delta$. Then we have $\|f(x+\alpha y)\| \leqq K(x)$, when $0<|\alpha|<\delta$. Thus we see that $\alpha=0$ is removable and $f(x+\alpha y)$ is analytic as to α for $|\alpha|<\infty$.

Then we have
$$f(x+\alpha y)=\sum_0^\infty h_n(x,y)\alpha^n,$$
where
$$h_n(x,y)=\frac{1}{2\pi i}\int_C \frac{f(x+\zeta y)}{\zeta^{n+1}}\,d\zeta, \text{ for } n=0,1,2,\cdots$$

Clearly, $h_n(x,y)$ is a homogeneous function as to y with a singular subspace L_1, because y is an arbitrary outside point of L_1. For an arbitrary point x_1 in L_1 and an arbitrary outside point y_1 of L_1,
$$h_n(x,x_1+\alpha y_1)=\frac{1}{2\pi i}\int_C \frac{f(x+\zeta(x_1+\alpha y_1))}{\zeta^{n+1}}\,d\zeta.$$

Put $\zeta=e^{i\theta}(0\leqq\theta\leqq 2\pi)$, then the point $x+x_1e^{i\theta}(0\leqq\theta\leqq 2\pi)$ lies on L_1 and so we have $\|f(x+x_1e^{i\theta}+y)\| \leqq K(\theta,)$ where $0\leqq\theta\leqq 2\pi$, for any y in a suitable neighbourhood $V(\theta)$ of $x+x_1e^{i\theta}$ and a constant $K(\theta)$ for $0\leqq\theta\leqq 2\pi$. By the covering theorem of Borel, there exist a system of neighbourhoods $V(\theta_1), V(\theta_2),\cdots, V(\theta_p)$ such that $x+x_1e^{i\theta} \subset \sum_1^p V(\theta_j)$, if $0\leqq\theta\leqq 2\pi$. Moreover, for a suitable neighbourhood $U(x_1)$ of x_1, we have $x+U(x_1)e^{i\theta} \subset \sum_{j=1} V(\theta_j)$, for $0\leqq\theta\leqq 2\pi$. Put $\underset{1\leq j\leq p}{Max.}\, K(\theta_j)=K$, then $\|f(x+U(x_1)e^{i\theta})\| \leqq K$. Let $U(x_1)\supset U(x_1,\delta)$ and $|\alpha|<\dfrac{\delta}{\|y_1\|}$, then $x+x_1e^{i\theta}+e^{i\theta}\alpha y_1 \subset x+U(x_1)e^{i\theta}$.
Then
$$\|h_n(x,x_1+\alpha y)\| \leqq \frac{1}{2\pi}\int_0^{2\pi}\|f(x+e^{i\theta}(x_1+\alpha y_1))\|d\theta$$
$$\leqq K.$$

Appealing to Theorem B, $h_n(x,y)$ is analytic as to y and we see that $h_n(x,y)$ is a homogeneous polynomial of degree n as to y. As well as the proof of Theorem 1, we see that the power series $\sum_{n=0}^\infty h_n(x,y)$ is convergent uniformly in a neighbourhood of x. Since x is arbitrary in L_1, L_1 is removable.

Thorem 3. *Let L_1 be a singular subspace of $f(x)$. If, for an arbitrary point x in L_1 and* *an arbitrary outside point y of L_1,*

$$\overline{\lim_{\alpha\to 0}}\ \|f(x+\alpha y)\| \leqq K(y),$$

where $K(y)$ is a constant depending upon y, then L_1 is removable.

Proof. Let x_1 be an arbitrary point of L_1 and y_1 be an arbitrary outside point of L_1. Since $\overline{\lim_{\alpha\to 0}}\ \|f(x_1+\alpha y_1)\| \leqq K(y_1)$, there exists a positive number δ for a given positive number ε such that $\|f(x_1+\alpha y_1)\| \leqq K(y_1)+\varepsilon$ for $|\alpha| \leqq \frac{\delta}{\|y_1\|}$. If y is an arbitrary outside point of L_1, we have $y=x_0+\alpha_0 y_1$, for a suitable x_0 in L_1 and a complex number α_0, because, there exists only one vector essentially being independent of L_1. Then

$$x_1+\alpha y = x_1+\alpha(x_0+\alpha_0 y_1) = (x_1+\alpha x_0)+\alpha\alpha_0 y_1.$$

Since $K(y)$ is independent of x,

$$\|f(x_1+\alpha y)\| \leqq K(y_1)+\varepsilon, \text{ when } \|\alpha\alpha_0 y_1\| \leqq \delta.$$

Let $|\alpha_1| = \frac{\delta}{\|y_1\|}$ and d be a distance between $\alpha_1 y_1$ and the singular subspace L_1. Clearly, $d>0$. If $d=0$, $\alpha_1 y_1$ is a limiting point of points derived from L_1 and so $\alpha_1 y_1$ must be a point of L_1 contradicting to the fact that y_1 is an outside point, since L_1 is closed. Let $d>\|y\|$. Since $\|y\|=\|x_0+\alpha_0 y_1\| \geqq \text{Dis. } (\alpha_0 y_1, L_1) \geqq |\alpha_0| \cdot \text{Dis. } (y_1, L_1) = |\alpha_0| \cdot \frac{d}{|\alpha_1|}$, $|\alpha| \geqq |\alpha_0|$. Then, $\|\alpha_0 y_1\| \leqq \|\alpha_1 y_1\| = \delta$ (the cace of $\alpha=1$). That is, $\|f(x_1+y)\| \leqq K(y_1)+\varepsilon$, when $\|y\| \leqq d$. Appealing to Theorem 2, L_1 is removable, since x_1 is an arbitrary point of L_1.

§ 3. Reciprocal homogeneous function.

If a singular set S of $f(x)$ is composed of some singlar subspaces such as L_1, the characters of $f(x)$ are not simple. Prior to the discussion of this chapter, we must define some functions.

Definition 1. *If $P(x)$ is a (reciprocal) homogeneous function of degree n with the singular subspace L_1 whose orders of singularity is m, $P(x)$ is called (n,m)-function with the singular subspace L_1. (If n is a negative integer, $P(x)$ is a reciprocal homogeneous function.)*

For example, put $x=(x_1,x_2)$, where x_1 and x_2 are complex numbers, and $P(x)=\frac{x_1^{n+m}}{x_2^m}$. Then $P(x)$ is a (n,m)-function with a singular subspace L_1, which is defined as $x_2=0$.

Definition 2. *Let S be composed of L_1 and L_2, and $R_n(x)$ be analytic at outside points of S and satisfies $R_n(\alpha x)=\frac{1}{\alpha^n}R_n(x)$ there.*

Moreover, $\overline{\lim_{\alpha\to\infty}}\ \|R_n(\alpha X+y)\| \leqq K(L_i, y)$, for an arbitrary x in L_i and an arbitrary outside point y of L_1. Of course, $\alpha x+y$ lies in the outside of S and $i=1,2$. Then, $R_n(x)$ is called R-function of degree n.

As well as $R_n(x)$, we can define $P_n(x)$, which is called *P-function of degree n* with the singular set S. That is, (1) *$P_n(x)$ is analytic on the outside of S*, (2) *$P_n(\alpha x)=\alpha^n P(x)$ for any complex number α and an arbitrary point x in the outside of S*, (3) $\overline{\lim_{\alpha\to 0}}\|P_n(x+\alpha y)\|\frac{1}{|\alpha|^n}\leqq K(L_i,y)$ *for an arbitrary point x in L_i and an arbitrary outside point y of L_i, where $i=1,2$.*

Theorem 4. *Let a point x on L_i lie on the outside of L_2. Then, for an arbitrary outside point y of L_i, we have*

$$R_n(x+y)=\sum_{m=-n}^{\infty} R_{m,n}(x,y),$$

where $R_{m,n}(x,y)$ is (m,n)- function with respect to y.

Proof. Since $R_n(x)$ is anaylytic on the outside of S and $x+\alpha y$ lies in the outside of S for a suitable α, we have

$$R_n(x+y)=\sum_{-\infty}^{\infty} R_{m,n}(x,y),$$

where $R_{m,n}(x,y)=\frac{1}{2\pi i}\int_C R_n(x+\alpha y)\alpha^{-m-1}d\alpha$, for $m=0,\pm 1,\pm 2,\cdots$ C is a circe $|\alpha|=r$, which satifies $0<r\cdot\|y\|<d(x,L_2)$, where $d(x,L_2)$ is the distance between x and L_2. If y lies on the outside of L_2, there exists z on L_2 such that $x=\lambda y+\mu z$ for suitable complex numbers λ and μ. Then $x+\alpha y=(\lambda+\alpha)y+\mu z$. This shows that $x+\alpha_0 y$ lies on L_2, if $\alpha_0=-\lambda$. $x+\alpha_0 y$ is an only point lying on L_2 for $|\alpha|<\infty$, because if there exist α such that $x+\alpha y\in L_2$, $(\alpha-\alpha_0)y=x+\alpha y-(x+\alpha_0 y)\in L_2$ contradicting to the fact that $y\overline{\in} L_2$. Since $\frac{x_0}{\alpha}+y=\frac{1}{\alpha_0}(x+\alpha_0 y)\in L_2$, we see that $\frac{x}{\alpha}+y\overline{\in} L$, if $|\alpha|\leqq r$, which is naturally smaller than $|\alpha_0|$. Thus we see that $R_n(x)$ is analytic at $\frac{x}{\alpha}+y$ for $|\alpha|\leqq r$ and we have

$$R_{m,n}(x,y)=\frac{1}{2\pi i}\int_C R_n(x+\alpha y)\alpha^{-m-1}d\alpha.$$

$$=\frac{1}{2\pi i}\int_C R_n\left(\frac{x}{\alpha}+y\right)\alpha^{-m-n-1}d\alpha$$

Put $\frac{1}{\alpha}=\beta$, and $\beta=\rho e^{i\theta}$, then $d\beta=i\rho e^{i\theta}d\theta$ and so

$$R_{m,n}(x,y)=\frac{-1}{2\pi}\int_0^{2\pi} R_n(\rho e^{i\theta}x+y)(\rho e^{i\theta})^{n+m}d\theta.$$

Then,

$$\|R_{m,n}(x,y)\|\leqq\frac{1}{2\pi}\int_0^{2\pi}\|R_n(\rho e^{i\theta}x+y)\|\rho^{n+m}d\theta,$$ and so we have

$$\|R_{m,n}(x,y)\|\leqq\frac{1}{2\pi}\int_0^{2\pi}\overline{\lim_{\beta\to\infty}}\|R_n(\beta x+y)\beta^{n+m}\|d\theta$$

$$=0, \text{ if } m<-n,$$

since $\overline{\lim_{\beta\to\infty}}\|R_n(\beta x+y)\|\leqq K(L_1,y)$ by the definition.
If y lies on L_2, $\frac{\alpha}{x}+y$ does not lie on L_1 nor L_2 and so we see that $R_{m,n}(x,y)=0$, when y lies on L_2 and $m<-n$.
Since y is arbitrary, we can easily see that $R_{m,n}(x,y)\equiv 0$, if $m<-n$. $R_{m,n}(x,y)$ is clearly analytic as to y on the outside of L_1 and satisfies $R_{m,n}(x,\alpha y)=\alpha^m R_{m,n}(x,y)$. Now, let x' be a point on L_1 and y be an outside point of L_1. Then

$$R_{m,n}(x,x'+\beta y)=\frac{1}{2\pi i}\int_C R_n(x+\alpha(x'+\beta y))\alpha^{-m-1}d\alpha$$

and so we have

$$\overline{\lim_{\beta\to 0}}\,|\beta|^n\cdot\|R_{m,n}(x,x'+\beta y)\|$$

$$\leqq \overline{\lim_{\beta\to 0}}\frac{1}{2\pi}\int_0^{2\pi}\|R_n(\frac{x}{\beta}+\alpha(\frac{x'}{\beta}+y))\|\,|\alpha|^{-m}d\theta$$

$$\leqq \frac{1}{2\pi}\int_0^{2\pi}\overline{\lim_{\beta\to 0}}\|R_n(\frac{1}{\beta}(\frac{x}{\alpha}+x')+y)\|\cdot|\alpha|^{-m-n}d\theta$$

$$\leqq \frac{1}{2\pi}\int_0^{2\pi}K(L_1,y)|\alpha|^{-m-n}d\theta$$

$$=K(L_1,y)|\alpha|^{-m-n}.$$

This shows that $R_{m,n}(x,y)$ has a singular subspace L_1 of degree n generally and so $R_{m,n}(x,y)$ is the (m,n)-function generally. This completes the proof.

The follwing example shows exactly this fact. Put $x=(x_1,x_2)$ and $\|x\|=\text{Max.}(|x_1|,|x_2|)$, where x_1 and x_2 are complex numbers. Then the set of x forms complex-Banach-spaces Ω. Let $f(x)=\frac{1}{x_1x_2}$ and $S=L_1\cup L_2$, where L_i is a set of points such that $x_i=0$ in Ω for $i=1,2$. Then $f(x)$ is analytic on the outside of S and satisfies there $f(\alpha x)=\frac{1}{\alpha^2}f(x)$, for an arbitrary complex number α. That is, $f(x)$ is the R-function of degree 2. Put $x=(0,x_2,)$ where $x_2\neq 0$ and $y=(y_1,y_2)$. Then we have

$$f(x+\alpha y)=\frac{1}{\alpha y_1(x_2+\alpha y_2)}=\sum_0^\infty\frac{(-1)^n}{x_2^{n+1}}\cdot\frac{y_2^n}{y_1}\alpha^{n-1}.$$

That is $f_{n,1}(x,y)=\frac{(-1)^{n+1}}{x_2^{n+1}}\cdot\frac{y_2^{n+1}}{y_1}$, which has a singular subspace L_1 of degree 1.

References

1). I. Shimoda: Notes on general analysis (V), Journal of Gakugei, Tokushima University, Vol. VI,

1955.

2). M. A. Zorn: Characterization of analytic functions in Banach spaces. Annals of Math. (2). 46(1945).

3). E. Hille: Functional analysis and semigroups, 1948. I. Shimoda: On power series in abstract spaces, 1948. If $\frac{1}{r} = \text{Sup.}_{\|x\|=1} \overline{\lim}_{n\to\infty} \sqrt[n]{\|h_n(x)\|}$, r is called "radius of analyticity".

4) $f(x)$ is locally bounded on L_1.