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as a Memorial Volume for their scientific Achievements

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Professor Tsuruichi Hayashi

Professor Jitsuo Yoshikawa

Tsuruichi Hayashi (1873–1935)

He was born at Kamikurachô in Tokushima, Japan, as the first son of Tadashi Hayashi, who was a teacher of a primary school. After graduating successively from the Tomida Primary School, the Tokushima Middle School and the Third High School in Kyôto, he entered Tôkyô University.

Through the influence given by Ushitarô Takeda in the middle school and Jûtarô Kawai in the high school, both, teachers of mathematics, he became deeply interested in mathematics and studied it at the Mathematical Institute under the directions of Prof. Dairoku Kikuchi and Prof. Rikitarô Fujisawa, and graduated from the Faculty of Science of Tôkyô University in 1897. His class consisting of him and Teiji Takagi, Takuji Yoshie was the most brilliant one in the Mathematical Institute.

Continuing mathematical study in Daigakuin, he became Lecturer in the Tôkyô Higher Normal School, and in 1898 was appointed assistant professor of Kyôto University, which he resigned in 1899. Then he became a lecturer in the Matsuyama Middle School, but returned to Tôkyô in 1901, as lecturer of the Tôkyô Higher Normal School, and there he was promoted to the professorship in 1907.

The experience of practice in the secondary education and teaching mathematics in the normal college had caused him to write text books of mathematics for the secondary school. His text books at that time were most widespread in Japan. He published also a series of monographs of elementary methematics, which served very promotively to the promotion of the level of the mathematical knowledge in Japan.

In 1911, when the Tôhoku University in Sendai was founded, he was selected as the head of the Mathematical Institute, and had held this position until he retired from his professorship in 1929. In 1912 he received the degree of Rigakuhakushi, and in 1929 he was conferred the title of Honorary Professor.

The most prominent achievement of him was the foundation of the Tôhoku Mathematical Journal in 1911. This was undertaken by him on his own expence at first. This publication marked a striking epoch in the history of mathematics in Japan. Afterwards the management of the journal was transferred to the Mathematical Institute from his own hand, yet he remained as the

Chief Editor until his last day.

His achievement for the promotion of mathematical education in the secondary school was also great. He was one of the founders of the Mathematical Association of Japan for Secondary Education, and worked ardently as its president.

In such a busy and active life he wrote several hundreds of papers, whose subjects are very vast, covering infinitesimal calculus, theory of infinite series, theory of differential equations, number theory, algebra, complex and real function theory, practical and applied mathematics, history of Japanese mathematics and astronomy. Among them the most noteworthy investigation is that of the Japanese mathematics.

His character was frank and energetic; he had strong will and executive power and was a real organizer. He was highly esteemed by his friends and deeply fond of by his students and pupils.

Although for Prof. Hayashi two huge, elegant and comprehensive memorial volumes of Tôhoku Mathematical Journal, dedicated to him on his 60th birthday by his friends and pupils, were published, yet there is none for Prof. Yoshikawa. With this reason we hope that the present small memorial volume issued to honor of both Professors Hayashi and Yoshikawa will remain for a long time, as an undecaying monument for their scientific achievements.

(Tokushima University, October 22, 1955, on the mass read for both Professors.)

Jitsuo Yoshikawa (1878–1915)

Er wurde in Kamikurachô zu Tokushima Stadt in Japan als der erste Sohn von Nobutomo Yoshikawa geboren, der ein Tokushima-Hanshi (Lehnmann) war, aber als Kaufmann lebte. Als er die Tomida-Volksschule, die Tokushima Mittelschule und Yamaguchi Höherschule, nacheinander durchmachte, war er zwanzig Jahre alt, und sogleich trat in die Tôkyô Universität ein. Es ist gesagt, daß seine innere Bestreibungskraft sich vom Schlafe erweckte, als er im Examen von Mathematik des anfänglichen Jahres der Mittelschule sich durchfiel. Seitdem war er immer der Erste seiner Klasse. In der Yamaguchi Höhereschule empfing er die Diszipline des Herrn Tokiyoshi Hôjyô, der später als der als der Direktor des Hiroshima Höheren Lehrerseminares sich berühmt machte, und den Aufschrift des Steinmonumentes J. Yoshikawas im Daianji-Hügel zu Tokushimas ausschrieb.

Mit vorzüglicher Vollendung wurde er im Juli 1901 von der naturwischenschaftliche Fakultät der Tôkyô Universität graduierte und gerade im September wurde ihm die erledigte Stelle des ausserordentlichen Professors der Kyôto Universität übertragen. Im Jahre 1908 wurde er bestellt, im Deutschland drei Jahre lang Aufenthalt zu machen und Mathematik zu studieren. Er ging zur Göttingen Universität, wo er unter Prof. David Hilbert Integralgleichungen untersuchte und einige Abhandlungen über Anwendungen davon auf zwei- und drei-parametrigen Randwertaufgaben fertig machte. Auch während seines damaligen eifrigen Lernens empfing er die freundliche Ratschläge und Unterstützungen von Prof. H. A. Schwarz und Dr. Weyl.

Im Jahre 1911 begab er sich heim, und wurde zur Professur erhoben, und im nächsten Jahre den Grad des Rigakuhakushi erhielt. Er hielt einseits Vorlesungen über Integralgleichungen, Funktionentheorie, und Projektive Geometrie usw. und anderseits setzte seine Forschungsarbeiten zu fertigen fort.

Schon bevor seiner Auslandsreise verfasste er ein musterhaftiges Lehrbuch über moderne synthetische Geometrie, und nach seinem Heimzug eins über die Funktionentheorie. In der Tat gab es bis damals kein auf Japanisch geschriebenes Buch über die Funktionentheorie. Daher erschien seine Werke zwar zum ersten Male, doch als die besten. Kläre und Eleganz waren seines Motto. In seiner reputirlichen Vorlesung am vom Unterrichtsministerium beistandenen Kolloquium für Mittelschulslehrer hat er gern im Buch von Prof. Felix Klein

"Elementarmathematik vom höheren Standpunkte aus" nachgeschlagen.

Also ging alles nach Wunsch sowohl in Gelehrtenwelt als in Gesellschaftsleben, und man erwartete sein fünftiges fruchtbares Gelingen. Dessenungeachtet war sein physyscher Körperleider gar nicht wohl. Im Alter von siebenunddreizig Jahren, noch vorm Mittelalter, starb er in Kyôto am sechsten April 1916.

Er war vielseitig aktiv und vorzüglich, seien es reine und angewandte Mathematik, z. B. mathematische Physik, Versicherungsmathematik, sei es auch Deutsch, sei es Schreiben, sei es Gespräch, sei es Sport wie Spazierenreiten oder Unterhaltung und Kartenspiele.

Sein Charakter war klar und ernst, sogar herzlich, wie man daraus vermuten kann, daß Herr Takujirô Tsurumoto, sein vertrauliche Freund, der späterhin der hintergebliebenen Frau und zwei Söhnen verstorbenen Freundes sehr lange Zeit half, als er seine Leichenrede las, trauervoll nur weinen konnte.

Obgleich zu Prof. Hayashi zwei sehr große, elegante und umfassende Gedächtnisbände in Tôhoku Mathematical Journal, an seinem sechzigsten Geburtstage von seinen Freunden und Schülern gewidmet waren, doch geben es keine zu Prof. Yoshikawa. Es versteht sich, warum wir darauf hoffen, daß dieser zum Andenken zur Ehre der beiden Professoren Hayashi und Yoshikawa herausgegebene gegenwärtige kleine Band als ein unverwelkliches Monument für ihre wissenschaftlichen Erlangung langer Zeit bleiben werde.

(Tokushima Universität, den 22. Oktober 1955, bei der Messe für beide Professoren.)

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NOTES ON GENERAL ANALYSIS (V)

Singular subspaces

By

Isae SHIMODA

(Received September 30, 1955)

In the preceding paper,[1] we discussed the isolated singular point of an analytic function in complex Banach spaces. The states of analytic functions at singular points are very complicated in complex Banach spaces. The isolated singular point does not exist generally in complex Banach spaces. If the set of singular points of an analytic function in complex Banach spaces is a subspace*), then we call the subspace "*the singular subspace*" of an analytic function. In this paper, we investigate mainly the characters of functions which have the singular subspaces.

In the chapter 1, we discuss homogeneous functions and reciprocal homogeneous functions of degree n which are analytic on whole spaces except their singular subspaces. The conditions, under which homogeneous functions of degree n are homogeneous polynomials of degree n, are stated.

In the chapter 2, removable singular subspaces of analytic functions and another theorems of functions which have singular subspaces are stated. Finally, some of the general theorems in complex Banach spaces is applied to the case of functions of several complex variables.

§ 1. Homogeneous functions and reciprocal homogeneous functions

Let E_1, E_2, E_3, $\cdots$ be complex Banach spaces and L_0 be a subspace of E_1.

Definition 1. *Let an E_2-valued function $f_n(x)$ defined in the outside of L_0 in E_1 be analytic and satisfy $f_n(\alpha x) = \alpha^n f(x)$ in the outside of L_0 in E_1, where α is an arbitrary complex number. $f_n(x)$ is called a homogeneous function of degree n, if n is a positive integer. $f_n(x)$ is called a reciprocal homogeneous function of degree $-n$, if n is a negative integer. L_0 is called their singular subspaces.*

Definition 2. *If x_0 and y_0 do not belong to L_0 and $y_0 \neq \alpha x_0 + \beta y$ for any complex number α, any complex number β and any y in L_0, then x_0 and y_0 are called independent mutually of L_0. That is, y_0 does not belong to the subspace $L(x_0, L_0)$ which is spun by x_0 and L_0.*

Theorem 1. *If there exist two vectors at least which are independent mutually of L_0,* *a homogeneous function* $f_n(x)$ *of degree* n *is a homogeneous polynomial of degree* n, *where* L_0 *is a singular subspace of* $f_n(x)$.

Proof. Let x_0 be an arbitrary point which does not belong to L_0. Since $f_n(x)$ is analytic at x_0, we have

$$f_n(x) = \frac{1}{2\pi i}\int_C \frac{f_n(x_0+\alpha(x-x_0))}{\alpha-1}\,d\alpha = \sum_{m=0}^{\infty} h_m(x),$$

where $h_m(x) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f_n(x_0+\alpha(x-x_0))}{\alpha^{m+1}}d\alpha$ for $m=0, 1, 2, \cdots$ and C is a circle whose radius $\rho>1$ and $\rho\|x-x_0\|<d$, which is the distance between x_0 and L_0. Since $f_n(\alpha x)=\alpha^n f_n(x)$,

$$h_m(x) = \frac{1}{2\pi i}\int_C \frac{f_n(x_0+\alpha(x-x_0))}{\alpha^{m+1}}\,d\alpha = \frac{1}{2\pi i}\int_C \frac{f_n\left(\dfrac{1}{\alpha}x_0+x-x_0\right)}{\alpha^{m-n+1}}\,d\alpha.$$

Put $\dfrac{1}{\alpha}=\beta$, then $d\alpha=-\dfrac{1}{\beta^2}d\beta$ and we have

$$h_m(x) = \frac{1}{2\pi i}\int_{C'} f_n(\beta x_0+x-x_0)\,\beta^{m-n-1}\,d\beta,$$

where C' is a circle whose radius is $\dfrac{1}{\rho}$.

Let $L(x_0, L_0)$ be the subspace which is spun by x_0 and L_0. By the assumption, there exists at least a point x which does not belong to $L(x_0, L_0)$. If x does not belong to $L(x_0, L_0)$, βx_0+x-x_0 does not belong to L_0, because, if $\beta x_0+x-x_0\in L_0$, put $\beta x_0+x-x_0=y$, then $x=y+(1-\beta)x_0$, contradicting to that x does not belong to $L(x_0, L_0)$. Therefore, $f_n(\beta x_0+x-x_0)$ is analytic in $|\beta|\leqq\dfrac{1}{\rho}$, and we have

$$h_m(x) = \frac{1}{2\pi i}\int_{C'} f_n(\beta x_0+x-x_0)\,\beta^{m-1-n}d\beta = 0, \quad \text{for } m\geqq n+1.$$

On the other hand, since $h_m(x)=h_m(x_0, x-x_0)$, $h_m(x)$ is a homogeneous polynomial of degree n with respect to $x-x_0$ and we see that $h_m(x)$ is a polynomial of degree m. As a polynomial of degree m is continuous, $h_m(x)=0$, even if $x\in L(x_0, L_0)$. Then we have

$$f_n(x) = \sum_0^n h_m(x).$$

This shows that $f_n(x)$ is a polynomial of degree n and we see that $f_n(x)$ is analytic on whole spaces. On the other hand, since $f_n(x)$ is homogeneous, that is, $f_n(\alpha x)=\alpha^n f_n(x)$ for $x\overline{\in}L_0$, we have $f_n(\alpha x)=\alpha^n f_n(x)$ for every x, because $f_n(x)$ is continuous.

Thus we see that $f_n(x)$ is a homogeneous polynomial of degree n[2].

Theorem 2. *If there exist two vectors x_0 and y_0 at least which are independent mutually of L_0, then there does not exist a reciprocal homogeneous function $f_{-n}(x)$ of degree n, whose singular subspace is L_0, where $n=1, 2, 3, \cdots$.*

Proof. Let x_0 be an arbitrary point which does not belong to L_0. Since $f_{-n}(x)$ is analytic at x_0, we have

$$f_{-n}(x) = \sum_{m=0}^{\infty} h_m(x_0, x-x_0),$$

where

$$h_m(x_0, x-x_0) = \frac{1}{2\pi i}\int_C \frac{f_{-n}(x_0+\alpha(x-x_0))}{\alpha^{m+1}}d\alpha, \quad \text{for} \quad m=0, 1, 2, \cdots,$$

and C is a circle whose radius $\rho>1$ and satisfies $\rho\|x-x_0\|<d$, which is the distance between x_0 and L_0.

$$h_m(x_0, x-x_0) = \frac{1}{2\pi i}\int_C \frac{f_{-n}\left(\frac{1}{\alpha}x_0+x-x_0\right)}{\alpha^{n+m+1}}d\alpha$$

$$= \frac{1}{2\pi i}\int_{C'} f_{-n}(\beta x_0+x-x_0)\beta^{n+m-1}d\beta,$$

where C' is a circle whose radius is $\frac{1}{\rho}$ and $\beta=\frac{1}{\alpha}$. Then

$$h_m(x_0, x-x_0) = 0,$$

for $m=0, 1, 2, \cdots$, if $x \in L(x_0, L_0)$.

On the other hand, there exists at least two vectors which are independent mutually of L_0, $h_m(x_0, x-x_0)\equiv 0$, from its continuity. Then we have $f_{-n}(x)\equiv 0$. That is, there do not exist reciprocal homogeneous functions in our cases.

Let $L(x_0, y_0, L_0)$ be a subspace spun by x_0, y_0 and L_0, where x_0 and y_0 are independent mutually of L_0. If the space $E_1 \supseteq L(x_0, y_0, L_0)$, then there do not exist reciprocal homogenous functions which have L_0 as their singular subspaces and homogeneous functions which have L_0 as their singular subspaces are homogeneous polynomials. But, when there do not exist two vectors which are independent mutually of a subspace L_0, these theorems are false as the following examples show.

Put $x=(x_1, x_2)$, whose norm $\|x\|=\max(|x_1|, |x_2|)$. Then we have the complex Banach spaces of 2 dimensions with respect to complex numbers. Put

$$h(x) = x_2^n e^{\frac{x_1}{x_2}}.$$

$h(x)$ is analytic at outside points of the closed linear subspace L_1 which is defined by $x_2=0$. Since $\alpha x=(\alpha x_1, \alpha x_2)$,

$$\begin{aligned} h(\alpha x) &= (\alpha x_2)^n e^{\frac{\alpha x_1}{\alpha x_2}} \\ &= \alpha^n x_2^n\, e^{\frac{x_1}{x_2}} \\ &= \alpha^n h(x). \end{aligned}$$

Thus, we see that $h(x)$ is a homogeneous function of degree n which has a singular subspace L_1. But, $h(x)$ is not a homogeneous polynomial of degree n.

From now on, let L_1 be a proper closed linear subspace of E_1 such that $E_1 = L(x_0, L_1)$ for an arbitrary outside point x_0 of L_1.

Theorem 3. *Let $h(x)$ be a homogeneous function of degree n whose singular subspace is L_1,*

(1) *If $y \overline{\in} L_1$ and $h(y) \neq 0$, $\|h(x+\alpha y)\| = |\alpha|^n(0)$ as $|\alpha| \to \infty$, for an arbitrary point x.*

(2) *If $x \in L_1$ and $y \overline{\in} L_1$, $h_m(y, x) = h_{-(m-n)}(x, y)$ and $h_m(y, \alpha x) = \alpha^m h_m(y, x)$, $h_m(\alpha y, x) = \alpha^{n-m} h_m(y, x)$.*

Proof of (1). $\lim\limits_{|\alpha|\to\infty} \dfrac{\|h(x+\alpha y)\|}{|\alpha|^n} = \lim\limits_{|\alpha|\to\infty} \left\| h\left(\dfrac{x}{\alpha} + y\right) \right\|$, since $h(x)$ is analytic at $x = y$.

Proof of (2). Since $y \overline{\in} L_1$, $h(x)$ is analytic at $x = y$ and we have

$$h(y + \alpha x) = \sum_{m=0}^{\infty} h_m(y, x) \alpha^m,$$

where $h_m(y, x) = \dfrac{1}{2\pi i} \displaystyle\int_{C'} \dfrac{h(y+\alpha x)}{\alpha^{m+1}} d\alpha$, for $m = 0, 1, 2, \cdots$ $h_m(y, x)$ is a homogeneous polynomial of degree m with respect to x. Clearly, $h_m(y, \alpha x) = \alpha^m h_m(y, x)$.

$$\begin{aligned} h_m(\beta y, \beta x) &= \frac{1}{2\pi i} \int_C \frac{h(\beta y + \alpha \beta x)}{\alpha^{m+1}} d\alpha \\ &= \frac{1}{2\pi i} \int_C \beta^n \frac{h(y+\alpha x)}{\alpha^{m+1}} d\alpha \\ &= \beta^n h_m(y, x). \end{aligned}$$

On the other hand, $h_m(\beta y, \beta x) = \beta^n h_m(\beta y, x)$. Then we have

$$\beta^m h_m(\beta y, x) = \beta^n h_m(y, x).$$

Dividing by β^m, we have $h_m(\beta y, x) = \beta^{n-m} h_m(y, x)$.

Since $h(y + \alpha x)$ is an analytic function of y lying in the outside of L_1, $h_m(y, x)$ is an analytic function of y lying on the outside of L_1 by uniform convergence of the integral. Then $h_m(y, x)$ is a homogeneous function of degree $n-m$ whose singular subspace is L_1, if $n \geqslant m$. If $n < m$, $h_m(y, x)$ is a reciprocal homogeneous function of degree $m-n$ whose singular subspace is L_1.

$$h_m(y, x) = \frac{1}{2\pi i}\int_C \frac{h(y+\alpha x)}{\alpha^{m+1}}\, d\alpha$$

$$= \frac{1}{2\pi i}\int_C \frac{h\left(\frac{1}{\alpha} y + x\right)}{\alpha^{m-n+1}}\, d\alpha$$

$$= \frac{1}{2\pi i}\int_{C'} h(\beta y + x)\beta^{m-n-1} d\beta,$$

where C' is a circle whose radius is $\frac{1}{|\alpha|}$ and $\beta = \frac{1}{\alpha}$, then

$$h_m(y, x) = h_{-(m-n)}(x, y).$$

This completes the proof.

Theorem 4. *Let $h(x)$ be a homogeneous function of degree n whose singular subspace is L_1. The necessary and sufficient condition that $h(x)$ should be a homogeneous polynomial is that*

$$\|h(x+\alpha y)\| \leqq K(x, y), \quad \textit{as } |\alpha| \textit{ tends to } 0,$$

for an arbitrary point x in L_1 and an arbitrary point y lying on the outside of L_1, where $K(x, y)$ is a positive constant with respect to α and is defined by x and y.

Proof. If $f(x)$ is a homogeneous polynomial of degree n, $h(x)$ is continuous and we have $\lim_{\alpha\to 0} \|h(x+\alpha y)\| = \|h(x)\|$, for arbitrary points x and y.

Suppose that $\|h(x+\alpha y)\| \leqq K(x, y)$ as $|\alpha|$ tends to 0, where x is an arbitrary point of L_1 and y is an arbitrary point which lies on the outside of L_1 and $K(x, y)$ is a constant with respect to α being defined by x and y. Let f^* be an arbitrary complex valued bounded linear functional in the conjugate space E_1^* of E_1,

$$|f^*(h(x+\alpha y))| \leqq M\|h(x+\alpha y)\|, \quad \text{where} \quad M = \|f^*\|.$$

For an arbitrary positive number ε, there exists a positive number δ such that $\|h(x+\alpha y)\| \leqq K(x, y) + \varepsilon$, for $|\alpha| < \delta$. Then we have $|f^*(h(x+\alpha y))| \leqq M(K(x, y)+\varepsilon)$ for $|\alpha| < \delta$. On the other hand, if $|\alpha| > 0$, $x+\alpha y \overline{\in} L_1$ and $h(x+\alpha y)$ is an analytic function of α for $|\alpha| > 0$ and we see that $f^*(h(x+\alpha y))$ is regular for $|\alpha| > 0$. Thus we see that $\alpha = 0$ is a removable singular point and $f^*(h(x+\alpha y))$ is regular at $\alpha = 0$. Since f^* is an arbitrary point of the conjugate space E_1^*, we see that $h(x+\alpha y)$ is analytic at $\alpha = 0$ [3] that is $h(x+\alpha y)$ is G-differentiable at x on L_1, if $y \overline{\in} L_1$.

Now, if x and y are arbitrary points lying on the outside of L_1, there exists only one complex number α_0 which satisfies $y + \alpha_0 x \in L_1$. Since $E_1 = L(y, L_1)$, there exists x' in L_1 which satisfies $x = \beta' y + \alpha' x'$, where α', β' are complex numbers. Put $-\frac{1}{\beta'} = \alpha_0$, $y + \alpha_0 x = -\frac{\alpha'}{\beta'} x' \in L_1$. If $y + \alpha_1 x \in L_1$ for $\alpha_1 \neq \alpha_0$, $y + \alpha_1 x - (y + \alpha_0 x) =$

$(\alpha_1-\alpha_0)x \in L_1$ and we have $x \in L_1$ contradicting to the assumption $x \overline{\in} L_1$. Then

$$h(y+\alpha x) = h(y+\alpha_0 x + (\alpha-\alpha_0)x).$$

Put $y+\alpha_0 x = x_0$ which belongs to L_1. $h(y+\alpha x) = h(x_0 + (\alpha-\alpha_0)x)$. This shows that $h(y+\alpha x)$ is an analytic function of α for $|\alpha|<\infty$. If $y \overline{\in} L_1$ and $x \in L_1$, $y+\alpha x \overline{\in} L_1$ for $|\alpha|<\infty$ and we see that $h(y+\alpha x)$ is an analytic function of α for $|\alpha|<\infty$ if y does not belong to L_1. Then we see that $h(y+\alpha x)$ is an analytic function of α, if only $y \overline{\in} L_1$, and we have

$$h(y+\alpha x) = \sum_{m=0}^{\infty} h_m(y,x)\,\alpha^m,$$

where $h_m(y,x)$ is a homogeneous polynomial of degree m with respect to x and satisfies

$$h_m(y,x) = \frac{1}{2\pi i}\int_C \frac{h(y+\alpha x)}{\alpha^{m+1}}\,d\alpha, \quad \text{for} \quad m = 0, 1, 2, \cdots.$$

Since $h(y+\alpha x)$ is analytic for $|\alpha|<\infty$, the radius of the circle C can be taken as large as we like. Then we have

$$\|h_m(y,x)\| \leqq \frac{1}{2\pi}\int_0^{2\pi} \frac{\|h(y+re^{i\theta}x)\|}{r^m}\,d\theta.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\|h(\frac{e^{-i\theta}}{r}y+x\|)}{r^{m-n}}\,d\theta$$

If $m>n$,

$$\|h_m(y,x)\| \leqq \overline{\lim_{r\to\infty}}\,\frac{1}{2\pi}\int_0^{2\pi} \frac{\|h(\frac{e^{-i\theta}}{r}y+x)\|}{r^{m-n}}\,d\theta$$

$$\leqq \frac{1}{2\pi}\int_0^{2\pi} \overline{\lim_{r\to\infty}}\,\frac{\|h(\frac{e^{-i\theta}}{r}y+x)\|}{r^{m-n}}\,d\theta \qquad 4)$$

$$= 0.$$

Because, $\lim_{r\to\infty} \|h(\frac{e^{-i\theta}}{r}y+x)\| = \|h(x)\|$, if $x \overline{\in} L_1$ and $\overline{\lim_{r\to\infty}} \|h(\frac{e^{-i\theta}}{r})y+x\| \leqq K(y,x)$, if $x \in L_1$.
Since x is an arbitrary point, we have $h_m(y,x) \equiv 0$ for $m>n$.
Therefore, $h(y+\alpha x) = \sum_0^n h_m(y,x)$.

$\sum_0^n h_m(y,x)$ is a polynomial of degree n. This shows that $h(x)$ is analytic on whole spaces. If $x \overline{\in} L_1$, $h(\alpha x) = \alpha^n h(x)$. Since $h(x)$ is analytic, $\lim_{x\to x'} h(\alpha x) = \lim_{x\to x'} \alpha^n h(x)$ for $x' \in L_1$ and we have

$$h(\alpha x') = \alpha^n h(x').$$

Thus we see that $h(x)$ is a homogeneous polynomial of degree n.

Theorem 5. *Let $h(x)$ be a homogeneous function of degree n whose singular subspace is L_1. The necessary and sufficient condition that $h(x)$ should be a homogeneous polynomial of degree n is that $\overline{\lim\limits_{\|x\|\to\infty}} \dfrac{\|h(x)\|}{\|x\|^n} \leqq K$, where K is a constant.*

Rroof. If $h(x)$ is a homogeneous polynomial of degree n, we have $\sup\limits_{\|x\|=1} \|h(x)\| < \infty$. Then

$$\overline{\lim_{\|x\|\to\infty}} \frac{\|h(x)\|}{\|x\|^n} = \overline{\lim_{\|x\|\to\infty}} \left\| h\left(\frac{x}{\|x\|}\right) \right\| \leqq \sup_{\|x\|=1} \|h(x)\| < \infty.^{5)}$$

Suppose that $\overline{\lim\limits_{\|x\|\to\infty}} \dfrac{\|h(x)\|}{\|x\|^n} \leqslant K$, where K is a constant. Let x be an arbitrary point of L_1 and y be an arbitrary point which does not belong to L_1. Then, $x+\alpha y \overline{\in} L_1$ and we have

$$\begin{aligned}
\overline{\lim_{\alpha\to 0}} \|h(x+\alpha y)\| &= \overline{\lim_{\alpha\to 0}} |\alpha|^n \left\| h\left(\frac{1}{\alpha}x+y\right) \right\| \\
&\leqq \overline{\lim_{\alpha\to 0}} |\alpha|^n \cdot \frac{\left\| h\left(\frac{1}{\alpha}x+y\right) \right\|}{\left\| \left(\frac{1}{\alpha}x+y\right) \right\|^n} \left\| \frac{1}{\alpha}x+y \right\|^n \\
&= \overline{\lim_{\alpha\to 0}} \frac{\left\| h\left(\frac{1}{\alpha}x+y\right) \right\|}{\left\| \left(\frac{1}{\alpha}x+y\right) \right\|^n} \cdot \|x+\alpha y\|^n \\
&= K \|x\|^n,
\end{aligned}$$

since $\lim\limits_{\alpha\to 0} \|\frac{1}{\alpha}x+\alpha y\| = +\infty$. Then Theorem 4 is applicable and we see that the condition $\overline{\lim\limits_{\|x\|\to\infty}} \dfrac{\|h(x)\|}{\|x\|^n} \leqq K$ is sufficient.

Theorem 6. *If $h_n(x)$ is an E_1-valued homogeneous polynomial of degree n defined on E_1 and $h_m(x)$ is an E_1-valued homogeneous polynomial of degree m defined on E_1, then $h_n(h_m(x))$ and $h_m(h_n(x))$ is a homogeneous polynomial of degree mn, but $h_n(h_m(x)) \neq h_m(h_n(x))$ generally.*

Proof. $h_n(h_m(x))$ is clearly an analytic function.

$$h_n(h_m(\alpha x)) = h_n(\alpha^m h_m(x)) = \alpha^{nm} h_n(h_m(x)).$$

This shows that $h_n(h_m(x))$ is a homogeneous polynomial of degree mn. On the same way, $h_n(h_m(x))$ is a homogeneous polynomial of degree mn.

Let $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ be a matrix of 2-2-types of complex numbers, and $\|x\| = \max(|x_{11}|, |x_{12}, |x_{21}|, |x_{22}|)$. Then the set of such X is complex Banach spaces. Let $f(X) = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ and $g(X) = \begin{pmatrix} 0 & 1 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$. Then

$$f(g(X)) = \begin{pmatrix} 4 & 11 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \quad g(f(X)) = \begin{pmatrix} 3 & 0 \\ 17 & 4 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}.$$

This shows that generally $f(g(x)) \neq g(f(x))$.

Theorem 7. *Let $R(x)$ be a reciprocal homogeneous function whose singular subspace is L_1. If $\overline{\lim}_{|\alpha|\to 0} \|R(x+\alpha y)\| \cdot |\alpha|^n \leqq K(x, y)$, for an arbitrary point x on L_1 and an arbitrary point y which does not belong to L_1, then $R(x+y)=R(y)$.*

Proof. For an arbitrary x on L_1 and an arbitrary y which does not belong to L_1, $R(x+\alpha y)$ is analytic when $|\alpha|>0$. Then we have

$$R(x+y) = \sum_{-\infty}^{\infty} R_m(x, y),$$

as well as the Laurent expansion of the complex valued function of complex variables, where

$$R_m(x, y) = \frac{1}{2\pi i}\int_C \frac{R(x+\alpha y)}{\alpha^{m+1}}\, d\alpha, \quad \text{for } m=0, \pm 1, \pm 2, \cdots.$$

$$R_m(x, y) = \frac{1}{2\pi i}\int_C \frac{R\left(\frac{1}{\alpha}x+y\right)}{\alpha^{n+m+1}}\, d\alpha$$

$$= \frac{1}{2\pi i}\int_{C'} R(\zeta x + y)\, \zeta^{n+m-1}\, d\zeta,$$

where $\zeta = \frac{1}{\alpha}$ and C' is a circle whose radius is $\frac{1}{|\alpha|}$. Since clearly $\zeta x + y \overline{\in} L_1$, $R(\zeta x+y)$ is analytic with respect to ζ for $|\zeta|<\infty$. Then

$$R_m(x, y) = 0, \quad \text{when} \quad n+m-1 \geqq 0.$$

Since $R_m(x, y)=0$ for an arbitrary y which does not belong to L_1, by the analytic continuation $R_m(x, y)\equiv 0$ for all y in E_1, where x is arbitrarily fixed. Since x is arbitrary, $R_m(x, y)\equiv 0$ for $m \geqq -n+1$.

Now, since

$$R_m(x, y) = \frac{1}{2\pi i}\int_C R(x+\alpha y)\, \alpha^{-m-1}\, d\alpha,$$

$$\|R_m(x, y)\| \leqq \frac{1}{2\pi}\int_0^{2\pi} \|R(x+re^{i\theta}y)\|\, r^{-m}\, d\theta$$

where $\alpha = re^{i\theta}$. Thus we have

$$\|R_m(x, y)\| \leqq \overline{\lim}_{\alpha\to 0} \int_0^{2\pi} \|R(x+\alpha y)\|\, r^{-m}\, d\theta$$

$$\leqq \int_0^{2\pi} \overline{\lim}_{\alpha\to 0} \|R(x+\alpha y)\| \cdot |\alpha|^n \cdot r^{-m-n}\, d\theta$$

$$\leqq \int_0^{2\pi} K(x, y) \lim_{\alpha\to 0} r^{-m-n}\, d\theta$$

$$= 0, \quad \text{if } -n > m.$$

As well as the above case, $R_m(x, y)\equiv 0$ for $m<-n$. Thus we have

$$R(x+y) = R_{-n}(x, y).$$

Since $x+y\overline{\in}L_1$, $R(\alpha(x+y))=\frac{1}{\alpha^n}R(x+y)$.

On the other hand, $R(\alpha(x+y))=R(\alpha x+\alpha y)=R_{-n}(\alpha x, \alpha y)=\frac{1}{\alpha^n}R_{-n}(\alpha x, y)$.
Then we have $R(x+y)=R_{-n}(\alpha x, y)=R(\alpha x+y)$. Since $R(\alpha x+y)$ is analytic as to α, we have

$$R(x+y) = \lim_{\alpha\to 0} R(x+y) = \lim_{\alpha\to 0} R(\alpha x+y) = R(y).$$

This completes the proof.

From this theorem,

$$\begin{aligned}\lim_{\alpha\to 0} \|R(x+\alpha y)\| &= \lim_{\alpha\to 0} \|R(\alpha y)\| \\ &= \lim_{\alpha\to 0} \frac{1}{|\alpha|^n}\|R(y)\| \\ &= +\infty,\end{aligned}$$

since $R(y)\neq 0$.**) The order of infinity of $R(x)$ is n.

Let $x=(x_1, x_2)$ and $\|x\|=\max(|x_1|, |x_2|)$. Then the set of x is a complex Banach spaces Ω. The Ω-valued reciprocal homogeneous function whose singular subspace is $x_1=0$, defined on Ω

$$f(x) = \left(\frac{1}{x_1^n}, 0\right)$$

satisfies the condition of Theorem 7. The complex valued reciprocal homogeneous function of degree n whose singular subspace is $x_2=0$, defined on Ω, $\frac{1}{x_2^n}e^{\frac{x_1}{x_2}}$ does not satisfy the condition of Theorem 7.

§ 2. Analytic functions

Let L_0 be a linear subspace of E_1.

Theorem 8. *If there exist at least two vectors which independent mutually of* L_0 *and an* E_2*-valued function* $f(x)$ *is analytic on the outside of* L_0 *in* E_1, *then* $f(x)$ *is analytic on whole space* E_1.

Proof. For an arbitrary point x which does not belong to L_0, $f(\alpha x)$ is analytic when $|\alpha|>0$. As well as the Laurent expansion of the complex valued function of complex variables, we have

$$f(\alpha x) = \sum_{-\infty}^{+\infty} f_m(x)\alpha^m,$$

where

$$f_m(x) = \frac{1}{2\pi i}\int_C \frac{f(\alpha x)}{\alpha^{m+1}}\, d\alpha, \quad \text{for } m = 0, \pm 1, \pm 2, \cdots.$$

By the uniformity of the integral, we see that $h_m(x)$ is analytic if x lies on the outside of L_0. Moreover, we can easily see that

$$f_m(\beta x) = \beta^m f_m(x), \quad \text{for } m = 0, \pm 1, \pm 2, \cdots.$$

This shows that $h_m(x)$ is a homogeneous function of degree m, whose singular subspace is L_0, when m is positive, and $h_m(x)$ is a reciprocal homogeneous function of degree $(-m)$, whose singular subspace is L_0, when m is a negative integer.

Appealing to Theorem 2, $f_m(x) \equiv 0$ if $m < 0$. Then wee have

$$f(x) = \sum_0^\infty f_m(x).$$

Appealing to Theorem 1, $f_m(x)$ is a homogeneous polynomial of degree m. Put $f_m(x) = h_m(x)$. Thus we see that $f(x)$ is a power series, that is $f(x) = \sum_0^\infty h_m(x)$.

Let x_0 be an arbitrary point which does not belong to L_0, and $d = \text{dis.}(x_0, L_0)$. Since $f(x)$ is analytic at x_0, for an arbitrary positive number ε there exists a positive number δ which satisfies

$$\|f(x) - f(x_0)\| < \varepsilon, \quad \text{if} \quad \|x - x_0\| < \delta (< d).$$

Let $U(x_0, \delta)$ be a set of x which satisfies $\|x - x_0\| < \delta$. On the same way, we have

$$\|f(x) - f(e^{i\theta} x_0)\| < \varepsilon, \quad \text{if } x \in U(e^{i\theta} x_0, \delta_\theta),$$

where $U(e^{i\theta}x_0, \delta_\theta) \frown L_0 = 0$. Appealing to the covering theorem of Borel, we have $\theta_1, \theta_2, \cdots, \theta_k$, such that the set $\sum_1^k U\left(e^{i\theta_j}x_0, \frac{\delta_{\theta j}}{2}\right)$ includes the set $x_0 e^{i\theta}$ $(0 \leqq \theta \leqq 2\pi)$.

Put $M = \max_{1 \leq j \leq k} (\|f(e^{i\theta_j}x_0)\| + \varepsilon)$, then if x lies in $\sum_1^k U(e^{i\theta_j} x_0, \delta_{\theta_j})$,

$$\|f(x)\| \leqq M.$$

When δ_0 is a small positive number such that $0 < \delta_0 \ll \operatorname{Min}_{1 \leq j \leq m} \left(\frac{\delta_{\theta j}}{2}\right)$, we have

$$e^{i\theta} U(x_0, \delta_0) \subset \sum_1^k U(x_0 e^{i\theta_j}, \delta_{\theta j}), \quad \text{for } 0 \leqq \theta \leqq 2\pi.$$

Then

$$\begin{aligned} \|h_m(x)\| &= \left\| \frac{1}{2\pi i}\int_C \frac{f(\alpha x)}{\alpha^{m+1}}\, d\alpha \right\| \\ &= \left\| \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{i\theta}x)}{e^{im\theta}}\, d\theta \right\| \\ &\leqq M, \end{aligned}$$

where C is a circle whose radius is 1, for $m=0, 1, 2, \cdots$ and $x \in U(x_0, \delta_0)$. Appealing to the lemma of Zorn[6], we see that

$$\|h_m(x)\| \leqq M, \quad \text{when} \quad \|x\| < \delta_0, \quad \text{for } m=0, 1, 2, 3, \cdots.$$

Thus we have

$$\begin{aligned}
&\sup_{\|y\|=1} \overline{\lim_{m\to\infty}} \sqrt[m]{\|h_m(y)\|}\ ^{7)} \\
&= \sup_{\|y\|=1} \overline{\lim_{m\to\infty}} \sqrt[m]{\left\|h_m\left(\frac{\delta y}{\delta}\right)\right\|}, \quad \text{for } 0<\delta<\delta_0, \\
&= \frac{1}{\delta} \sup_{\|y\|=1} \overline{\lim_{m\to\infty}} \sqrt[m]{\|h_m(\delta y)\|}, \\
&\leqq \frac{1}{\delta} \sup_{\|y\|=1} \overline{\lim_{m\to\infty}} \sqrt[m]{M}, \quad \text{because } \|\delta y\| = \delta < \delta_0, \\
&= \frac{1}{\delta}.
\end{aligned}$$

This shows that the radius of analyticity of $f(x)$ is not smaller than δ and we see that $f(x)$ is analytic in the neighbourhood of 0. On the same method, we see that $f(x)$ is analytic at an arbitrary point of L_0. This completes the proof.

Corollary. *If a complex valued function $f(z_1, z_2, \cdots, z_n)$ of n-complex variables is regular on the outside of the subspace $L(z_1, z_2, \cdots, z_{n-2})$ of $(n-2)$-dimensions, then $f(z_1, z_2, \cdots, z_n)$ is regular on whole spaces.*[8]

Proof. Since $f(z_1, z_2, \cdots, z_n)$ is regular on the outside of L, $f(z_1, z_2, \cdots, z_n)$ is continuous at the point of the outside of L. Let $z=(z_1, z_2, \cdots, z_n)$ be an arbitrary point in the outside of L and $w=(w_1, w_2, \cdots, w_n)$ be an arbitrary point.

$$\begin{aligned}
&\lim_{\alpha\to 0} \frac{f(z+\alpha w)-f(z)}{\alpha} \\
&= \lim_{\alpha\to 0} \sum_{i=1}^{n} \frac{f(z_1, \cdots, z_{i-1}, z_i+\alpha w_i, \cdots, z_n+\alpha w_n) - f(z_1, \cdots, z_i, z_{i+1}+\alpha w_{i+1}, z_n+\alpha w_n)}{\alpha} \\
&= \sum_{i=1}^{n} \frac{\partial f(z_1, \cdots, z_n)}{\partial z_i} w_i.
\end{aligned}$$

This shows that $f(z_1, z_2, \cdots, z_n)$ is G-differentiable on the outside of L. Appealing to Theorem 8, $f(z_1, z_2, \cdots, z_n)$ is analytic on whole spaces. Then $f(z_1, z_2, \cdots, z_n)$ is partially differentiable, because it is G-differentiable, and we see that $f(z_1, z_2, \cdots, z_n)$ is regular on whole spaces. If the dimension of L is smaller than n-2, this theorem is clearly true.

Let exist only one vector which is independent of a subspace L_1 in E_1, that is, $E_1 = L(x, L_1)$ for an arbitrary point x in the outside of L_1.

Theorem 9. *If an E_2-valued function $f(x)$ defined on the outside of L_1 in E_1 is analytic in E_1 removing L_1 and*

$$\overline{\lim_{|\alpha|\to\infty}} \|f(\alpha x+y)\| \leqq K(x, y),$$

for an arbitrary point x *of* L_1 *and an arbitrary* y *in the outside of* L_1 *in* E_1, *where* $K(x, y)$ *is a constant as to* α, *then*

$$f(x+y)=f(y).$$

Proof. Since y lies in the outside of L_1, $f(x)$ is analytic at y and so we have

$$f(y+\alpha x) = \sum_0^\infty h_n(y, x)\alpha^n,$$

$$h_n(y, x) = \frac{1}{2\pi i}\int_C \frac{f(y+\alpha x)}{\alpha^{n+1}}\, d\alpha, \quad \text{for } n=0, 1, 2, \cdots$$

Clearly, $y+\alpha x \overline{\in} L_1$ and we see that $f(y+\alpha x)$ is analytic for $|\alpha|<\infty$. By the assumption, $\lim_{|\alpha|\to\infty} \|f(y+\alpha x)\| \leqq K(x, y)$, we have

$$\|f(y+\alpha x)\| \leqq K(x, y)+\varepsilon, \quad \text{for } |\alpha|>R,$$

where ε is an arbitrary positive number and a positive number R is determined by ε. Since $f(y+\alpha x)$ is continuous on $|\alpha|\leqq R$, $\|f(y+\alpha x)\|$ is bounded on $|\alpha|\leqq R$. That is, for a suitable positive number M, we have

$$\|f(y+\alpha x)\| \leqq M, \quad \text{for } |\alpha|\leqq R.$$

Then we have

$$\|f(y+\alpha x)\| \leqq \max(M, K(x, y)+\varepsilon) \quad \text{when } |\alpha|<\infty.$$

Appealing to the extended theorem of Liouville, $f(y+\alpha x)=c(x, y)$, where $c(x, y)$ is a constant as to α. Then, for $\alpha=0$ and $\alpha=1$, we have $f(y+x)=f(y)$. Since x and y are arbitrary, this completes the proof.

Theorem 10. *If an* E_2*-valued function* $f(x)$ *defined on the outside of* L_1 *is analytic there and satisfies the following inequality*

$$\overline{\lim_{|\alpha|\to\infty}} \|f(y+\alpha x)\| \leqq K,$$

where K *is a constant and* x *is an arbitrary point in* L_1 *and* y *is an arbitrary outside point of* L_1, *then* $f(y)$ *is a constant.*

Proof. Appealing to Theorem 9, we have $f(y+x)=f(y)$, for an arbitrary x in L_1 and an arbitrary y in the outside of L_1. Then

$$\|f(y)\| = \lim_{|\alpha|\to\infty} \|f(y)\| = \lim_{|\alpha|\to\infty} \|f(y+\alpha x)\| \leqq K.$$

That is, $\|f(y)\|\leqq K$. This inequality is true for an arbitrary y in the outside of L_1. Since $f(\beta y)$ is analytic for $|\beta|>0$ and $\|f(\beta y)\|\leqq K$ for $|\beta|<\infty$, $\beta=0$ is a removable singular point. Appealing to the extended theorem of Liouville, we see that $f(\beta y)=c(y)$, where $c(y)$ is a constant with respect to β. On the same way, since $\alpha y+x\overline{\in}L_1$, for $\alpha\neq 0$, $\|f(\alpha y+x)\|\leqq K$ and then we see that $f(\alpha y+x)$ is a constant with respect to α. Let y_1 and y_2 be arbitrary points in the outside of L_1. If $y_1=y_2+\beta x$ for a suitable point x in L_1 and a suitable complex number β, $f(y_1)=f(y_2+\beta x)=f(y_2)$. If $y_1\neq y_2+\beta x$, since $E_1=L(y_2, L_1)$, $y_1=\alpha y_2+\beta x$ for suitable complex number α, β and a suitable x in L_1, where $\alpha\neq 1$. Then, $y_2+\gamma(y_1-y_2)$ $=y_2+\gamma(\alpha y_2+\beta x-y_2)=\gamma\beta x+(1+\gamma(\alpha-1))y_2$. For $\gamma_0=\frac{1}{1-\alpha}$, $y_2+\gamma_0(y_1-y_2)=$ $\alpha\beta x\in L_1$. Put $y_2+\gamma_0(y_1-y_2)=x_0$, then $y_2+\gamma(y_1-y_2)-x_0=(\gamma-\gamma_0)(y_1-y_2)$ and we have $y_2+\gamma(y_1-y_2)=x_0+(\gamma-\gamma_0)(y_1-y_2)$. Since $y_1-y_2\overline{\in}L_1$, $f(y_2+\gamma(y_1-y_2))=$ $f(x_0+(\gamma-\gamma_0)(y_1-y_2))$ is constant with respect to $\gamma-\gamma_0$ and we have $f(y_2)=f(y_1)$, for $\gamma=0$ and $\gamma=1$. From this we can easily see that $f(y)$ is a constant if $y\overline{\in}L_1$. By the analytic continuation, $f(y)$ in a constant on E_1.

Corollary. *If an E_2-valued function $f(x)$ defined on the outside of L_1 is analytic there and satisfies the following inequality*

$$\|f(y+\alpha x)\|\leqq K,$$

for an arbitrary x in L_1 and an arbitrary y in the outside of L_1, where K is a constant, then $f(y)$ is a contant.

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*) A subspace is, of course, closed and linear.

**) Let $R(y_1)=0$ for a y_1, which lies in the outside of L_1. Since there is an element which is linearly independent mutually of L_1, an arbitrary point $z=\beta x_1+\alpha y_1$, if $z\overline{\in}L_1$, for a suitable point x_1

in L_1 and suitable complex numbers α, β. Clearly $\beta x_1 \in L_1$, then we have

$$R(\beta x_1+\alpha y_1)=R(\alpha y_1)=\frac{1}{\alpha^n}R(y_1)=0.$$

Thus we see that $R(y)=0$ on the outside of L_1 and we have $R(x)\equiv 0$ on L_1 by the analytic continuation, contradicting to the fact that $R(x)$ is not a constant.

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ALL SEMIGROUPS OF ORDER AT MOST 5

By

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The main object of this paper is to show a list of all semigroups of order 5, which have been obtained after long computation by hand. However all the discussions as to construction method are remained in another papers. (See references.) For convenience' sake semigroups of order 2, 3, and 4 will be again listed.

In 1951, T. Sakuragi and T. Tamura calculated all semigroups of order 2 and 3 [1] [2], and M. Yamamura computed all semigroups of order 4 for September 1952—January 1953 by elementary method, but T. Akazawa and R. Shibata computed them again in August 1954 by use of a comparatively new method [3]. According to [4], we know that Poole listed some of distinct commutative semigroups and Carman, Harden, and Posey corrected some errors and added some distinct non-commutative semigroups, but their results were incomplete. G.E. Forsythe [4] computed all semigroups of order 4 by electronic method in May and July 1954, and his result is equivalent to ours. Hewitt and Zuckerman also got all semigroups of order 3 and others.

For order 5, unipotent semigroups were gotten by K. Tetsuya and T. Hashimoto in May 1953, the computation of all commutative semigroups were continued by T. Akazawa and R. Shibata for September 1954—January 1955, and all types which include unipotent ones and commutative ones have been obtained by T. Inui and others[1] a very large calculation for April—July 1955. Independently from us, T.S. Motzkin and J.L. Selfridge computed all semigroups of order 5 with the electronic computer in May 1955, and we received the list which they sent to us three days after we computed them. Our results are completely equivalent to theirs.[2]

In the previous paper [3] we classified all commutative semigroups by types of greatest semilattice decomposition, and non-commutative semigroups by types of greatest commutativity decomposition, but in the present paper all semigroups are

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2) It took a month to compute to make their tables correspond to ours by hand. We owe correction of our tables to Prof. Motkin and Prof. Selfridge.

classified into categories of types of greatest semilattice decomposition and we use new expression inductively to show multiplication table so that their structures are clarified. In the greatest s-decomposition of a semigroup S, S is decomposed into class sum of s-indecomposable semigroups [9]. S is constructed by types of semilattice, s-indecomposable semigroup, and suitable translations [10] [11]. Further s-indecomposable semigroups are classified into the four categories: c-indecomposable semigroups, unipotent ones with zero, unipotent ones with group, and c-decomposable ones to unipotent. As shown in the list, we have 1160 isomorphically and anti-isomorphically distinct semigroups.

We express our thoughtful thanks to Prof. T.S. Motzkin and Prof. J.L. Selfridge for their kind advice about checking of our list.

References

With respect to each list, see the reference as following.

number of lists		number of references
1		[1]
3		[3]
2, 5		[10] [11]
6 I	c-ind.	[18]
	others	[14] [15] [16] [17]
6 II ~		[7] [9] [10] [11] [12]
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[16] T. Tamura, On finite unipotent semigroup with zero, to appear.

[17] T. Tamura, Note on s-indecomposable semigroups, to appear.

[18] It is proved that c-indecomposable semigroups of order 5 are completely simple. About the structure of completely simple semigroups, see [13].

Errata of the previous paper

"Note on finite semigroups and determination of semigroups of order 4", by T. Tamura, Journal of Gakugei, Tokushima University, Vol. V, 1954, pp. 17-27.

p. 23, line 5 from the bottom, read 188 for 194.

It is for this reason that there are 6 semigroups which are anti-isomorphic to themselves.

I express my thoughtful thanks to Prof. T.S. Motzkin and Prof. J.L. Selfridge for their pointing out this miss.

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Introductory Remarks

General Rules

k_2	the k-th semigroup of order 2,
k_n ( )	the k-th semigroup of order n ($\geqq 3$), the bracket giving its structure.
k ( )	But the suffix n is often omitted if there is no fear of confusion.
k'	transpose of k.
×	commutative,
*	self-dual (anti-isomorphic to itself),
c. i.	c-indecomposable semigroup,
u. z.	unipotent semigroup with zero,
u. g.	unipotent semigroup with group,
c. d. u.	c-decomposable semigroup to unipotent one.

Semilattices are represented by diagrams, for example,

$a<\begin{smallmatrix}b\\c\end{smallmatrix}$ .... a is a greatest upper bound of b and c, where the ordering $a \geqq b$ means $a=bx$ for some x.

c. i. $k_n(a_1^{(1)}a_2^{(1)}\cdots a_n^{(1)}, a_1^{(2)}a_2^{(2)}\cdots a_n^{(2)}, \cdots, a_1^{(n)}a_2^{(n)}a_n^{(n)}$ ....

$$\begin{array}{|cccc|}
a_1^{(1)} & a_2^{(1)} & \cdots\cdots & a_n^{(1)} \\
a_1^{(2)} & a_2^{(2)} & \cdots\cdots & a_n^{(2)} \\
\multicolumn{4}{|c|}{\cdots\cdots\cdots\cdots} \\
a_1^{(n)} & a_2^{(n)} & \cdots\cdots & a_n^{(n)}
\end{array}$$

u. z. $k_n(a_1^{(1)}\cdots a_{n-2}^{(1)}, a_1^{(2)}\cdots a_{n-2}^{(2)}, \cdots, a_1^{(n-2)}\cdots a_{n-2}^{(n-2)})$ …………

$$\begin{array}{|cccc|} a & a & \cdots\cdots & a \\ a & a & & a \\ \vdots & \vdots & a_1^{(1)}\cdots\cdots & a_{n-2}^{(1)} \\ \vdots & \vdots & \vdots & \\ a & a & a_1^{(n-2)}\cdots & a_{n-2}^{(n-2)} \end{array}$$

u. g. $k_n(j_m, l_{n-m+1}, a_{i_1}a_{i_2}\cdots a_{i_{n-m}})$ a unipotent semigroup whose greatest group is j_m, its difference semigroup modulo j_m is l_{n-m+1}, and a mapping f of elements $a_{m+1}, \cdots, a_n$ not contained in j_m into j_m: $f(a_{m+t})=a_{m+t}a_1=a_1a_{m+t}=a_{i_t}$.

c. d. u. $k_n(j_{n-1}; a_1\cdots a_{n-1}, b_1\cdots b_{n-1}; p)$ …………………………

$$\begin{array}{|c|c|} j_{n-1} & \begin{matrix} b_1 \\ \vdots \\ b_{n-1} \end{matrix} \\ \hline a_1\cdots a_{n-1} & p \end{array}$$

$k_n(i_{n-1})$ …………………………See examples in remarks of each list.

$k_n(i_{n-1}; a_{t_1}\cdots a_{t_{n-1}}, a_{s_1}\cdots a_{s_{n-1}})$ ……………………………………

$$\begin{array}{c|c|c|} & a_1\cdots a_{n-1} & a_n \\ \hline \begin{matrix} a_1 \\ \vdots \\ a_{n-1} \end{matrix} & i_{n-1} & \begin{matrix} a_{s_1} \\ \vdots \\ a_{s_{n-1}} \end{matrix} \\ \hline a_n & a_{t_1}\cdots a_{t_{n-1}} & a_n \\ \hline \end{array}$$

$\left(i_l, j_m; \begin{matrix} a_1^{(1)}\cdots\cdots a_l^{(1)} & b_1^{(1)}\cdots\cdots b_l^{(1)} \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ a_1^{(m)}\cdots\cdots a_l^{(m)} & b_1^{(m)}\cdots\cdots b_l^{(m)} \end{matrix} \right)$ ………………………

$$\begin{array}{|c|c|} i_l & \begin{matrix} b_1^{(1)} & \cdots\cdots & b_1^{(m)} \\ \vdots & \vdots & \vdots \\ b_l^{(1)} & \cdots\cdots & b_l^{(m)} \end{matrix} \\ \hline \begin{matrix} a_1^{(1)} & \cdots\cdots & a_l^{(1)} \\ \vdots & \cdots\cdots & \\ a_1^{(m)} & \cdots\cdots & a_l^{(m)} \end{matrix} & j_m \\ \hline \end{array}$$

As far as automorphisms are concerned, we show all ones, if exist, except an identical mapping.

k_n $a_{i_1}a_{i_2}\cdots\cdots a_{i_n}$……………an automorphism $\begin{pmatrix} a_1 & a_2 & \cdots\cdots a_n \\ a_{i_1} & a_{i_2} & \cdots\cdots a_{i_n} \end{pmatrix}$ of a semigroup k_n where a_t is mapped to a_{i_t},

k_n $a_{i_1}a_{i_2}\cdots\cdots a_{i_n}{}^*$ …………dual-automorphism of k_n.

Translations If a semigroup S is a right (left) unit, any right (left) translation f_z (g_z) of S is given by $f_z(x)=xz$ $(g_z(x)=zx)$. Such translations are omitted in the list.

List 1

a—b ……………………………the diagram of the semilattice of order 2.

Automorphism 1_2 b a……$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$, automorphism of 1_2.

c. i. 1_3(abc abc abc) …………

a	b	c
a	b	c
a	b	c

u. z. 3_3(b) ………………………

a	a	a
a	a	a
a	a	b

u. g. $5_3(3_2, 2_2, b)$………………

a	b	b
b	a	a
b	a	a

u. g. 6_3(group)…………………

a	b	c
b	c	a
c	a	b

c. d. u. $7_3(1_2$; ab aa; a) ……

a	b	a
a	b	a
a	b	a

$8_3(1_2)$ ……………………………semigroup whose greatest s-decomposition is a-(b, c)$_1$; i.e.

a	a	a
a	b	c
a	b	c

$12_3(1_2$; ab bb)…………………semigroup whose greatest s-decomposition is (a, b)$_1$-c; i.e.

a	b	b
a	b	b
a	b	c

Automorphism

1_3 xyz ……………………the automorphisms of 1_3 where x, y, z are arbitrary, showing 6 permutations of (a, b, c).

11_3 b a c …………………the automorphism of 11_3, $\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}$

List 2

$1_2 r.$ aa ab ba bb ……………right translations of 1_2 are

$$\begin{pmatrix} a & b \\ a & a \end{pmatrix}, \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \begin{pmatrix} a & b \\ b & b \end{pmatrix}.$$

$2_2 r.$ & $l.$ aa ab ………………right translations as well as left translations of 2_2;

$$\begin{pmatrix} a & b \\ a & a \end{pmatrix}, \begin{pmatrix} a & b \\ a & b \end{pmatrix}.$$

7_3 $l.$ aba abb abc ……………left translations of 7_3 are

$$\begin{pmatrix} a & b & c \\ a & b & a \end{pmatrix}, \begin{pmatrix} a & b & c \\ a & b & b \end{pmatrix}, \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}.$$

$1_3 r.$ xyz…………………………arbitrary mappings of (abc) into itself are right translations of 1_3.

List 3

20 (cyclic group)………………

a	b	c	d
b	c	d	a
c	d	a	b
d	a	b	c

21 ($3_2 \times 3_2$ group) ……………

a	b	c	d
b	a	d	c
c	d	a	b
d	c	b	a

24 ($1_2 \times 3_2$)………………………

a	b	c	d
a	b	c	d
c	d	a	b
c	d	a	b

a-(b, c, d)$_i$ 28 (1) ……………

a	a	a	a
a	b	c	d
a	b	c	d
a	b	c	d

$a<^{(b,\ c)_i}_{d}$ 91 (1) ……………

a	a	a	a
a	b	c	a
a	b	c	a
a	a	a	a

$\overbrace{(a,\ b)<^{(c)}_{d}}^{i}$ $k(i;\ a_1 a_2 a_3,\ b_1 b_2 b_3)$ ……

$$\begin{array}{|c|c|} \hline i_3 & \begin{array}{c} b_1 \\ b_2 \\ b_3 \end{array} \\ \hline a_1\ a_2\ a_3 & d \\ \hline \end{array}$$

where i_3 is the form (a, b)-c.

a-b-(c, d)$_i$ $k(i)$ ………………

$$\begin{array}{|cccc|} \hline a & a & a & a \\ a & b & b & b \\ \cline{3-4} a & b & \multicolumn{2}{|c|}{i} \\ a & b & \multicolumn{2}{|c|}{} \\ \hline \end{array}$$

a-$\overbrace{(b,\ c)\text{-}(d)}^{i}$ $k(i)$ ………………

$$\begin{array}{|c|ccc|} \hline a & a & a & a \\ \cline{2-4} a & & & \\ a & & i & \\ a & & & \\ \hline \end{array}$$

where i is the form (b, c)-(d).

$\overbrace{(a,\ b)\text{-}(c)}^{i}$-d $k(i:\ a_1 a_2 a_3,\ b_1 b_2 b_3)$………

$$\begin{array}{|c|c|} \hline i & \begin{array}{c} b_1 \\ b_2 \\ b_3 \end{array} \\ \hline a_1\ a_2\ a_3 & d \\ \hline \end{array}$$

The final table in the List 3 shows the numbers written in the [3] for the numbers in the present list, for example, No. 1 here is No. 20 in the former.

List 4

1 xyzu …………………………every mapping of (abcd) into (abcd) is an automorphism of 1_4.

2 badc 2 cdab 2 dcba 2 acbd*……all automorphisms of 2_4 are $\begin{pmatrix} a & b & c & d \\ b & a & d & c \end{pmatrix}$ $\begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix}$

$\begin{pmatrix} a & b & c & d \\ d & c & b & a \end{pmatrix}$ and dual automorphism of 2_4 is $\begin{pmatrix} a & b & c & d \\ a & c & b & d \end{pmatrix}$

3 axyz ………………………x, y, z vary through a, b, c, d.

List 5

Since it is complicated to list all translations, we show only the contractions of translations to a one-sided base.

2*r*. $\left(\begin{array}{cc} c & d \\ \frown & \frown \\ c\ d & c\ d \end{array}\right)$ ………………contractions of right translations to a right base; $\begin{pmatrix} c & d \\ c & c \end{pmatrix}$, $\begin{pmatrix} c & d \\ c & d \end{pmatrix}$, $\begin{pmatrix} c & d \\ d & c \end{pmatrix}$, $\begin{pmatrix} c & d \\ d & d \end{pmatrix}$.

2*l*. $\left(\begin{array}{cc} b & d \\ \frown & \frown \\ b\ d & b\ d \end{array}\right)$ ………………contractions of left translations to a left base; $\begin{pmatrix} b & d \\ b & b \end{pmatrix}$, $\begin{pmatrix} b & d \\ b & d \end{pmatrix}$, $\begin{pmatrix} b & d \\ d & b \end{pmatrix}$, $\begin{pmatrix} b & d \\ d & d \end{pmatrix}$.

1*r*. $\begin{pmatrix} a & b & c & d \\ x & y & z & u \end{pmatrix}$…………………x, y, z, u are arbitrary.

3 $\begin{pmatrix} b & c & d \\ x & y & z \end{pmatrix}$ ……………………there is no distinction between right and left because 3_4 is commutative.

List 6

135 (cyclic group) …………

$$\begin{array}{|ccccc|} a & b & c & d & e \\ b & c & d & e & a \\ c & d & e & a & b \\ d & e & a & b & c \\ e & a & b & c & d \end{array}$$

$a < \begin{matrix} (b,\ c)_i \\ (d,\ e)_j \end{matrix}$ 596 (1, 2)……

$$\begin{array}{|c|cc|cc|} \hline a & a & a & a & a \\ \cline{2-3} a & i & & a & a \\ a & & & a & a \\ \cline{2-5} a & a & a & j & \\ a & a & a & & \\ \hline \end{array}$$

$(a,\ b)_i - c - (d,\ e)_j$ $\quad k\left(i,\ j;\ \begin{matrix} a_1 a_2 & a_1' a_2' \\ b_1 b_2 & b_1' b_2' \\ c_1 c_2 & c_1' c_2' \end{matrix} \right)$ ………

$$\begin{array}{|cc|c|cc|} \hline i & & a_1' & b_1' & c_1' \\ & & a_2' & b_2' & c_2' \\ \hline a_1 & a_2 & c & c & c \\ \cline{4-5} b_1 & b_2 & c & j & \\ c_1 & c_2 & c & & \\ \hline \end{array}$$

$\overbrace{(a,\ b) < \begin{matrix} (c) \\ (d) \end{matrix}}^{i} \!\!\searrow e$ $\quad k(i;\ a_1 a_2 a_3 a_4,\ b_1 b_2 b_3 b_4)$ ……

$$\begin{array}{|cccc|c|} \hline i & & & & b_1 \\ & & & & b_2 \\ & & & & b_3 \\ & & & & b_4 \\ \hline a_1 & a_2 & a_3 & a_4 & e \\ \hline \end{array}$$

where *i* is the form

$(a,\ b) < \begin{matrix} (c) \\ (d) \end{matrix}$

$a < \overbrace{\begin{matrix} (b,\ c) \\ e \end{matrix} > (d)}^{i}$ $\quad k(i)$ ……………………………

$$\begin{array}{|c|ccc|c|} \hline a & a & a & a & a \\ \cline{2-4} a & & & & a \\ a & & i & & a \\ a & & & & e \\ \cline{2-4} a & a & a & e & e \\ \hline \end{array}$$

where *i* is the form

$(b,\ c) - (d)$

$\overbrace{(a,\ b) < \begin{matrix} (c) \\ (d) \end{matrix}}^{i} > e$ $\quad k(i;\ a_1 a_2 a_3 a_4,\ b_1 b_2 b_3 b_4)$ ……*i* is the form $(a,\ b) < \begin{matrix} (c) \\ (d) \end{matrix}$

List 7

Our number 1 is equivalent to Motzkin's number 994, and our 12 to his 17. Here "equivalent" means to "isomorphic or anti-isomorphic"

List 8

We show *l*-ordering and *r*-ordering of u.z. of order 5, and greatest c-decomposition of c.d.u., for example, 136∼139 are homomorphic to a commutative 2 and this c-decomposition is greatest.

List 1 Semigroups of Order 2, 3

Order 2

c.i.

a	b
a	b

1_2

u.z.

a	a
a	a

$2_2^{\times}$

u.g.

a	b
b	a

$3_2^{\times}$

a–b

a	a
a	b

$4_2^{\times}$

Automorphism 1_2 ba

Order 3

I s-indecomposable

c.i. 1_3(abc abc abc) u.z. $2_3^{\times}$(a) u.z. $3_3^{\times}$(b) u.g. $4_3^{\times}$(3_2, 2_2, a) u.g. $5_3^{\times}$(3_2, 2_2, b) u.g. $6_3^{\times}$(group) c.d.u. 7_3(1_2; ab aa; a)

II a–(b, c)$_i$

8_3(1_2) $9_3^{\times}$(2_2) $10_3^{\times}$(3_2)

III (a, b)$_i$–c

11_3(1_2; ab ab) 12_3(1_2; ab bb) $13_3^{\times}$(2_2; aa aa) 14_3(2_2; aa ab) $15_3^{\times}$(2_2; ab ab) $16_3^{\times}$(3_2; ab ab)

IV Semilattices

$17_3^{\times}$ $a <^{b}_{c}$ $18_3^{\times}$ a—b—c

Automorphisms

1_3 x y z 2_3 a c b 6_3 a c b 8_3 a c b 11_3 b a c 17_3 a c b

List 2 Translations of Semigroups of Order 2, 3

1_2 *r.* aa ab ba bb, 2_2 *r.* & *l.* aa ab 3_2 *r.* & *l.* ab ba 4_2 *r.* & *l.* aa ab 1_3 *r.* xyz 2_3 *r.* & *l.* aaa aab aac aba abb abc aca acb acc 3_3 *r.* & *l.* aaa aab abc 4_3 *r.* & *l.* aba bab abc 5_3 *r.* & *l.* abb baa abc 7_3 *r.* aaa bab aac aba bbb abc 7_3 *l.* aba abb abc 8_3 *r.* aaa aab aac aba abb abc aca acb acc 9_3 *r.* & *l.* aaa abb abc 11_3 *r.* aaa abb abc baa bbb bbc 13_3 *r.* & *l.* aaa aac aba abc 14_3 *l.* aaa aab aac aba abb abc aca acb acc 17_3 *r.* & *l.* aaa aac aba abc

List 3 Semigroups of Order 4

I s-indecomposable

c-ind.

1 (abcd abcd abcd abcd) 2* (abab abab cdcd cdcd)

u.z.

$3^{\times}$(aa aa) $4^{\times}$(aa ab) 5*(aa ba) 6(aa bb) $7^{\times}$(ab ba) $8^{\times}$(ab bb) $9^{\times}$(ba ab) 10*(ba bb) $11^{\times}$(bb bb) $12^{\times}$(ab bc)

u.g.

$13^{\times}$(3_2, 2_3, aa) $14^{\times}$(3_2, 2_3, ab) $15^{\times}$(3_2, 2_3, bb) $16^{\times}$(3_2, 3_3, aa) $17^{\times}$(3_2, 3_3, ab) $18^{\times}$(6_3, 2_2, a) $19^{\times}$(6_3, 2_2, b) $20^{\times}$(cyclic group) $21^{\times}$($3_2 \times 3_2$ group)

c.d.u.

22(1_3; abc aaa; a) 23(1_3; abc aab; a) 24($1_2 \times 3_2$) 25(7_3; aba aaa; a) 26(7_3; aba aaa; c) 27(7_3; aba bbb; b)

II a–(b, c, d)$_i$

28(1) $29^{\times}$(2) $30^{\times}$(3) $31^{\times}$(4) $32^{\times}$(5) $33^{\times}$(6) 34(7)

III $(\mathbf{a}, \mathbf{b})_i\text{-}(\mathbf{c}, \mathbf{d})_j$

$35\left(1,1;\ \begin{matrix}ab & aa\\ ab & aa\end{matrix}\right)$ $36\left(1,1;\ \begin{matrix}ab & aa\\ ab & bb\end{matrix}\right)$ $37\left(1,1;\ \begin{matrix}ab & ab\\ ab & ab\end{matrix}\right)$ $38\left(1,1';\ \begin{matrix}ab & aa\\ ab & aa\end{matrix}\right)$ $39\left(1,1';\ \begin{matrix}ab & ab\\ ab & ab\end{matrix}\right)$

$40\left(1,2;\ \begin{matrix}ab & ab\\ ab & ab\end{matrix}\right)$ $41\left(1,2;\ \begin{matrix}ab & bb\\ ab & bb\end{matrix}\right)$ $42\left(1,3;\ \begin{matrix}ab & ab\\ ab & ab\end{matrix}\right)$ $43\left(1,3;\ \begin{matrix}ab & ab\\ ab & ba\end{matrix}\right)$ $44\left(1,3;\ \begin{matrix}ab & bb\\ ab & bb\end{matrix}\right)$

$45\left(2,1;\ \begin{matrix}aa & aa\\ aa & aa\end{matrix}\right)$ $46\left(2,1;\ \begin{matrix}ab & aa\\ ab & aa\end{matrix}\right)$ $47\left(2,1;\ \begin{matrix}aa & ab\\ aa & ab\end{matrix}\right)$ $48\left(2,1;\ \begin{matrix}ab & ab\\ ab & ab\end{matrix}\right)$ $49^{\times}\left(2,2;\ \begin{matrix}aa & aa\\ aa & aa\end{matrix}\right)$

$50\left(2,2;\ \begin{matrix}aa & ab\\ aa & ab\end{matrix}\right)$ $51^{\times}\left(2,2;\ \begin{matrix}ab & ab\\ ab & ab\end{matrix}\right)$ $52^{\times}\left(2,3;\ \begin{matrix}aa & aa\\ aa & aa\end{matrix}\right)$ $53\left(2.3;\ \begin{matrix}ab & aa\\ ab & aa\end{matrix}\right)$ $54^{\times}\left(2,3;\ \begin{matrix}ab & ab\\ ab & ab\end{matrix}\right)$

$55\left(3,1;\ \begin{matrix}ab & ab\\ ab & ab\end{matrix}\right)$ $56^{\times}\left(3,2;\ \begin{matrix}ab & ab\\ ab & ab\end{matrix}\right)$ $57^{\times}\left(3,3;\ \begin{matrix}ab & ab\\ ab & ab\end{matrix}\right)$ $58^{\times}\left(3,3;\ \begin{matrix}ab & ab\\ ba & ba\end{matrix}\right)$

IV $(\mathbf{a}, \mathbf{b}, \mathbf{c})_i\text{–}\mathbf{d}$

59(1; abc aaa) 60(1; abc aba) 61(1; abc abc) $62^{\times}$(2; aaa aaa) 63(2; aaa aba)
64(2; aaa abb) 65(2; aaa abc) $66^{\times}$(2; aac aac) 67*(2; aac aba) 68(2; aac abc)
$69^{\times}$(2; abb abb) 70(2; abc acc) $71^{\times}$(2; abc abc) $72^{\times}$(3; aaa aaa) $73^{\times}$(3; abc abc)
$74^{\times}$(4; aba aba) 75(4; aba abc) $76^{\times}$(4; abc abc) $77^{\times}$(5; abb abb) 78(5; abb abc)
$79^{\times}$(5; abc abc) $80^{\times}$(6; abc abc) 81(7; aba aaa) 82(7; aba aac) 83(7; aba aba)
84(7; aba abc) 85(7; aba bbb) 86(7; abc aaa) 87(7; abc aac) 88(7; abc aba)
89(7; abc abc) 90(7; abc bbb)

V $\mathbf{a}<^{(\mathbf{b},\ \mathbf{c})_i}_{\mathbf{d}}$

91(1) $92^{\times}$(2) $93^{\times}$(3)

VI $\overbrace{(\mathbf{a}, \mathbf{b})<^{(\mathbf{c})}_{\mathbf{d}}}^{i}$

94(11; aba aaa) 95(12; abb aaa) 96(12; abb bbb) $97^{\times}$(13; aaa aaa) 98(14; aaa aaa)
99(14; aba aaa) 100(14; abb aaa) $101^{\times}$(15; aaa aaa) $102^{\times}$(16; aba aba)

VII $\mathbf{a}\text{-}\mathbf{b}\text{-}(\mathbf{c}, \mathbf{d})_i$

103(1) $104^{\times}$(2) $105^{\times}$(3)

VIII $\mathbf{a}\text{-}(\overbrace{\mathbf{b}, \mathbf{c})\text{-}(\mathbf{d}}^{i})$

106(11) 107(12) $108^{\times}$(13) 109(14) $110^{\times}$(15) $111^{\times}$(16)

IX $(\overbrace{\mathbf{a}, \mathbf{b})\text{-}(\mathbf{c}}^{i})\text{-}\mathbf{d}$

112(11; abc abc) 113(12; abc abc) 114(12; abc bbc) $115^{\times}$(13; aac aac) 116(13; aac abc)
$117^{\times}$(13; abc abc) 118(14; aac abc) 119(14; abc abc) $120^{\times}$(15; abc abc) $121^{\times}$(16; abc abc)

X Semilattice

$122^{\times}\ a\begin{smallmatrix}\diagup b\\ - c\\ \diagdown d\end{smallmatrix}$ $123^{\times}\ a<^{b}_{c}>d$ $124^{\times}\ a<^{b—d}_{c}$ $125^{\times}\ a—b<^{c}_{d}$ $126^{\times}$ a—b—c—d

Former Numbers for New Numbers

	0	1	2	3	4	5	6	7	8	9
000		020	021	001	003	010	004	009	006	008
010	007	005	002	013	014	011	012	015	016	017
020	018	019	036	037	029	044	043	045	035	125
030	126	124	123	122	084	022	024	023	027	028
040	082	081	046	047	048	080	025	026	052	110
050	083	109	108	049	107	056	106	104	105	032
060	033	034	118	078	079	030	119	031	053	120
070	054	121	117	116	115	055	114	112	057	113
080	111	075	038	077	039	076	041	051	040	050
090	042	074	103	102	068	070	069	101	071	072
100	073	100	099	067	097	098	064	065	096	066
110	095	094	058	059	060	093	063	092	061	062
120	091	090	089	088	087	086	085			

List 4 Automorphisms of Semigroups of Order 4

1 xyzu 2 badc 2 cdab 2 dcba 2 acbd* 3 axyz 5 abdc* 7 abdc
9 abdc 10 abdc* 11 abdc 13 abdc 15 abdc 18 acbd 20 adcb 21 axyz
22 acbd 24 badc 25 abdc 27 badc 28 axyz 29 abdc 33 abdc 35 abdc
36 badc 37 abdc 37 badc 37 bacd 38 abdc 39 bacd 39 abdc 39 badc
40 bacd 42 bacd 43 bacd 45 abdc 46 abdc 47 abdc 48 abdc* 55 abdc
59 acbd 61 xyzd 62 acbd 65 acbd 67 acbd* 71 acbd 80 acbd 91 acbd
95 abdc 96 badc 97 abdc 99 abdc* 100 abdc* 102 abdc 103 abdc 106 acbd
112 bacd 122 axyz 123 acbd 125 abdc

List 5 Translations of Semigroup of Order 4

$1r\begin{pmatrix} abcd \\ xyzu \end{pmatrix}$ $2r\begin{pmatrix} c & d \\ \overparen{cd} & \overparen{cd} \end{pmatrix}$ $2l\begin{pmatrix} b & d \\ \overparen{bd} & \overparen{bd} \end{pmatrix}$ $3\begin{pmatrix} bcd \\ xyz \end{pmatrix}$ $4\begin{pmatrix} c & d \\ \overparen{abc} & \overparen{abcd} \end{pmatrix}$ $5r\begin{pmatrix} c & d \\ \overparen{abcd} & \overparen{abd} \end{pmatrix}$ $5l\begin{pmatrix} c & d \\ \overparen{abc} & \overparen{abcd} \end{pmatrix}$

$6r\begin{pmatrix} c & d \\ \overparen{ab} & \overparen{ab} \end{pmatrix}\begin{pmatrix} c & d \\ \overparen{cd} & \overparen{cd} \end{pmatrix}$ $6l\begin{pmatrix} c & d \\ \overparen{abc} & \overparen{abcd} \end{pmatrix}$ $7\begin{pmatrix} c & d \\ \overparen{ab} & \overparen{ab} \end{pmatrix}\begin{pmatrix} c & d \\ c & d \end{pmatrix}$ $8\begin{pmatrix} c & d \\ \overparen{ab} & \overparen{ab} \end{pmatrix}\begin{pmatrix} c & d \\ c & d \end{pmatrix}$ $9\begin{pmatrix} c & d \\ \overparen{ab} & \overparen{ab} \end{pmatrix}\begin{pmatrix} c & d \\ c & d \end{pmatrix}$

$10r\begin{pmatrix} c & d \\ \overparen{ab} & \overparen{ab} \end{pmatrix}\begin{pmatrix} c & d \\ c & d \end{pmatrix}$ $10l\begin{pmatrix} c & d \\ \overparen{ab} & \overparen{ab} \end{pmatrix}\begin{pmatrix} c & d \\ c & d \end{pmatrix}$ $11\begin{pmatrix} c & d \\ \overparen{ab} & \overparen{ab} \end{pmatrix}\begin{pmatrix} c & d \\ \overparen{cd} & \overparen{cd} \end{pmatrix}$ $12\begin{pmatrix} d \\ \overparen{abcd} \end{pmatrix}$ $13\begin{pmatrix} c & d \\ \overparen{acd} & \overparen{acd} \end{pmatrix}\begin{pmatrix} c & d \\ b & b \end{pmatrix}$

$14\begin{pmatrix} c & d \\ \overparen{ac} & \overparen{bd} \end{pmatrix}\begin{pmatrix} c & d \\ \overparen{bd} & \overparen{ac} \end{pmatrix}$ $15\begin{pmatrix} c & d \\ \overparen{bcd} & \overparen{bcd} \end{pmatrix}\begin{pmatrix} c & d \\ a & a \end{pmatrix}$ $16\begin{pmatrix} d \\ \overparen{abcd} \end{pmatrix}$ $17\begin{pmatrix} d \\ \overparen{abcd} \end{pmatrix}$ $18\begin{pmatrix} d \\ \overparen{abcd} \end{pmatrix}$ $19\begin{pmatrix} d \\ \overparen{abcd} \end{pmatrix}$

$22r\begin{pmatrix} b & c & d \\ \overparen{abc} & \overparen{abc} & \overparen{abcd} \end{pmatrix}$ $22l\begin{pmatrix} d \\ \overparen{abcd} \end{pmatrix}$ $23r\begin{pmatrix} b & c & d \\ b & \overparen{abc} & \overparen{bd} \end{pmatrix}\begin{pmatrix} b & c & d \\ a & \overparen{abc} & a \end{pmatrix}\begin{pmatrix} b & c & d \\ c & \overparen{abc} & c \end{pmatrix}$ $23l\begin{pmatrix} d \\ \overparen{abd} \end{pmatrix}$ $24r\begin{pmatrix} c & d \\ \overparen{abcd} & \overparen{abcd} \end{pmatrix}$

$25r\begin{pmatrix} b & c & d \\ \overparen{ab} & b & b \end{pmatrix}\begin{pmatrix} b & c & d \\ \overparen{ab} & \overparen{acd} & \overparen{acd} \end{pmatrix}$ $25l\begin{pmatrix} c & d \\ \overparen{abcd} & \overparen{abcd} \end{pmatrix}$ $26r\begin{pmatrix} b & d \\ \overparen{ab} & \overparen{abcd} \end{pmatrix}$ $26l\begin{pmatrix} d \\ \overparen{abcd} \end{pmatrix}$ $27r\begin{pmatrix} c & d \\ \overparen{abcd} & \overparen{abcd} \end{pmatrix}$

$27l\begin{pmatrix} c & d \\ \overparen{abcd} & \overparen{abcd} \end{pmatrix}$ $28r\begin{pmatrix} b & c & d \\ \overparen{abcd} & \overparen{abcd} & d \end{pmatrix}$ $29\begin{pmatrix} c & d \\ a & a \end{pmatrix}\begin{pmatrix} c & d \\ \overparen{bcd} & \overparen{bcd} \end{pmatrix}$ $30\begin{pmatrix} d \\ \overparen{abcd} \end{pmatrix}$ $31\begin{pmatrix} d \\ \overparen{abcd} \end{pmatrix}$ $32\begin{pmatrix} d \\ \overparen{abcd} \end{pmatrix}$

$34r\begin{pmatrix} c & d \\ \overparen{abc} & \overparen{abcd} \end{pmatrix}$ $34l\begin{pmatrix} d \\ \overparen{abcd} \end{pmatrix}$ $35r\begin{pmatrix} b & c & d \\ \overparen{ab} & \overparen{acd} & \overparen{acd} \end{pmatrix}\begin{pmatrix} b & c & d \\ \overparen{ab} & b & b \end{pmatrix}$ $36r\begin{pmatrix} c & d \\ \overparen{abcd} & \overparen{abcd} \end{pmatrix}$ $37r\begin{pmatrix} c & d \\ a & a \end{pmatrix}\begin{pmatrix} c & d \\ b & b \end{pmatrix}\begin{pmatrix} c & d \\ \overparen{cd} & \overparen{cd} \end{pmatrix}$

$38r\begin{pmatrix} b & c \\ \overparen{ab} & \overparen{abc} \end{pmatrix}$ $38l\begin{pmatrix} c & d \\ \overparen{acd} & \overparen{acd} \end{pmatrix}$ $39l\begin{pmatrix} c & d \\ \overparen{abcd} & \overparen{abcd} \end{pmatrix}$ $40r\begin{pmatrix} d \\ \overparen{abcd} \end{pmatrix}$ $40l\begin{pmatrix} d \\ \overparen{abcd} \end{pmatrix}$ $41r\begin{pmatrix} a & d \\ \overparen{ab} & \overparen{abcd} \end{pmatrix}$ $41l\begin{pmatrix} d \\ \overparen{abcd} \end{pmatrix}$

$44r\begin{pmatrix} a & c \\ a & \overparen{abcd} \end{pmatrix}\begin{pmatrix} a & c \\ b & \overparen{bcd} \end{pmatrix}$ $45r\begin{pmatrix} b & c & d \\ \overparen{ab} & \overparen{acd} & \overparen{acd} \end{pmatrix}$ $45l\begin{pmatrix} b & d \\ \overparen{ab} & \overparen{ad} \end{pmatrix}$ $46r\begin{pmatrix} b & c & d \\ \overparen{abcd} & \overparen{abcd} & \overparen{abcd} \end{pmatrix}$ $47r\begin{pmatrix} c & d \\ a & a \end{pmatrix}\begin{pmatrix} c & d \\ \overparen{cd} & \overparen{cd} \end{pmatrix}$

$47l\begin{pmatrix} b & d \\ \overparen{ab} & \overparen{abd} \end{pmatrix}$ $48r\begin{pmatrix} c & d \\ \overparen{ab} & \overparen{ab} \end{pmatrix}\begin{pmatrix} c & d \\ \overparen{cd} & \overparen{cd} \end{pmatrix}$ $49\begin{pmatrix} b & d \\ \overparen{ab} & \overparen{abcd} \end{pmatrix}$ $50r\begin{pmatrix} d \\ \overparen{abcd} \end{pmatrix}$ $50l\begin{pmatrix} b & d \\ \overparen{abc} & \overparen{abcd} \end{pmatrix}$ $51\begin{pmatrix} d \\ \overparen{abcd} \end{pmatrix}$

$52\begin{pmatrix} b & d \\ \overparen{ab} & \overparen{acd} \end{pmatrix}$ $53r\begin{pmatrix} b & d \\ \overparen{ab} & \overparen{abcd} \end{pmatrix}$ $55r\begin{pmatrix} c & d \\ \overparen{acd} & \overparen{acd} \end{pmatrix}\begin{pmatrix} c & d \\ b & b \end{pmatrix}$ $56\begin{pmatrix} d \\ \overparen{abcd} \end{pmatrix}$ $59r\begin{pmatrix} b & c & d \\ \overparen{abc} & \overparen{abc} & \overparen{abcd} \end{pmatrix}$ $60r\begin{pmatrix} c & d \\ \overparen{abc} & \overparen{abcd} \end{pmatrix}$

$62\begin{pmatrix} b & c & d \\ \overparen{abc} & \overparen{abc} & \overparen{ad} \end{pmatrix}$ $63r\begin{pmatrix} c & d \\ \overparen{abc} & \overparen{ad} \end{pmatrix}$ $63l\begin{pmatrix} b & c & d \\ \overparen{ad} & \overparen{ac} & \overparen{abd} \end{pmatrix}$ $64r\begin{pmatrix} c & d \\ \overparen{abc} & \overparen{ad} \end{pmatrix}$ $64l\begin{pmatrix} c & d \\ \overparen{abcd} & \overparen{abd} \end{pmatrix}$ $65l\begin{pmatrix} bcd \\ xyz \end{pmatrix}$

$66\begin{pmatrix} b & d \\ \overparen{ab} & \overparen{acd} \end{pmatrix}$ $67r\begin{pmatrix} c & d \\ \overparen{ac} & \overparen{acd} \end{pmatrix}$ $67l\begin{pmatrix} b & d \\ \overparen{ab} & \overparen{abd} \end{pmatrix}$ $68l\begin{pmatrix} b & d \\ \overparen{abc} & \overparen{abcd} \end{pmatrix}$ $69\begin{pmatrix} c & d \\ a & \overparen{ab} \end{pmatrix}\begin{pmatrix} c & d \\ \overparen{bc} & d \end{pmatrix}$ $70r\begin{pmatrix} b & d \\ \overparen{abc} & \overparen{abcd} \end{pmatrix}$

$72\begin{pmatrix} c & d \\ \overparen{abc} & \overparen{ad} \end{pmatrix}$ $74\begin{pmatrix} c & d \\ \overparen{ac} & \overparen{ad} \end{pmatrix}\begin{pmatrix} c & d \\ b & b \end{pmatrix}$ $75l\begin{pmatrix} c & d \\ \overparen{acd} & \overparen{acd} \end{pmatrix}\begin{pmatrix} c & d \\ b & b \end{pmatrix}$ $77\begin{pmatrix} c & d \\ a & b \end{pmatrix}\begin{pmatrix} c & d \\ \overparen{bc} & \overparen{ad} \end{pmatrix}$ $78l\begin{pmatrix} c & d \\ \overparen{ad} & \overparen{bc} \end{pmatrix}\begin{pmatrix} c & d \\ \overparen{bc} & \overparen{ad} \end{pmatrix}$

$81r\begin{pmatrix} b & c & d \\ \overparen{ab} & \overparen{ac} & \overparen{ad} \end{pmatrix}\begin{pmatrix} b & c & d \\ \overparen{ab} & b & b \end{pmatrix}$ $81l\begin{pmatrix} c & d \\ \overparen{abc} & \overparen{ad} \end{pmatrix}$ $82r\begin{pmatrix} b & d \\ \overparen{ab} & \overparen{abd} \end{pmatrix}$ $82l\begin{pmatrix} c & d \\ \overparen{acd} & \overparen{acd} \end{pmatrix}$ $83r\begin{pmatrix} c & d \\ \overparen{ac} & \overparen{ad} \end{pmatrix}\begin{pmatrix} c & d \\ b & b \end{pmatrix}$ $83l\begin{pmatrix} c & d \\ \overparen{ac} & \overparen{abd} \end{pmatrix}$

$84l\begin{pmatrix} c & d \\ \overparen{abcd} & \overparen{abcd} \end{pmatrix}$ $85r\begin{pmatrix} c & d \\ \overparen{abc} & \overparen{abd} \end{pmatrix}$ $85l\begin{pmatrix} c & d \\ \overparen{abc} & \overparen{bd} \end{pmatrix}$ $86r\begin{pmatrix} b & c & d \\ \overparen{ab} & \overparen{abcd} & \overparen{abcd} \end{pmatrix}$ $87r\begin{pmatrix} b & d \\ \overparen{ab} & \overparen{abcd} \end{pmatrix}$

$88r\begin{pmatrix} c & d \\ \overparen{ac} & \overparen{acd} \end{pmatrix}\begin{pmatrix} c & d \\ b & b \end{pmatrix}$ $90r\begin{pmatrix} c & d \\ \overparen{abcd} & \overparen{abcd} \end{pmatrix}$ $91r\begin{pmatrix} c & d \\ \overparen{ac} & \overparen{ad} \end{pmatrix}$ $91l\begin{pmatrix} b & c & d \\ \overparen{abc} & \overparen{abc} & \overparen{ad} \end{pmatrix}$ $92\begin{pmatrix} c & d \\ \overparen{abc} & \overparen{ad} \end{pmatrix}$ $93\begin{pmatrix} c & d \\ \overparen{abc} & \overparen{ad} \end{pmatrix}$

$94r\begin{pmatrix} c & d \\ \overparen{ac} & \overparen{ad} \end{pmatrix}\begin{pmatrix} c & d \\ b & b \end{pmatrix}$ $94l\begin{pmatrix} c & d \\ \overparen{abc} & \overparen{ad} \end{pmatrix}$ $95r\begin{pmatrix} a & c & d \\ \overparen{ab} & a & a \end{pmatrix}\begin{pmatrix} a & c & d \\ \overparen{ab} & \overparen{bc} & \overparen{bd} \end{pmatrix}$ $95l\begin{pmatrix} c & d \\ \overparen{bc} & \overparen{bd} \end{pmatrix}$ $96r\begin{pmatrix} c & d \\ \overparen{abc} & \overparen{abd} \end{pmatrix}$ $96l\begin{pmatrix} c & d \\ \overparen{bc} & \overparen{ad} \end{pmatrix}$

$97\begin{pmatrix} b & c & d \\ \overparen{ab} & \overparen{ac} & \overparen{ad} \end{pmatrix}$ $98r\begin{pmatrix} c & d \\ \overparen{ac} & \overparen{ad} \end{pmatrix}$ $98l\begin{pmatrix} b & c & d \\ \overparen{abc} & \overparen{abc} & \overparen{ad} \end{pmatrix}$ $99r\begin{pmatrix} c & d \\ \overparen{ac} & \overparen{abd} \end{pmatrix}$ $99l\begin{pmatrix} c & d \\ \overparen{abc} & \overparen{ad} \end{pmatrix}$ $100r\begin{pmatrix} c & d \\ \overparen{ac} & \overparen{abd} \end{pmatrix}$

$100l\begin{pmatrix} c & d \\ \overparen{abc} & \overparen{ad} \end{pmatrix}$ $101\begin{pmatrix} c & d \\ \overparen{abc} & \overparen{ad} \end{pmatrix}$ $102\begin{pmatrix} c & d \\ \overparen{ac} & \overparen{ad} \end{pmatrix}\begin{pmatrix} c & d \\ b & b \end{pmatrix}$ $103r\begin{pmatrix} c & d \\ a & a \end{pmatrix}\begin{pmatrix} c & d \\ \overparen{bcd} & \overparen{bcd} \end{pmatrix}$ $104\begin{pmatrix} d \\ \overparen{abcd} \end{pmatrix}$ $107r\begin{pmatrix} b & d \\ \overparen{abc} & \overparen{abcd} \end{pmatrix}$

$108\begin{pmatrix} c & d \\ a & a \end{pmatrix}\begin{pmatrix} c & d \\ \overparen{bc} & \overparen{bd} \end{pmatrix}$ $109l\begin{pmatrix} c & d \\ a & a \end{pmatrix}\begin{pmatrix} c & d \\ \overparen{bcd} & \overparen{bcd} \end{pmatrix}$ $113r\begin{pmatrix} a & d \\ \overparen{ab} & \overparen{abcd} \end{pmatrix}$ $115\begin{pmatrix} b & d \\ \overparen{ab} & \overparen{acd} \end{pmatrix}$ $116l\begin{pmatrix} b & d \\ \overparen{ab} & \overparen{abcd} \end{pmatrix}$

$118l\begin{pmatrix} b & d \\ \overparen{abc} & \overparen{abcd} \end{pmatrix}$ $122\begin{pmatrix} b & c & d \\ \overparen{ab} & \overparen{ac} & \overparen{ad} \end{pmatrix}$ $124\begin{pmatrix} c & d \\ \overparen{ac} & \overparen{abd} \end{pmatrix}$ $125\begin{pmatrix} c & d \\ a & a \end{pmatrix}\begin{pmatrix} c & d \\ \overparen{bc} & \overparen{bd} \end{pmatrix}$

List 6 Semigroups of Order 5

I s-indecomposable

c-ind.

1(abcde abcde abcde abcde abcde) 2*(aaaaa aaacb abcaa acdaa aaade)
3*(aaaaa abcaa abccb aeddc acdaa)

u. z.

4$^{\times}$(aaa aaa aaa) 5(aaa aaa abc) 6$^{\times}$(aaa aba aac) 7*(aaa aab aca) 8(aaa aba acc)
9(aaa aab acc) 10(aaa abb acc) 11$^{\times}$(aaa abc acc) 12$^{\times}$(aaa abc aca) 13*(aaa acb acc)
14*(aaa acb aac) 15$^{\times}$(aaa abc acb) 16$^{\times}$(aaa aab abd) 17$^{\times}$(aaa aaa aab) 18*(aaa aaa aba)
19(aaa aaa abb) 20(aaa aaa bba) 21(aaa aaa bbb) 22(aab aaa bbc) 23$^{\times}$(aab aab bbc)
24$^{\times}$(aab aba bac) 25(aab aba bbc) 26$^{\times}$(aab abb bbc) 27$^{\times}$(abb bcc bcc) 28$^{\times}$(aaa aab aba)
29$^{\times}$(aaa aab abb) 30$^{\times}$(aaa aba aab) 31*(aaa aba abb) 32$^{\times}$(aaa abb abb) 33*(aaa aab baa)
34*(aaa bab baa) 35*(aaa aab bab) 36*(aaa bab bab) 37*(aaa baa aab) 38(aaa baa bab)
39(aaa aab bba) 40(aaa bab bba) 41(aaa abb bba) 42(aaa bbb aba) 43(aaa bbb bba)
44(aaa abb baa) 45(aaa abb bbb) 46(aaa baa bbb) 47(aaa bba aab) 48(aaa bba bab)
49(aaa bba abb) 50(aaa bbb aab) 51(aaa bba bbb) 52(aaa bbb bbb) 53*(aba aab baa)
54*(aba aab bba) 55$^{\times}$(aba bab aba) 56*(aba bab bba) 57$^{\times}$(abb bab bba) 58$^{\times}$(baa aab aba)
59(baa aab bba) 60(baa bab bba) 61$^{\times}$(bab aab bba) 62*(bab baa aba) 63*(bab bab aba)
64(bab baa bba) 65(bab bab bba) 66$^{\times}$(bbb baa baa) 67*(bbb baa bba) 68$^{\times}$(bbb bab bba)
69$^{\times}$(baa abb aba) 70*(baa abb baa) 71(baa abb bba) 72(baa bbb aba) 73*(baa bbb baa)
74(baa bbb bba) 75$^{\times}$(bab abb bba) 76*(bab bba aba) 77(bab bbb aba) 78*(bab bbb bba)
79$^{\times}$(bba bbb aba) 80*(bba bbb baa) 81(bba bbb bba) 82$^{\times}$(bbb bbb bba) 83$^{\times}$(baa aba aab)
84*(baa aba abb) 85$^{\times}$(baa abb abb) 86(baa aba bbb) 87*(baa abb bab) 88(baa abb bbb)
89*(baa bba bbb) 90(baa bbb bbb) 91$^{\times}$(bab abb bbb) 92*(bab bba abb) 93*(bab bbb bbb)
94*(bab bbb bbb) 95$^{\times}$(bbb bbb bbb) 96$^{\times}$(aab abc bcd)

u. g.

97$^{\times}$(3_2, 3_4, aaa) 98$^{\times}$(3_2, 3_4, aab) 99$^{\times}$(3_2, 3_4, abb) 100$^{\times}$(3_2, 3_4, bbb) 101$^{\times}$(3_2, 4_4, aaa)
102$^{\times}$(3_2, 4_4, aba) 103$^{\times}$(3_2, 4_4, aab) 104$^{\times}$(3_2, 4_4, abb) 105*(3_2, 5_4, aaa) 106*(3_2, 5_4, abb)
107(3_2, 5_4, bab) 108(3_2, 6_4, aaa) 109(3_2, 6_4, abb) 110$^{\times}$(3_2, 7_4, aaa) 111$^{\times}$(3_2, 7_4, bab)
112$^{\times}$(3_2, 7_4, abb) 113$^{\times}$(3_2, 8_4, aaa) 114$^{\times}$(3_2, 8_4, abb) 115$^{\times}$(3_2, 9_4, aaa) 116$^{\times}$(3_2, 9_4, aab)
117$^{\times}$(3_2, 9_4, abb) 118*(3_2, 10_4, aaa) 119*(3_2, 10_4, abb) 120$^{\times}$(3_2, 11_4, aaa) 121$^{\times}$(3_2, 11_4, abb)
122$^{\times}$(3_2, 12_4, aaa) 123$^{\times}$(3_2, 12_4, bab) 124$^{\times}$(6_3, 2_3, aa) 125$^{\times}$(6_3, 2_3, ab) 126$^{\times}$(6_3, 2_3, bb)
127$^{\times}$(6_3, 2_3, bc) 128$^{\times}$(6_3, 3_3, aa) 129$^{\times}$(6_3, 3_3, bc) 130$^{\times}$(20_4, 2_2, a) 131$^{\times}$(20_4, 2_2, b)
132$^{\times}$(20_4, 2_2, c) 133$^{\times}$(21_4, 2_2, a) 134$^{\times}$(21_4, 2_2, b) 135$^{\times}$(cyclic group)

c. d. u.

136(1_4, abcd aaaa; a) 137(1_4, abcd aaba; a) 138(1_4, abcd aabb; a) 139*(2_4, abab aacc; a)
140(24_4, abcd aacc; a) 141(24_4, cdab ccaa; a) 142(22_4, abca aaaa; a) 143(22_4, abca aaba; a)
144(22_4, abca bbab; b) 145(22_4, abca bbbb; b) 146(23_4, abca aaba: a) 147(23_4, abca bbab; b)
148(23_4, abcb cccc; c) 149(25_4, abaa aaac; a) 150(25_4, abac bbbb; b) 151(25_4, abac aaaa; c)
152(25_4, abaa aaac; c) 153(26_4, abac bbbb; b) 154(26_4, abaa aaac; c) 155(22_4, abca aaaa; d)
156(22_4, abca aaba; d) 157(25_4, abaa aaaa; a) 158(25_4, abaa bbbb; b) 159(25_4, abaa aaaa; c)
160(27_4, abab aaaa; c) 161(25_4, abac aaac; a) 162(26_4, abac aaac; a) 163(26_4, abaaa aaa; c)
164(26_4, abac aaac; c) 165(25_4, abac aaac; d)

II $\mathbf{a}\text{-}(\mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e})_i$

$166(1)$ $167^{*}(2)$ $168^{\times}(3)$ $169^{\times}(4)$ $170^{*}(5)$ $171(6)$ $172^{\times}(7)$ $173^{\times}(8)$ $174^{\times}(9)$ $175^{*}(10)$

$176^{\times}(11)$ $177^{\times}(12)$ $178^{\times}(13)$ $179^{\times}(14)$ $180^{\times}(15)$ $181^{\times}(16)$ $182^{\times}(17)$ $183^{\times}(18)$ $184^{\times}(19)$ $185^{\times}(20)$

$186^{\times}(21)$ $187(22)$ $188(23)$ $189(24)$ $190(25)$ $191(26)$ $192(27)$

III $(\mathbf{a}, \mathbf{b})_i\text{-}(\mathbf{c}, \mathbf{d}, \mathbf{e})_j$

$193\left(1, 1;\ \begin{matrix} ab & aa \\ ab & aa \\ ab & aa \end{matrix}\right)$ $194\left(1, 1;\ \begin{matrix} ab & aa \\ ab & bb \\ ab & aa \end{matrix}\right)$ $195\left(1, 1;\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$ $196\left(1, 1';\ \begin{matrix} ab & aa \\ ab & aa \\ ab & aa \end{matrix}\right)$ $197\left(1, 1';\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$

$198\left(1, 2;\ \begin{matrix} ab & aa \\ ab & aa \\ ab & aa \end{matrix}\right)$ $199\left(1, 2;\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$ $200\left(1, 3;\ \begin{matrix} ab & aa \\ ab & aa \\ ab & aa \end{matrix}\right)$ $201\left(1, 3;\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$ $202\left(1, 4;\ \begin{matrix} ab & aa \\ ab & aa \\ ab & aa \end{matrix}\right)$

$203\left(1, 4;\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$ $204\left(1, 4;\ \begin{matrix} ab & ab \\ ab & ba \\ ab & ab \end{matrix}\right)$ $205\left(1, 5;\ \begin{matrix} ab & aa \\ ab & aa \\ ab & aa \end{matrix}\right)$ $206\left(1, 5;\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$ $207\left(1, 5;\ \begin{matrix} ab & ab \\ ab & ba \\ ab & ba \end{matrix}\right)$

$208\left(1, 6;\ \begin{matrix} ab & aa \\ ab & aa \\ ab & aa \end{matrix}\right)$ $209\left(1, 6;\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$ $210\left(1, 7;\ \begin{matrix} ab & aa \\ ab & aa \\ ab & aa \end{matrix}\right)$ $211\left(1, 7;\ \begin{matrix} ab & aa \\ ab & bb \\ ab & aa \end{matrix}\right)$ $212\left(1, 7;\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$

$213\left(1, 7';\ \begin{matrix} ab & aa \\ ab & aa \\ ab & aa \end{matrix}\right)$ $214\left(1, 7';\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$ $215\left(2, 1;\ \begin{matrix} aa & aa \\ aa & aa \\ aa & aa \end{matrix}\right)$ $216\left(2, 1;\ \begin{matrix} ab & aa \\ ab & aa \\ ab & aa \end{matrix}\right)$ $217\left(2, 1;\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$

$218\left(2, 1';\ \begin{matrix} ab & aa \\ ab & aa \\ ab & aa \end{matrix}\right)$ $219^{\times}\left(2, 2;\ \begin{matrix} aa & aa \\ aa & aa \\ aa & aa \end{matrix}\right)$ $220\left(2, 2;\ \begin{matrix} aa & ab \\ aa & ab \\ aa & ab \end{matrix}\right)$ $221^{\times}\left(2, 2;\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$ $222^{\times}\left(2, 3;\ \begin{matrix} aa & aa \\ aa & aa \\ aa & aa \end{matrix}\right)$

$223\left(2, 3;\ \begin{matrix} aa & ab \\ aa & ab \\ aa & ab \end{matrix}\right)$ $224^{\times}\left(2, 3;\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$ $225^{\times}\left(2, 4;\ \begin{matrix} aa & aa \\ aa & aa \\ aa & aa \end{matrix}\right)$ $226\left(2, 4;\ \begin{matrix} aa & ab \\ aa & ab \\ aa & ab \end{matrix}\right)$ $227^{\times}\left(2, 4;\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$

$228^{\times}\left(2, 5;\ \begin{matrix} aa & aa \\ aa & aa \\ aa & aa \end{matrix}\right)$ $229\left(2, 5;\ \begin{matrix} aa & ab \\ aa & ab \\ aa & ab \end{matrix}\right)$ $230^{\times}\left(2, 5;\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$ $231^{\times}\left(2, 6;\ \begin{matrix} aa & aa \\ aa & aa \\ aa & aa \end{matrix}\right)$ $232\left(2, 6;\ \begin{matrix} aa & ab \\ aa & ab \\ aa & ab \end{matrix}\right)$

$233^{\times}\left(2, 6;\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$ $234\left(2, 7;\ \begin{matrix} aa & aa \\ aa & aa \\ aa & aa \end{matrix}\right)$ $235\left(2, 7;\ \begin{matrix} ab & aa \\ ab & aa \\ ab & aa \end{matrix}\right)$ $236\left(2, 7;\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$ $237\left(2, 7';\ \begin{matrix} ab & aa \\ ab & aa \\ ab & aa \end{matrix}\right)$

$238\left(3, 1;\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$ $239^{\times}\left(3, 2;\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$ $240^{\times}\left(3, 3;\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$ $241^{\times}\left(3, 4;\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$ $242^{\times}\left(3, 4;\ \begin{matrix} ab & ab \\ ba & ba \\ ab & ab \end{matrix}\right)$

$243^{\times}\left(3, 5;\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$ $244^{\times}\left(3, 5;\ \begin{matrix} ab & ab \\ ba & ba \\ ba & ba \end{matrix}\right)$ $245^{\times}\left(3, 6;\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$ $246\left(3, 7;\ \begin{matrix} ab & ab \\ ab & ab \\ ab & ab \end{matrix}\right)$

IV $(\mathbf{a}, \mathbf{b}, \mathbf{c})_i\text{-}(\mathbf{d}, \mathbf{e})_j$

$247\left(1, 1;\ \begin{matrix} abc & aaa \\ abc & aaa \end{matrix}\right)$ $248\left(1, 1;\ \begin{matrix} abc & aaa \\ abc & bbb \end{matrix}\right)$ $249\left(1, 1;\ \begin{matrix} abc & aba \\ abc & aba \end{matrix}\right)$ $250\left(1, 1;\ \begin{matrix} abc & aba \\ abc & cbc \end{matrix}\right)$

$251\left(1, 1;\ \begin{matrix} abc & abc \\ abc & abc \end{matrix}\right)$ $252\left(1, 1';\ \begin{matrix} abc & aaa \\ abc & aaa \end{matrix}\right)$ $253\left(1, 1';\ \begin{matrix} abc & aba \\ abc & aba \end{matrix}\right)$ $254\left(1, 1';\ \begin{matrix} abc & aba \\ abc & abb \end{matrix}\right)$

$255\left(1, 1';\ \begin{matrix} abc & abc \\ abc & abc \end{matrix}\right)$ $256\left(1, 2;\ \begin{matrix} abc & aaa \\ abc & aaa \end{matrix}\right)$ $257\left(1, 2;\ \begin{matrix} abc & aaa \\ abc & aab \end{matrix}\right)$ $258\left(1, 2;\ \begin{matrix} abc & aba \\ abc & aba \end{matrix}\right)$

$259\left(1, 2;\ \begin{matrix} abc & abc \\ abc & abc \end{matrix}\right)$ $260\left(1, 3;\ \begin{matrix} abc & aaa \\ abc & aaa \end{matrix}\right)$ $261\left(1, 3;\ \begin{matrix} abc & aba \\ abc & aba \end{matrix}\right)$ $262\left(1, 3;\ \begin{matrix} abc & aba \\ abc & bab \end{matrix}\right)$

$263\left(1, 3;\ \begin{matrix} abc & abc \\ abc & abc \end{matrix}\right)$ $264\left(1, 3;\ \begin{matrix} abc & abc \\ abc & acb \end{matrix}\right)$ $265\left(2, 1;\ \begin{matrix} aaa & aaa \\ aaa & aaa \end{matrix}\right)$ $266\left(2, 1;\ \begin{matrix} aaa & abb \\ aaa & acc \end{matrix}\right)$

$267\left(2, 1;\ \begin{matrix} aba & aaa \\ aba & aaa \end{matrix}\right)$ $268\left(2, 1;\ \begin{matrix} aba & aac \\ aba & aac \end{matrix}\right)$ $269\left(2, 1;\ \begin{matrix} aba & aba \\ aba & aba \end{matrix}\right)$ $270\left(2, 1;\ \begin{matrix} abb & aaa \\ abb & aaa \end{matrix}\right)$

$271\left(2, 1;\ \begin{matrix} abb & abb \\ abb & abb \end{matrix}\right)$ $272\left(2, 1;\ \begin{matrix} abc & aaa \\ abc & aaa \end{matrix}\right)$ $273\left(2, 1;\ \begin{matrix} abc & aba \\ abc & aba \end{matrix}\right)$ $274\left(2, 1;\ \begin{matrix} abc & abb \\ abc & abb \end{matrix}\right)$

$275\left(2, 1;\ \begin{matrix} abc & abb \\ abc & acc \end{matrix}\right)$ $276\left(2, 1;\ \begin{matrix} abc & abc \\ abc & abc \end{matrix}\right)$ $277\left(2, 1';\ \begin{matrix} aaa & aba \\ aaa & abb \end{matrix}\right)$ $278\left(2, 1';\ \begin{matrix} aba & aaa \\ aba & aaa \end{matrix}\right)$

$279\left(2, 1';\ \begin{matrix} abb & aaa \\ abb & aaa \end{matrix}\right)$ $280\left(2, 1';\ \begin{matrix} abc & aaa \\ abc & aaa \end{matrix}\right)$ $281\left(2, 1';\ \begin{matrix} abc & aba \\ abc & aba \end{matrix}\right)$ $282\left(2, 1';\ \begin{matrix} abc & aba \\ abc & abb \end{matrix}\right)$

$283\left(2, 1';\ \begin{matrix} abc & abb \\ abc & abb \end{matrix}\right)$ $284^{\times}\left(2, 2;\ \begin{matrix} aaa & aaa \\ aaa & aaa \end{matrix}\right)$ $285\left(2, 2;\ \begin{matrix} aaa & aaa \\ aaa & aab \end{matrix}\right)$ $286\left(2, 2;\ \begin{matrix} aaa & aba \\ aaa & aba \end{matrix}\right)$

$287\left(2, 2;\ \begin{matrix} aaa & abb \\ aaa & abb \end{matrix}\right)$ $288\left(2, 2;\ \begin{matrix} aaa & abc \\ aaa & abc \end{matrix}\right)$ $289^{\times}\left(2, 2;\ \begin{matrix} aaa & aaa \\ aab & aab \end{matrix}\right)$ $290\left(2, 2;\ \begin{matrix} aaa & abc \\ aab & abc \end{matrix}\right)$

$291^{*}\left(2, 2;\ \begin{matrix} aac & aba \\ aac & aba \end{matrix}\right)$ $292^{\times}\left(2, 2;\ \begin{matrix} aba & aba \\ aba & aba \end{matrix}\right)$ $293\left(2, 2;\ \begin{matrix} aba & abc \\ aba & abc \end{matrix}\right)$ $294^{\times}\left(2, 2;\ \begin{matrix} abb & abb \\ abb & abb \end{matrix}\right)$

295$\left(2, 2;\ \begin{matrix}abb & abc\\ abb & abc\end{matrix}\right)$
296$^{\times}\left(2, 2;\ \begin{matrix}abc & abc\\ abc & abc\end{matrix}\right)$
297$^{\times}\left(2, 3;\ \begin{matrix}aaa & aaa\\ aaa & aaa\end{matrix}\right)$
298$\left(2, 3;\ \begin{matrix}aaa & aba\\ aaa & aba\end{matrix}\right)$
299$\left(2, 3;\ \begin{matrix}aaa & abb\\ aaa & abb\end{matrix}\right)$
300$\left(2, 3;\ \begin{matrix}aaa & abc\\ aaa & abc\end{matrix}\right)$
301$\left(2, 3;\ \begin{matrix}aaa & abc\\ aaa & acb\end{matrix}\right)$
302$^{*}\left(2, 3;\ \begin{matrix}aac & aba\\ aac & aba\end{matrix}\right)$
303$^{\times}\left(2, 3;\ \begin{matrix}aba & aba\\ aba & aba\end{matrix}\right)$
304$\left(2, 3;\ \begin{matrix}aba & abc\\ aba & abc\end{matrix}\right)$
305$^{\times}\left(2, 3;\ \begin{matrix}abb & abb\\ abb & abb\end{matrix}\right)$
306$\left(2, 3;\ \begin{matrix}abb & abc\\ abb & abc\end{matrix}\right)$
307$^{\times}\left(2, 3;\ \begin{matrix}abc & abc\\ abc & abc\end{matrix}\right)$
308$\left(2, 3;\ \begin{matrix}abc & abc\\ abc & acb\end{matrix}\right)$
309$^{\times}\left(2, 3;\ \begin{matrix}abc & abc\\ acb & acb\end{matrix}\right)$
310$\left(3, 1;\ \begin{matrix}aaa & aaa\\ aaa & aaa\end{matrix}\right)$
311$\left(3, 1;\ \begin{matrix}abc & abc\\ abc & abc\end{matrix}\right)$
312$^{\times}\left(3, 2;\ \begin{matrix}aaa & aaa\\ aaa & aaa\end{matrix}\right)$
313$\left(3, 2;\ \begin{matrix}aaa & aaa\\ aaa & aab\end{matrix}\right)$
314$^{\times}\left(3, 2;\ \begin{matrix}aaa & aaa\\ aab & aab\end{matrix}\right)$
315$^{\times}\left(3, 2;\ \begin{matrix}abc & abc\\ abc & abc\end{matrix}\right)$
316$^{\times}\left(3, 3;\ \begin{matrix}aaa & aaa\\ aaa & aaa\end{matrix}\right)$
317$^{\times}\left(3, 3;\ \begin{matrix}abc & abc\\ abc & abc\end{matrix}\right)$
318$\left(4, 1;\ \begin{matrix}aba & aba\\ aba & aba\end{matrix}\right)$
319$\left(4, 1;\ \begin{matrix}abc & aba\\ abc & aba\end{matrix}\right)$
320$\left(4, 1;\ \begin{matrix}abc & abc\\ abc & abc\end{matrix}\right)$
321$\left(4, 1';\ \begin{matrix}abc & aba\\ abc & aba\end{matrix}\right)$
322$^{\times}\left(4, 2;\ \begin{matrix}aba & aba\\ aba & aba\end{matrix}\right)$
323$\left(4, 2;\ \begin{matrix}aba & abc\\ aba & abc\end{matrix}\right)$
324$^{\times}\left(4, 2;\ \begin{matrix}abc & abc\\ abc & abc\end{matrix}\right)$
325$^{\times}\left(4, 3;\ \begin{matrix}aba & aba\\ aba & aba\end{matrix}\right)$
326$\left(4, 3;\ \begin{matrix}aba & abc\\ aba & abc\end{matrix}\right)$
327$^{\times}\left(4, 3;\ \begin{matrix}aba & aba\\ bab & bab\end{matrix}\right)$
328$^{\times}\left(4, 3;\ \begin{matrix}abc & abc\\ abc & abc\end{matrix}\right)$
329$\left(5, 1;\ \begin{matrix}abb & abb\\ abb & abb\end{matrix}\right)$
330$\left(5, 1;\ \begin{matrix}abc & abb\\ abc & abb\end{matrix}\right)$
331$\left(5, 1;\ \begin{matrix}abc & abc\\ abc & abc\end{matrix}\right)$
332$\left(5, 1';\ \begin{matrix}abc & abb\\ abc & abb\end{matrix}\right)$
333$^{\times}\left(5, 2 \cdot\ \begin{matrix}abb & abb\\ abb & abb\end{matrix}\right)$
334$\left(5, 2;\ \begin{matrix}abb & abc\\ abb & abc\end{matrix}\right)$
335$^{\times}\left(5, 2 \cdot\ \begin{matrix}abc & abc\\ abc & abc\end{matrix}\right)$
336$^{\times}\left(5, 3;\ \begin{matrix}abb & abb\\ abb & abb\end{matrix}\right)$
337$\left(5, 3;\ \begin{matrix}abb & abc\\ abb & abc\end{matrix}\right)$
338$^{\times}\left(5, 3;\ \begin{matrix}abc & abc\\ abc & abc\end{matrix}\right)$
339$^{\times}\left(5, 3;\ \begin{matrix}abb & abb\\ baa & baa\end{matrix}\right)$
340$\left(6, 1;\ \begin{matrix}abc & abc\\ abc & abc\end{matrix}\right)$
341$^{\times}\left(6, 2;\ \begin{matrix}abc & abc\\ abc & abc\end{matrix}\right)$
342$^{\times}\left(6, 3;\ \begin{matrix}abc & abc\\ abc & abc\end{matrix}\right)$
343$\left(7, 1;\ \begin{matrix}aba & aaa\\ aba & aaa\end{matrix}\right)$
344$\left(7, 1;\ \begin{matrix}aba & aaa\\ aba & bbb\end{matrix}\right)$
345$\left(7, 1;\ \begin{matrix}aba & aac\\ aba & aac\end{matrix}\right)$
346$\left(7, 1;\ \begin{matrix}aba & aba\\ aba & aba\end{matrix}\right)$
347$\left(7, 1;\ \begin{matrix}aba & abc\\ aba & abc\end{matrix}\right)$
348$\left(7, 1;\ \begin{matrix}aba & bbb\\ aba & bbb\end{matrix}\right)$
349$\left(7, 1;\ \begin{matrix}abc & aaa\\ abc & aaa\end{matrix}\right)$
350$\left(7, 1;\ \begin{matrix}abc & aaa\\ abc & bbb\end{matrix}\right)$
351$\left(7, 1;\ \begin{matrix}abc & aac\\ abc & aac\end{matrix}\right)$
352$\left(7, 1;\ \begin{matrix}abc & aba\\ abc & aba\end{matrix}\right)$
353$\left(7, 1;\ \begin{matrix}abc & abc\\ abc & abc\end{matrix}\right)$
354$\left(7, 1;\ \begin{matrix}abc & bbb\\ abc & bbb\end{matrix}\right)$
355$\left(7, 1';\ \begin{matrix}aba & aaa\\ aba & aaa\end{matrix}\right)$
356$\left(7, 1';\ \begin{matrix}aba & aac\\ aba & aac\end{matrix}\right)$
357$\left(7, 1';\ \begin{matrix}aba & aba\\ aba & aba\end{matrix}\right)$
358$\left(7, 1';\ \begin{matrix}aba & abc\\ aba & abc\end{matrix}\right)$
359$\left(7, 1';\ \begin{matrix}aba & bbb\\ aba & bbb\end{matrix}\right)$
360$\left(7, 1';\ \begin{matrix}abc & aaa\\ abc & aaa\end{matrix}\right)$
361$\left(7, 1';\ \begin{matrix}abc & aac\\ abc & aac\end{matrix}\right)$
362$\left(7, 1';\ \begin{matrix}abc & aba\\ abc & aba\end{matrix}\right)$
363$\left(7, 1';\ \begin{matrix}abc & abc\\ abc & abc\end{matrix}\right)$
364$\left(7, 1';\ \begin{matrix}abc & bbb\\ abc & bbb\end{matrix}\right)$
365$\left(7, 2;\ \begin{matrix}aba & aaa\\ aba & aaa\end{matrix}\right)$
366$\left(7, 2;\ \begin{matrix}aba & aac\\ aba & aac\end{matrix}\right)$
367$\left(7, 2;\ \begin{matrix}aba & aba\\ aba & aba\end{matrix}\right)$
368$\left(7, 2;\ \begin{matrix}aba & abc\\ aba & abc\end{matrix}\right)$
369$\left(7, 2;\ \begin{matrix}aba & bbb\\ aba & bbb\end{matrix}\right)$
370$\left(7, 2;\ \begin{matrix}abc & aaa\\ abc & aaa\end{matrix}\right)$
371$\left(7, 2;\ \begin{matrix}abc & aac\\ abc & aac\end{matrix}\right)$
372$\left(7, 2;\ \begin{matrix}abc & aba\\ abc & aba\end{matrix}\right)$
373$\left(7, 2;\ \begin{matrix}abc & abc\\ abc & abc\end{matrix}\right)$
374$\left(7, 2;\ \begin{matrix}abc & bbb\\ abc & bbb\end{matrix}\right)$
375$\left(7, 3;\ \begin{matrix}aba & aaa\\ aba & aaa\end{matrix}\right)$
376$\left(7, 3;\ \begin{matrix}aba & aac\\ aba & aac\end{matrix}\right)$
377$\left(7, 3;\ \begin{matrix}aba & aba\\ aba & aba\end{matrix}\right)$
378$\left(7, 3;\ \begin{matrix}aba & aba\\ aba & bab\end{matrix}\right)$
379$\left(7, 3;\ \begin{matrix}aba & abc\\ aba & abc\end{matrix}\right)$
380$\left(7, 3;\ \begin{matrix}aba & bbb\\ aba & bbb\end{matrix}\right)$
381$\left(7, 3;\ \begin{matrix}abc & aaa\\ abc & aaa\end{matrix}\right)$
382$\left(7, 3;\ \begin{matrix}abc & aac\\ abc & aac\end{matrix}\right)$
383$\left(7, 3;\ \begin{matrix}abc & aba\\ abc & aba\end{matrix}\right)$
384$\left(7, 3;\ \begin{matrix}abc & aba\\ abc & bab\end{matrix}\right)$
385$\left(7, 3;\ \begin{matrix}abc & abc\\ abc & abc\end{matrix}\right)$
386$\left(7, 3;\ \begin{matrix}abc & bbb\\ abc & bbb\end{matrix}\right)$

V (a, b, c, d)$_i$–e

387(1; abcd aaaa)
388(1; abcd abaa)
389(1; abcd abab)
390(1; abcd abca)
391(1; abcd abcd)
392*(2; abab aacc)
393(2; abcd aacc)
394*(2; abcd abcd)
395$^{\times}$(3; aaaa aaaa)
396(3; aaaa abaa)
397(3; aaaa abab)
398(3; aaaa abbb)
399(3; aaaa abca)
400(3; aaaa abcb)
401(3; aaaa abcd)
402*(3; aaca abaa)
403(3; aaca abab)
404*(3; aacc abab)
405(3; aacd abaa)
406*(3; aacd abca)
407$^{\times}$(3; abaa abaa)
408(3; abaa abca)
409(3; abaa abcd)
410$^{\times}$(3; abab abab)
411(3; abab abcb)
412(3; abab abcd)
413(3; abad abab)
414$^{\times}$(3; abbb abbb)
415(3; abbb abcb)
416(3; abbb abcd)
417(3; abbd abca)
418*(3; abbd abcb)
419$^{\times}$(3; abca abca)
420(3; abca abcd)
421$^{\times}$(3; abcb abcb)
422(3; abcb abcd)
423(3; abcc abbb)
424$^{\times}$(3; abcd abcd)
425(3; abdd abaa)
426$^{\times}$(4; aaaa aaaa)
427(4; aaaa aaca)
428(4; aaaa aacc)
429$^{\times}$(4; aaca aaca)
430$^{\times}$(4; aacc aacc)
431$^{\times}$(4; abad abad)
432(4; abad abcd)
433$^{\times}$(4; abbd abbd)
434(4; abbd abcd)
435$^{\times}$(4; abcd abcd)
436*(5; aaaa aaaa)
437(5; aaaa abca)
438(5; aaaa abcb)
439*(5; aaca aaad)
440(5; aaca abcd)
441*(5; abad abca)
442(5; abad abcb)

443*(5; abbd abcb)	444*(5; abcd abcd)	445(6; aaaa aaaa)	446(6; aaaa abcc)
447(6; aacc abcd)	448(6; abcd abcd)	449×(7; aaaa aaaa)	450×(7; abcd abcd)
451×(8; aaaa aaaa)	452×(8; abcd abcd)	453×(9; aaaa aaaa)	454×(9; abcd abcd)
455*(10; aaaa aaaa)	456*(10; abcd abcd)	457×(11; aaaa aaaa)	458×(11; abcc abcc)
459(11; abcc abcd)	460×(11; abcd abcd)	461×(12; aaaa aaaa)	462×(12; abcd abcd)
463×(13; abaa abaa)	464(13; abaa abca)	465(13; abaa abcc)	466(13; abaa abcd)
467×(13; abca abca)	468*(13; abca abad)	469(13; abca abcd)	470×(13; abcc abcc)
471(13; abcc abcd)	472×(13; abcd abcd)	473×(14; abab abab)	474(14; abab abad)
475(14; abab abcb)	476(14; abab abcd)	477×(14; abad abad)	478(14; abad abcb)
479(14; abad abcd)	480×(14; abcb abcb)	481(14; abcb abcd)	482×(14; abcd abcd)
483×(15; abbb abbb)	484(15; abbb abcb)	485(15; abbb abcc)	486(15; abbb abcd)
487*(15; abbd abcb)	488×(15; abcb abcb)	489(15; abcb abcd)	490×(15; abcc abcc)
491(15; abcc abcd)	492×(15; abcd abcd)	493×(16; abaa abaa)	494×(16; abcd abcd)
495×(17; abab abab)	496×(17; abcd abcd)	497×(18; abca abca)	498(18; abca abcd)
499×(18; abcd abcd)	500×(19; abcb abcb)	501(19; abcb abcd)	502×(19; abcd abcd)
503×(20; abcd abcd)	504×(21; abcd abcd)	505(22; abca aaaa)	506(22; abca aaad)
507(22; abca abaa)	508(22; abca abad)	509(22; abca abba)	510(22; abca abbd)
511(22; abca abca)	512(22; abca abcd)	513(22; abca bbbb)	514(22; abca bbcb)
515(22; abcd aaaa)	516(22; abcd aaad)	517(22; abcd abaa)	518(22; abcd abad)
519(22; abcd abba)	520(22; abcd abbd)	521(22; abcd abca)	522(22; abcd abcd)
523(22; abcd bbbb)	524(22; abcd bbcb)	525(23; abca aaaa)	526(23; abca abad)
527(23; abca abbd)	528(23; abca bbbb)	529(23; abcb cccc)	530(23; abcd aaca)
531(23; abcd abcd)	532(23; abcd bbcb)	533(24; abcd aacc)	534(24; abcd abcd)
535(25; abaa aaaa)	536(25; abaa aaca)	537(25; abaa aacc)	538(25; abaa aacd)
539(25; abaa abaa)	540(25; abaa abca)	541(25; abaa abcc)	542(25; abaa abcd)
543(25; abaa bbbb)	544(25; abad aaca)	545(25; abad abca)	546(25; abca aaaa)
547(25; abca aaca)	548(25; abca aacd)	549(25; abca abaa)	550(25; abca abca)
551(25; abca abcd)	552(25; abca bbbb)	553(25; abcc aaaa)	554(25; abcc aacc)
555(25; abcc aacd)	556(25; abcc abaa)	557(25; abcc abcc)	558(25; abcc abcd)
559(25; abcc bbbb)	560(25; abcd aaaa)	561(25; abcd aaca)	562(25; abcd aacc)
563(25; abcd aacd)	564(25; abcd abaa)	565(25; abcd abca)	566(25; abcd abcc)
567(25; abcd abcd)	568(25; abcd bbbb)	569(26; abaa aaaa)	570(26; abaa abaa)
571(26; abaa bbbb)	572(26; abcd aacd)	573(26; abcd abcd)	574(27; abab aaaa)
575(27; abab aaca)	576(27; abab aacc)	577(27; abab abab)	578(27; abab abcb)
579(27; abab abcd)	580(27; abad aaaa)	581(27; abad aaca)	582(27; abad abcb)
583(27; abcb aaaa)	584(27; abcb aaca)	585(27; abcb abab)	586(27; abcb abcb)
587(27; abcb abcd)	588(27; abcd aaaa)	589(27; abcd aaca)	590(27; abcd aacc)
591(27; abcd abab)	592(27; abcd abcb)	593(27; abcd abcd)	

VI $\mathbf{a}<^{(\mathbf{b},\,\mathbf{c})_i}_{(\mathbf{d},\,\mathbf{e})_j}$

594(1,1) 595*(1,1′) 596(1,2) 597(1,3) 598×(2,2) 599×(2,3) 600×(3,3)

VII $\mathrm{a}<^{(\mathbf{b},\,\mathbf{c},\,\mathbf{d})_i}_{\mathbf{e}}$

601(1) 602×(2) 603×(3) 604×(4) 605×(5) 606×(6) 607(7)

VIII $\overbrace{(\mathbf{a},\,\mathbf{b})<^{(\mathbf{c},\,\mathbf{d})}_{\mathbf{e}}}^{i}$

608(35; abaa aaaa)	609(35; abaa abaa)	610(35; abaa bbbb)	611(36; abab aaaa)
612(36; abab abab)	613(37; abaa aaaa)	614(38; abaa aaaa)	615(38; abaa abaa)
616(38; abaa bbbb)	617(39; abaa aaaa)	618(40; abaa aaaa)	619(41; abbb aaaa)
620(41; abbb abbb)	621(41; abbb bbbb)	622(42; abaa aaaa)	623(43; abab aaaa)
624(44; abbb aaaa)	625(44; abbb abbb)	626(44; abbb bbbb)	627(45; aaaa aaaa)
628(45; aaaa abaa)	629(45; abaa aaaa)	630(45; abaa abaa)	631(46; aaaa aaaa)
632(46; aaaa abaa)	633(46; aaaa abab)	634(46; aaaa abbb)	635(47; aaaa aaaa)
636(47; abaa aaaa)	637(47; abbb aaaa)	638(48; aaaa aaaa)	639×(49; aaaa aaaa)

640(49; aaaa abaa) 641(49; aaaa abab) 642×(49; abaa abaa) 643×(49; abab abab)
644(50; aaaa aaaa) 645(50; abaa aaaa) 646(50; abbb aaaa) 647×(51; aaaa aaaa)
648×(52; aaaa aaaa) 649(52; aaaa abaa) 650×(52; abaa abaa) 651(53; aaaa aaaa)
652(53; aaaa abaa) 653(53; aaaa abbb) 654×(54; aaaa aaaa) 655(55; abaa abaa)
656×(56; abaa abaa) 657×(57; abaa abaa) 658×(58; abab abab)

IX $\overbrace{(\mathbf{a}, \mathbf{b}, \mathbf{c})<^{(\mathbf{d})}_{\mathbf{e}}}^{i}$

659(59; abca aaaa) 660(59; abca abaa) 661(59; abca abba) 662(59; abca abca)
663(59; abca bbbb) 664(59; abca bbcb) 665(60; abca aaca) 666×(62; aaaa aaaa)
667(62; aaaa abaa) 668(62; aaaa abba) 669(62; aaaa abca) 670*(62; aaca abaa)
671×(62; abaa abaa) 672(62; abaa abca) 673×(62; abba abba) 674(62; abba abca)
675×(62; abca abca) 676(63; aaaa aaca) 677*(63; aaca aaaa) 678(63; aaca aaca)
679*(63; abaa aaaa) 680(63; abaa aaca) 681*(63; abab aaaa) 682(63; abab aaca)
683(63; abca aaaa) 684(63; abca aaca) 685(63; abcb aaaa) 686(63; abcb aaca)
687*(64; abba aaaa) 688*(64; abbb aaaa) 689(64; abca aaaa) 690(65; abbb aaaa)
691*(65; abca aaaa) 692*(65; abcb aaaa) 693×(66; abaa abaa) 694*(67; abaa aaca)
695*(67; abab aaca) 696*(67; abab aacc) 697×(72; aaaa aaaa) 698×(72; abca abca)
699×(74; abaa abaa) 700(74; abaa abca) 701×(74; abca abca) 702*(75; abca abaa)
703*(75; abcc abaa) 704×(77; abba abba) 705(77; abca abba) 706×(77; abca abca)
707*(78; abca abba) 708×(80; abca abca) 709(81; abaa aaaa) 710(81; abca aaaa)
711(81; abca aaca) 712(81; abca abaa) 713(81; abca abca) 714(81; abca bbbb)
715(82; abaa aaaa) 716(82; abaa abaa) 717(82; abaa bbbb) 718(82; abca aaaa)
719(82; abca abaa) 720(82; abca bbbb) 721(82; abcc aaaa) 722(82; abcc abaa)
723(82; abcc bbbb) 724(83; abaa aaaa) 725(83; abab bbbb) 726(83; abca aaaa)
727(83; abca aaca) 728(83; abcb bbbb) 729(84; abaa aaaa) 730(84; abab bbbb)
731(84; abca aaaa) 732(84; abcb bbbb) 733(84; abcc aaaa) 734(85; abab aaaa)
735(85; abab bbbb) 736(85; abcb aaaa) 737(85; abcb aaca) 738(85; abcb abab)
739(85; abcb abcb) 740(85; abcb bbbb)

X $\mathbf{a}\text{–}\mathbf{b}\text{–}(\mathbf{c}, \mathbf{d}, \mathbf{e})_i$

741(1) 742×(2) 743×(3) 744×(4) 745×(5) 746×(6) 747(7)

XI $\mathbf{a}\text{–}(\overbrace{\mathbf{b}, \mathbf{c})\text{–}(\mathbf{d}, \mathbf{e}}^{i})$

748(35) 749(36) 750(37) 751(38) 752(39) 753(40) 754(41) 755(42) 756(43) 757(44)
758(45) 759(46) 760(47) 761(48) 762×(49) 763(50) 764×(51) 765×(52) 766(53) 767×(54)
768(55) 769×(56) 770×(57) 771×(58)

XII $\mathbf{a}\text{–}\overbrace{(\mathbf{b}, \mathbf{c}, \mathbf{d})\text{–}(\mathbf{e})}^{i}$

772(59) 773(60) 774(61) 775×(62) 776(63) 777(64) 778(65) 779×(66) 780*(67) 781(68)
782×(69) 783(70) 784×(71) 785×(72) 786×(73) 787×(74) 788(75) 789×(76) 790×(77) 791(78)
792×(79) 793×(80) 794(81) 795(82) 796(83) 797(84) 798(85) 799(86) 800(87) 801(88)
802(89) 803(90)

XIII $(\mathbf{a}, \mathbf{b})_i\text{–}\mathbf{c}\text{–}(\mathbf{d}, \mathbf{e})_j$

804$\left(1, 1;\ \begin{matrix}ab & ab\\ ab & ab\\ ab & ab\end{matrix}\right)$ 805$\left(1, 1;\ \begin{matrix}ab & bb\\ ab & ab\\ ab & ab\end{matrix}\right)$ 806$\left(1, 1;\ \begin{matrix}ab & bb\\ ab & bb\\ ab & bb\end{matrix}\right)$ 807$\left(1, 1';\ \begin{matrix}ab & ab\\ ab & ab\\ ab & ab\end{matrix}\right)$ 808$\left(1, 1';\ \begin{matrix}ab & bb\\ ab & ab\\ ab & ab\end{matrix}\right)$

809$\left(1, 1';\ \begin{matrix}ab & bb\\ ab & bb\\ ab & bb\end{matrix}\right)$ 810$\left(1, 2;\ \begin{matrix}ab & ab\\ ab & ab\\ ab & ab\end{matrix}\right)$ 811$\left(1, 2;\ \begin{matrix}ab & bb\\ ab & ab\\ ab & ab\end{matrix}\right)$ 812$\left(1, 2;\ \begin{matrix}ab & bb\\ ab & bb\\ ab & bb\end{matrix}\right)$ 813$\left(1, 3;\ \begin{matrix}ab & ab\\ ab & ab\\ ab & ab\end{matrix}\right)$

814$\left(1, 3;\ \begin{matrix}ab & bb\\ ab & ab\\ ab & ab\end{matrix}\right)$ 815$\left(1, 3;\ \begin{matrix}ab & bb\\ ab & bb\\ ab & bb\end{matrix}\right)$ 816$\left(2, 1;\ \begin{matrix}aa & aa\\ aa & aa\\ aa & aa\end{matrix}\right)$ 817$\left(2, 1;\ \begin{matrix}aa & aa\\ aa & ab\\ aa & ab\end{matrix}\right)$ 818$\left(2, 1;\ \begin{matrix}aa & ab\\ aa & ab\\ aa & ab\end{matrix}\right)$

819$\left(2, 1;\ \begin{matrix}aa & aa\\ ab & aa\\ ab & aa\end{matrix}\right)$ 820$\left(2, 1;\ \begin{matrix}aa & aa\\ ab & ab\\ ab & ab\end{matrix}\right)$ 821$\left(2, 1;\ \begin{matrix}aa & ab\\ ab & ab\\ ab & ab\end{matrix}\right)$ 822$\left(2, 1;\ \begin{matrix}ab & aa\\ ab & aa\\ ab & aa\end{matrix}\right)$ 823$\left(2, 1;\ \begin{matrix}ab & aa\\ ab & ab\\ ab & ab\end{matrix}\right)$

$824\left(2,\ 1;\ \begin{matrix}\mathrm{ab}&\mathrm{ab}\\ \mathrm{ab}&\mathrm{ab}\\ \mathrm{ab}&\mathrm{ab}\end{matrix}\right)$ $825^{\times}\left(2,\ 2;\ \begin{matrix}\mathrm{aa}&\mathrm{aa}\\ \mathrm{aa}&\mathrm{aa}\\ \mathrm{aa}&\mathrm{aa}\end{matrix}\right)$ $826\left(2,\ 2;\ \begin{matrix}\mathrm{aa}&\mathrm{aa}\\ \mathrm{aa}&\mathrm{ab}\\ \mathrm{aa}&\mathrm{ab}\end{matrix}\right)$ $827\left(2,\ 2;\ \begin{matrix}\mathrm{aa}&\mathrm{ab}\\ \mathrm{aa}&\mathrm{ab}\\ \mathrm{aa}&\mathrm{ab}\end{matrix}\right)$ $828^{\times}\left(2,\ 2;\ \begin{matrix}\mathrm{aa}&\mathrm{aa}\\ \mathrm{ab}&\mathrm{ab}\\ \mathrm{ab}&\mathrm{ab}\end{matrix}\right)$

$829\left(2,\ 2;\ \begin{matrix}\mathrm{aa}&\mathrm{ab}\\ \mathrm{ab}&\mathrm{ab}\\ \mathrm{ab}&\mathrm{ab}\end{matrix}\right)$ $830^{\times}\left(2,\ 2;\ \begin{matrix}\mathrm{ab}&\mathrm{ab}\\ \mathrm{ab}&\mathrm{ab}\\ \mathrm{ab}&\mathrm{ab}\end{matrix}\right)$ $831^{\times}\left(2,\ 3;\ \begin{matrix}\mathrm{aa}&\mathrm{aa}\\ \mathrm{aa}&\mathrm{aa}\\ \mathrm{aa}&\mathrm{aa}\end{matrix}\right)$ $832\left(2,\ 3;\ \begin{matrix}\mathrm{aa}&\mathrm{aa}\\ \mathrm{aa}&\mathrm{ab}\\ \mathrm{aa}&\mathrm{ab}\end{matrix}\right)$ $833\left(2,\ 3;\ \begin{matrix}\mathrm{aa}&\mathrm{ab}\\ \mathrm{aa}&\mathrm{ab}\\ \mathrm{aa}&\mathrm{ab}\end{matrix}\right)$

$834^{\times}\left(2,\ 3;\ \begin{matrix}\mathrm{aa}&\mathrm{aa}\\ \mathrm{ab}&\mathrm{ab}\\ \mathrm{ab}&\mathrm{ab}\end{matrix}\right)$ $835\left(2,\ 3;\ \begin{matrix}\mathrm{aa}&\mathrm{ab}\\ \mathrm{ab}&\mathrm{ab}\\ \mathrm{ab}&\mathrm{ab}\end{matrix}\right)$ $836^{\times}\left(2,\ 3;\ \begin{matrix}\mathrm{ab}&\mathrm{ab}\\ \mathrm{ab}&\mathrm{ab}\\ \mathrm{ab}&\mathrm{ab}\end{matrix}\right)$ $837\left(3,\ 1;\ \begin{matrix}\mathrm{ab}&\mathrm{ab}\\ \mathrm{ab}&\mathrm{ab}\\ \mathrm{ab}&\mathrm{ab}\end{matrix}\right)$ $838^{\times}\left(3,\ 2;\ \begin{matrix}\mathrm{ab}&\mathrm{ab}\\ \mathrm{ab}&\mathrm{ab}\\ \mathrm{ab}&\mathrm{ab}\end{matrix}\right)$

$839^{\times}\left(3,\ 3;\ \begin{matrix}\mathrm{ab}&\mathrm{ab}\\ \mathrm{ab}&\mathrm{ab}\\ \mathrm{ab}&\mathrm{ab}\end{matrix}\right)$

XIV $\overbrace{(\mathbf{a},\ \mathbf{b})\text{-}(\mathbf{c},\ \mathbf{d})}^{i}\text{-}\mathbf{e}$

840(35 ; abcd aacc) 841(35 ; abcd aacd) 842(35 ; abcd abcc) 843(35 ; abcd abcd)
844(36 ; abcd aacc) 845(36 ; abcd abcd) 846(37 ; abcd abcc) 847(37 ; abcd abcd)
848(38 ; abcc aacd) 849(38 ; abcc abcd) 850(38 ; abcd aacd) 851(38 ; abcd abcd)
852(39 ; abcc abcd) 853(39 ; abcd abcd) 854(40 ; abcc abcc) 855(40 ; abcc abcd)
856(40 ; abcd abcc) 857(40 ; abcd abcd) 858(41 ; abcc abcc) 859(41 ; abcc abcd)
860(41 ; abcc bbcc) 861(41 ; abcc bbcd) 862(41 ; abcd abcc) 863(41 ; abcd abcd)
864(41 ; abcd bbcc) 865(41 ; abcd bbcd) 866(42 ; abcd abcd) 867(43 ; abcd abcd)
868(44 ; abcd abcd) 869(44 ; abcd bbcd) 870(45 ; aacd aacc) 871(45 ; aacd aacd)
872(45 ; aacd abcc) 873(45 ; aacd abcd) 874(45 ; abcd aacc) 875(45 ; abcd aacd)
876(45 ; abcd abcc) 877(45 ; abcd abcd) 878(46 ; aacd aacc) 879(46 ; abcd aacd)
880(46 ; abcd abcc) 881(46 ; abcd abcd) 882(47 ; aacd abcc) 883(47 ; aacd abcd)
884(47 ; abcd abcc) 885(47 ; abcd abcd) 886(48 ; abcd abcc) 887(48 ; abcd abcd)
$888^{\times}$(49 ; aacc aacc) 889(49 ; aacc aacd) 890(49 ; aacc abcc) 891(49 ; aacc abcd)
$892^{\times}$(49 ; aacd aacd) 893(49 ; aacd abcc) 894(49 ; aacd abcd) $895^{\times}$(49 ; abcc abcc)
896(49 ; abcc abcd) $897^{\times}$(49 ; abcd abcd) 898(50 ; aacc abcc) 899(50 ; aacc abcd)
900(50 ; aacd abcc) 901(50 ; aacd abcd) 902(50 ; abcc abcc) 903(50 ; abcc abcd)
904(50 ; abcd abcc) 905(50 ; abcd abcd) $906^{\times}$(51 ; abcc abcc) 907(51 ; abcc abcd)
$908^{\times}$(51 ; abcd abcd) $909^{\times}$(52 ; aacd aacd) 910(52 ; aacd abcd) $911^{\times}$(52 ; abcd abcd)
912(53 ; abcd aacd) 913(53 ; abcd abcd) $914^{\times}$(54 ; abcd abcd) 915(55 ; abcd abcc)
916(55 ; abcd abcd) $917^{\times}$(56 ; abcc abcc) 918(56 ; abcc abcd) $919^{\times}$(56 ; abcd abcd)
$920^{\times}$(57 ; abcd abcd) $921^{\times}$(58 ; abcd abcd)

XV $\overbrace{(\mathbf{a},\ \mathbf{b},\ \mathbf{c})\text{-}(\mathbf{d})}^{i}\text{-}\mathbf{e}$

922(59 ; abcd aaad) 923(59 ; abcd abad) 924(59 ; abcd abbd) 925(59 ; abcd abcd)
926(60 ; abcd abad) 927(60 ; abcd abcd) 928(61 ; abcd abcd) $929^{\times}$(62 ; aaad aaad)
930(62 ; aaad abad) 931(62 ; aaad abbd) 932(62 ; aaad abcd) 933*(62 ; aacd abad)
$934^{\times}$(62 ; abad abad) 935(62 ; abad abcd) $936^{\times}$(62 ; abbd abbd) 937(62 ; abbd abcd)
$938^{\times}$(62 ; abcd abcd) 939(63 ; aaad abad) 940(63 ; aaad abcd) 941(63 ; aacd abad)
942(63 ; aacd abcd) 943(63 ; abad abad) 944(63 ; abad abcd) 945(63 ; abcd abad)
946(63 ; abcd abcd) 947(64 ; aaad abbd) 948(64 ; aaad abcd) 949(64 ; abbd abbd)
950(64 ; abbd abcd) 951(64 ; abcd abbd) 952(64 ; abcd abcd) 953(65 ; aaad abcd)
954(65 ; abad abcd) 955(65 ; abbd abcd) 956(65 ; abcd abcd) $957^{\times}$(66 ; aacd aacd)
958(66 ; aacd abcd) $959^{\times}$(66 ; abcd abcd) 960*(67 ; aacd abad) 961(67 ; aacd abcd)
962*(67 ; abcd abcd) 963(68 ; aacd abcd) 964(68 ; abcd abcd) $965^{\times}$(69 ; abbd abbd)
966(69 ; abbd abcd) $967^{\times}$(69 ; abcd abcd) 968(70 ; abcd abcd) 969(70 ; abcd accd)
$970^{\times}$(71 ; abcd abcd) $971^{\times}$(72 ; aaad aaad) $972^{\times}$(72 ; abcd abcd) $973^{\times}$(73 ; abcd abcd)
$974^{\times}$(74 ; abad abad) 975(74 ; abad abcd) $976^{\times}$(74 ; abcd abcd) 977(75 ; abad abcd)
978(75 ; abcd abcd) $979^{\times}$(76 ; abcd abcd) $980^{\times}$(77 ; abbd abbd) 981(77 ; abbd abcd)
$982^{\times}$(77 ; abcd abcd) 983(78 ; abbd abcd) 984(78 ; abcd abcd) $985^{\times}$(79 ; abcd abcd)
$986^{\times}$(80 ; abcd abcd) 987(81 ; abad aaad) 988(81 ; abad aacd) 989(81 ; abad abad)
990(81 ; abad abcd) 991(81 ; abcd aaad) 992(81 ; abcd aacd) 993(81 ; abcd abad)
994(81 ; abcd abcd) 995(82 ; abad aacd) 996(82 ; abad abcd) 997(82 ; abcd aacd)

998(82; abcd abcd) 999(83; abad abad) 1000(83; abad abcd) 1001(83; abcd abad)
1002(83; abcd abcd) 1003(84; abad abcd) 1004(84; abcd abcd) 1005(85; abad abad)
1006(85; abad abcd) 1007(85; abad bbbd) 1008(85; abcd abad) 1009(85; abcd abcd)
1010(85; abcd bbbd) 1011(86; abcd aaad) 1012(86; abcd aacd) 1013(86; abcd abad)
1014(86; abcd abcd) 1015(87; abcd aacd) 1016(87; abcd abcd) 1017(88; abcd abad)
1018(88; abcd abcd) 1019(89; abcd abcd) 1020(90; abcd abad) 1021(90; abcd abcd)
1022(90; abcd bbbd)

XVI $\mathbf{a}<\begin{matrix}(\mathbf{b},\mathbf{c})_i\\ \mathbf{d}\\ \mathbf{e}\end{matrix}$

1023(1) 1024$^{\times}$(2) 1025$^{\times}$(3)

XVII $\overbrace{(\mathbf{a},\mathbf{b})<\begin{matrix}(\mathbf{c})\\ (\mathbf{d})\\ \mathbf{e}\end{matrix}}^{i}$

1026(94; abaa aaaa) 1027(94; abba bbbb) 1028(95; abbb aaaa) 1029(95; abbb bbbb)
1030$^{\times}$(97; aaaa aaaa) 1031(97; aaaa abaa) 1032*(98; abaa aaaa) 1033*(98; abba aaaa)
1034$^{\times}$(101; aaaa aaaa) 1035$^{\times}$(102; abaa abaa)

XVIII $\mathbf{a}<\begin{matrix}\mathbf{b}\text{-}(\mathbf{c},\mathbf{d})_i\\ \mathbf{e}\end{matrix}$

1036(1) 1037$^{\times}$(2) 1038$^{\times}$(3)

XIX $\mathbf{a}<\begin{matrix}(\mathbf{b},\mathbf{c})_i\\ \mathbf{d}\text{-}\mathbf{e}\end{matrix}$

1039(1) 1040$^{\times}$(2) 1041$^{\times}$(3)

XX $\mathbf{a}<\begin{matrix}\overbrace{(\mathbf{b},\mathbf{c})\text{-}(\mathbf{d})}^{i}\\ \mathbf{e}\end{matrix}$

1042(11) 1043(12) 1044$^{\times}$(13) 1045(14) 1046$^{\times}$(15) 1047$^{\times}$(16)

XXI $\overbrace{(\mathbf{a},\mathbf{b})<\begin{matrix}(\mathbf{c})\text{-}(\mathbf{d})\\ \mathbf{e}\end{matrix}}^{i}$

1048(112; abaa aaaa) 1049(113; abbb aaaa) 1050(113; abbb abbb) 1051(113; abbb bbbb)
1052(114; abba aaaa) 1053(114; abbb bbbb) 1054$^{\times}$(115; aaaa aaaa) 1055(115; aaaa abaa)
1056$^{\times}$(115; abaa abaa) 1057(116; aaaa aaaa) 1058(116; abaa aaaa) 1059(116; abab aaaa)
1060$^{\times}$(117; aaaa aaaa) 1061(118; aaaa aaaa) 1062(118; abaa aaaa) 1063(118; abbb aaaa)
1064(119; aaaa aaaa) 1065$^{\times}$(120; aaaa aaaa) 1066$^{\times}$(121; abaa abaa)

XXII $\mathbf{a}<\begin{matrix}\mathbf{b}\\ \mathbf{c}\end{matrix}>(\mathbf{d},\mathbf{e})_i$

1067(1) 1068$^{\times}$(2) 1069$^{\times}$(3)

XXIII $\mathbf{a}<\overbrace{\begin{matrix}(\mathbf{b},\mathbf{c})\\ \mathbf{e}\end{matrix}>(\mathbf{d})}^{i}$

1070(11) 1071(12) 1072$^{\times}$(13) 1073(14) 1074$^{\times}$(15) 1075$^{\times}$(16)

XXIV $\overbrace{(\mathbf{a},\mathbf{b})<\begin{matrix}(\mathbf{c})\\ (\mathbf{d})\end{matrix}}^{i}>\mathbf{e}$

1076(94; abcd abcd) 1077(95; abcd abcd) 1078(95; abcd bbcd) 1079(96; abcd abcd)
1080$^{\times}$(97; aacd aacd) 1081(97; abcd aacd) 1082$^{\times}$(97; abcd abcd) 1083(98; aacd abcd)
1084(98; abcd abcd) 1085*(99; abcd abcd) 1086*(100; abcd abcd) 1087$^{\times}$(101; abcd abcd)
1088$^{\times}$(102; abcd abcd)

XXV $\mathbf{a}\text{-}\mathbf{b}<\begin{matrix}(\mathbf{c},\mathbf{d})_i\\ \mathbf{e}\end{matrix}$

1089(1) 1090$^{\times}$(2) 1091$^{\times}$(3)

XXVI $a\text{-}(\overbrace{b, c)<^{(d)}_{(e)}}^{i}$

1092(94) 1093(95) 1094(96) 1095$^{\times}$(97) 1096(98) 1097*(99) 1098*(100) 1099$^{\times}$(101) 1100$^{\times}$(102)

XXVII $\overbrace{(a, b)\text{-}(c)<^{(d)}_{e}}^{i}$

1101(112; abcc abcc) 1102(113; abcc abcc) 1103(113; abcc bbcc) 1104$^{\times}$(115; aacc aacc)
1105(115; abcc aacc) 1106$^{\times}$(115; abcc abcc) 1107*(116; abcc aacc) 1108(118; abcc abcc)
1109(118; aacc abcc) 1110$^{\times}$(120; abcc abcc) 1111$^{\times}$(121; abcc abcc)

XXVIII $a\text{-}b\text{-}c\text{-}(d, e)_i$

1112(103) 1113$^{\times}$(104) 1114$^{\times}$(105)

XXIX $a\text{-}(\overbrace{b)\text{-}(c, d)\text{-}(e}^{i})$

1115(106) 1116(107) 1117$^{\times}$(108) 1118(109) 1119$^{\times}$(110) 1120$^{\times}$(111)

XXX $a\text{-}(\overbrace{b, c)\text{-}(d)\text{-}(e}^{i})$

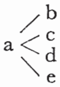
1121(112) 1122(113) 1123(114) 1124$^{\times}$(115) 1125(116) 1126$^{\times}$(117) 1127(118) 1128(119)
1129$^{\times}$(120) 1130$^{\times}$(121)

XXXI $(\overbrace{a, b)\text{-}(c)\text{-}(d}^{i})\text{-}e$

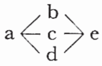
1131(112; abcd abcd) 1132(113; abcd abcd) 1133(113; abcd bbcd) 1134(114; abcd abcd)
1135$^{\times}$(115; aacd aacd) 1136(115; abcd aacd) 1137$^{\times}$(115; abcd abcd) 1138(116; aacd abcd)
1139(116; abcd abcd) 1140$^{\times}$(117; abcd abcd) 1141(118; aacd abcd) 1142(118; abcd abcd)
1143(119; abcd abcd) 1144$^{\times}$(120; abcd abcd) 1145$^{\times}$(121; abcd abcd)

XXXII Semilattice

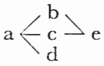
1146$^{\times}$



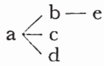
1147$^{\times}$



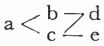
1148$^{\times}$



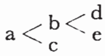
1149$^{\times}$



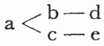
1150$^{\times}$



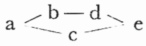
1151$^{\times}$



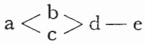
1152$^{\times}$



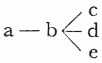
1153$^{\times}$



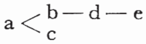
1154$^{\times}$



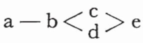
1155$^{\times}$



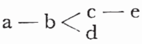
1156$^{\times}$



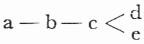
1157$^{\times}$



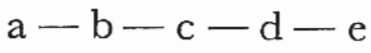
1158$^{\times}$



1159$^{\times}$



1160$^{\times}$



List 7 Motzkin's Numbers for Our Numbers

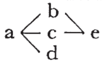
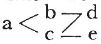
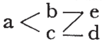
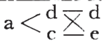
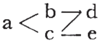
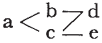
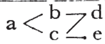
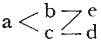
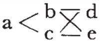
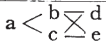
	0	1	2	3	4	5	6	7	8	9
0		0994	0654	0655	0000	0005	0034	0019	0037	0020
10	0049	0047	0017	0048	0038	0054	0018	0001	0003	0004
20	0007	0008	0025	0117	0120	0123	0126	0203	0015	0016
30	0033	0036	0046	0021	0071	0022	0076	0039	0041	0023
40	0077	0052	0024	0078	0050	0053	0072	0040	0081	0051
50	0042	0082	0084	0132	0134	0115	0136	0198	0118	0121
60	0145	0124	0133	0141	0135	0147	0116	0137	0199	0119
70	0138	0140	0122	0143	0146	0200	0139	0142	0201	0125
80	0144	0148	0202	0221	0223	0226	0224	0227	0228	0238
90	0240	0244	0247	0248	0249	0253	0127	0995	1062	1119
100	1152	0996	1064	1063	1120	0998	1121	1123	0999	1122
110	1003	1129	1124	1004	1125	1009	1065	1126	1011	1127
120	1012	1128	1005	1130	1109	1144	1153	1154	1110	1155
130	1149	1157	1158	1148	1156	1159	0404	0580	0660	1075
140	1076	1143	0114	0197	0391	0392	0574	0578	0403	0075
150	0155	0151	0083	0251	0239	0243	0252	0014	0112	0045
160	0241	0152	0153	0225	0250	0154	0935	0968	0834	0835
170	0837	0838	0843	0844	0849	0851	0853	0845	0936	0938
180	0962	0963	0982	0972	0984	0990	0989	0872	0887	0969
190	0842	0852	0870	0810	0811	0961	0827	0986	0779	0937
200	0780	0939	0813	0971	0973	0825	0983	0985	0831	0992

	0	1	2	3	4	5	6	7	8	9
210	0785	0786	0944	0812	0970	0313	0383	0674	0370	0294
220	0355	0661	0295	0356	0662	0321	0372	0675	0330	0379
230	0677	0337	0386	0678	0298	0371	0665	0359	1057	1044
240	1045	1058	1099	1060	1141	1061	1048	0766	0767	0823
250	0824	0981	0772	0828	0829	0987	0760	0762	0814	0974
260	0773	0830	0832	0988	0991	0064	0100	0091	0213	0208
270	0101	0583	0108	0219	0589	0593	0596	0093	0089	0098
280	0107	0218	0573	0588	0062	0065	0087	0096	0105	0131
290	0150	0211	0206	0216	0581	0586	0594	0069	0092	0102
300	0109	0110	0215	0210	0220	0585	0590	0597	0598	0653
310	0234	0601	0232	0235	0246	0599	0236	0602	1017	1023
320	1036	1022	1015	1020	1034	1019	1024	1070	1037	1096
330	1097	1103	1102	1089	1090	1098	1104	1105	1106	1136
340	1117	1115	1118	0284	0287	0351	0325	0378	0399	0621
350	0625	0638	0630	0644	0650	0291	0352	0335	0384	0400
360	0626	0640	0632	0645	0651	0281	0348	0322	0373	0397
370	0619	0628	0636	0642	0648	0292	0353	0336	0339	0385
380	0401	0627	0641	0633	0634	0646	0652	0732	0774	0826
390	0833	0993	1086	1087	1107	0002	0006	0009	0011	0010
400	0012	0013	0027	0029	0158	0031	0162	0026	0028	0032
410	0156	0159	0160	0157	0405	0407	0408	0163	0410	0161
420	0164	0409	0411	0406	0432	0030	0035	0043	0044	0165
430	0166	0167	0168	0412	0413	0433	0056	0073	0074	0079
440	0080	0436	0437	0438	0439	0057	0149	0204	0440	0128
450	0458	0129	0459	0222	0468	0230	0471	0245	0474	0475
460	0476	0130	0460	0997	1000	1001	1002	1006	1007	1008
470	1025	1026	1027	1066	1068	1071	1073	1069	1072	1074
480	1077	1078	1079	1131	1132	1133	1134	1137	1135	1138
490	1139	1140	1142	1010	1028	1067	1080	1111	1112	1113
500	1145	1146	1147	1151	1150	0273	0293	0346	0354	0334
510	0382	0342	0390	0396	0402	0563	0564	0567	0568	0565
520	0569	0566	0570	0571	0572	0575	0576	0577	0579	0659
530	0656	0657	0658	1088	1108	0061	0086	0095	0104	0070
540	0189	0103	0111	0113	0192	0193	0188	0094	0194	0190
550	0191	0455	0196	0425	0427	0429	0426	0428	0430	0431
560	0449	0451	0453	0195	0450	0452	0454	0456	0457	0231
570	0237	0242	0472	0473	0270	0348	0347	0331	0338	0387
580	0272	0345	0381	0393	0394	0333	0340	0388	0549	0552
590	0550	0553	0551	0554	0741	0749	0738	0752	0736	0750
600	0769	0710	0697	0698	0711	0716	0721	0701	0692	0755
610	0728	0693	0758	0715	0694	0768	0729	0719	0712	0726
620	0753	0689	0720	0722	0730	0770	0695	0259	0288	0278
630	0498	0267	0537	0538	0541	0266	0525	0528	0605	0257
640	0276	0279	0495	0496	0264	0523	0526	0603	0261	0289
650	0508	0268	0539	0542	0606	1041	1039	1042	1084	0687
660	0696	0718	0724	0725	0731	0771	0055	0058	0059	0060
670	0170	0169	0171	0414	0415	0434	0085	0181	0182	0175
680	0176	0177	0178	0185	0441	0186	0442	0418	0419	0422
690	0423	0446	0447	0461	0462	0463	0464	0229	0469	1013
700	1014	1029	1031	1032	1081	1082	1083	1085	1114	0256
710	0482	0484	0483	0485	0486	0263	0290	0344	0515	0516
720	0517	0518	0519	0520	0262	0332	0509	0510	0514	0269
730	0380	0540	0548	0543	0271	0395	0555	0557	0556	0558
740	0559	0923	0910	0911	0924	0929	0932	0914	0905	0906
750	0928	0907	0930	0925	0902	0931	0933	0908	0859	0867
760	0866	0890	0857	0864	0888	0861	0868	0891	0955	0951
770	0958	0967	0900	0909	0934	0836	0839	0840	0841	0846
780	0847	0848	0873	0874	0875	0850	0876	0940	0942	0946
790	0964	0965	0966	0978	0856	0863	0862	0869	0871	0882
800	0884	0883	0885	0886	0956	0805	0797	0957	0806	0799
810	0952	0803	0795	0959	0807	0800	0309	0318	0366	0319
820	0610	0616	0367	0617	0671	0307	0316	0364	0608	0614
830	0669	0311	0320	0368	0611	0618	0672	1054	1052	1055
840	0793	0801	0802	0808	0794	0809	0950	0960	0815	0819
850	0816	0820	0975	0976	0941	0945	0943	0947	0783	0784
860	0781	0782	0789	0790	0787	0788	0977	0979	0821	0817
870	0305	0312	0314	0315	0326	0327	0504	0505	0374	0375
880	0544	0545	0363	0369	0535	0536	0668	0673	0296	0297
890	0299	0300	0301	0302	0303	0499	0500	0501	0357	0358
900	0360	0361	0529	0530	0531	0532	0663	0664	0666	0323

	0	1	2	3	4	5	6	7	8	9
910	0328	0511	0376	0546	0676	1051	1056	1046	1047	1049
920	1059	1101	0761	0763	0764	0765	0818	0822	0980	0063
930	0066	0067	0068	0173	0172	0174	0416	0417	0435	0088
940	0090	0183	0184	0179	0180	0205	0443	0097	0099	0420
950	0421	0444	0445	0106	0187	0424	0448	0207	0209	0465
960	0212	0214	0466	0217	0467	0582	0584	0591	0592	0587
970	0595	0233	0470	0600	1016	1018	1030	1021	1033	1035
980	1091	1092	1094	1093	1095	1100	1116	0282	0285	0283
990	0286	0491	0492	0493	0494	0349	0521	0350	0522	0324
1000	0329	0512	0513	0377	0547	0341	0389	0398	0560	0561
1010	0562	0620	0622	0623	0624	0637	0639	0629	0631	0643
1020	0635	0647	0649	0683	0681	0684	0685	0717	0686	0680
1030	0254	0255	0479	0480	0477	1038	0707	0705	0708	0740
1040	0737	0751	0704	0709	0699	0700	0702	0713	0714	0727
1050	0754	0690	0723	0691	0258	0277	0497	0260	0488	0489
1060	0478	0265	0527	0524	0481	0604	1040	0777	0775	0778
1070	0747	0748	0742	0743	0744	0756	0757	0735	0734	0759
1080	0274	0275	0487	0280	0490	0612	0613	0607	1043	0896
1090	0894	0897	0898	0893	0899	0854	0855	0879	0880	0877
1100	0948	0949	0792	0791	0304	0306	0502	0506	0533	0362
1110	0667	1050	0920	0918	0921	0922	0917	0912	0913	0915
1120	0926	0927	0903	0904	0858	0860	0878	0865	0881	0889
1130	0953	0954	0798	0796	0804	0308	0310	0503	0317	0507
1140	0609	0365	0534	0615	0670	1053	0679	0733	0688	0682
1150	0745	0703	0739	0746	0776	0892	0706	0901	0895	0916
1160	0919									

List 8 Structure of u.z. and c.d.u.

u.z.

left ordering	right ordering	No.	left ordering	right ordering	No.
$a<\begin{smallmatrix}b\\c\\d\end{smallmatrix}$	$a<\begin{smallmatrix}b\\c\\d\end{smallmatrix}$	4		$a<\begin{smallmatrix}b-d\\c-e\end{smallmatrix}$	5
$a<\begin{smallmatrix}b-d\\c-e\end{smallmatrix}$	$a<\begin{smallmatrix}b-d\\c-e\end{smallmatrix}$	6	$a<\begin{smallmatrix}b-d\\c-e\end{smallmatrix}$	$a<\begin{smallmatrix}b-e\\c-d\end{smallmatrix}$	7
$a<\begin{smallmatrix}b-d\\c-e\end{smallmatrix}$		8	$a<\begin{smallmatrix}b-d\\c-e\end{smallmatrix}$		9
$a<\begin{smallmatrix}b-d\\c-e\end{smallmatrix}$		10			11, 12
		13, 14			15
$a<\begin{smallmatrix}b-d-e\\c\end{smallmatrix}$	$a<\begin{smallmatrix}b-d-e\\c\end{smallmatrix}$	16	$a<\begin{smallmatrix}b-e\\c\\d\end{smallmatrix}$	$a<\begin{smallmatrix}b-e\\c\\d\end{smallmatrix}$	17
$a<\begin{smallmatrix}b-e\\c\\d\end{smallmatrix}$	$a<\begin{smallmatrix}b-d\\c\\e\end{smallmatrix}$	18	$a<\begin{smallmatrix}b-e\\c\\d\end{smallmatrix}$	$a<\begin{smallmatrix}b<\begin{smallmatrix}d\\e\end{smallmatrix}\\c\end{smallmatrix}$	19
$a<\begin{smallmatrix}b-e\\c\\d\end{smallmatrix}$	$a<\begin{smallmatrix}b<\begin{smallmatrix}c\\d\end{smallmatrix}\\e\end{smallmatrix}$	20	$a<\begin{smallmatrix}b-e\\c\\d\end{smallmatrix}$	$a-b<\begin{smallmatrix}c\\d\\e\end{smallmatrix}$	21
$a<\begin{smallmatrix}b-c-e\\d\end{smallmatrix}$	$a-b<\begin{smallmatrix}c-e\\d\end{smallmatrix}$	22	$a-b<\begin{smallmatrix}c-e\\d\end{smallmatrix}$	$a-b<\begin{smallmatrix}c-e\\d\end{smallmatrix}$	23~26
$a-b-c<\begin{smallmatrix}d\\e\end{smallmatrix}$	$a-b-c<\begin{smallmatrix}d\\e\end{smallmatrix}$	27	$a<\begin{smallmatrix}b<\begin{smallmatrix}d\\e\end{smallmatrix}\\c\end{smallmatrix}$	$a<\begin{smallmatrix}b<\begin{smallmatrix}d\\e\end{smallmatrix}\\c\end{smallmatrix}$	28~32
$a<\begin{smallmatrix}b<\begin{smallmatrix}d\\e\end{smallmatrix}\\c\end{smallmatrix}$	$a<\begin{smallmatrix}b<\begin{smallmatrix}c\\e\end{smallmatrix}\\d\end{smallmatrix}$	33~38	$a<\begin{smallmatrix}b<\begin{smallmatrix}d\\e\end{smallmatrix}\\c\end{smallmatrix}$	$a-b<\begin{smallmatrix}c\\d\\e\end{smallmatrix}$	39~52
$a-b<\begin{smallmatrix}c\\d\\e\end{smallmatrix}$	$a-b<\begin{smallmatrix}c\\d\\e\end{smallmatrix}$	53~95	$a-b-c-d-e$	$a-b-c-d-e$	96

c. d. u.

greatest c-decomp.	No.	greatest c-decomp.	No.
2_2	136～139	4_3	140
5_3	141	2_3	142～154
3_3	155, 156	3_4	157, 158
4_4	159, 160	7_4	161
8_4	162	9_4	163
11_4	164	12_4	165

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AUFGABEN BETREFFEND DAS IRRFAHRTPROBLEM

Von

Yoshikatsu WATANABE

(Eingegangen am 30 September, 1955)

Pólya handelt, in seiner interessante Arbeit „Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz", Math. Ann. 84 (1921), S. 141–160, von der Wahrscheinlichkeit Ω_n, dafür, daß der im d–dimensionalen Geradennetz herumwandernde Punkt, welcher zur Zeit $t=0$ im Anfangspunkt des Koordinatensystems sein Irrfahrt beginnt mit der Geschwindigkeit 1 und jedem Zeitpunkt $t=0, 1, 2, \cdots$ mit der Wahrscheinlichkeit $1/2d$ für eine der d–Koordinatenaxen parallelen $2d$ Richtungen sich entscheidet, innerhalb der Zeitspann $0<t\leqq 2n$ mindestens einmal wieder den Anfangspunkt zurückkehrt. Offenbar wächst Ω_n mit n, und zwar strebt es gegen 1 für $n\rightarrow\infty$ bei $d=1$ oder $d=2$, während dagegen bei $d\geqq 3$, gegen eine Bruchzahl <1, und also der Wanderer gewiß ins Unendliche entgeht. Also stellt der wesentliche Unterschied sich beim Übergang von der Ebene zu dem dreidimensionalen Raum ein. Jetzt betrachte ich drei betreffende Aufgaben: Wie ist es beschaffen, **1.** wenn das ebene Straßennetz, an statt von Rechtkreuzung, sich sechswinklig konstruirt, so daß nun eine vom Zufall geleitete Entscheidung unter 6 gleichmöglichen Richtungen fällt?[1] **2.** Oder, für zwei parallele ebene ähnliche Straßennetze, die durch Elevator in jedem Kreuz kombiniert werden, so daß 5 Richtungen gleichmöglich sind? **3.** Schließlich für den Fall, daß außer Bewegungen an $2d$–gleichmöglichen Richtungen noch die Ruhesitzung in demselben Punkt während nächster Zeiteinheit ebenso auch wahrscheinlich ist?

§ 1.

Wir stellen uns ein hexagonales (dreieckiges) Straßennetz, und dreilineare α–, β– und γ– Koordinatenachsen (Fig 1), O, die Ausgangsstelle des Wanderers als Anfangspunkt des Koordinatensystems vor. Die gewöhnlichen obliquen Koordinaten jedes Knotenpunktes M, d.h. Punktes mit ganzzahligen Koordinaten, mögen

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[1] Dies wird schon von A. Dvoretzky und P. Erdös gefragt und ohne Beweis antwortet: Some Problems on Random Walk in Space, Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, held at the statistical laboratory, Department of Mathematics, University of California, 1951, p. 367. Auch für ähnliches hexagonales Straßennetz aber mit nur einseitigem Wege ist gelöst von R. Sherman Lehman: A Problem on Random Walk, ditto, p. 263.

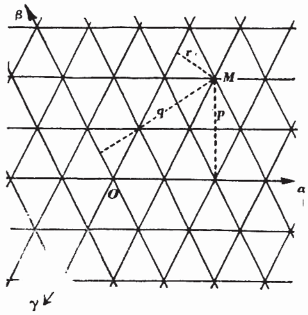

Fig.1.

entweder durch $(\alpha_\gamma, \beta_\gamma)$ in bezug auf α–, β–Achsen, dabei γ–Achse verlassen ist, oder gleicherweise durch $(\beta_\alpha, \gamma_\alpha)$, oder abermals durch $(\gamma_\beta, \alpha_\beta)$ angegeben werden. Wenn der Wanderer an jedem neuen Knotenpunkt angelangt, soll er sich mit der Wahrscheinlichkeit 1/6 für eine der möglichen 6 Richtungen entscheiden.

Es seien p, q und r die Längen der auf je α–, β–, und γ–Achse gafällten Lote, deren Vorzeichen folgendermaßen bestimmt werden soll.

Angenommen die Reihenfolge $\alpha\beta\gamma\alpha$, halbiert z.B. volle α–Achse die ganze Ebene zu zwei Halbebenen, deren eine zwar positive β–Achse enthält und hier $p>0$ sei, während die andere aber negative β–Achse enthält und dort $p<0$, usw. Damit erhält man

$$\frac{2}{\sqrt{3}}p=\beta_\gamma=-\gamma_\beta,\quad \frac{2}{\sqrt{3}}q=\gamma_\alpha=-\alpha_\gamma,\quad \frac{2}{\sqrt{3}}r=\alpha_\beta=-\beta_\alpha.$$

Es ist aber $p+q+r=0$ und mithin $\alpha_\beta+\beta_\gamma+\gamma_\alpha=0$. Also, obgleich dreilineare Koordinaten eines Punktes durch $(\alpha_\beta, \beta_\gamma, \gamma_\alpha)$—oder, kurz als (α, β, γ)—gegeben werden mögen, können sie durch je zwei von α, β, γ, schon urteilt werden. Danach sind die zu (α, β, γ) nächstliegenden Knotenpunkte nicht $(\alpha\pm1, \beta, \gamma)$, $(\alpha, \beta\pm1, \gamma)$, $\cdots$, sondern eben

$$(1)\qquad (\alpha\pm1, \beta, \gamma\mp1),\quad (\alpha, \beta\pm1, \gamma\mp1)\quad \text{und}\quad (\alpha\pm1, \beta\mp1, \gamma);$$

insbesondere sind die Nachbarspunkte zu $O(0, 0, 0)$ zwar $(\pm1, 0, \mp1)$, $(0, \pm1, \mp1)$ und $(\pm1, \mp1, 0)$.

Wir betrachten einen Wanderer, der zur Zeit $t=0$ von O aus anbrechend auf der oben beschreibenen Weise im hexagonalen Straßennetz herumirrt. Die Wahrscheinlichkeit dafür, daß im Zeitpunkt $t=m$ der Wanderer im Knotenpunkt (α, β, γ) sich findet, sei mit $P_m(\alpha, \beta, \gamma)$ bezeichnet. Dann ist $6^m P_m(\alpha, \beta, \gamma)$ die Anzahl sämtlicher Zickzackwege im Netz, die aus m Stücken von der Länge 1 zusammengesetzt ist und vom Punkt $(0, 0, 0)$ zum Punkt (α, β, γ) führt. Daher ist

$$(2)\qquad 6^m P_m(\alpha, \beta, \gamma)=\sum_{\nu=1}^{6} 6^{m-1} P_{m-1}(\alpha'_\nu, \beta'_\nu, \gamma'_\nu),$$

die Summe über die zu (α, β, γ) nächstliegenden 6 Punkte $(\alpha'_\nu, \beta'_\nu, \gamma'_\nu)$ erstreckt. Die Wahrscheinlichkeit läßt sich mittels dreifaches Integrals durch

$$(3)\qquad P_m(\alpha,\beta,\gamma)\equiv P_m(\alpha,\beta)$$
$$=\frac{1}{(2\pi)^3}\iiint\left(\frac{\cos(\varphi_1-\varphi_3)+\cos(\varphi_2-\varphi_3)+\cos(\varphi_1-\varphi_2)}{3}\right)^m$$
$$\exp\{i\alpha(\varphi_1-\varphi_3)+i\beta(\varphi_2-\varphi_3)\}\,d\varphi_1 d\varphi_2 d\varphi_3$$

darstellen, wobei das Integrationsgebiet aus einem Würfel $W: 0\leqq\varphi_k\leqq 2\pi$, $(k=1, 2, 3)$ besteht.

Beweis. Bei $m=0$ ist es klar, da nach Anfangsbedingung $P_0(0, 0)=1$ so wie $P_0(\alpha, \beta)=0$ für $(\alpha, \beta)\neq(0, 0)$ sind. Wenn (3) bei $t=m$ gilt, so ist auch (3) richtig bei $t=m+1$. Denn, wegen (2) entsteht die Rekursionsformel $P_{m+1}(\alpha, \beta, \gamma)=\frac{1}{6}\sum P_m(\alpha', \beta', \gamma')$, worin für $(\alpha', \beta', \gamma')$ die Werte (1) eingesetzt, sodann nach (3) bei $t=m$ berechnet, die resultierende Summe mit (3) für $t=m+1$ übereinkommt. Also ist der Beweis vollständig geleistet. Insbesondere gilt

$$(4)\qquad P_m(0, 0, 0)\equiv P_m(0, 0)$$
$$=\frac{1}{(2\pi)^3}\iiint_W\left[\frac{1}{3}\Big(\cos(\varphi_1-\varphi_3)+\cos(\varphi_2-\varphi_3)+\cos(\varphi_1-\varphi_2)\Big)\right]^m d\varphi_1 d\varphi_2 d\varphi_3.$$

Für $m=1$ ist notwendig $P_m(0, 0)=0$, jedoch wird für jedes ganzzahliges $m\geqq 2$ zwar $P_m(0, 0)>0$, was bei gerades m aus (4) klar ist, während bei ungerades m wegen (2) auch giltig.[2]

Wir werden nun $P_m(0, 0)$ für großes m abschätzen, und dazu den Integrand in (3) nach absoluten Betrage

$$(5)\qquad \left|\frac{1}{3}\Big(\cos(\varphi_1-\varphi_3)+\cos(\varphi_2-\varphi_3)+\cos(\varphi_1-\varphi_2)\Big)\right|^m$$

erwögen. Dies erreicht sein Maximum 1 längs Diagonales D des Würfels $W: \varphi_1=\varphi_2=\varphi_3$. Überdies erlangt (5) auch sein Maximum 1 bei am Ende von D nicht liegenden 6 Sheiteln. Es seien V das rechtwinklige Prism, D als Achse mit Quadratbasis der Kantenlänge $2a$, und $W_j (j=1, 2, \cdots, 6)$, die kleine Würfel mit Kantenlänge a, je habende gemeine Scheitel mit W. Im Bereich $U=W-V-\Sigma W_j$ hat der Ausdruck (5) eine obere Grenz $\rho(<1)$, und demnach geht das über U erstreckte Integral J unten $O(\rho^m)=0\left(\frac{1}{n^N}\right)$ mit beliebig großes N bei $m\to\infty$ nieder. Um die über V und W_j erstreckte Integrale zu berechnen, transformiere man die rechtwinklige Koordinaten $(\varphi_1, \varphi_2, \varphi_3)$ in neue rechtwinklige Koordinaten (ψ_1, ψ_2, ψ_3), wie etwa

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[2] Es ist anschaulich ersichtlich, daß z.B. $P_m(1, -1, 0)>0$ für jedes $m>1$ ist. Denn, es bräche der Wanderer zur Zeit $t=0$ vom Punkt $(1, -1, 0)$ auf, und irre nur längs des Perimeters des Hexagon mit Mittelpunkt $O(0, 0, 0)$ bis zur Zeit $t=m-1$ herum, alsdann ging zu O. Der umkehrte Zigzagweg gilt eben denjenigen der ausgehend von O und zur Zeit $t=m$ am Punkt $(1, -1, 0)$ erlangt, und also ist gewiß $P_m(1, -1, 0)>0$. Daraus aber folgt, daß $P_m(0, 0, 0)>0$ für jedes $m\geqq 2$.

$$\begin{aligned} \varphi_1 &= -\frac{1}{\sqrt{2}}\psi_1+\frac{1}{\sqrt{6}}\psi_2+\frac{1}{\sqrt{3}}\psi_3, \\ \varphi_2 &= \qquad\qquad -\frac{2}{\sqrt{6}}\psi_2+\frac{1}{\sqrt{3}}\psi_3, \qquad \text{mit Jakobien} = \frac{\partial(\varphi_1,\varphi_2,\varphi_3)}{\partial(\psi_1,\psi_2,\psi_3)} = 1. \\ \varphi_3 &= \frac{1}{\sqrt{2}}\psi_1+\frac{1}{\sqrt{6}}\psi_2+\frac{1}{\sqrt{3}}\psi_3, \end{aligned}$$

Infolgedessen ergibt sich

$$\begin{aligned} J_V &= \frac{1}{(2\pi)^3}\iiint\limits_V \left[\frac{\cos(\varphi_1-\varphi_2)+\cos(\varphi_2-\varphi_3)+\cos(\varphi_1-\varphi_2)}{3}\right]^m d\varphi_1 d\varphi_2 d\varphi_3 \\ &= \frac{1}{8\pi^3}\int_0^{2\pi\sqrt{3}} d\psi_3 \int_{-a}^{a}\int_{-a}^{a}\left[\frac{1}{3}\left(\cos\frac{2\psi_1}{\sqrt{2}}+\cos\frac{\psi_1+\sqrt{3}\psi_2}{\sqrt{2}}+\cos\frac{\psi_1-\sqrt{3}\psi_2}{\sqrt{2}}\right)\right]^m d\psi_1 d\psi_2. \end{aligned}$$

Dabei wird etwaig am Ende sich erhebender Fehler $=O(a^3)$ und folglich $\rightarrow 0$ bei $a \rightarrow 0$. Setzt man ferner $\psi_k=t_k/\sqrt{m}$ $(k=1, 2)$, so gilt asymptotisch bei genügend großem m

$$\begin{aligned} J_V &= \frac{\sqrt{3}}{4\pi^2}\int_{-\sqrt{m}a}^{\sqrt{m}a}\int_{-\sqrt{m}a}^{\sqrt{m}a}\left[\frac{1}{3}\left(\cos\frac{2t_1}{\sqrt{2m}}+\cos\frac{t_1+\sqrt{3}t_2}{\sqrt{2m}}+\cos\frac{t_1-\sqrt{3}t_2}{\sqrt{2m}}\right)\right]^m \frac{dt_1 dt_2}{m} \\ &\cong \frac{\sqrt{3}}{4m\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left\{-\frac{t_1^2}{2}-\frac{t_2^2}{2}\right\}dt_1\,dt_2 \cong \frac{\sqrt{3}}{2m\pi}.^{3)} \end{aligned}$$

Ich habe noch die Beitragen aus W_j zu abschätzen. Z.B. mache ich für W_1: $2\pi-a<\varphi<2\pi$, $0<\varphi_2<a$, $0<\varphi_3<a$, die Transformation

$$\xi_1 = 2\pi+\sqrt{2}\psi_1, \quad \xi_2 = -2\pi+\sqrt{6}\psi_2, \quad \xi_3 = -2\pi+\sqrt{3}\psi_3.$$

so ergeben sich $0<\xi_1<2a$, $-3a<\xi_2<a$, $-a<\xi_3<2a$, mti Jakobien$=\dfrac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(\psi_1, \psi_2, \psi_3)}=6$, und gilt

$$J_{W_1}=\frac{1}{48\pi^3}\int_{-a}^{2a}\int_{-3a}^{a}\int_0^{2a}\left[\frac{1}{3}\left(\cos\xi_1+\cos\frac{\xi_1+\xi_2}{2}+\cos\frac{\xi_1-\xi_2}{2}\right)\right]^m d\xi_1 d\xi_2 d\xi_3,$$

was durch nochmalige Ersetzung $\xi_k=t_k/\sqrt{m}$ $(k=1, 2)$ bei $m\rightarrow\infty$

$$\cong \frac{a}{16\pi^3 m}\int_{-\infty}^{\infty}\int_0^{\infty}\exp\left\{-\frac{3t_1^2}{4}-\frac{t_2^2}{4}\right\}dt_1\,dt_2 \cong \frac{a}{8\sqrt{3}\pi^2 m}$$

wird, und dgl. für andre W_j. Da aber a beliebig klein gemacht werden kann.[4] so kommt

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[3] Vgl. dazu Y. Watanabe und Y. Ichijô, Über die Laplacesche asymptotische Formel für das Integrale von Potenz mit großen Indexe, dieses Journ. S. 63.

[4] Wir wollen am Ende den Schluß ziehen, daß $\sum P_m(0,0,0)=\infty$, wofür Abschätzung (6) bereits genug thut, geschweige denn dazu Hinzusetzung der nicht negativen J_{W_j}, wie beschaffen es seien, je mehr divergieren läßt.

$$(6) \qquad P_m(0,\ 0,\ 0) \simeq \frac{\sqrt{3}}{2m\pi}\,.$$

Obwohl dies etwas verschieden von Pólyasche Resultaten $P_{2n}(0,\ 0)=\frac{1}{n\pi}$, $P_{2n+1}(0,0)=0$ scheint, doch divergiert gleichviel gegenwärtige Reihe $\sum P_m(0,\ 0)$ und stimmt weiterer Schluß mit Pólyaschem überein.[5] Seien nämlich, Ω_n die Wahrscheinlichkeit dafür, daß der Wanderer innerhalb der Zeitspann $0<t\leqq n$ den Anfangspunkt zurückkehrt, und ω_m bei $0<m\leqq n$, die Wahrscheinlichkeiten dafür, daß der Wanderer erstemals in $t=m$ den Anfangspunkt zurückkehrt, so gilt

$$(7) \qquad \Omega_n = \sum_{m=1}^{n} \omega_m,$$

während $P_m(0,\ 0.\ 0)$ nichts anders als der Koeffizient von z^m in die Entwickelung des Bruches $(1-\omega_1 z-\omega_2 z^2-\cdots)^{-1}$ ist, und deswegen lautet

$$(8) \qquad f(z) \equiv 1-\sum_{m=1}^{\infty}\omega_m z^m = \frac{1}{\sum P_m(0,\ 0,\ 0)z^m}\,.$$

Daraus aber folgt $f(1-0)=0$ und $\sum_{m=1}^{\infty}\omega_m=1$, wenn $\sum P_m(0,\ 0,\ 0)=\infty$.

§ 2.

Ich lege mir zwei parallele Kreuzförmige Pólyasche Straßennetz über, deren ein auf Ebene $z=0$, aber das andere auf Ebene $z=1$ gebaut, und entsprechende Kreuzpunkte durch Elevator verbindet worden sind, so daß an jedem Knotenpunkte 5 Richtungen deren vier horizontale Kreuzen und eine vertikales Auf- oder Abgehen sind, aufs Geratwohl ausgewählt werden können. Für solches Straßennetz ist die betreffende Wahrscheinlichkeit $P_m(x,\ y,\ z)$, die ebenso als (1.3) definiert ist, in folgendermaßen dargestellt:

$$(1) \qquad P_m(x,\ y,\ z) = \frac{1}{(2\pi)^3}\iiint_{-\pi/2}^{3\pi/2}\left(\frac{2\cos\varphi+2\cos\psi+\cos\theta+i\sin\theta}{5}\right)^m \exp\{-ix\varphi-iy\psi-iz\theta\}\,d\varphi\,\alpha\,\psi\,d\theta,$$

wobei x und y irgend ganze Zahlen, aber $z=0$ oder 1 bedeuten. Nach Anfangsbedingung sind $P_0(0,\ 0,\ 0)=1$ und $P_0(x,\ y,\ z)=0$ bei $(x,\ y,\ z)\neq(0,\ 0,\ 0)$, wofür (1) eben richtig bestehen. Allgemein kann (1) mit hilfe der zu (1.2) gleichartigen Rekursionsformel

$$P_m(x,\ y,\ z) = \frac{1}{5}\sum P_{m-1}(x',\ y',\ z')$$

durch vollständige Induktion bewiesen werden.

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[5] Siche Pólya, l.c. S. 156.

Nun will ich $P_m(0, 0, 0)$ für genügend großes m ausrechnen. Bei ungerades m ist ersichtlich

$$P_{2n+1}(0, 0, 0) = 0, \tag{2}$$

während für gerades $m=2n$

$$P_{2n}(0, 0, 0) = \frac{1}{(2\pi)^3} \iiint_{-\pi/2}^{3\pi/2} \left(\frac{2\cos\varphi + 2\cos\psi + \cos\theta + i\sin\theta}{5}\right)^{2n} d\varphi\, d\psi\, d\theta \tag{3}$$

gilt. Da aber die Größe

$$R = \left|\frac{2\cos\varphi + 2\cos\psi + \cos\theta + i\sin\theta}{5}\right|$$

für $\varphi=\psi=\theta=0$ oder π innerhalb des ganzen Integrationsgebietes W ihre Maxmum 1 erreicht, so beträgt

$$R \leqq \rho < 1 \quad \text{in} \quad W - W_0 - W_\pi = U,$$

wobei W_0 und W_π zwei offenen Würfel mit jede Mittelpunkt $(0, 0, 0)$, (π, π, π) von der Kantenlänge $2a$ $(0<a<1)$ bedeuten. Damit wird

$$\left|\iiint_U\right| < \rho^m = 0\left(\frac{1}{m^N}\right)$$

übergezeugt. Anderseits ist der Beitrage aus W_0

$$\iiint_{W_0} = \frac{1}{8\pi^3} \iiint_{-a}^{a} \left[\frac{1}{5}\left(2\cos\frac{t_1}{\sqrt{n}} + 2\cos\frac{t_2}{\sqrt{n}} + \cos\frac{t_3}{\sqrt{n}} + i\sin\frac{t_3}{\sqrt{n}}\right)\right]^{2n} \frac{dt_1 dt_2 dt_3}{n\sqrt{n}}$$

$$(\varphi = t_1/\sqrt{n}\,,\ \psi = t_2/\sqrt{n}\,,\ \theta = t_3/\sqrt{n}\,)$$

$$\cong \frac{1}{8\pi^3 n\sqrt{n}} \iiint_{-\sqrt{n}a}^{\sqrt{n}a} \left[1 - \frac{2t_1^2 + 2t_2^2 + t_3^2}{10n} + \frac{it_3}{5\sqrt{n}} + 0\left(\frac{1}{n}\right)\right]^{2n} dt_1\, dt_2\, dt_3.$$

Darin gilt der Integrand

$$\exp\left[2n \log\left\{1 - \frac{2t_1^2 + 2t_2^2 + t_3^2}{10n} + \frac{it_3}{5\sqrt{n}} + 0\left(\frac{1}{n}\right)\right]$$

$$\cong \exp\left\{-\frac{2t_1^2 + 2t_2^2 + t_3^2}{5} + \frac{2it_3}{5}\sqrt{n} + 0(1)\right\}$$

für $n\to\infty$. Hiermit erhält man

$$\iiint_{W_0} \cong \frac{1}{8\pi^3 n\sqrt{n}} \iint_{-\infty}^{\infty} \exp\left\{-\frac{2t_1^2 + 2t_2^2}{5}\right\} dt_1 dt_2 \int_{-\sqrt{n}a}^{\sqrt{n}a} \exp\left\{-\frac{t_3^2}{5} + \frac{2i}{5} t_3\sqrt{n}\right\} dt_3$$

$$\cong \frac{5}{16\pi^2} \int_{-\sqrt{n}a}^{\sqrt{n}a} \exp\left(-\frac{t_3^2}{5}\right) \left[\cos\frac{2}{5} t_3\sqrt{n} + i\sin\frac{2}{5} t_3\sqrt{n}\right] dt_3,$$

und derselbe gilt für W_π, und daraus folgt

$$(4) \qquad P_{2n}(0,\,0,\,0) \simeq \frac{5}{4\pi^2 n\sqrt{n}} \int_0^{\sqrt{n}a} \exp\left\{-\frac{t_3^2}{5}\right\} \cos\frac{2}{5}\, t_3\sqrt{n}\, dt_3$$
$$= \int_0^\infty - \int_{\sqrt{n}a}^\infty = \text{(i)} - \text{(ii)},$$

worin, wie funktionentheoretisch ermittelt werden kann,

$$\text{(i)} \simeq \left(\frac{5}{n\pi}\right)^{\frac{3}{2}} e^{-\frac{n}{5}} \quad \text{für} \quad n \to \infty.$$

In bezug auf (ii) setze ich $\frac{2}{5} t_3\sqrt{n} = t$, so dann hervorgeht bei $p\pi - \frac{\pi}{2} \leqq \frac{2na}{5} < p\pi + \frac{\pi}{2}$

$$|\text{(ii)}| \leqq \frac{1}{2}\left(\frac{5}{2n\pi}\right)^2 \left| \int_{2na/5}^\infty \exp\left\{-\frac{5t^2}{4n}\right\} \cos t\, dt \right|$$
$$\leqq \frac{1}{2}\left(\frac{5}{2n\pi}\right)^2 \int_{p\pi-\pi/2}^{pn+\pi/2} \exp\left\{-\frac{5t^2}{4n}\right\} |\cos t|\, dt$$
$$< \frac{25}{8n^2\pi} \exp\left\{-\frac{5}{4n}\left(p\pi - \frac{\pi}{2}\right)^2\right\}$$
$$\simeq \frac{25}{8n^2\pi} \exp\left\{-\frac{na^2}{5}\right\} \leqq \frac{25}{8n^2\pi}.$$

Daher

$$(5) \qquad P_{2n}(0,\,0,\,0) < \left(\frac{5}{n\pi}\right)^{\frac{3}{2}} + \frac{25}{8n^2\pi},$$

womit, gewählt passend großes $n_0 = n_0(\varepsilon)$,

$$\sum_{n=n_0}^\infty P_{2n}(0,\,0,\,0) \leqq \sum_{n=n_0}^\infty \left(\frac{5}{n\pi}\right)^{\frac{3}{2}} + \sum_{n=n_0}^\infty \frac{25}{8n^2\pi} < \varepsilon,$$

und damit ist die Konvergenz der Reihe $\sum_{n=1}^\infty P_{2n}(0,0,0)$ schon gesichert worden. Daraus aber folgt, daß die Reihe (1.8) bei $z \to 1-0$ gegen gewißes positives Bruch <1 konvergiert, und folglich aus (1.7) $\lim_{n\to\infty} \Omega_n$ unter gewißem Bruch <1 bleibt. Also tritt wesentlicher Unterschied, was sich beim Übergang von der Ebene zum dreidimensionalen Raum einstellt, schon beim ebenen Straßennetz, das durch nur eine unterirdishe Straße erweitert ist, auf.

§ 3.

Y. Hayashi befragte mir mündlich, ob das Auf- oder Abgehen bei unterirdish erstreckte Straßennetz etwas gleichgültig gegen die Ruhesitzung an demselben Punkt in einfache Straßennetze sei. Diese Frage ist bemerkenswürdig, aber die Beantwortung ist nein.

Es seien q die Wahrscheinlichkeit dafür, daß der auf den d-dimensionalen geradennetz herumwandernde Punkt an jedem ganzzahligen Zeitpunkte $t=m$ sich im Ruhe in denselben Punkt während nächsten Zeiteinheit läßt, und $p/2d$ mit $p=1-q$ die Wahrscheinlichkeit dafür, daß der Punkt sich längs einer der gleichmäßig möglichen $2d$ Richtungen bewegt. Offenbar ist in diesem Falle $P_m(0, \cdots, 0)>0$ für jedes ganze m, also schon verschieden als (2.2). Im allgemeinen gilt

$$(1) \qquad P_m(x_1, x_2, \cdots, x_d) = \frac{1}{(2\pi)^d}\iint\cdots\int\left[\frac{p}{d}\sum_{\nu=1}^{d}(\cos\varphi_\nu + q\right]^m \exp\left[-i\sum_{\nu=1}^{d}x_\nu\varphi_\nu\right]d\varphi_1 d\varphi_2\cdots d\varphi_d,$$

dessen Integral über den Würfel $W: -\alpha\leqq\varphi_\nu\leqq 2\pi-\alpha$ $(\nu=1, 2, \cdots, d)$ erstreckt ist, dabei sei $0<\alpha<\pi$, so daß $O(0, 0, \cdots, 0)$ ein innere Punkt des W ist. Denn, bei $m=0$ ist $P_0(0, 0, \cdots, 0)=1$ nach Anfangsbedingung, und für $(x_1, x_2, \cdots, x_d)\neq(0, 0 \cdots, 0)$ besteht

$$P_0(x_1, \cdots, x_0) = \frac{1}{(2\pi)^d}\iint\cdots\int\exp\{-i\sum x_\nu\varphi_\nu\}\, d\varphi_1 \cdots d\varphi_d = 0.$$

Also ist (1) bei $m=0$ bereits wahr, und aus der Richtigkeit für m folgt sie für $m+1$, da nach der Rekursionsformel

$$P_{m+1}(x_1, \cdots, x_d) = \frac{p}{2d}\sum_{\nu=1}^{d}P_m(x_1, \cdots, x_\nu \pm 1, \cdots, x_d) + q\,P_m(x_1, \cdots, x_d),$$

dessen rechten Seite durch Einsetzung von (1)

$$= \frac{1}{(2\pi)^d}\iint\cdots\int\left(\frac{p}{d}\sum_{\nu=1}^{d}\cos\varphi_\nu + q\right)^m\left[\sum_{\nu=1}^{d}\frac{p}{2d}(e^{i\varphi_\nu}+e^{-i\varphi_\nu})+q\right]$$
$$\exp\left[-i\sum_{\nu}x_\nu\varphi_\nu\right]d\varphi_1 d\varphi_2\cdots d\varphi_d$$
$$= \frac{1}{(2\pi)^d}\iint\cdots\int\left(\frac{p}{d}\sum_{\nu=1}^{d}\cos\varphi_\nu + q\right)^{m+1}\exp\left[-i\sum_{\nu}x_\nu\varphi_\nu\right]d\varphi_1\cdots d\varphi_d$$

liefert. Insbesondere

$$(2) \qquad P_m(0, 0, \cdots, 0) = \frac{1}{(2\pi)^d}\iint\cdots\int\left(\frac{p}{d}\sum_{\nu=1}^{d}\cos\varphi_\nu + q\right)^m d\varphi_1\cdots d\varphi_d,$$

das abschätzt zu werden braucht. Um das Maximum des Integrandes nach absolute Betrage zu ausfinden, betrachte man $Y=\frac{p}{d}\sum_{\nu=1}^{d}\cos\varphi_\nu+q$, und setze Ableitungen $\frac{\partial Y}{\partial\varphi_\nu}=\frac{p}{d}\sin\varphi_\nu=0$ $(\nu=1, 2, \cdots, d)$, so dann werden $\varphi_\nu=0, \pi$ erhält. Tatsächlich erreicht Y den maximale Wert 1 bei alle $\varphi_\nu=0$ während für alle $\varphi_\nu=\pi$ nicht so, da $|q-p/d|<1$ ist, und dgl. wenn $\varphi_\nu=0$, $\varphi_\mu=\pi$ gleichzeitig stattfinden. Somit erhalten wir nur einzige maximale Punkt $O(0, 0, \cdots, 0)$, sofern $q\neq 0$ ist.

Es sei W_0 offener Würfel von der Kantenlänge $2a$, O als Mittelpunkt. In das abgeschossene Gebiet $W-W_0=V$ hat Y ein bestimmtes Maximum ρ $(0<\rho<1)$ und der von diesem Gebiet herrühende Teil des Integrales (2) ist $<\rho^m$, also wird $o\left(\frac{1}{m^N}\right)$ und vernachlässigbar. Daher braucht man nur über W_0 erstreckter Teil J_{W_0} des Integrales (2) zu betrachten. Dies ist

$$J_{W_0}=\frac{1}{(2\pi)^d}\int\int_{\varphi_\nu=-a}^{\varphi_\nu=a}\cdots\int\left(\frac{p}{d}\sum_{\nu=1}^{d}\cos\varphi_\nu+q\right)^m d\varphi_1\cdots d\varphi_d \quad \left(\varphi_\nu=\frac{t_\nu}{\sqrt{m}}\right)$$

$$=\frac{1}{(2\pi\sqrt{m})^d}\int\int_{-a\sqrt{m}}^{a\sqrt{m}}\cdots\int\exp m\log\left[\frac{p}{d}\sum\cos\frac{t_\nu}{\sqrt{m}}+q\right]dt_1\cdots dt_d.$$

Es ist aber der Exponent im Integrande bei genügend großes m

$$\exp\left\{m\log\left[\frac{p}{d}\sum_{\nu=1}^{d}\left(1-\frac{t_\nu{}^2}{2m}\right)+q+O\left(\frac{1}{m^2}\right)\right]\right\}$$

$$=\exp\left\{m\log\left[1-\frac{p}{2dm}\sum_{\nu=1}^{d}t_\nu^2+O\left(\frac{1}{m^2}\right)\right]\right\}$$

$$\cong\exp\left\{-\frac{p}{2d}\sum_{\nu=1}^{d}t_\nu^2\right\}.$$

Daraus läßt sich zeigen

$$J_{W_0}\cong\frac{1}{(2\pi\sqrt{m})^d}\prod_{\nu=1}^{d}\int_{-\infty}^{\infty}\exp\left\{-\frac{p}{2d}t_\nu^2\right\}dt_\nu=\left(\frac{d}{2\pi mp}\right)^{\frac{d}{2}},$$

und also entsteht für $m\rightarrow\infty$

$$P_m(0,0,\cdots,0)\cong\left(\frac{d}{2\pi mp}\right)^{\frac{d}{2}}. \tag{3}$$

Die oben gestellte Hayashis Frage entspricht zum besonderen Falle, daß $d=2$, $q=\frac{1}{5}$, $p=\frac{4}{5}$, und darauf

$$P_m(0,0)\cong\frac{5}{4m\pi} \tag{4}$$

gegenüber (2.5) antwortet.

Pólyasche Resultat war

$$P_m(0,0,\cdots,0)\cong 2\left(\frac{d}{4n\pi}\right)^{d/2} \text{ oder } 0$$

je nach dem $m=2n$ oder $2n-1$, so daß durchschnittlich

$$\left(\frac{d}{2\pi m}\right)^{a/2},$$

was etwas kleiner als obiges (3) ist. Also ist die Gelegenheit des Zurückkehren mit der Hinzufügung des Ruhezustandes ein wening vermehrt. Da aber die Konvergenzeigenschaft der Reihe $\sum P_m$ so beschaffen wie Pólyasche ist, bleibt Pólyasche Schluß durchaus unveränderlich, ob die Ruhesitzung stattfände oder nicht.

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ON THE LINEAR PARTIAL DIFFERENTIAL EQUATION OF SECOND ORDER IN N INDEPENDENT VARIABLES WITH CONSTANT COEFFICIENT

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§1. The proposed equation is of the form:

$$\sum_{i,\,j=1}^{n} a_{ij}\frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial w}{\partial x_i} + a_0 w = f(x_1,\, x_2,\, \cdots,\, x_n), \tag{1.1}$$

where a_0, a_i, $a_{ij}(=a_{ji})$, $i=1, 2, \cdots, n$ are given real constant, and $f(x_1, x_2, \cdots, x_n)$ is a given integrable function.

First we intend to find the complementary function, *i.e.* the general integral of

$$\sum_{i,\,j=1}^{n} a_{ij}\frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial w}{\partial x_i} + a_0 w = 0. \tag{1.2}$$

We commence with a particular case, such that the lefthanded side is resoluble into linear factors as

$$\left(\sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + b_0\right)\left(\sum_{j=1}^{n} c_j \frac{\partial}{\partial x_j} + c_0\right) w = 0, \tag{1.3}$$

where b's, c's are constants, and, since (1.1) is assumed to be really of second order, at least one among a_{ij} and accordingly one of b_i and c_j should be non-zero, so that conveniently let it be $b_1 c_1 \neq 0$.[1]
Since the factors of product in (1.3) are commutative, the required complementary function shall be found by solving

$$\left(\sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + b_0\right) w = 0, \tag{1.4}$$

or

$$\left(\sum_{i=1}^{n} c_j \frac{\partial}{\partial x_i} + c_0\right) w = 0. \tag{1.5}$$

On writing the subsidiary equation of the partial differential equation of first order (1.4)

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[1] If all $a_{ii}=0$, this assumption becomes absurd, to speak more we must say that some $b_i c_j \neq 0$. But the matter being trivial, only for the sake of brevity we have assumed as above.

$$\frac{dx_1}{b_1}=\frac{dx_2}{b_2}=\cdots=\frac{dx_n}{b_n}=\frac{dw}{-b_0 w},$$

where $b_1 \neq 0$, we see immediately that their solutions are

$$x_i-\frac{b_i}{b_1}x_1=\text{const.},\qquad i=2,\ 3,\ \cdots,\ n,$$

and

$$w\exp\left\{\frac{b_0}{b_1}x_1\right\}=\text{const.},$$

so that the general integral of (1.4) is

$$w=\exp\left\{-\frac{b_0}{b_1}x_1\right\}\varPhi\left(x_2-\frac{b_2}{b_1}x_1,\ \cdots,\ x_n-\frac{b_n}{b_1}x_1\right), \tag{1.6}$$

where $\varPhi$ denotes any arbitrary function. Quite similarly with (1.5) we get

$$w=\exp\left\{-\frac{c_0}{c_1}x_1\right\}\psi\left(x_2-\frac{c_2}{c_1}x_1,\ \cdots,\ x_n-\frac{c_n}{c_1}x_1\right). \tag{1.7}$$

Therefore the required general integral of (1.2) is given by

$$w=e^{-\frac{b_0}{b_1}x_1}\varPhi\left(x_2-\frac{b_2}{b_1}x_1,\ \cdots,\ x_n-\frac{b_n}{b_1}x_1\right)+e^{-\frac{c_0}{c_1}x_1}\psi\left(x_2-\frac{c_2}{c_1}x_1,\ \cdots,\ x_n-\frac{c_n}{c_1}x_1\right), \tag{1.8}$$

where $\varPhi$ and ψ are arbitrary functions.

In the case, that all $b_i=c_i$, however (1.3) becomes

$$\left(\sum_{i=1}^{n}b_i\frac{\partial}{\partial x_i}+b_0\right)^2 w=0, \tag{1.9}$$

and the corresponding solution (1.8) contains essentially only one arbitrary function, so that it ceases to be general. To obtain the general integral, let us put

$$\left(\sum_{i=1}^{n}b_i\frac{\partial}{\partial x_i}+b_0\right)w=v,$$

and solve

$$\left(\sum_{i=1}^{n}b_i\frac{\partial}{\partial x_i}+b_0\right)v=0.$$

In view of (1.6) the latter's general integral is

$$v=\exp\left\{-\frac{b_0}{b_1}x_1\right\}\varPhi\left(x_2-\frac{b_2}{b_1}x_1,\ \cdots,\ x_n-\frac{b_n}{b_1}x_1\right),$$

and accordingly we have to solve

$$\left(\sum_{i=1}^{n}b_i\frac{\partial}{\partial x_i}+b_0\right)w=\exp\left\{-\frac{b_0}{b_1}x_1\right\}\varPhi\left(x_2-\frac{b_2}{b_1}x_1,\ \cdots,\ x_n-\frac{b_n}{b_1}x_1\right).$$

With regard to this linear partial differential equation of first order the subsidiary equations become

$$\frac{dx_1}{b_1}=\frac{dx_2}{b_2}=\cdots=\frac{dx_n}{b_n}=\frac{dw}{\exp\left\{-\frac{b_0}{b_1}x_1\right\}\Phi\left(x_2-\frac{b_2}{b_1}x_1,\cdots,x_n-\frac{b_n}{b_1}x_1\right)-b_0w}\cdot$$

whose solutions are $n-1$ equations

$$x_i-\frac{b_i}{b_1}x_1=k_i \qquad (i=2,3,\cdots,n),$$

where k_i are arbitrary constants, and one more equation that is obtainable from

$$\frac{dw}{dx_1}+\frac{b_0}{b_1}w=\frac{1}{b_1}\exp\left\{-\frac{b_0}{b_1}x_1\right\}\Phi(k_2,\cdots,k_n),$$

i. e.

$$w\exp\left\{\frac{b_0}{b_1}x_1\right\}-\frac{x_1}{b_1}\Phi(k_2,\cdots,k_n)=k_1,$$

where $k_i=x_i-\frac{b_i}{b_1}x_1$ $(i=2,3,\cdots,n)$, and $\frac{1}{b_1}\Phi$ can be written simply Φ as an arbitrary function. Therefore the general integral of (1.9) is

$$w=\exp\left\{-\frac{b_0}{b_1}x_1\right\}\left[x\Phi\left(x_2-\frac{b_2}{b_1}x_1,\cdots,x_n-\frac{b_n}{b_1}x_1\right)\right.$$
$$\left.+\psi\left(x_2-\frac{b_2}{b_1}x_1,\cdots,x_n-\frac{b_n}{b_1}x_1\right)\right], \tag{1.10}$$

where Φ and ψ are arbitrary functions.

Next we proceed to find a particular integral of (1.1). For this purpose we put again in view of (1.3)

$$\left(\sum_{i=1}^{n}c_i\frac{\partial}{\partial x_i}+c_0\right)w=u \tag{1.11}$$

and

$$\left(\sum_{i=1}^{n}b_i\frac{\partial}{\partial x_i}+b_0\right)u=f(x_1,x_2,\cdots,x_n)\,. \tag{1.12}$$

Now the subsidiary equations of the latter being

$$\frac{dx_1}{b_1}=\frac{dx_2}{b_2}=\cdots=\frac{dx_n}{b_n}=\frac{du}{f(x_1,\cdots,x_n)-b_0u},$$

their solutions are again

$$x_i-\frac{b_i}{b_1}x_1=k_i \qquad (i=2,3,\cdots,n),$$

and the solution of

$$\frac{du}{dx_1}+\frac{b_0}{b_1}u=\frac{1}{b_1}f(x_1,\cdots,x_n),$$

i. e.

$$u \exp\left\{\frac{b_0}{b_1}x_1\right\} = \frac{1}{b_1}\int \exp\left\{\frac{b_0}{b_1}x_1\right\} f\left(x_1, \frac{b_2}{b_1}x_1+k_2, \cdots, \frac{b_n}{b_1}x_1+k_n\right) dx_1 + k_1 .$$

Hence, on setting the additional arbitrary constant $k_1=0$, we obtain as a particular solution of (1.12)

$$(1.13) \qquad u = \frac{1}{b_1} \exp\left\{-\frac{b_0}{b_1}x_1\right\} \int \exp\left\{\frac{b_0}{b_1}x_1\right\} f\left(x_1, \frac{b_2}{b_1}x_1+k_2, \cdots, \frac{b_n}{b_1}x_1+k_n\right) dx_1 ,$$

where constants k_i should be replaced by $x_i - \frac{b_i}{b_1}x_1$ after integration, so that it yields

$$u = v\,(x_1, x_2, \cdots, x_n).$$

Substituting this in (1.11) and solving it, we get a solution of (1.11), namely, a particular integral of the linear partial differential equation of second order (1.11)

$$(1.14) \qquad w = \frac{1}{c_1} \exp\left\{-\frac{c_0}{c_1}x_1\right\} \int \exp\left\{\frac{c_0}{c_1}x_1\right\} v\left(x_1, \frac{c_2}{c_1}x_1+l_2, \cdots, \frac{c_n}{c_1}x_1+l_n\right) dx_1 ,$$

where again l_i must be replaced by $x_i - \frac{c_i}{c_1}x_1$ after integration.

Example 1.

$$2\frac{\partial^2 w}{\partial x^2} + 3\frac{\partial^2 w}{\partial y^2} - 4\frac{\partial^2 w}{\partial z^2} - 5\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y \partial z} - 2\frac{\partial^2 w}{\partial x\, \partial z}$$
$$+ 8\frac{\partial w}{\partial x} - 10\frac{\partial w}{\partial y} - 4\frac{\partial w}{\partial z} + 8w = yze^{-2x}.$$

Factorizing the left-handed member, we get

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + \frac{\partial}{\partial z} + 2\right)\left(2\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y} - 4\frac{\partial}{\partial z} + 4\right) w = yze^{-2x},$$

and thus $b_1=1$, $b_2=-1$, $b_3=1$, $b_0=2$, $c_1=2$, $c_2=-3$, $c_3=-4$, $c_0=4$. Hence the complementary function becomes by virtue of (1.8)

$$w = e^{-2x}\{\Phi(y+x, z-x) + \psi(2y+3x, z+2x)\},$$

while the particular integral is obtained by means of (1.13) and (1.14) as follows:

$$u = e^{-2x}\int (k_2-x)(k_3+x)dx = e^{-2x}\left\{k_2k_3x + \frac{1}{2}(k_2-k_3)x^2 - \frac{x^3}{3}\right\}$$
$$= e^{-2x}\left[xyz - \frac{1}{2}x^2y + \frac{1}{2}x^2z - \frac{1}{3}x^3\right],$$

and

$$w = \frac{1}{2}e^{-2x}\int e^{2x}u\,dx$$
$$= \frac{1}{2}e^{-2x}\int e^{2x}e^{-2x}\left\{x(l_2-\frac{3}{2}x)(l_3-2x) - \frac{1}{2}x^2(l_2-\frac{3}{2}x) + \frac{1}{2}x^2(l_3-2x) - \frac{x^3}{3}\right\} dx$$

$$= \frac{1}{2}e^{-2x}\left[\frac{x^2}{2}l_2l_3 - \frac{x^3}{3}\left(\frac{5}{2}l_2 + l_3\right) + \frac{29}{48}x^4\right]$$

$$= \frac{1}{2}e^{-2x}\left[\frac{3}{32}x^4 + \frac{1}{12}x^3y + \frac{5}{24}x^3z + \frac{1}{4}x^2yz\right].$$

It is easy to check that

$$u = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + \frac{\partial}{\partial z} + 2\right)w = e^{-2x}\left[\frac{x^3}{2} + 2x^2y + \frac{3}{8}x^2z + \frac{1}{2}xyz\right],$$

$$\left(2\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y} - 4\frac{\partial}{\partial z} + 4\right)u = e^{-2x}yz.$$

Example 2. Our method might be repeatedly applied *e.g.* for a linear partial differential equation of third order as

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + \frac{\partial}{\partial z} + 2\right)\left(2\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y} - 4\frac{\partial}{\partial z} + 4\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} + 2\right)v = e^{-2x}yz.$$

Putting $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} + 2\right)v = w$, the problem reduces to *Ex.* 1, and w is rendered by the above result. Hence the particular integral is found similarly, as before,

$$v = e^{-2x}\int w\,dx \quad (y = x + h_2,\ z = x + h_3)$$

$$= e^{-2x}\left[\frac{1}{12}h_2h_3x^3 + \frac{1}{96}(8h_2 + 11h_3)x^4 + \frac{61}{480}x^5\right]$$

$$= e^{-2x}\left[\frac{x^5}{80} + \frac{7}{12}x^4y + \frac{1}{32}x^4z - \frac{1}{12}x^3yz\right],$$

while the complementary function is easily found to be

$$V = e^{-2x}\{\Phi(y+z,\ z-x) + \psi(2y+3x,\ z+2x) + \Theta(y-x,\ z-x)\},$$

where Φ, Ψ, Θ are arbitrary functions.

§2. Next we shall treat the case that does not permit any factorization like (1.3). In this case we will write in a standard form the given linear partial differential equation of second order

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i\frac{\partial w}{\partial x_i} + c_0 w = f(x_1, x_2, \cdots, x_n), \tag{2.1}$$

where $a_{ij}(=a_{ji})$, b_i, c_0 $(i, j = 1, 2, \cdots, n)$ are given real constants.

As well known, the symmetric matrix $A=(a_{ij})$ can be brought into a diagonal matrix by operating a suitably chosen orthogonal matrix T:

$$\bar{T}AT = A' = \begin{pmatrix} a_1' & & 0 \\ & \ddots & \\ 0 & & a_n' \end{pmatrix}, \tag{2.2}$$

where

$$T=\begin{pmatrix} l_{11}\cdots\cdots\cdots l_{1n} \\ \vdots \quad \cdots\cdots\cdots\cdots \\ l_{n1}\cdots\cdots\cdots l_{nn} \end{pmatrix}, \quad \bar{T}=\begin{pmatrix} l_{11}\cdots\cdots\cdots l_{n1} \\ \vdots \quad \cdots\cdots\cdots\cdots \\ l_{1n}\cdots\cdots\cdots l_{nn} \end{pmatrix}$$

with $\bar{T}=T^{-1}$, $T\bar{T}=I$. If $\mathfrak{x}$ is transformed into $\mathfrak{x}'$ by T:

$$T\mathfrak{x}=\begin{pmatrix} l_{11}\cdots\cdots\cdots l_{1n} \\ \vdots \quad \cdots\cdots\cdots\cdots \\ l_{n1}\cdots\cdots\cdots l_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}=\begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix}=\mathfrak{x}', \quad \text{or } \mathfrak{x}=\bar{T}\mathfrak{x}', \tag{2.3}$$

the binary quadratic form

$$Q=\sum_{i,\,j=1}^{n} a_{ij}x_ix_j=\bar{\mathfrak{x}}A\mathfrak{x} \tag{2.4}$$

$$=(x_1,\ \cdots,\ x_n)\begin{pmatrix} a_{11}\cdots\cdots a_{1n} \\ \vdots \\ a_{n1}\cdots\cdots a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

shall be transformed into the standard form

$$\sum_{i=1}^{r} a'_i\, x'^2_i \tag{2.5}$$

where r denotes the rank of matrix A (or it may be written still $\sum_{i=1}^{n} a'_i x'^2_i$, but now some a'_i are allowed to be 0). By transformation (2.3) we get $x_j=\sum l_{ij}x'_i$, so that $\frac{\partial w}{\partial x'_i}=\sum_j \frac{\partial w}{\partial x_j}\frac{\partial x_j}{\partial x'_i}=\sum_j l_{ij}\frac{\partial w}{\partial x_j}$ and similarly $\frac{\partial w}{\partial x_j}=\sum_i l_{ij}\frac{\partial w}{\partial x'_i}$. Thus

$$\begin{cases} \dfrac{\partial}{\partial x'_i}=l_{j1}\dfrac{\partial}{\partial x_1}+l_{i2}\dfrac{\partial}{\partial x_2}+\cdots+l_{in}\dfrac{\partial}{\partial x_n} \quad (i=1,\ 2,\ \cdots,\ n), \\[2ex] \dfrac{\partial}{\partial x_j}=l_{1j}\dfrac{\partial}{\partial x'_1}+l_{2j}\dfrac{\partial}{\partial x'_2}+\cdots+l_{nj}\dfrac{\partial}{\partial x'_n} \quad (j=1,\ 2,\ \cdots,\ n). \end{cases} \tag{2.6}$$

By the transformations, the quadratic differential form $\sum_{i,\,j=1}^{n} a_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}w$ in (2.1) could be brought into $\sum_{i=1}^{n} a'_i\frac{\partial^2 w}{\partial x'^2_i}$. At the same time, the linear differential form in (2.1) would be transformed into

$$\sum_{j=1}^{n} b_j\frac{\partial w}{\partial x_j}=\sum_j b_j\sum_i l_{ij}\frac{\partial w}{\partial x'_i}=\sum_i\Big(\sum_j l_{ij}b_j\Big)\frac{\partial w}{\partial x'_i}=\sum_i b'_i\frac{\partial w}{\partial x'_i} \tag{2.7}$$

where $\mathfrak{b}'=T\mathfrak{b}$, *i.e.*

$$\begin{pmatrix} b'_1 \\ \vdots \\ b'_n \end{pmatrix}=\begin{pmatrix} l_{11}\cdots\cdots\cdots l_{1n} \\ \vdots \quad \cdots\cdots\cdots\cdots \\ l_{n1}\cdots\cdots\cdots l_{nn} \end{pmatrix}\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \tag{2.8}$$

and lastly either $c_0'=c_0$, or it might be written as $c_0=\sum_{i=1}^{n} c_i'$, where c_i' are chosen arbitrariry, so far as their sum become c_0.

Thus, on performing T, the partial differential equation (2.1) reduces to

$$\sum_{i=1}^{n}\left(a_i'\frac{\partial^2 w}{\partial x_i'^2}+b_i'\frac{\partial w}{\partial x_i'}+c_i' w\right)=f_1(x_1, x_2, \cdots, x_n), \tag{2.9}$$

which form we have to treat below.

To find a complete integral, we write $f_1=0$:

$$\sum_{i=1}^{n}\left(a_i'\frac{\partial^2 w}{\partial x_i'^2}+b_i'\frac{\partial w}{\partial x_i'}+c_i' w\right)=0. \tag{2.10}$$

Now, as usually made in Harmonic Analysis, we assume that

$$w=\prod_{i=1}^{n} u_i(x_i), \tag{2.11}$$

where every u_i is a function of x_i only. On substituting (2.11) in (2.10), and dividing out by w, we obtain

$$\sum_{i=1}^{n}\frac{1}{u_i}\left(a_i'\frac{d^2 u_i}{dx_i'^2}+b_i'\frac{du_i}{dx_i'}+c_i' u_i\right)=0,$$

so that each summand ought to vanish reparately:

$$a_i'\frac{d^2 u_i}{dx_i'^2}+b_i'\frac{du_i}{dx_i'}+c_i' u_i=0, \qquad (i=1, 2, \cdots, n), \tag{2.12}$$

with auxiliary equation

$$a_i' m^2+b_i' m+c_i'=0. \tag{2.13}$$

If the rank of matrix A be n, namely the determinant $|A|=|A'|=\prod_{i=1}^{n} a_i'\neq 0$, no a_i' could be zero, and consequently (2.13) should have two roots;

$$\alpha_i, \beta_i=\frac{1}{2a_i'}\{b_i'\pm\sqrt{b_i'^2-4a_i'c_i'}\} \qquad (i=1, 2, \cdots, n)$$

and we obtain, as solutions

$$u_i(x_i)=A_i\exp\alpha_i x_i+B_i\exp\beta_i x_i \qquad \text{if} \quad \alpha_i\neq\beta_i,$$

or else

$$=(A_i x_i+B_i)\exp\alpha_i x_i \qquad \text{if} \quad \alpha_i=\beta_i.$$

Thus we get, as a complete integral of (2.9),

$$w=u_1(x_1)\,u_2(x_2)\cdots u_n(x_n),$$

wich contains $3n-1$ arbitrary constants A_i, B_i and c_i' with condition $\sum_{i=1}^{n} c_i'=c_0$.

If the rank of matrix A be $1\leqq r<n$, then equations (2.12) becomes

$$
(2.14)\qquad \begin{cases} \dfrac{1}{u_i}\left(a'_i\dfrac{d^2u_i}{dx'^2_i}+b'_i\dfrac{du_i}{dx'_i}+c'_iu_i\right)=0 & (i=1,\ 2,\ \cdots,\ r), \\[2ex] b'_i\dfrac{du_i}{dx'_i}+c'_iu_i=0 & (i=r+1,\ \cdots,\ n). \end{cases}
$$

In the latter equations, every coefficients b'_i surely $\neq 0$, since, otherwise, the very variable x'_i does disappear, what contradicts our assumption of n independent variables, and those solutions are

$$
u_i\,(x_i)=C_i\exp\left\{\frac{c'_i}{b'_i}x_i\right\} \qquad (i=r+1,\ \cdots,\ n).
$$

However, in a complete integral $w=\prod_{i=1}^{n}u_i(x_i)$, the constants C_i may be mingled in some A_j, B_j, so it contains only $n+2r-1$ arbitrary constants.

In the present case, it is somewhat difficult to discuss in general how to obtain a particular integral. We ought to find it ingeneously by problem. However, if it occurs that

$$
f(x_1,\ \cdots,\ x_n)\sim f(x_1^{(0)},\ \cdots,\ x_n^{(0)})+\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i^{(0)}}\right)(x_i-x_i^{(0)})=d_0+\sum_{i=1}^{n}d_ix_i,
$$

as seen in the small oscillation about equilibrium position, then upon writing

$$
d_0\,\delta_i^1+d_ix_i, \quad \text{with} \quad \delta_i^1=0\ (i=1)\ \text{or}\ =0\ (i\neq 1)
$$

in the right-handed side of (2.12) or (2.14), we may find the required particular integral.

§3. To illustrate how the above mentioned transformation to be executed actually, let us consider the case n=3:

$$
(3.1)\qquad \begin{aligned} A'\frac{\partial^2 w}{\partial x^2}+B'\frac{\partial^2 w}{\partial y^2}+C'\frac{\partial^2 w}{\partial z^2}+2F\frac{\partial^2 w}{\partial y\,\partial z}+2G\frac{\partial^2 w}{\partial z\partial x}+2H\frac{\partial^2 w}{\partial x\,\partial y}& \\ +2K\frac{\partial w}{\partial x}+2L\frac{\partial w}{\partial y}+2M\frac{\partial w}{\partial z}+Nw=f(x,\,y,\,z),& \end{aligned}
$$

where $A', B', \cdots, N$ are given (real) constants, and $f(x, y, z)$ a given function. We conceive the problem of principal axes of the corresponding quadratic surface:

$$
A'x^2+B'y^2+C'z^2+2Fyz+2Gzx+2Hxy=R.
$$

To reduce this to the standard form we solve the characteristic equation

$$
(3.2)\qquad \Delta\,(\lambda)=\begin{vmatrix} A'-\lambda & H & G \\ H & B'-\lambda & F \\ G & F & C'-\lambda \end{vmatrix}=0,
$$

and let its *3* characteristic roots be λ_1, λ_2, λ_3.

i) *When 3 roots are all different.* Then, among the simultaneous equations

$$(3.3)\qquad \begin{cases} (A'-\lambda_i)\,l_i + Hm_i + Gn_i = 0 \\ Hl_i + (B'-\lambda_i)m_i + Fn_i = 0 \\ Gl_i + Fm_i + (C'-\lambda_i)\,n_i = 0 \end{cases} \qquad (i = 1,\ 2,\ 3),$$

there only two being independent, we have to take

$$\begin{aligned} &(A'-\lambda_i)\,l_i + Hm_i + Gn_i = 0 \\ &(H+G)l_i + (B'+F-\lambda_i)m_i + (F+C'-\lambda_i)n_i = 0. \end{aligned}$$

Whence the ratios $l_i : m_i : n_i$, and further on combing them with $l_i^2+m_i^2+n_i^2=1$, the respective values l_i, m_i, n_i $(i=1, 2, 3)$ could be determined; moreover selecting the root-signs ± for l_i, m_i, n_i adequately, it is always possible to make the Jacobian

$$J = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = 1.$$

ii) *When 2 roots of* (2.3) *are equal, say* $\lambda_2=\lambda_3$. We can determined l_1, m_1, n_1 as in i). As to λ_2, λ_3, we have

$$\begin{aligned} &(A'-\lambda_2)\,l_2 + Hm_2 + Gn_2 = 0, \\ &(A'-\lambda_3)\,l_3 + Hm_3 + Gn_3 = 0, \end{aligned}$$

and

$$l_2l_3 + m_2m_3 + n_2n_3 = 0,$$

of which first two assure that the directions (l_2, m_2, n_2), (l_3, m_3, n_3) are perpendicular to (l_1, m_1, n_1). Here we ought to determine *4* ratio's $l_2 : m_2 : n_2$, $l_3 : m_3 : n_3$ from the above *3* equations, so that one unknown may be assumed at will. Hence, *e.g.* on taking $l_2=0$, we get $m_2 : n_2 = -G : H$, and consequently

$$(A'-\lambda_2)l_3 + Hm_3 + Gn_3 = 0, \quad -Gm_3 + Hn_3 = 0,$$

whence

$$l_3 : m_3 : n_3 = -(G^2+H^2) : H(A'-\lambda_2) : G(A'-\lambda_3).$$

Thus in the above two cases we have already found a triple orthogonal system with Jacobien $J=1$. Hence making transformations

$$(3.4)\qquad \begin{cases} \xi = l_1x + m_1y + n_1z \\ \eta = l_2x + m_2y + n_2z \\ \zeta = l_3x + m_3y + n_3z \end{cases} \quad \text{or} \quad \begin{cases} x = l_1\xi + l_2\eta + l_3\zeta \\ y = m_1\xi + m_2\eta + m_3\zeta \\ z = n_1\xi + n_2\eta + n_3\zeta \end{cases}$$

i.e.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = T \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \quad \text{with } T = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix}, \quad |T| = 1,$$

we have

$$A'x^2 + B'y^2 + C'z^2 + 2Fyz + 2Gzx + 2Hxy = \lambda_1\xi^2 + \lambda_2\eta^2 + \lambda_3\zeta^2.$$

Hence also by transformation

$$\begin{cases} \dfrac{\partial}{\partial x} = l_1\dfrac{\partial}{\partial\xi} + l_2\dfrac{\partial}{\partial\eta} + l_3\dfrac{\partial}{\partial\zeta} \\ \dfrac{\partial}{\partial y} = m_1\dfrac{\partial}{\partial\xi} + m_2\dfrac{\partial}{\partial\eta} + m_3\dfrac{\partial}{\partial\zeta} \\ \dfrac{\partial}{\partial z} = n_1\dfrac{\partial}{\partial\xi} + n_2\dfrac{\partial}{\partial\eta} + n_3\dfrac{\partial}{\partial\zeta}, \end{cases} \quad J = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = 1,$$

the linear partial differential equation (3.1) becomes

$$\lambda_1\frac{\partial^2 w}{\partial\xi^2} + \lambda_2\frac{\partial^2 w}{\partial\eta^2} + \lambda_3\frac{\partial^2 w}{\partial\zeta^2} + 2K_1\frac{\partial w}{\partial\xi} + 2L_1\frac{\partial w}{\partial\eta} + 2M_1\frac{\partial w}{\partial\zeta} + N_1 w = f_1(\xi, \eta, \zeta).$$

Remark. When (3.2) has *3* equal roots $\lambda_1=\lambda_2=\lambda_3=\lambda$, we have $\Delta(\lambda)=0$, $\Delta'(\lambda)=0$, $\Delta''(\lambda)=2(A'+B'+C')-6\lambda=0$. Hence $\lambda=\frac{1}{3}(A'+B'+C')$ and this being substituted in $\Delta'(\lambda)=0$, we get

$$(A'-B')^2+(B'-C')^2+(C'-A')^2+2F^2+2G^2+2H^2=0,$$

so that, for real coefficients, we must have $A'=B'=C'$, $F=G=H=0$. Therefore the quadratic differential form in (3.1) becomes

$$A'\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)w,$$

i.e. Laplace's form and there is no need of transformation.

Example 3.

$$6\frac{\partial^2 w}{\partial y^2} - 18\frac{\partial^2 w}{\partial y\partial z} - 6\frac{\partial^2 w}{\partial x\partial z} + 2\frac{\partial^2 w}{\partial x\partial y} - 9\frac{\partial w}{\partial x} + 5\frac{\partial w}{\partial y} - 5\frac{\partial w}{\partial z} + w = 0.$$

Here

$$|A| = \begin{vmatrix} 0 & 1 & -3 \\ 1 & 6 & -9 \\ -3 & -9 & 0 \end{vmatrix} = 0, \text{ and the matrix } A \text{ is of rank } 2.$$

Also from

$$\Delta(\lambda) = \begin{vmatrix} -\lambda & 1 & -3 \\ 1 & 6-\lambda & -9 \\ -3 & -9 & -\lambda \end{vmatrix} = -\lambda(\lambda+7)(\lambda-13) = 0,$$

we get *3* differents roots *0*, *−7*, *13*, and correspondingly

$$\begin{array}{l} l_1 : m_1 : n_1 = -9 : 3 : 1 \\ l_2 : m_2 : n_2 = 1 : 2 : 3 \\ l_3 : m_3 : n_3 = 1 : 4 : -3 \end{array} \qquad \text{so} \quad T = \begin{pmatrix} -9/\sqrt{91} & 3/\sqrt{91} & 1/\sqrt{91} \\ 1/\sqrt{14} & 2/\sqrt{14} & 3/\sqrt{14} \\ 1/\sqrt{26} & 4/\sqrt{26} & -3/\sqrt{26} \end{pmatrix}.$$

Therefore by transformation

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = T \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{so also} \quad \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{pmatrix} = \bar{T} \begin{pmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \\ \dfrac{\partial}{\partial \zeta} \end{pmatrix}$$

the given partial differential equation is reduced to

$$7\frac{\partial^2 w}{\partial \eta^2} - 13\frac{\partial^2 w}{\partial \zeta^2} + \sqrt{91}\frac{\partial w}{\partial \xi} - \sqrt{14}\frac{\partial w}{\partial \eta} + \sqrt{26}\frac{\partial w}{\partial \zeta} + w = 0.$$

This becomes, on assuming $w = X(\xi)Y(\eta)Z(\zeta)$

$$\left(\frac{\sqrt{91}}{X}\frac{dX}{d\xi} + 1\right) + \frac{1}{Y}\left(\frac{d^2Y}{d\eta^2} - \sqrt{14}\frac{dY}{d\eta}\right) - \frac{1}{Z}\left(13\frac{d^2Z}{d\zeta^2} - \sqrt{26}\frac{dZ}{d\zeta}\right) = 0.$$

Putting the expression under every bracket $=0$, we get

$$X = A_1 \exp\left\{-\frac{\xi}{\sqrt{91}}\right\}, \quad Y = A_2 \exp\left\{\sqrt{\frac{2}{7}}\eta\right\} + B_2, \quad Z = A_3 \exp\left\{\sqrt{\frac{2}{13}}\zeta\right\} + B_3,$$

so that a complete integral of the given partial differential equation is

$$w = \exp\left\{\frac{1}{91}(-9x+3y+z)\right\}\left[A_2 \exp\left\{\frac{1}{7}\;(x+y+z)\right\} + B_2\right] \times$$
$$\left[A_3 \exp\left\{\frac{1}{13}(x+4y-3z)\right\} + B_3\right],$$

where A_2, B_2, A_3, B_3 are arbitrary constants.

Example 4.

$$\frac{\partial^2 w}{\partial z^2} = 3\frac{\partial w}{\partial x} + 4\frac{\partial w}{\partial y}$$

Here

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and its rank is *1*. Assuming $w = X(x)Y(y)Z(z)$, the given equation becomes

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = \frac{3}{X}\frac{dX}{dx} + \frac{4}{Y}\frac{dY}{dy}.$$

Putting

$$\frac{3}{X}\frac{dX}{dx}=c_1, \quad \frac{4}{Y}\frac{dY}{dy}=c_2, \quad \frac{1}{Z}\frac{d^2Z}{dz^2}=c_1+c_2=c,$$

we obtain

$$\begin{aligned} X &= k_1 \exp\left\{\frac{c_1}{3}x\right\}, \quad Y = k_2 \exp\left\{\frac{c_2}{4}y\right\}, \\ Z &= A_1 e^{\sqrt{c}z} + B_1 e^{-\sqrt{c}z}, && \text{if} \quad c>0, \\ &= A_1 \cos\sqrt{-c}z + B_1 \sin\sqrt{-c}z, && \text{if} \quad c<0, \\ &= A_1 + B_1 z, && \text{if} \quad c=0, \end{aligned}$$

where $c=c_1+c_2$. Therefore, a complete integral of the given partial differential epuation is

$$w = Z\exp\left\{\frac{c_1}{3}x+\frac{c_2}{4}y\right\},$$

which contains *4* arbitrary constants A_1, B_1, c_1, c_2.

The writer closes his paper by expressing his hearty thanks to Professor Y. Watanabe for his interest in this work and valuable suggestions.

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ÜBER DIE LAPLACESCHE ASYMPTOTISCHE FORMEL FÜR DAS INTEGRAL VON POTENZE MIT GROßEM INDEXE

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(Eingegangen am 30, 1955)

§ 1

Wir wollen eine bekannte Laplacesche Formel verallgemeinern wie der

Satz 1. *Die reellen Funktionen* $\varphi(x) \equiv \varphi(x_1, x_2, \cdots, x_m)$ *und* $f(x) \equiv f(x_1, x_2, \cdots, x_m)$ *seien in* $B: (a_i \leqq x_i \leqq b_i;\ i = 1, 2, \cdots, m)$ *Teilbereiche m–dimensionalen Euklidischen Raumes definiert und dem forgenden Bedingungen unterworfen:*

1° *Die Funktionen* $f(x) \geqq 0$ *und* $\varphi(x) \cdot [f(x)]^n$ $(n = 1, 2. \cdots)$ *seien absolut integrabel in B.*

2° $F(x) = \log f(x)$ *sei regular innerhalb B, also in der Umgebung* W_2 *einer Stelle* ξ *im Innern von B in Taylorsche Reihe entwickelbar, und sogar dort erreihe ihr Maximum in starken Sinn, d. h. quadratische Glieder in dortige Taylorsche Reihe bilden eine negative definite quadratische Form.*

3° $\varphi(x)$ *erlaube auch Taylorsche Entwickelung in W und sei* $\varphi(\xi) \neq 0$.

Dann gilt für $n \to \infty$ *die folgende asymptotische Formel*[1]:

$$\int_{a_m}^{b_m}\int_{a_{m-1}}^{b_{m-1}}\cdots\cdots\int_{a_1}^{b_1}\varphi(x_1, x_2, \cdots, x_m)\,[f(x_1, x_2, \cdots, x_m)]^n dx_1 dx_2 \cdots dx_m \tag{1}$$

$$\cong \frac{(2\pi/n)^{m/2}}{\sqrt{(-1)^m D}}\,[f(\xi_1, \xi_2, \cdots, \xi_m)]^n \{\varphi(\xi_1, \xi_2, \cdots, \xi_m) + O(1/n)\}$$

wobei

$$D = \begin{vmatrix} a_{11} & a_{12} \cdots\cdots & a_{1m} \\ a_{21} & a_{22} \cdots\cdots & a_{2m} \\ \cdots & \cdots\cdots\cdots & \cdots \\ a_{m1} & a_{m2} \cdots\cdots & a_{mm} \end{vmatrix}, \qquad a_{ij} = \left[\frac{\partial^2 F(x_1, x_2. \cdots, x_m)}{\partial x_i \partial x_j}\right]_{x_k = \xi_k} (k = 1, 2, \cdots, m) \tag{2}$$

Beweis. Betrachte man einen m–dimensionalen offenen innerhalb B liegenden

---

[1] Für $m = 1$ lautet

$$\int_a^b \varphi(x)\,[f(x)]^n dx \cong \sqrt{\frac{2\pi}{-nF'(\xi)}} \cdot [f(\xi)]^n \cdot [\varphi(\xi) + O(1/n)]$$

die bekannte Laplacesche Formel (vgl. Pólya und Szegö, Aufgaben und Lehrsätze, Bd. 1, S. 78 und S. 244.)

Würfer W_0 von der Kantenlänge 2δ, ξ als Mittelpunkt, so erreihe die Funktion $F(x)-F(\xi)$ im abgeschlossenen Bereich $B-W_0$ ein maximum Wert ($<\rho$), und deshalb $\exp\{F(x)-F(\xi)\}$ auch ein maximum wert ρ, wo $0<\rho<1$ ist. Setzt man

$$\text{(3)} \qquad \int_B\cdots\int \varphi(x)\exp\{n[F(x)-F(\xi)]\}dx_1\cdots dx_m = \int_{W_0}\cdots\int + \int_{B-W_0}\cdots\int = \text{(i)} + \text{(ii)}$$

so erhält man

$$|\text{(ii)}| < \rho^n \int_{B-W}\cdots\int |\varphi(x)|\,dx_1\cdots dx_m = O(\rho^n) = o\left(\frac{1}{n^\omega}\right),$$

bei genügend großes n, wieviel groß (jedoch bestimmt) ω genommen werden mag. Daher hat man nur (1) allein zu abschätzen. Dazu schreibe man

$$\varphi(x) = \varphi(\xi) + \sum_{i=1}^m (x_i-\xi_i)\varphi_i(\xi) + \frac{1}{2}\sum_{i=1}^m\sum_{j=1}^m (x_i-\xi_i)(x_j-\xi_j)\varphi_{ij}(x'),$$

$$F(x) = F(\xi) + \frac{1}{2}\sum_{i=1}^m\sum_{j=1}^m (x_i-\xi_i)(x_j-\xi_j)F_{ij}(\xi) + \frac{1}{6}\sum_{i=1}^m\sum_{j=1}^m\sum_{k=1}^m (x_i-\xi_i)(x_j-\xi_j)(x_k-\xi_k)F_{ijk}(\xi)$$
$$+\frac{1}{24}\sum_{i=1}^m\sum_{j=1}^m\sum_{k=1}^m\sum_{l=1}^m (x_i-\xi_i)(x_j-\xi_j)(x_k-\xi_k)(x_l-\xi_l)F_{ijkl}(x''),$$

wobei
$$\varphi_{ij}(x') = \left[\frac{\partial^2\varphi(x)}{\partial x_i\partial x_j}\right]_{x_k=x'_k} \quad (k=1,\ 2,\ \cdots,\ m,)$$

mit
$$x'_k = \xi_k + \vartheta'_k(x_k-\xi_k) \quad , \quad 0<\vartheta'_k<1$$

und
$$F_{ijkl}(x'') = \left[\frac{\partial^4 F(x)}{\partial x_i\partial x_j\partial x_k\partial x_l}\right]_{x_p=x''_p} \quad (p=1,\ 2,\ \cdots,\ m,)$$

mit
$$x''_p = \xi_p + \vartheta''_p(x_p-\xi_p) \quad , \quad 0<\vartheta''_p<1.$$

Nach Annahme, daß $F(x)$ am Punkte ξ ein stärke Maximum hat, soll die quadratische Form $\sum_{i=1}^m\sum_{j=1}^m (x_i-\xi_i)(x_j-\xi_j)F_{ij}(\xi)$ definit negativ sein und folglich

$$(-1)^k\begin{vmatrix} a_{11}\cdots\cdots a_{1k} \\ \vdots \qquad\quad \vdots \\ a_{k1}\cdots\cdots a_{kk}\end{vmatrix} > 0 \qquad (k=1,\ 2,\ \cdots,\ m)$$

Insbesondere ist bei $k=m$, $D_m=D$ und $(-1)^m\cdot D>0$. Wenn man neue Integrationsvariablen $u_1, \cdots, u_m$ dadurch einführt, daß

$$N^2 = (-1)^m D\frac{n}{2} \quad (N>0) \quad \text{und} \quad x_i-\xi_i = \frac{u_i}{N} \qquad (i=1,\ 2,\ \cdots,\ m,),$$

so geht

$$\text{(5)} \qquad \text{(i)} = \int_{-N\delta}^{N\delta}\cdots\int_{-N\delta}^{N\delta}\left\{\varphi(\xi) + \frac{1}{N}\sum_{i=1}^m\varphi_i(\xi)\cdot u_i + \frac{1}{2N^2}\sum_{i=1}^m\sum_{j=1}^m\varphi_{ij}(x')\cdot u_i\cdot u_j\right\}$$
$$\times\exp\left\{\frac{(-1)^m}{D}\sum_{i=1}^m\sum_{j=1}^m a_{ij}\cdot u_i\cdot u_j + \frac{n}{6N^3}\sum_{i=1}^m\sum_{j=1}^m\sum_{k=1}^m\varphi_{ijk}(\xi)\cdot u_i\cdot u_j\cdot u_k\right.$$

$$+\frac{n}{24N^4}\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m}\sum_{l=1}^{m}F_{ijkl}(x'')u_iu_ju_ku_l\Big\}\frac{du_1du_2\cdots du_m}{N^m}\;.$$

Bezeichnen wir nun den ersten Faktor des Integrandes von (5) mit A, und den zweiten mit $\exp B$, und weiter sukzessiven Glieder in B mit B_1, B_2, B_3, so daß

$$\exp B=\exp B_1\exp B_2\exp B_3,$$

$$\exp B_2=1+B_2+\cdots=1+\frac{n}{6N^3}\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m}F_{ijk}(\xi)u_iu_ju_k+\cdots,$$

$$\exp B_3=1+\cdots,$$

so ergibt sich

$$A\exp B=\varphi(\xi)\exp B_1\cdot\Big[1+\frac{1}{N\varphi(\xi)}\sum_i\varphi_i(\xi)\cdot u_i$$

$$+\frac{1}{2N^2\varphi(\xi)}\sum_i\sum_j\varphi_{ij}(x')\cdot u_iu_j\Big]\cdot[1+B_2+\cdots]\cdot[1+\cdots]$$

Da aber die sämtlichen Zahlen $\cdots$ unten $O\Big(\frac{1}{n}\Big)$ herab sinken, mögen diese zwar vernachlässigt werden,[2] und haben wir

$$(6)\qquad A\exp B=\varphi(\xi)\exp B_1\Big[1+\frac{1}{N\phi(\xi)}\sum_{i=1}^{m}\varphi_i(\xi)u_i$$

$$+\frac{n}{6N^3}\sum_i\sum_j\sum_k F_{ijk}(\xi)u_iu_ju_k+R_1+R_2+R_3\Big]$$

wobei

$$R_1=\frac{1}{2N^2\varphi(\xi)}\sum_i\sum_j\varphi_{ij}(x')u_iu_j,$$

$$R_2=\frac{B_2}{2N^2\varphi(\xi)}\sum_i\sum_j\varphi_{ij}(x')u_iu_j=\frac{n}{12N^5\varphi(\xi)}\Big[\sum_i\sum_j\varphi_{ij}(x')u_iu_j\Big]\Big[\sum_i\sum_j\sum_k F_{ijk}(\xi)u_iu_ju_k\Big]$$

und

$$R_3=\frac{B_2}{N\varphi(\xi)}\sum\varphi_i(\xi)u_i=\frac{n}{6N^4\varphi(\xi)}\Big[\sum_i\sum_j\sum_k F_{ijk}(\xi)u_iu_ju_k\Big]\cdot\Big[\sum_i\varphi_i(\xi)u_i\Big].$$

Gesetzt hier $c_{ij}=(-1)^{m+1}a_{ij}/D$ so lautet

$$B_1=\frac{(-1)^m}{D}\sum_i\sum_j a_{ij}u_iu_j=-\sum_i\sum_j c_{ij}u_iu_j\equiv-c_1$$

und folglich wird C_1 ein positive definite quadratische Form. Es seien die Matrix

---

[2] Freilich bei Annaherung an $\pm N\delta$ zunehmen diese bis auf $O(n)$ und die Vernachlässigungen scheinen als zweielhaft. Trotzdem noch existieren und bleiben beschränkt die Integralen

$$\int_{-N\delta}^{N\delta}\cdots\int_{-N\delta}^{N\delta}u_i\cdot u_j\cdot\exp B_1du_1\cdots du_m$$

etc., wie nachträglich gezeigt werden wird, z.B. ist für B_2 wegen

$$\frac{n}{6N^3}=O\Big(\frac{1}{\sqrt{n}}\Big),\;\frac{1}{\underline{|2}}B_2^2+\frac{1}{\underline{|3}}|B_2^3|+\cdots<K\Big[\frac{1}{n}+\frac{1}{n\sqrt{n}}+\frac{1}{n^2}+\cdots\Big]=\frac{K}{n-\sqrt{n}},$$

wo K die obere Grenze gewissen Polynome von $u_1,u_2,\cdots,u_m$ in W_0 bedeutet und endlich bleibt, und folglich wird das Integral noch $O\Big(\frac{1}{n}\Big)$.

$C=(c_{ij})$ und ihre Determinante $c=|c_{ij}|$ so daß

$$c_1=(u_1, u_2, \cdots, u_m)\,(c_{ij})\begin{pmatrix}u_1\\u_2\\\vdots\\u_m\end{pmatrix}.$$

Es seien $\lambda_1, \lambda_2, \cdots, \lambda_m$ die Wurzeln der charakteristische Gleichung $|C-\lambda E|=0$, welche sämtlich reel und sogar positiv sein sollen, weil $c_{ij}=c_{ji}$ und alle c_{ij} reel und $c\neq 0$, und sogar c_1 definit-positiv ist. Es gibt folglich derartige orthogonale Matrix

$$T=(\tau_{ij}), \quad |T|=1$$

daß bei linearen Transformation

$$\begin{pmatrix}u_1\\u_2\\\vdots\\u_m\end{pmatrix}=T\begin{pmatrix}u_1'\\u_2'\\\vdots\\u_m'\end{pmatrix}$$

die quadratische Form c_1 in die Form

$$\lambda_1 u_1'^2+\lambda_2 u_2'^2+\cdots+\lambda_m u_m'^2$$

übergeht. Dabei reduziert der Jacobien der Transformation

$$J=\frac{\partial(u_1, \cdots, u_m)}{\partial(u_1', \cdots, u_m')}=|\tau_{ij}|=1.$$

Daher verwandelt das Integral sich

$$\int_{-N\delta}^{N\delta}\cdots\int_{-N\delta}^{N\delta}\exp B_1\,du_1\cdots du_m=\int_{-\alpha_1}^{\alpha_1}\cdots\int_{-\alpha_m}^{\alpha_m}\exp\{-\sum_i\lambda_i u_i'^2\}\,du_1'\cdots du_m'$$
$$=\prod_{i=1}^{m}\int_{-\alpha_i}^{\alpha_i}\exp\{-\lambda_i u_i'^2\}\,du_i',$$

wobei α_i die u_i'-Ordinate derjeniges Punktes bedeutet, woran die u_i'-Achse mit Wurfelfläche $|u_j|=N\delta$ durchschneidet, unt folglich bei wachsendem n auch zu ∞ strebt. Daher strebt dus obige Integral gegen

$$\prod_{i=1}^{m}\int_{-\infty}^{\infty}\exp\{-\lambda_i u_i'^2\}\,du_i'+O\left(\frac{1}{n^\omega}\right)=\prod_{i=1}^{m}\sqrt{\frac{\pi}{\lambda_i}}+O\left(\frac{1}{n^\omega}\right)$$

Es ist aber

$$T^{-1}CT=\begin{pmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_m\end{pmatrix}$$

und deswegen

$$|c_{ij}| = \prod_{i=1}^{m} \lambda_i = \left(\frac{(-1)^{m+1}}{D}\right)^m \cdot |a_{ij}| = D^{-m+1}.$$

Also erhält man

$$\int_{-N\delta}^{N\delta} \cdots \int_{-N\delta}^{N\delta} \exp B_1 \, du_1 \cdots du_m \cong \pi^{\frac{m}{2}} D^{\frac{m-1}{2}} + O\left(\frac{1}{n^{\omega}}\right) \tag{7}$$

bei $n \to \infty$. Aus dieserben Betrachtung folgen auch sowohl

$$\begin{aligned} &\int_{-N\delta}^{N\delta} \cdots \int_{-N\delta}^{N\delta} \exp B_1 \cdot \sum_{i=1}^{m} \varphi_i(\xi) \cdot u_i \, du_1 \cdots du_m \\ &= \sum_i \varphi_i(\xi) \int_{-N\delta}^{N\delta} \cdots \int_{-N\delta}^{N\delta} u_i \exp B_1 \, du_1 \cdots du_m \\ &= \sum_i \varphi_i(\xi) \sum_k \tau_{ik} \int_{-a_1}^{\alpha_1} \cdots \int_{-\alpha_m}^{a_m} u'_k \exp \{-\sum_j \lambda_j u'^2_j\} \, du'_1 \cdots du'_m \\ &= 0 \end{aligned} \tag{8}$$

als

$$\int_{-N\delta}^{N\delta} \cdots \int_{-N\delta}^{N\delta} \exp B_1 \sum_i \sum_j \sum_k F_{ijk}(\xi) \, u_i u_j u_k \, du_1 \cdots du_m = 0. \tag{9}$$

Es bleibt noch Integrale der Restgliedern in (6) zu abschätzen. Zunächst

$$\begin{aligned} &\left| \int_{-N\delta}^{N\delta} \cdots \int_{-N\delta}^{N\delta} \exp B_1 \cdot R_1 \cdot du_1 \cdots du_m \right| \\ &\leqq \frac{1}{2N^2 |\varphi(\xi)|} \sum_i \sum_j \int_{-N\delta}^{N\delta} \cdots \int_{-N\delta}^{N\delta} |\varphi_{ij}(x') u_i u_j| \exp B_1 \, du_1 \cdots du_m \\ &\leqq \frac{M}{2N^2 |\varphi(\xi)|} \sum_i \sum_j \int_{-N\delta}^{N\delta} \cdots \int_{-N\delta}^{N\delta} |u_i| \, |u_j| \exp B_1 \, du_1 \cdots du_m \\ &= \frac{M}{2N^2 |\varphi(\xi)|} \sum_i \sum_j \int_{-\alpha_1}^{\alpha_1} \cdots \int_{-\alpha_m}^{\alpha_m} \left| \sum_k \sum_l \tau_{ik} \tau_{jl} u'_k u'_l \right| \exp [-\sum_p \lambda_p u'^2_p] \, du'_1 \cdots du'_m. \end{aligned}$$

Da aber

$$\begin{aligned} &\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |u'_h u'_l| \exp \{-\sum_p \lambda_p u'^2_p\} \, du'_1 \cdots du'_m \\ &= 4 \int_0^{\infty} u'_k \exp \{-\lambda_k u'^2_k\} \, du_k \cdot \int_0^{\infty} u'_l \exp \{-\lambda_l u'^2_l\} \, du_l \cdot \prod_{p \neq k, l}^{m} \int_{-\infty}^{\infty} \exp \{-\lambda_p u'^2_p\} \, du_p \end{aligned}$$

beschränkt bleibt, so gilt

$$\int_{-N\delta}^{N\delta} \cdots \int_{-N\delta}^{N\delta} (\exp B_1) \, R_1 \, du_1 \cdots du_m = O\left(\frac{1}{n}\right) \tag{10}$$

Ganz ebenso erhält man

$$\int_{-N\delta}^{N\delta} \cdots \int_{-N\delta}^{N\delta} (\exp B_1) \, R_2 \, du_1 \cdots du_m = O\left(\frac{1}{n\sqrt{n}}\right) \tag{11}$$

$$\int_{-N\delta}^{N\delta} \cdots \int_{-N\delta}^{N\delta} (\exp B_1) \, R_3 \, du_1 \cdots du_m = O\left(\frac{1}{n^2}\right) \tag{12}$$

Zussammengesetzt (4), (5) und (7)-(12), resultiert es, daß

$$\int \cdots \int_B \varphi(x) \cdot \exp\{n(F(x) - F(\xi))\} dx_1 \cdots dx_m \cong \frac{(2\pi/n)^{m/2}}{\sqrt{(-1)^m D}} \left\{\varphi(\xi) + O\left(\frac{1}{n}\right)\right\}.$$

Multiplizieren wir diese Gleichung mit $\{f(\xi)\}^n$, so kommt die Formel (1) sogleich.

Bemerkung. Wenn es mehre Maxima, etwa an den Punkte $\xi_1, \xi_2, \cdots, \xi_p$ geben, so braucht man den entsprechenden Wert (1) an jedem Punkt ξ_i zu schreiben und alle diese zu zussammenaddieren.

Satz 2. Die Funktionen $\varphi(x) = \varphi(x_1, \cdots, x_m)$ und $f(x) = f(x_1, \cdots, x_m)$ seien in B $(a_i \leq x_i \leq b_i\,;\ i = 1, 2, \cdots, m)$ definiert und den folgenden Bedingungen unterworfen:

1° $f(x) \geq 0$ und $\varphi(x)[f(x)]^n$ $(n = 0, 1, 2, \cdots)$ seien absolut integrabel in B.

2° Die Funktion $F(x) = \log f(x)$ erreihe denselben maximalen Wert K langs gewisser Kurve C in B: $x_1 = x_1$, $x_2 = g_2(x_1)$, $\cdots$, $\mathrm{x}_m = g_m(x_1)$, $(a_1 \leqq \alpha \leqq x_1 \leqq \beta \leqq b)$ aber in jedem Hyperplne $x_1 = Konst.$ werde $F(x)$ am Punkte $(x_1 = Konst., x_2 = g_2(x_1), \cdots, x_m = g_m(x_1))$ maximal in stärken Sinn, und in der Umgebung dieses Punktes ihre Taylorsche Entwickelung möglich.

3° $\varphi(x)$ erlaube auch Taylorsche Entwickelung in der Nähe von C.

Dann besteht

$$\int \cdots \int_B \varphi(x) [f(x)]^n dx_1 \cdots dx_m \cong \left(\frac{2\pi}{n}\right)^{\frac{m-1}{2}} K^n \left[\int_a^\beta \frac{\varphi(x_1, g_2(x_1), \cdots, g_m(x_1))}{\sqrt{(-1)^{m-1} D}} dx_1 + O\left(\frac{1}{n}\right)\right], \tag{13}$$

wobei

$$D = |a_{ij}|, \quad a_{ij} = \left[\frac{\partial^2 F(x_1, x_2, \cdots, x_m)}{\partial x_i \partial x_j}\right]_{x_k = g_k(x_1)}, \quad (2 \leq i,\ j \leq m) \quad (k = 2, 3, \cdots, m). \tag{14}$$

Beweis. Gemäß Bedingungen 1° und 2°

$$\int_{a_m}^{b_m} \cdots \int_{a_1}^{b_1} \varphi(x) [f(x)]^n dx_1 \cdots dx_m$$

$$= \int_{a_1}^{b_1} dx_1 \left[\int_{a_m}^{b_m} \cdots \int_{a_2}^{a_1} \varphi(x_1, \cdots, x_m)\right] [f(x_1, \cdots, x_m]^n dx_1 \cdots dx_m = \int_{a_1}^{b_1} I(x_1)\, dx_1$$

Dabei

$$I(x_1) = \int_{a_m}^{b_m} \cdots \int_{a_2}^{b_2} \varphi(x) [f(x)]^n dx_2 \cdots dx_m \qquad (x_1 = fest)$$

zwar genugt Bedingungen des *Satzes 1* und daraus folgt

$$I(x_1) \cong \frac{(2\pi/n)^{\frac{m-1}{2}}}{\sqrt{(-1)^{m-1} D}} K^n \left[\varphi(x_1, g_2(x_1), \cdots, g_m(x_1)) + O\left(\frac{1}{n}\right)\right], \quad \alpha \leq x_1 \leq \beta$$

Da aber für $a_1 \leq x_1 < \alpha$ oder $\beta < x_1 \leqq b_1$ ebenso wie (4), $I(x_1) = O\left(\frac{1}{n^\omega}\right)$ ist, so erhalten wir sofort (13).

Wiederholt man den Gebrauch des *Satzes 2*, so erhält man den

Satz 3. Die Funktionen $\varphi(x)$ und $f(x)$ seien in $B \subset R^m$ definiert, und

1° $f(x) > 0$, $\varphi(x)[f(x)]^n$ seien absolut integrabel in B;

2° Die Funktion $F(x) = \log f(x)$ erreihe demselben maximal Wert K in schwachen Sinn an sämtliche Punkte einer innerhalb B enthaltenen p–dimensionalen Fläche S $(p < m)$, und erlaube in der Nahe W von S ihre Taylorsche Entwickelung; aber, wenn $x_1, x_2, \cdots, x_p$ als fest in S bestimmt werden, so erreihe $F(x)$ als Funktion von $m-p$ Veränderlichen $x_{p+1}, \cdots, x_m$ ihre Maximum K in stärken Sinn am Punkt

$$x_{p+1} = g_{p+1}(x_1, x_2, \cdots, x_p), \cdots, x_m = g_m(x_1, x_2, \cdots, x_p)$$

3° $\varphi(x)$ erlaube Taylorche Entwickelung in W.

Dann gilt

$$\int_B \cdots \int \varphi(x)[f(x)]^n dx_1 \cdots dx_m \cong \left(\frac{2\pi}{n}\right)^{\frac{m-p}{2}} K^n \tag{15}$$
$$\cdot \left[\int_B \cdots \int \frac{\varphi(x_1, \cdots, x_p, g_{p+1}, \cdots, g_m)}{\sqrt{(-1)^{m-p} D}} dx_1 \cdots dx_p + O\left(\frac{1}{n}\right)\right],$$

wobei

$$g_j = g_j(x_1, \cdots, x_p), \quad D = |a_{ij}|$$
$$a_{ij} = \left[\frac{\partial^2 F(x_1, \cdots, x_p, x_{p+1}, \cdots, x_m)}{\partial x_i \partial x_j}\right]_{x_k = g_k \quad (k=p+1, \cdots, m)} \qquad p+1 \leqq i,\ j \leqq m.$$

§ 2

Wir wollen nächstens das in *Satz 1* erhaltenen $O\left(\frac{1}{n}\right)$ für den Fall $m=1$ oder 2 ausführlich ausrechnen. Bei $m=2$ sei $au^2 + 2buv + cv^2$ eine positive definite quadratische Forme mit Determinante

$$D' = \begin{vmatrix} a & b \\ b & c \end{vmatrix} > 0,$$

und braucht man den Wert des Integrales

$$J_{m,n} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u^m \cdot v^n \cdot \exp\{-au^2 + 2buv + cv^2)\} du dv \tag{16}$$

zu abschätzen, Nach einfacher Rechnung schließt man, daß

$$J_{m,n} = 0 \qquad \text{falls} \quad m+n = \text{ungerad}, \tag{17}$$

während, falls $m+n$=gerad, also m, n beide gerad oder ungerad sind,

$$(18) \qquad J_{2m',2n'} = \sum_{p=0}^{m'} \binom{2m'}{2p} a^{n'-m'} \cdot b^{2p} \frac{\pi}{(D')^{n'+p-1/2}} \cdot \frac{(2m'-2p+1)\cdots 3\cdot 1\cdot(2n'+p-1)\cdots 3\cdot 1}{2^{m'+n'}}$$

$$(19) \qquad J_{2m'+1,2m'-1} = -\sum_{p=0}^{m'} \binom{2m'+1}{2p+1} a^{m'+n'-1} b^{2p+1} \frac{\pi}{(D')^{n'+p+1/2}} \cdot \frac{(2m'-2p-1)(2m'-2p-3)\cdots 3\cdot 1\cdot(2n'+2p-1)(2n'+2p-3)\cdots 3\cdot 1}{2^{m'+n'}}$$

gelten. In *Satz 1* seien die betreffenden Funktionen $\varphi(x, y)$, $f(x, y)$ und $F(x, y) = \log f(x, y)$, so daß

$$\varphi(x, y) = \varphi(\xi, \eta) + \left\{(x-\xi)\frac{\partial}{\partial x} + (y-\eta)\frac{\partial}{\partial y}\right\} \varphi(\xi, \eta) + \frac{1}{\underline{|2}} \left\{(x-\xi)\frac{\partial}{\partial x} + (y-\eta)\frac{\partial}{\partial y}\right\}^2 \varphi(\xi, \eta) + \frac{1}{\underline{|3}} \left\{(x-\xi)\frac{\partial}{\partial x} + (y-\eta)\frac{\partial}{\partial y}\right\}^3 \varphi(x', y')$$

$$F(x, y) - F(\xi, \eta) = \frac{1}{\underline{|2}} \left\{(x-\xi)\frac{\partial}{\partial x} + (y-\eta)\frac{\partial}{\partial y}\right\}^2 F(\xi, \eta) + \frac{1}{\underline{|3}} \left\{(x-\xi)\frac{\partial}{\partial x} + (y-\eta)\frac{\partial}{\partial y}\right\}^3 F(\xi, \eta) + \frac{1}{\underline{|4}} \left\{(x-\xi)\frac{\partial}{\partial x} + (y-\eta)\frac{\partial}{\partial y}\right\}^4 F(\xi, \eta) + \frac{1}{\underline{|5}} \left\{(x-\xi)\frac{\partial}{\partial x} + (y-\eta)\frac{\partial}{\partial y}\right\}^5 F(x'', y'')$$

wobei (x', y') so wie (x'', y'') gewisse zwischene Punkte ausdrücken. Ferner setze man

$$x - \xi = u/N, \quad y - \eta = v/N, \quad \text{mit} \quad N = \sqrt{nD(\xi, \eta)/2},$$

wobei

$$D(x, y) = \begin{vmatrix} F_{xx}(x, y), & F_{xy}(x, y) \\ F_{yx}(x, y), & F_{yy}(x, y) \end{vmatrix},$$

und

$$D(\xi, y) > 0,$$

weil $\left\{(x-\xi)\frac{\partial}{\partial x} + (y-\eta)\frac{\partial}{\partial y}\right\}^2 F(\xi, \eta)$ eine negative definite quadratische Form ist. In ahnlicher Weise wie (3) zerlegt man das Integrationsbereich

$$(20) \qquad \int_{a_1}^{b_1}\int_{a_2}^{b_2} \varphi(x, y) \exp\{n[F(x, y) - F(\xi, \eta)]\} dx dy = \iint_W + \iint_{B-W} \equiv \text{(i)} + \text{(ii)},$$

so strebt (i) gegen $O\left(\frac{1}{n^\omega}\right)$ nach (4), und deswegen hat man nur (i) zu betrachten. Wird dies ausführlich geschlieben, so

$$\text{(i)} = \int_{-N\delta}^{N\delta}\int_{-N\delta}^{N\delta} \left[\varphi(\xi, \eta) + \frac{1}{N}\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\varphi(\xi, \eta) + \frac{1}{2N^2}\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)^2 \varphi(\xi, y) + \frac{1}{6N^3}\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)^3 F(\xi, \eta)\right] \cdot \exp\left[\frac{n}{2N^2}\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)^2 F(\xi, \eta) + \frac{n}{6N^3}\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)^3 F(\xi, \eta)\right.$$

$$+\frac{n}{24N^4}\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)^4 F(\xi,\eta)+\frac{n}{120N^5}\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)^5 F(x'',y'')\Big]\frac{du\,dv}{N^2}$$

Der Kürze halber bezeichne man den ersten und zweiten Faktor des Integrandes mit A und $\exp B$ bzw., und weiter

$$A=\varphi(\xi,\eta)\,[1+A_1+A_2+A_3],\quad \exp B=\exp B_1\cdot\exp B_2\cdot\exp B_3\cdot\exp B_4.$$

Erstens

$$B_1=\frac{1}{D(\xi,\eta)}\left\{F_{xx}(\xi,\eta)\cdot u^2+2F_{xy}(\xi,\eta)u\cdot v+F_{yy}(\xi,\eta)v^2\right\},$$

was wegen Bedingung 2° negativ definit ist, und folglich wird

$$(21)\qquad -B_1=au^2+2buv+cv^2 \text{ mit } a=\frac{-F_{xx}(\xi,\eta)}{D(\xi,\eta)},\quad b=\frac{-F_{xy}(\xi,\eta)}{D(\xi,\eta)},\quad c=\frac{-F_{yy}(\xi,\eta)}{D(\xi,\eta)},$$

eine positive definite quadratische Form. Daher hat man

$$(22)\qquad D'=ac-b^2=\frac{F_{xx}\cdot F_{yy}-F_{xy}^2}{[D(\xi,\eta)]^2}=\frac{1}{D(\xi,\eta)}>0 \text{ und } a<0,\ c<0.$$

Zweitens ist[3]

$$\exp B_2=1+B_2+\tfrac{1}{2}B_2^2+\cdots,\quad \exp B_3=1+B_3+\cdots,\quad \exp B_4=1+\cdots,$$

und deshalb[3]

$$\exp B=\exp B_1[1+B_2+\tfrac{1}{2}B_2^2+B_3+B_2B_3+\tfrac{1}{2}B_2^2\cdot B_3+\cdots].$$

Folglich gilt[3]

$$A\exp B=\varphi(\xi,\eta)\cdot\exp B_1[1+B_2+\tfrac{1}{2}B_2^2+B_3+A_1+A_1B_2+A_2+R_1+R_2]$$

wobei

$$\begin{aligned}R_1&=B_2B_3+\tfrac{1}{2}B_2^2B_3+\tfrac{1}{2}B_2^2A_1+A_1B_2+A_1B_2B_3+\tfrac{1}{2}B_2^2B_3A_1\\&\quad+A_2B_2+\tfrac{1}{2}A_2B_2^2+A_2B_3+A_2B_2B_3+\tfrac{1}{2}A_2B_2^2A_3,\end{aligned}$$

und

$$R_2=A_3[1+B_2+\tfrac{1}{2}B_2^2+B_3+B_2B_3+\tfrac{1}{2}B_2^2B_3+\cdots].$$

Es ist aber

$$\int_{-N\delta}^{N\delta}\int_{-N\delta}^{N\delta}R_1\exp B_1\,du\,dv=O\left(\frac{1}{n^2}\right),$$

wie man mit Benutzung von (17), (18), (19) leicht bestätigen kann: z.B., da, B_2, B_3 ein Polynom fünften Grades ist, schon verschwindet sein Integral wegen (17); auch für $A_2\cdot A_3$ gilt

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[3] Hier sind die Zahlen⋯vernachlässigbar, da, wegen denserben Grunde wie unter Futznot 2) geschliebt worden ist, die Reste $\leqq O\left(\frac{1}{n\sqrt{n}}\right)$. Im jetzigen Falle soll der Koeffizient von $O\left(\frac{1}{n\sqrt{n}}\right)$ zwar $u^m\cdot v^n$ mit ungerades $m+n$ sein, und das betreffende Integral verschwinden, so daß ⋯ sogar biszu $O\left(\frac{1}{n^2}\right)$ herabsinken.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_2 B_3 \exp B_1\, du\cdot dv$$
$$= \frac{n}{24N^6}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)^2 \varphi(\xi,\eta)\cdot\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)^2 F(\xi,\eta)\right]\exp B_1 b u dv$$

dessen Integrand ein Polynom sechsten Grades als Faktor enthält und nach (18) wird das Integral $O\left(\frac{1}{n^2}\right)$. Sowohl auch gelten

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_2\cdot\exp B_1\, du dv = O\left(\frac{1}{n^2}\right)$$

und wegen (17)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_2\cdot\exp B_1\, du dv = O, \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_1 \exp B_1\, du dv = 0 \quad \text{u. s. w.}$$

Daher, wenn man diejenigen als $O\left(\frac{1}{n}\right)$ bleibenden Glieder zussammenschreibt, so

$$\text{(i)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[G(x,y)\exp B_1 du dv + O\left(\frac{1}{n^2}\right)\right]\frac{\varphi(\xi,\eta)}{N^2}$$

wobei

$$G(x,y) = 1 + A_2 + A_1B_2 + B_2 + \tfrac{1}{2}B_2^2$$
$$= 1 + \frac{1}{nD\varphi(\xi,\eta)}\{\varphi_{xx}u^2 + 2\varphi_{xy}uv + \varphi_{yy}v^2\} + \frac{1}{6D^2 n\varphi(\xi,\eta)}[(\varphi F_{xxxx} + 4\varphi_x F_{xxx})u^4$$
$$+ (3\varphi F_{xxxy} + 12F_{xxy}\varphi_x + 4F_{xxx}\varphi_y)u^3v + (6F_{xxyy}\varphi + 12F_{xyy}\varphi_x + 12F_{xxy}\varphi_y)u^2v^2$$
$$+ (3\varphi F_{xyyy} + 4F_{yyy}\varphi_x + 12F_{xyy}\varphi_y)uv^3 + (\varphi F_{yyyy} + 4F_{yyy})v^4]$$
$$+ \frac{1}{9D^3n}[F_{xxx}^2u^6 + 6F_{xxx}u^5v + (9F_{xxy}^2 + 6F_{xxx}F_{yyy})u^4v^2 + (2F_{xxx}F_{yyy} + 18F_{xxy}F_{xyy})u^2v^3 +$$
$$(9F_{xyy}^2 + 6F_{xxy}F_{yyy})u^2v^4 + 6F_{xyy}F_{yyy}uv^5 + F_{yyy}^2v^6].$$

Alle diese zussammengefasst, kann man den wert des Integrales (20) ausfinden, wir folgt.

$$\iint \varphi(x,y)\exp[n(F(x,y) - F(\xi,\eta))]dxdy$$
$$\cong \frac{\varphi(\xi,y)}{N^2}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp B_1\, du dv + \frac{1}{n}M + O\left(\frac{1}{n^2}\right),\right.$$

wobei

$$M = \frac{1}{\varphi D}\left[\varphi_{xx}\left\{\frac{\pi}{2a(D')^{1/2}} + \frac{b^2\pi}{2a(D')^{3/2}}\right\} - 2\varphi_{xy}\frac{b\pi}{2(D')^{3/2}} + \varphi_{yy}\frac{a\pi}{2(D')^{3/2}}\right]$$
$$+ \frac{1}{6D^2\varphi}\left[(\varphi F_{xxxx} + 4\varphi_x F_{xxx})\left(\frac{3}{4}\,\frac{\pi}{a^2}\,\frac{1}{(D')^{1/2}} + \frac{3}{2}\,\frac{b^2}{a^2}\,\frac{\pi}{(D')^{3/2}} + \frac{3}{4}\,\frac{b^4}{a^2}\,\frac{\pi}{(D')^{5/2}}\right)\right.$$
$$- (3\varphi F_{xxxy} + 12F_{xxy}\varphi_x + 4F_{xxx}\varphi_y)\left\{\frac{3}{4}\,\frac{b}{a}\,\frac{\pi}{(D')^{3/2}} + \frac{3}{4}\,\frac{b^3}{a}\,\frac{\pi}{(D')^{5/2}}\right\} + (6\varphi F_{xxyy}$$

$$+12\varphi_x F_{xyy}+12F_{xxy}\varphi_y)\left\{\frac{1}{4}\frac{\pi}{(D')^{3/2}}+\frac{3}{4}b^2\frac{\pi}{(D')^{5/2}}\right\}-(3\varphi F_{xyyy}+4\varphi_x F_{yyy}$$

$$+12\varphi_y F_{yyy})\frac{3}{4}ab\frac{\pi}{(D')^{5/2}}+(\varphi F_{yyyy}+4F_{yyy}\varphi_y)\frac{3}{4}\frac{\pi}{a^2}\frac{1}{(D')^{3/2}}\Bigg]-\frac{1}{9D^3}$$

$$\times\Bigg[F^2_{xxx}\left\{\frac{15}{8}\frac{\pi}{a^3}\frac{1}{(D')^{1/2}}+\frac{45}{8}\frac{b^2}{a^3}\frac{\pi}{(D')^{3/2}}+\frac{45}{8}\frac{b^4}{a^3}\frac{\pi}{(D')^{5/2}}+\frac{15}{8}\frac{b^6}{a^3}\frac{\pi}{(D')^{7/2}}\right\}$$

$$-6F_{xxx}\cdot F_{xxy}\left\{\frac{15}{8}\frac{ab\pi}{(D')^{3/2}}+\frac{30}{8}\frac{a\,b^3\,\pi}{(D')^{5/2}}+\frac{15}{8}\frac{ab^5\pi}{(D')^{7/2}}\right\}+(9F_{xxy}^{\,2}+6F_{xxx}F_{xyy})$$

$$\times\left\{\frac{3}{8}\frac{1}{a}\frac{\pi}{(D')^{3/2}}+\frac{18}{8}\frac{b^2}{a}\frac{\pi}{(D')^{5/2}}+\frac{15}{8}\frac{b^4}{a}\frac{\pi}{(D')^{7/2}}\right\}-(2F_{xxx}F_{yyy}+18F_{xxy}F_{xyy})$$

$$\times\left\{\frac{9}{8}\frac{a^2b\pi}{(D')^{5/2}}+\frac{15}{8}\frac{a^2b^3\pi}{(D')^{7/2}}\right\}+(9F^2_{yyy}+6F_{xxy}F_{yyy})\left\{\frac{3}{8}\frac{a\pi}{(D')^{5/2}}+\frac{30}{8}ab^2\frac{\pi}{(D')^{7/2}}\right\}$$

$$-\frac{9}{4}F_{xyy}F_{yyy}\frac{a^2b\pi}{(D')^{7/2}}+F^2_{yyy}\frac{15}{8}\frac{\pi}{a_3}\frac{1}{(D')^{1/2}}\Bigg].$$

Es ist nach (7)

$$\int_{-\infty}^{\infty}\exp B_1\,dudv=\sqrt{\pi}\;D$$

und, noch mit Benutzung von (21) (22), kann das Resultät folgendenmaßen formuliert werden.

Satz 4. Das Integral (1) des *Satzes 1.* bei $m=2$, ausgerechnet genau bis zu $O\left(\frac{1}{n}\right)$, lautet

$$(23)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(x,y)\,[f(x,y)]^n dxdy\cong\frac{2\pi[f(\xi,\eta)]^n}{n\sqrt{D(\xi,\eta)}}\left[\varphi(\xi,\eta)+\frac{1}{n}\psi(\xi,\eta)+O\left(\frac{1}{n^2}\right)\right],$$

wobei

$$\psi(\xi,\eta)=\left[-\frac{1}{2}\frac{\varphi_{xx}}{F_{xx}}+\frac{1}{8}(\varphi F_{xxxx}+4\varphi_x F_{xxx})\frac{1}{F^2_{xx}}+\frac{5}{24}F^2_{xxx}\frac{\varphi}{(F_{xx})^3}\right]$$

$$+\frac{1}{D}\Bigg[-\frac{\varphi_{xy}}{2}\cdot\frac{F^2_{xy}}{F_{xx}}+\varphi_{xy}F_{xy}-\varphi_{yy}F_{xx}+\frac{F^2_{xy}}{4F^2_{xx}}(\varphi F_{xxxx}+4\varphi_x F_{xxx})-\frac{1}{2}\frac{F_{yy}}{F_{xx}}\bigg(\frac{3}{4}\varphi F_{xxxy}$$

$$+3F_{xxy}\varphi_x+F_{xxx}\varphi_y\bigg)+\frac{1}{2}\left(\frac{\varphi}{2}F_{xxyy}+\varphi_x F_{xyy}+F_{xxy}\varphi_y\right)+\frac{5}{8}\frac{F_{xy}^{\,2}}{F_{xx}^{\,3}}F_{xxx}^{\,2}\cdot\varphi$$

$$+\frac{1}{24}(9F_{xxy}^{\,2}+6F_{xxx}F_{xyy})\frac{\varphi}{F_{xx}}\Bigg]+\frac{1}{D^2}\Bigg[-\frac{F_{xy}^{\,3}}{2F_{xx}}\left(\frac{3}{4}\varphi F_{xxxy}+3F_{xxy}\varphi_x+F_{xxx}\varphi_y\right)$$

$$+\frac{1}{8}\frac{F_{xy}^{\,4}}{F_{xx}^{\,2}}(\varphi F_{xxxx}+4\varphi_x F_{xxx})+\frac{3}{2}F_{xy}^{\,2}(\tfrac{1}{2}\varphi F_{xxyy}+\varphi_x F_{yyy}+F_{xxy}\varphi_y)$$

$$-\tfrac{1}{2}F_{xx}F_{xy}\left(\frac{3}{4}\varphi F_{xyyy}+\varphi_x F_{yyy}+3F_{xxy}\varphi_y\right)+\frac{1}{2F_{xx}^{\,2}}(\varphi F_{yyyy}+F_{yyy}\varphi_y)$$

$$+\frac{5}{8}\frac{F_{xy}^{\,4}}{F_{xx}^{\,3}}F_{xxx}^{\,2}\varphi+\frac{3}{4}\frac{F_{xy}^{\,2}}{F_{xx}}(3F_{xxy}^{\,2}+2F_{xxx}F_{xyy})\varphi+\frac{F_{xx}}{8}(3F_{yyy}^{\,2}+2F_{xxy}F_{yyy})\varphi\Bigg]$$

$$+\frac{1}{D^3}\Bigg[\frac{5}{24}\frac{F_{xy}^{\,6}}{F_{xx}^{\,3}}F_{xxx}^{\,2}\varphi+\frac{5}{24}\frac{F_{yyy}^{\,2}}{F_{xx}^{\,3}}+\frac{5}{8}\frac{F_{xy}^{\,4}}{F_{xx}}(3F_{xxy}^{\,2}+2F_{xxx}F_{xyy})\varphi$$

$$+\frac{5}{4}F_{xx}F_{xy}^{3}(F_{yyy}^{2}+F_{xxy}F_{yyy})+\frac{9}{32}F_{xx}^{2}F_{xy}^{2}F_{xyy}F_{yyy}\Big]$$

$$+\frac{1}{D^4}\Big[\frac{5}{4}F_{xxx}F_{xxy}F_{xx}F_{xy}\varphi+\frac{1}{4}F_{xx}^{2}F_{xy}(F_{xxx}F_{yyy}+9F_{xxy}F_{yyy})\varphi\Big]$$

$$+\frac{1}{D^5}\Big[\frac{5}{2}F_{xxx}F_{xxy}F_{xx}F_{xy}^{3}\varphi+\frac{5}{12}F_{xx}^{2}F_{xy}^{3}(F_{xxx}F_{yyy}+9F_{xxy}F_{xyy})\varphi\Big]$$

$$+\frac{1}{D^6}\Big[\frac{5}{4}F_{xxx}F_{xxy}F_{xx}F_{xy}^{5}\varphi\Big]$$

Der Fall $m=1$. Wenn $\varphi(x, y)$ und $f(x, y)$ tatsächlich von y frei, also lediglich in $\varphi(x)$ und $f(x)$ ausarten, alle Ableitungen $\varphi_y, \varphi_{xy}, \varphi_{yy}, \cdots, f_y, f_{yx}, \cdots$ verschwinden. Somit erhalt man aus (23) den[4]

Satz 5. Die Funktionen $\varphi(x)$, $F(x)$ und $f(x)=\exp F(x)$ seien im endlichen oder unendlichen Intervall $a \leq x \leq b$ definiert und den folgenden Bedingungen unterworfen.

1° $\varphi(x)[f(x)]^n$ sei absolut integrabel in $a \leq x \leq b$, $n=0, 1, 2, \cdots$.

2° Die Funktion $F(x)=\log f(x)$ erreihe an einer Stelle ξ im Innern von a, b ihr Maximum, und zwar sei die obere Grenz von $F(x)$ in jedem abgeschlossenen Intervall, das ξ nicht enthalt, kleiner als $F(\xi)$, ferner gebe es eine Umgebung von ξ, wo $F''(x)$ existiert und stetig ist. Endlich sei $F''(\xi)<0$.

3° $\varphi(x)$ sei stetig für $x=\xi$, $\varphi(\xi) \neq 0$.

Dann gilt für n → ∞ die folgende asymptotische Formel:

$$\int_a^b \varphi(x)\,[f(x)]^n dx \cong \sqrt{\frac{-2\pi}{nF''(\xi)}}\,[f(\xi)]^n \left\{\varphi(\xi)+\frac{1}{n}\psi(\xi)+O\left(\frac{1}{n^2}\right)\right\}, \tag{25}$$

wobei

$$\psi(\xi)=\frac{1}{2F''(\xi)}\left[-\varphi''(\xi)+\frac{\varphi'(\xi)F'''(\xi)}{F''(\xi)}+\frac{\varphi(\xi)F''''(\xi)}{4F'''(\xi)}+\frac{5}{12}\varphi(\xi)\left(\frac{F'''(\xi)}{F''(\xi)}\right)^2\right].$$

Der schreiber beendigen diese Schrift mit seinen herzlichen Dank für Prof. Y. Watanabes lehrreiche Suggestion.

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[4] Die Aussage Satzes 5. ist aus Polya und Szegö, loc. cit. Kopiert, darin, aber unsere vorgedachte Annahme etwas verringert worden ist. Um die Abschätzung des $O\left(\frac{1}{n}\right)$, &.c. zu ausfuhren können, haben wir einigermaßen vermehrte Voraussetzung gemacht.

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ANALYSES OF BIMODAL DISTRIBUTIONS0)

(ON THE DECOMPOSITION OF A BIMODAL DISTRIBUTION INTO TWO NORMAL CURVES)

By

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§1. Introduction

As well known, if a random variable z be a sum of two independent normal variables x and y, so also the former shall be normal, and vice versa. Our present problem differs from this, and rather relates to the so-called general normal distribution.1) In the operation of convolution the problem is to add independent variables $x_1+x_2=x$, so to speak, while our problem is concerned with the super-

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0) This research was done under the sponsorship of the Kōraku Conference of Mr. Y. Miki.

1) Just as the general Poisson's distribution is defined as $F(\lambda)=\int\frac{\lambda^x e^{-\lambda}}{x!}p(\lambda)d\lambda$, e.g. K. Kunizawa, Modern Theories of Probabilities, (Japanese), 1951, p. 75, we may conceive the general normal distribution $F(x)=\int\int\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}p(a,\sigma)\,da\,d\sigma$.

positions $y_1+y_2=y$. The independent variable belongs to one or the other of the two normal distributions $N(x_1, a_1, \sigma_1)$ and $N(x_2, a_2, \sigma_2)$ with certain probabilities p_1 and $p_2(=1-p_1)$, so that the resulting statistics consists in a mixture of x_1 and x_2 with rates r_1 and r_2, each proportional to p_1 and p_2. From a given actual statistics we need to estimate those unknowns a_1, a_2, σ_1, σ_2, r_1, r_2 and in particular, when the given distribution is bimodal, although this is not sometimes apparently disclosed, if the difference $|a_1-a_2|$ is small enough, or one of r_1, r_2 quite large compared with the other.

§2. Preliminary Computations

In actual statistics frequently the distribution appears to be a superposition of two unimodal curves.[2] In the present note we shall mainly treat of the case, where two components are normal. So the presumed representation is of the form

$$\begin{aligned} y=Nf(x)&=\frac{n_1}{\sqrt{2\pi}\sigma_1}\exp\left\{-\frac{(x-a_1)^2}{2\sigma_1^2}\right\}+\frac{n_2}{\sqrt{2\pi}\sigma_2}\exp\left\{-\frac{(x-a_2)^2}{2\sigma_2^2}\right\} \qquad (1)\\ &=n_1f_1(x)+n_2f_2(x), \end{aligned}$$

where n_1, n_2 (unknown) and $n_1+n_2=$ the whole frequency N (known) denote the nomber of each component and the all, respectively. Since we are mainly concerned with the bimodal distribution, it shall be understood that $a_1 \neq a_2$. Also we are interested in the case when n_1, n_2 both >0, since, otherwise, the problem becomes an algebraical superposition, as difference, we exclude such sorts of representations. To estimate parameters, following Pearson's method of moments, it is usually said that we shall have only an enough number of moments equations to determine parameters—thus in the present case, up to the fifth moment, since there are six unknowns in (1). However, to speak more exactly, further (higher) moments equations should be also satisfied by thus determined values of parameters. Hence we ought to tackle necessarily to solve many moments equations by the method of least squares. Indeed, usual Pearson like treatment is nothing but its a first approximation. Nevertheless, since these calculations are intricate enough in the present state that culculating machines are of still lower capacity, we are obliged to put up with the first approximation.

Let ν_k be the k-th moment of (1) about $x=0$, thus

$$\nu_k=\sum_{i=1,2}\int_{-\infty}^{\infty} n_i x^k f_i(x)dx=\sum_{i=1,2} n_i\int_{-\infty}^{\infty}(x-a_i+a_i)^k f_i(x)dx=\sum_{i=1,2} n_i\sum_{h=0}^{k}\binom{k}{h}a_{i\,k-h}\,\mu_{ih},$$

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[2] Y. Watanabe, Bimodal Distributions, this Journal, vol. V (1954), p. 29.

where $\mu_{ih}=\int_{-\infty}^{\infty}(x-a_i)^h f_i(x)\,dx$, so that,

$$\mu_{i0}=1,\quad \mu_{i1}=0,\quad \mu_{i2}=\sigma_i^2,\quad \mu_{i3}=0,\ \mu_{i4}=3\sigma_i^4,\quad \mu_{i5}=0.$$

More in detail

$$\nu_0=n_1+n_2=N,\ \nu_1=n_1a_1+n_2a_2,\ \nu_2=n_1(a_1^2+\sigma_1^2)+n_2(a_2^2+\sigma_2^2),\ \nu_3=\sum_{i=1,2}n_i(a_i^3+3a_i\sigma_i^2),$$
$$\nu_4=\sum n_i(a_i^4+6a_i^2\sigma_i^2+3\sigma_i^4),\quad \nu_5=\sum_{i=1,2}n_ia_i(a_i^4+10a_i^2\sigma_i^2+15\sigma_i^4).$$

Further on setting $n_i/N=r_i$ and assuming that the mean of the whole distribution is taken as origin, the k-th moment about mean, μ_k, would be

$$\left.\begin{array}{l}\mu_0=r_1+r_2=1,\quad \mu_1=r_1a_1+r_2a_2=0,\quad \mu_2=r_1(a_1^2+\sigma_1^2)+r_2(a_2^2+\sigma_2^2),\\ \mu_3=r_1(a_1^3+3a_1\sigma_1^2)+r_2(a_2^3+3a_2\sigma_2^2),\quad \mu_4=\sum_{i=1,2}r_i(a_i^4+6a_i^2\sigma_i^2+3\sigma_i^4),\\ \mu_5=\sum_{i=1,2}r_ia_i(a_i^4+10a_i^2\sigma_i^2+15\sigma_i^4).\end{array}\right\}\tag{2}$$

Case I. If it happens that μ_3 and μ_5 are nearly zero, we may put simply $r_1=r_2=\frac{1}{2}$, $a_1=-a_2=a$, and $\sigma_1=\sigma_2=\sigma$ by symmetry. We have only to solve $\mu_2=a^2+\sigma^2$, $\mu_4=a^4+6a^2\sigma^2+3\sigma^4$, which yield immediately

$$a^4=\tfrac{3}{2}\mu_2^2-\tfrac{1}{2}\mu_4,\quad \sigma^2=\mu_2-a^2.\tag{3}$$

Case II. If the assumption $\sigma_1=\sigma_2=\sigma$ be still granted, but not symmetry $(r_1\neq r_2)$, we have five unknowns, and equations (2) degenerate into

$$\mu_2=\sigma^2+r_1a_1^2+r_2a_2^2,\quad \mu_3=r_2a_1^3+r_2a_2^3,\quad \mu_4=r_1a_1^4+r_2a_2^4+6\mu_2\sigma^2-3\sigma^4,$$

besides $r_1+r_2=1$, $r_1a_1+r_2a_2=0$. In this case, firstly the variance σ^2 shall be found from the cubic equation

$$2(\mu_2-\sigma^2)^3-(3\mu_2^2-\mu_4)(\mu_2-\sigma^2)-\mu_3^2=0,\tag{4}$$

i.e. $$(\sigma^2)^3-3\mu_2(\sigma^2)^2+\tfrac{1}{2}(\mu_4+3\mu_2^2)\sigma^2+\tfrac{1}{2}(\mu_3-\mu_2\mu_4+\mu_2^3)=0,\tag{5}$$

and secondly the proportion ratio $r_1/r_2=q(>0)$ from the quadratic equation

$$\frac{(q-1)^2}{q}=\frac{\mu_3^2}{(\mu_2-\sigma^2)^3},\tag{6}$$

and consequently r_1, r_2, a_1, a_2 can be all determined. Observing that the left-hand side of (6) is positive, the right-hand side must be the same, and accordingly $\mu_2>\sigma^2$ should hold, and whence by (4) it follows that

$$(\mu_2-\sigma^2)^2>\tfrac{1}{2}(3\mu_2^2-\mu_4).\tag{7}$$

If this inequality does not hold, such σ^2 should be abandoned. As to signs of a_1, a_2 we must choose such a pair as makes the equality in regard to μ_3 consistent.

Case III. In general $\sigma_1 \neq \sigma_2$. From the first two equations of (2) we get

$$r_1 = \frac{-a_1}{a_1 - a_2}, \quad r_2 = \frac{a_1}{a_1 - a_2} \qquad (a_1 \neq a_2). \tag{8}$$

Whence, putting

$$a_1 + a_2 = s, \quad a_1 a_2 = p, \tag{9}$$

yield

$$r_1 a_1^2 + r_2 a_2^2 = -p, \quad r_1 a_1^3 + r_1 a_2^3 = -sp, \quad r_1 a_1^4 + r_2 a_2^4 = -p(s^2 - p),$$

and

$$r_1 a_1^5 + r_2 a_2^5 = -sp(s^2 - 2p).$$

Further, upon writing

$$a_i^2 + \sigma_i^2 = b_i, \qquad (i = 1, 2) \tag{10}$$

the remaining equations of (2) reduce to

$$\left.\begin{aligned} &\mu_2 = r_1 b_1 + r_2 b_2. \ \ \mu_3 = 3(r_1 a_1 b_1 + r_2 a_2 b_2) + sp, \ \ \mu_4 = 3(r_1 b_1^2 + r_2 b_2^2) + p(s^2 - p), \\ &\mu_5 = 15(r_1 a_1 b_1^2 + r_2 a_2 b_2^2) - 20(r_1 a_1^3 + r_2 a_2^3) - 6sp(s^2 - 2p). \end{aligned}\right\} \tag{11}$$

From the first two of (11), we obtain

$$b_i = \mu_2 - \frac{1}{3a_j}(\mu_3 - 2sp), \qquad i, j = 1, 2 \quad (i \neq j) \tag{12}$$

and on substituting these in the last two equations of (11),

$$\left.\begin{aligned} &6p^3 - 2s^2p^2 + (3\mu_4 - 9\mu_2^2 - 4s\mu_3)p + \mu_3^2 = 0, \\ &4sp^3 - (2s^3 + 20\mu_3)p^2 + (3\mu_5 - 30\mu_2\mu_3)p + 5\mu_3^2 s = 0. \end{aligned}\right\} \tag{13}$$

First, eliminating p^3 from (13) and second, eliminating p^0, we have

$$\left.\begin{aligned} &\alpha p^2 + \beta p + \gamma \equiv 2(s^3 + 30\mu_3)p^2 - (8\mu_3 s^2 + 6Bs - 9C)p - 13\mu_3^2 s = 0, \\ &\alpha' p^2 + \beta' p + r' \equiv 26sp^2 - 4(2s^3 - 5\mu_3)p - (20\mu_3 s^2 + 15Bs - 3C) = 0. \end{aligned}\right\} \tag{14}$$

where

$$B = 3\mu_2^2 - \mu_4 \,(\text{Biquadratic}). \quad C = 10\,\mu_2\mu_3 - \mu_5 \,(\text{Cinq}). \tag{15}$$

Third, eliminating p^2 between (14), also p^0, respectively, we get

$$(\alpha\beta' - \alpha'\beta)p = \gamma\alpha' - \gamma'\alpha, \quad (\gamma\alpha' - \gamma\alpha')p^2 = (\beta\gamma' - \beta'\gamma)p, \tag{16}$$

and whence finally

$$(\alpha\beta' - \alpha'\beta)\,(\beta\gamma' - \beta'\gamma) = (r\alpha' - \gamma'\alpha)^2. \tag{17}$$

If $\alpha, \beta, \cdots, \gamma'$, coefficients in (14) be fully written up, (16) yields

$$p=\frac{20\mu_3s^5+15Bs^4-3Cs^3+431\mu_3^2s^2+450B\mu_3s-90C\mu_3}{-[8s^6+116\mu_3s^3-78Bs^2+117Cs-600\mu_3^2]}$$

$$=\frac{56\mu_3^2s^4+240\mu_3Bs^3+(90B^2-204C\mu_3)s^2+(260\mu_3^3-153BC)s+27C^2}{2(\text{the expression in numerator of first fraction})}, \tag{18}$$

which must be negative, since a_1, a_2 are to have different signs.

From (18) we obtain, as the detailed form of (17),

$$s=0 \quad \text{and} \quad \sum_{m=0}^{9} A_m s^{9-m}=0, \tag{19}$$

an equation of ninth degree in s, where

$$\left.\begin{array}{l} A_0=312\mu_3^2, \quad A_1=780\mu_3B, \quad A_2=292.5B^2-468\mu_3C, \\ A_3=10764\mu_3^3-351BC, \quad A_4=21333B\mu_3^2+58.5C^2, \\ A_5=4680B^2\mu_3-7371C\mu_3^2 \quad A_6=92020.5\mu_3^4+3861BC\mu_3-1755B^3, \\ A_7=152880B\mu_3^3+5616B^2C-4914\mu_3C^2, \\ A_8=87750B^2\mu_3^2-585C\mu_3^2-5001.75BC^2, \\ A_9=789.75C^3-17550\mu_3^2BC-39000\mu_3^5. \end{array}\right\} \tag{20}$$

Here all A_m are homogeneous expressions of degree $m+6$, because B and C are defined as (15). To compute these coefficients, it will be convenient to tabulate the requisite values of $B^lC^m\mu_3^n$ for $l, m, n=0, 1, 2, \cdots$, by preliminary calculations.

When a value of s is found from (19) and the corresponding values of p from (18), and if p be negative, a_1 and a_2 could be calculated by (9), and whence r_1, r_2 by (8); further b_1, b_2 by (12) and finally σ_1^2, σ_2^2 by (10). Thus all unknowns would be completely determind.

Lastly we should try the χ^2– or ω^2– test to examine the goodness of fit; these are illustrated in §5, §6 by examples. We have met $s=0$ at (19), namely $a_1+a_2=0$. Hence we have a special

Case IV. $r_1=r_2$, $a_1=-a_2=a$, $\sigma_1 \neq \sigma_2$.

In this case we obtain from (2)

$$\mu_2=a^2+\tfrac{1}{2}(\sigma_1^2+\sigma_2^2), \ \mu_3=\tfrac{3}{2}a(\sigma_1^2-\sigma_2^2), \ \mu_4=a^4+3a^2(\sigma_1^2+\sigma_2^2)+\tfrac{3}{2}(\sigma_1^4+\sigma_2^4). \tag{21}$$

Consequently $\sigma_1^2+\sigma_2^2=2(\mu_2-a_2)$, $\sigma_1^2-\sigma_2^2=\frac{3}{2}(\mu_3/a)$, and therefore

$$\sigma_1^2=\mu_2-a^2+\mu_3/3a, \quad \sigma_2^2=\mu_2-a^2-\mu_3/3a. \tag{22}$$

These being substituted in μ_4, we get an equation, cubic in a^2:

$$a^6+\tfrac{1}{2}(\mu_4-3\mu_2^2)a^2-\tfrac{1}{6}\mu_3^2=0, \tag{23}$$

from which a can be always found, and whence σ_1^2, σ_2^2 by (22).

§3. Alternative Formulas

We may also alternatively proceed as follows: Let

$$\left.\begin{aligned} &\frac{r_1}{r_2}=q\ (>0),\ \text{so that}\ r_1=\frac{q}{1+q},\ r_2=\frac{1}{1+q}, \\ \text{and}\qquad &a_1=a,\quad a_2=-aq. \end{aligned}\right\} \tag{24}$$

These being substituted in μ_2, μ_3 of (2), we have

$$\mu_2=qa^2+\frac{q\sigma_1^2+\sigma_2^2}{1+q},\quad \mu_3=q(1-q)a^3+\frac{3aq}{1+q}(\sigma_1^2-\sigma_2^2).$$

Whence

$$\sigma_1^2=\mu_2+\frac{\mu_3}{3aq}-\frac{a^2}{3}(1+2q), \tag{25}$$

and

$$\sigma_2^2=\mu_2-\frac{\mu_3}{3a}-\frac{a^2}{3}(2+q)q. \tag{26}$$

Again these being substituted in μ_4 of (2), we obtain

$$2q^2(1+q+q^2)a^6-4q(1-q)\mu_3a^3+3q(\mu_4-3\mu_2^2)a^2-\mu_3^2=0, \tag{27}$$

and $$2q(1-q)(1+q^2)a^6-15q(1-q)\mu_2a^4-20\mu_3a^3-3(10\mu_2\mu_3-\mu_5)a-5(1-q)\mu_3^2=0. \tag{28}$$

If we eliminate q between (27), (28), we shall obtain, besides $a=0$ an equation of 30-th degree in a, while, if a be eliminated, besides $q=0$, an equation of 27-th degree in q; thus both are impracticable, unless by means of electronic computer & *c*. However, from above we may deduce some special cases.

Case V. When q is known. In this case a can be found from (27), and consequently σ_1, σ_2 from (25), (26); of course r_1, r_2 from (24). It is noteworthy that there is no need of μ_5 here.

Case VI. When $a_1=a$ is known. Rewriting (27) in the form

$$q^4+q^3+\left(1+\frac{2\mu_3}{a^3}\right)q^2-\left(\frac{2\mu_3}{a^3}+\frac{3(3\mu_2^2-\mu_4)}{2a^4}\right)q-\frac{\mu_3^2}{2a^6}=0,$$

which permits at least one positive root. The remaining calculations are the same as Case V.

Case VII. When one of S.D. e.g. σ_2 is known. In this case, we may eliminate q between (26), (27) and obtain, after easy but somewhat lengthy calculations, an equation of 10-th degree in $a(=a_1)$:

$$a^{10}-\frac{\mu_3}{Q}a^9+\frac{11}{7}Qa^8-2\mu_3a^7+\frac{1}{28}\left[25Q^2-24B+\frac{26\mu_3^2}{Q}\right]a^6-\frac{25}{14}\mu_3Qa^5+\left(\frac{5}{4}\mu_3^2-\frac{45}{14}BQ\right)a^4$$
$$-\frac{3}{7}\left(\frac{\mu_3}{4Q}-12B\right)\mu_3a^3+\frac{9}{4}B\left(\frac{B}{28}-\frac{\mu_3^2}{Q}\right)a^2-\frac{9B^2\mu_3}{112Q}a-\frac{\mu_3^4}{112Q}=0,$$

where $Q=3(\mu_2-\sigma_2^2)$, $B=3\mu_2^2-\mu_4$. This equation has surely one positive and one negative root at least. With a thus obtained, q can be computed from (26), accordingly σ_1 from (25) and r_1, r_2, a_2 from (24). In general, formulas in this section are rather intricate except that in Case V, which is effective in some special example (cf. Ex. 7 in §6). But before we apply the above methods to actual examples, we shall still discuss the Case VIII, that $a_1=a_2$.

§4. The Case with Common Mean

In the foregoing we have assumed that $a_1\neq a_2$. Now let us treat briefly the case where two normal components have the same mean $a_1=a_2=a$, but with different variances. Taking the common mean as origin, the superposed one becomes

$$y=y_1+y_2=\frac{r_1}{\sqrt{2\pi}\sigma_1}\exp\left\{-\frac{x^2}{2\sigma_1^2}\right\}+\frac{r_2}{\sqrt{2\pi}\sigma_2}\exp\left\{-\frac{x^2}{2\sigma_2^2}\right\}, \tag{29}$$

where $r_1+r_2=1$ and $\sigma_1\neq\sigma_2$. The central moments become

$$\mu_{2k}=(2k-1)(2k-3)\cdots 3\cdot 1\,(r_1\sigma_1^{2k}+r_2\sigma_2^{2k}),\qquad \mu_{2k+1}=0.$$

Hence, to find r_1, r_2, σ_1, σ_2 we ought to utilize the first four even ordered moments:

$$r_1+r_2=\mu_0=1,\quad r_1\sigma_1^2+r_2\sigma_2^2=\mu_2,\quad r_1\sigma_1^4+r_2\sigma_2^4=\frac{1}{3}\mu_4,\quad r_1\sigma_1^6+r_2\sigma_2^6=\frac{1}{15}\mu_6.$$

Or. putting $\frac{r_1}{r_2}=q$, we have

$$r_1=\frac{q}{1+q},\quad r_2=\frac{1}{1+q}; \tag{30}$$

and
$$q\sigma_1^2+\sigma_2^2=(1+q)\mu_2,\quad q\sigma_1^4+\sigma_2^4=\frac{1+q}{3}\mu_4,\quad q\sigma_1^6+\sigma_2^6=\frac{1+q}{15}\mu_6.$$

Whence

$$q=\frac{\mu_2-\sigma_2^2}{\sigma_1^2-\mu_2}=\frac{\mu_4/3-\sigma_2^4}{\sigma_1^4-\mu_4/3}=\frac{\mu_6/15-\sigma_2^6}{\sigma_1^6-\mu_6/15}, \tag{31}$$

where $q\neq 0,\infty$, so that either numerator or denominator cannot vanish separately. and also simultaneously, since then $\sigma_1=\sigma_2$ contradictory to hyporhesis. Hence

$$(\sigma_1^2-\mu_2)\sigma_2^4-(\sigma_1^4-\frac{\mu_4}{3})\sigma_2^2+\mu_2\sigma_1^4-\frac{1}{3}\mu_4\sigma_1^2=0, \tag{32}$$

and
$$\left(\sigma_1^4-\frac{\mu_4}{3}\right)\sigma_2^4-\left[\sigma_1^6+\mu_2\sigma_1^4-\frac{1}{3}\mu_4\sigma_1^2-\frac{1}{15}\mu_6\right]\sigma_2^2+\mu_2\sigma_1^6-\frac{1}{15}\mu_6\sigma_1^2=0. \tag{33}$$

If we eliminate σ_2^4 between (32), (33), we shall obtain

$$(\sigma_1^2-\sigma_2^2)\left[\left(\mu_2^2-\frac{1}{3}\mu_4\right)\sigma_1^4+\left(\frac{1}{15}\mu_6-\frac{1}{3}\mu_2\mu_4\right)\sigma_1^2+\frac{1}{9}\mu_4^2-\frac{1}{15}\mu_2\mu_6\right]=0. \tag{34}$$

Since $\sigma_1 \neq \sigma_2$, the secoud factor must vanish. Again (32) can be written as

$$(\sigma_2^2-\mu_2)\sigma_1^4-\left(\sigma_2^4-\frac{\mu_4}{3}\right)\sigma_1^2+\mu_2\sigma_2^4-\frac{1}{3}\mu_4\sigma_2^2=0. \tag{35}$$

Now eliminating σ_1^4 between (34) and (35), we obtain

$$(\sigma_1^2-\mu_2)\left[\left(\mu_2^2-\frac{1}{3}\mu_4\right)\sigma_2^4-\left(\frac{1}{3}\mu_2\mu_4-\frac{1}{15}\mu_6\right)\sigma_2^2+\frac{1}{9}\mu_4^2-\frac{1}{15}\mu_2\mu_6\right]=0. \tag{36}$$

But, since $\sigma_1^2 \neq \mu_2$ by (31), and we see by (34) and (36) that σ_1 as well as σ_2 should be 2 roots of the same equation

$$\left(\mu_2^2-\frac{1}{3}\mu_4\right)\sigma^4+\left(\frac{1}{15}\mu_6-\frac{1}{3}\mu_2\mu_4\right)\sigma^2+\frac{1}{9}\mu_4^2-\frac{1}{15}\mu_2\mu_6=0. \tag{37}$$

Specially, provided every coefficient in (37) vanishes, then it follows that $\mu_4=3\mu_2^2$ and $\mu_6=15\mu_2^3$, which implies that the given distribution is already normal as a whole, and there is no need to be decomposed.

Ex. 1. In a certain sampling distribution of means, the moments were obtained as in the following table (odd ordered moments are known to be zero). To decompose it into two normal distributions.

$\bar{x}-\tilde{x}$ central value	f in %	$u=\dfrac{\bar{x}-\tilde{x}}{0.5}$	fu^2	fu^4	fu^6
0	21.94	0	0	0	0
± 0.5	17.97	± 1	17.97	17.97	17.97
± 1.0	10.63	± 2	42.52	170.08	680.32
± 1.5	5.48	± 3	49.32	443.88	3994.92
± 2.0	2.78	± 4	44.48	711.68	11386.88
± 2.5	1.33	± 5	33.25	831.25	20781.25
± 3.0	0.56	± 6	20.16	725.76	26127.36
± 3.5	0.20	± 7	9.80	480.20	23529.80
± 4.0	0.06	± 8	3.84	245.76	15728.64
± 4.5	0.02	± 9	1.62	131.22	10628.82
sum	$N=100.00$		222.96×2	3757.80×2	112875.96×2
$\div N$	$\mu_0'=1$		$\mu_2'=4.4592$	$\mu_4'=75.1560$	$\mu_6'=2257.5192$

Performing Sheppard's corrections, we get

$$\mu_2=\mu_2'-0.0833=4.3759, \quad \sqrt{\mu_2}=2.0919,$$

$$\mu_4=\mu_4'-\frac{1}{2}\mu_2-0.0125=72.9566,$$

$$\mu_6=\mu_6'-\frac{5}{4}\mu_4-\frac{3}{4}\mu_2-\frac{1}{448}=2166.3213.$$

Substituting these values in (37) we obtain

$$5.1704\sigma^4-38.0043\sigma^2+26.1226=0, \text{ or } \sigma^4-7.3504\sigma^2+5.0523=0,$$

whence $\sigma_1^2=6.651,\ \sigma_2^2=0.6984$ and $\sigma_1=2.579,\ \sigma_2=0.8357.$

Hence by (31) $q=\frac{\mu_2-\sigma_2^2}{\sigma_1^2-\mu_2}=1.616$ and by (30) $r_1=0.618$, $r_2=0.382$.

Thus the given distribution seems to be a mixture of two samples whose proportion is about 3 : 2 and with different variances, $\sigma_1^2 : \sigma_2^2=1 : 10$.

To test its legitimacy, we try e.g. χ^2–test. The above result gives as its representation

$$\begin{aligned}\tilde{y}&=\frac{61.8}{2.579\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{u}{2.579}\right)^2\right\}+\frac{38.2}{0.8357\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{u}{0.8357}\right)^2\right\}\\ &=2.396\,\varphi(t_1)+4.571\,\varphi(t_2)=\tilde{y}_1+\tilde{y}_2,\end{aligned}$$

where $t_1=\frac{u}{2.579}$, $t_2=\frac{u}{0.8357}$ and $\varphi(t)=\frac{1}{\sqrt{\pi 2}}e^{-t^2/2}$. By use of the $\varphi(t)$–Table we computed the values of $\tilde{y}_{1j}$, $\tilde{y}_{2j}$, and $\tilde{y}_j$ for $u=j=0, \pm 1, \pm 2, \cdots$,

u	0	± 1	± 2	± 3	± 4	± 5	± 6	± 7	± 8	± 9	total
obs. y	21.94	17.97	10.63	5.48	2.78	1.33	0.56	0.20	0.06	0.02	100.00
cal. $\tilde{y}$	27.80	17.82	8.12	4.89	2.87	1.46	0.64	0.24	0.08	0.02	100.08

Whence it is found that $\chi^2=\sum|y-\tilde{y}|^2/\tilde{y}=3.270$. Here degrees of freedom being $10-4=6$, $Pr(\chi^2\geqq 3.270)>Pr(\chi^2\geqq 3.83)=0.7>0.05$, and the representation is not to be rejected.

To speak more precisely, we ought to use the Table of normal integral $\int_{-\infty}^{t}\varphi(t)dt=\Phi(t)$ and to calculate $Nr_i\left[\Phi\left(t_{ij}+\frac{1}{2\sigma_i}\right)-\Phi\left(t_{ij}-\frac{1}{2\sigma_i}\right)\right]$ as the correct value of $\tilde{y}_{ij}$. But, assuming that the width $1/\sigma_i$ is small, this is nearly equal to $\frac{Nr_i}{\sigma_i}\varphi(t_{ij})$, and thus it will do merely to put $u=u_j$ in $\tilde{y}_i$.

§5. Applications to Pedagogical Statistics

Ex. 2. A result of certain estimation test for students in some middle school is given as the two first columns in the following table, in which x and y denote the respective mark and the percentage of number of the corresponding students,[3] $\tilde{y}$ in the last column being theoretical values calculated afterwards from the representation that we shall obtain below. The distribution being bimodal we try representation (1).

For the sake of convenience, instead of central values x we have taken $u=\frac{1}{5}(x-67.5)$ and worked out as usual:

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[3] Those numbers falling on ends of subintervals were bisected, and each half counted into both neighbouring subintervals.

c.v. x	y	u	yu	yu^2	yu^3	yu^4	yu^5	$\tilde{y}$
37.5	1	−6	− 6	36	−216	1296	−7776	0.6
42.5	2	−5	−10	50	−250	1250	−6250	2.2
47.5	6	−4	−24	96	−384	1536	−6144	5.5
52.5	8	−3	−24	72	−216	648	−1944	9.6
57.5	13	−2	−26	52	−104	208	− 416	12.2
62.5	12	−1	−12	12	− 12	12	− 12	12.0
67.5	11	0	0	0	0	0	0	11.0
72.5	12	1	12	12	12	12	12	11.5
77.5	13	2	26	52	104	208	416	12.4
82.5	10	3	30	90	270	810	2430	11.0
87.5	6	4	24	96	384	1536	6144	7.2
92.5	4	5	20	100	500	2500	12500	3.3
97.5	2	6	12	72	432	2592	15552	1.1
sum	$N=100$		22	740	520	12608	14512	99.6

Reducing the total to unity on dividing by $N=100$, we get the tabular moments about $u=0$ to be $\nu'_0=1$, $\nu'_1=d=0.22$, $\nu'_2=7.40$, $\nu'_3=5.20$, $\nu'_4=126.08$, $\nu'_5=145.12$; whence central moments (moments about mean $\bar{u}=d=0.22$, i.e. $v=u-d=0$) were obtained as $\mu'_0=1$, $\mu'_1=0$, $\mu'_2=\nu'_2-d^2=7.3156$, $\mu'_3=\nu'_3=3d\nu'_2+2d^3=0.3373$, $\mu'_4=\nu'_4-4d\nu'_3+6d^2\nu'_2-3d^4=123.6459$, $\mu'_5=\nu'_5-5d\nu'_4+10d^2\nu'_3-10d^3\nu'_2+4d^5=7.9499$. Finally Sheppard's corrections being made, they become[4] $\mu_0=1$, $\mu_1=0$, $\mu_2=\mu'_2-\frac{1}{12}=7.2683$, $\sqrt{\mu_2}=2.960$, $\mu_3=0.3373$, $\mu_4=\mu'_4-\frac{\mu_2}{2}-\frac{1}{80}=119.9992$, $\mu_5=\mu'_5-\frac{5}{6}\mu_3=7.6688$.

Here moments of odd order being comparatively small, Case I may be applied, and we get by (3) $a^4=\frac{1}{2}(3\mu_2^2-\mu_4)=19.23$, so that $a=\sqrt{4.385}=2.094$ and $\sigma^2=\mu_2-a^2=2.883$, $\sigma=1.698$. Therefore, the required representation becomes

$$\tilde{y}=\frac{50}{\sqrt{2\pi}\sigma}\left[\exp\left\{-\frac{(v-a)^2}{2\sigma^2}\right\}+\exp\left\{-\frac{(v+a)^2}{2\sigma^2}\right\}\right]$$

with $\sigma=\sigma_u=1.698$ and $a=2.094$. Or, on writing $v=u-d=\frac{1}{5}(x-67.5)-0.22=\frac{1}{5}(x-68.6)$,

$$\tilde{y}=\frac{50}{\sqrt{2\pi}\sigma_u}\left[\exp\left\{-\frac{(x-79)^2}{2\sigma_x^2}\right\}+\exp\left\{-\frac{(x-58)^2}{2\sigma_x^2}\right\}\right] \text{ nearly,}$$

where $\sigma_x=5\sigma_u=8.49$. Or, setting $\frac{x-79}{\sigma_x}=\frac{u-2.314}{\sigma_u}=t_1$, $\frac{x-58}{\sigma_x}=\frac{u+1.874}{\sigma_u}=t_2$,

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[4] These procedures are the usual way of calculations, namely first to compute central moments from tabular moments and second to make Sheppard's corrections. This way is mach more simple in calculations than the reverse procedures, i.e. to correct tabular moments by Sheppard at first, and then transform them into central moments. Though both manners give the same result, the former is preferable, because, even when Sheppard's corrections are found to be inapplicable after determination of representation, the calculated uncorrected central moments shall be still of use (compare §9)

$$\tilde{y} = \frac{50}{\sigma_u}\{\varphi(t_1) + \varphi(t_2)\},$$

where $\varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$, the standard normal density function. Using Table of $\varphi(t)$, we found the values of $\tilde{y}$ written in the above table, and whence $\chi^2 = \sum(y-\tilde{y})^2 \div \tilde{y} = 1.68$. As the number of degrees of freedom equals $12-2=10$, and Pr ($\chi^2 \geqq 4.87$) is $0.9 > 0.05$, our representation fits the given data very well enough.

Ex. 3. A similar estimation test as in Ex. 2, gave the following table, here y being the actual numbers, find its representation.

x (central value)	32.5	37.5	42.5	47.5	52.5	57.5	62.5	67.5	72.5	77.5	82.5	87.5	92.5	97.5	total
y (frequency)	3	14	60	161	263	264	171	102	127	187	114	42	10	2	1520

Transformed x into $u = \frac{1}{5}(x-62.5) = -6, -5, \cdots, 6, 7$ and reduced the total number to unity, tabular moments about $u=0$ are found to be $\nu_0'=1$, $\nu_1'=0.03684$, $\nu_2'=6.4526$, $\nu_3'=5.6618$, $\nu_4'=89.4658$, $\nu_5'=153.0566$; whence moments about mean $u=d$ (uncorrected and corrected by Sheppard) are obtained as $\mu_0'=1$, $\mu_1'=0$, $\mu_2'=6.4513$, $\mu_3'=4.9488$, $\mu_4'=88.6838$, $\mu_5'=136.6536$; as well as $\mu_0=1$, $\mu_1=0$, $\mu_2=6.3680$, $\mu_3=4.9488$, $\mu_4=85.4873$, $\mu_5=132.5296$.

Here μ_3, μ_5 being not so small, but dispersions somewhat alike, let us apply Case II. Equation (4) becomes $X^3 - 18.0835X - 12.2453 = 0$ ($X = \mu_2 - \sigma^2 > 0$), which has only one positive root 4.5438. So that we obtain $\sigma^2 = 1.8242$, which satisfies inequality (7) in fact. The corresponding proportions ratio equation (6) becomes $q^2 - 2.2611\,q + 1 = 0$, that gives $q = 1.6579$ or 0.6032. Hence $r_1 = 0.6238$ or 0.3762 whereas $r_2 = 1 - r_1$ and $a_2 = -qa_1$. Combining the last equation with $r_1a_1^2 + r_2a_2^2 = 4.5438$, we obtain $a_1 = \pm 1.6554$, $a_2 = \mp 2.7445$, or else $a_1 = \pm 2.7469$, $a_2 = \mp 1.6569$. But the inequality $r_1a_1^3 + r_2a_2^3 = q(1-q)a_1^3 = \mu_3 > 0$ repuires $a_1 \gtreqless 0$ according as $q \gtreqless 1$. Hence we have either (i) $a_1 = -1.6554$, $a_2 = 2.7445$, or (ii) $a_1 = 2.7445$, $a_2 = -1.6554$. Consequently we obtain the following two representation:

$$\tilde{y} = \frac{1}{\sqrt{2\pi}\sigma}\left[n_1 \exp\left\{-\frac{(u+1.6186)^2}{2\sigma^2}\right\} + n_2 \exp\left\{-\frac{(u-2.7813)^2}{2\sigma^2}\right\}\right],$$

where $\sigma = 1.3506$ and (i) $n_1 = 948.12$ or (ii) 571.88 while $n_2 = 1520 - n_1$. Which one will do, shall be decided by the χ^2– or ω^2– test.

However, having evaluated $\tilde{y}$, ordinates at $u = -6, -5, \cdots, 7$, and computed $\chi^2 = \sum(y-\tilde{y})^2/\tilde{y}$, similarly as in Ex. 2, we found extraordinarily large values $\chi^2 = 139.5$ and 416.1 for (i) and (ii) respectively, and as this shows that (i) is preferable to (ii), thereby, however, the acceptability of (i) is never ascertained. Hitherto we have consulted with ordinate values only as rough approximations of frequencies. To speak more exactly, we should compute the area under the normal density curve in every subclass. To do this we have to refer to the Table of $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt$, taking for x, the end values of u: $-6.5, -5.5, \cdots, 7.5$. Thus obtained values of frequencies, $\tilde{Y}$ say, are as in the following table:

u	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	total
obs. frequency	3	14	60	161	263	264	171	102	127	187	114	42	10	2	1520
cal. fr. by (i)	1.7	13.8	61.8	149.2	281.0	251.5	159.6	111.3	155.6	164.1	111.9	45.5	11.0	1.5	1519.5
cal. fr. by (ii)	1.0	8.3	37.3	89.9	170.0	155.7	118.2	136.9	248.5	271.0	185.4	75.4	18.3	2.6	1518.5

Whence $\chi^2=14.32$ for (i) and 372.9 for (ii). For 8 degrees of freedom it is $\Pr(\chi^2 \geqq 14.32) > \Pr(\chi^2 \geqq 15.51)=0.05$. Hence we may adopt the representation (i).

Ex. 4. A percertage result of entrance examination for mathematics held in some school was informed to have been as follows:

mark x	0~10	10~20	20~30	30~40	40~50	50~60	60~70	70~80	80~90	90~100	tatal
fr. y	0.5	2.8	9.4	17.3	17.3	10.6	12.2	19.2	9.5	1.2	100.00

Those falling to end marks were bisected, and each half distributed to the neighbouring subintervals.

Taking central values of subintervals $x=5, 15, \cdots, 95$ and putting $u=\frac{x-55}{10}$, the tabular moments about $u=0$ are found to be $\nu_0'=1$, $\nu_1'=-0.099=d$, $\nu_2'=4.221$, $\nu_3'=-1.521$, $\nu_4'=34.809$, $\nu_5'=-31.209$ and whence the central moments, corrected by Sheppard, $\mu_0=1$, $\mu_1=0$, $\mu_2=4.1279$, $\sqrt{\mu_2}=2.3172$, $\mu_3=-0.27221$, $\mu_4=32.8721$, $\mu_5=-12.8603$ (uncorrected μ_k' being $\mu_2'=4.2112$, $\mu_4'=34.9486$, &c.). Here μ_5 is not small, while $\sigma_1 \neq \sigma_2$ since two subranges $5<x<35$ and $75<x<95$ appear different in magnitude. So we have no choice but to solve equation (19) straightforwardly. It runs now

$$s^9-167.58s^8+4221.31s^7-459.206s^6+1254.28s^5-1838.47s^4$$
$$-462.492s^3+129040s^2+83231.68s-1517.79=0.$$

This equation has a root $s(=a_1+a_2)=0.5072$, which being substituted in (18), $p=a_1a_2=-3.358$ follows. Therefore $a_1, a_2=2.10, -1.60$ nearly. Accordingly by (8) $r_1=0.4324$, $r_2=0.5676$ and by (12) $b_1=0.6529$, $b_2=0.4975$ and by (9) $\sigma_1^2=0.3708$,

$\sigma_2^2=1.0704$. Hence we obtain

$$\tilde{y}=\frac{43.24}{\sigma_1}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(v-2.10)^2}{2\sigma_1^2}\right\}+\frac{56.76}{\sigma_2}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(v+1.60)^2}{2\sigma_2^2}\right\},$$

where $\sigma_1=0{,}6089$, $\sigma_2=1.0346$ and $v=u-d=0.1(x-54.01)$. Or, since $v-2.10=u-2.001=0.1(x-75)$; $v-1.60=u+1.699=0.1(x-38)$, we have

$$\tilde{y}=\frac{71.01}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-75}{6.089}\right)^2\right\}+\frac{54.86}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-38}{10.346}\right)^2\right\}.$$

To evaluate $\tilde{y}$, we set $t_1=(u-2.001)/\sigma_1$, $t_2=(u+1.699)/\sigma_2$ and

$$\tilde{y}=71.01\varphi(t_1)+54.86\varphi(t_2),\quad \text{where}\quad \varphi(t)=\frac{1}{\sqrt{2\pi}}e^{-t^2/2}.$$

On using Table of $\varphi(t)$, we find $\tilde{y}_i$ as the 4-th column in the following table:

i	obs. y_i	S_i	$\tilde{y}_i$	$\tilde{S}_i$	$\lvert S_i-\tilde{S}_i\rvert$	$\lvert S_i-\tilde{S}\rvert^2$	$\tilde{y}_i\lvert S_i-\tilde{S}_i\rvert^2$
−5	0.5	0.5	0.1	0.1	0.4	0.16	0.02
−4	2.8	3.3	1.9	2.0	1.2	1.44	2.74
−3	9.4	12.7	9.9	11.9	0.8	0.64	6.34
−2	17.3	30.0	21.0	32.9	2.9	8.41	176.61
−1	17.3	47.3	17.4	50.3	3.0	9.00	156.60
0	10.6	57.9	5.8	56.1	1.8	3.24	18.79
1	12.2	70.1	7.9	64.0	6.1	37.21	293.96
2	19.2	89.3	28.0	92.0	2.7	7.29	204.12
3	9.5	98.8	7.6	99.6	0.8	0.64	4.80
4	1.2	100.0	0.1	99.7	0.3	0.09	0.01
N=100.00							δ^2=864.05

To try the ω^2-test,[5] we proceed as follows: Since ω^2 is defined as

$$\omega^2=\frac{1}{N}\int_{-\infty}^{\infty}|S(u)-NF(u)|^2 f(u)\,du,$$

where $f(u)$ and $F(u)$ are the probability density function and cumulative distribution function, while $S(u)$ denotes the observed accumulated number, the approximate value of ω^2 is given by

$$\omega^2=\frac{h}{N}\sum_i|S_i-\tilde{S}_i|^2\frac{\tilde{y}_i}{N}=\frac{h\delta^2}{N^2},\tag{38}$$

where h denotes the width of one u-subinterval, (usually $h=1$), and

$$\delta^2=\sum_i|S-\tilde{S}_i|^2\tilde{y}_i,\tag{39}$$

where $S_i=\sum_{j=-5}^{i}y_j$, $\tilde{S}_i=\sum_{j=-5}^{i}\tilde{y}_j$. These being calculated as in the above table, we get

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5) Y. Watanabe, On the ω^2-Distributions, this Journal vol. II (1952), p. 21; also T. Kondō, Evaluation of some ω_n^2-Distribution, this Journal vol. III (1954) p. 46.

$$\omega^2 = 864.05/100^2 = 0.0864.$$

Entering Table of the $\Phi(\omega_\infty^2)$ loc. cit., we find that $\Phi(\omega_\infty^2=0.0864)=0.3440$, and $\Pr(\omega_\infty^2 \geqq 0.0864) = 1-\Phi(\omega_n^2 = 0.0864) = 0.6560 > 0.05$. Or, if referred to Kondō's Table of $\Phi(\omega_9^2)$, more approximately $\Phi(\omega_9^2=0.0865)==0.3602$, so that $\Pr(\omega_9^2 \geqq 0.0864)=1-\Phi(\omega_9^2)=0.6398>0.05$. Hence our representation is not to be rejected.

Here $\sigma_1=0.6089$ being somewhat small, we may compute $\tilde{Y}_j$ as remarked at the end of §4, and obtain $\omega^2=0.0553$, and correspondingly $Pr(\omega_9^2 \geqq 0.0553)=0.8146$. However with these $\tilde{Y}_j$, still χ^2–test does deny the above representation, since even when pooled at ends two by two, χ^2 amounts to 6.882, and with 2 degrees of freedom, $\Pr\{\chi^2 \geqq 6.882\}<0.05$.

For later comparison we shall add one more example, which seems rather inadequate to be expressed by (1).

Ex. 5. A similar estimation test as in Ex. 2 gave the following result in percentage:

x	0~10	10~20	20~30	30~40	40~50	50~60	60~70	70~80	80~90	90~100	total
c.v.	5	15	25	35	45	55	65	75	85	95	
u	−5	−4	−3	−2	−1	0	1	2	3	4	
y	0	3	11	8	6	10	28	31	3	0	100

For a later use, now we shall calculate moments about $u=0$, and first correct them by Sheppard as: $\nu_0=1$, $\nu_1=d=\bar{u}=0.32$, $\nu_2=3.55667$, $\nu_3=-2.10$, $\nu_4=23.80917$, $\nu_5=-40.85$, $\nu_6=220.8194$, $\nu_7=-618.9531$ (those moments of higher order shall be used later in §11). Whence central moments $\mu_2=3.4543$, $\mu_3=-5.4489$, $\mu_4=28.6824$, $\mu_5=-82.2471$, $\mu_6=336.7033$, $\mu_7=-1172.9612$, while uncorrected moments are $\mu_2'=3.5376$, $\mu_3'=-5.4489$, $\mu_4'=30.3220, \cdots$ (these shall be used in §9).

Here approximately we may apply Case II: $\sigma_1=\sigma_2=\sigma$. Taking equation (4), we have to solve $X^3-3.5570X-14.8453=0$. It has only one positive root $X=\mu_2-\sigma^2=2.953$, so that $\sigma^2=0.5190$ and $\sigma=0.7204$. Hence by (6) $q^2-3.1740q+1=0$, which gives $q=0.3547$ (or 1.5870), and whence $r_1=0.262$, $r_2=0.738$. Also $a_1^2=X/q=8.2754$, so that $a_1=\pm 2.8767$, $a_2=-qa_1=\mp 10.203$. Describing given data in a graph, it is seen that $a_1=-2.8767$, $a_2=1.0203$ are to be taken preferably. Therefore we obtain, as a rough representation,

$$\tilde{y}=\frac{100}{0.7204\sqrt{2\pi}}\left[0.262\exp\left\{-\frac{1}{2}\left(\frac{v+2.88}{0.7204}\right)^2\right\}+0.738\exp\left\{-\frac{1}{2}\left(\frac{v-1.02}{0.7204}\right)^2\right\}\right].$$

Or, setting

$$t_1=\frac{(u+2.88)}{\sigma}=\frac{u+2.88-0.32}{0.7204}=1.388u+3.554,$$

and

$$t_2=\frac{v-1.02}{\sigma}=\frac{u-1.02-0.32}{0.7204}=1.388u-1.860,$$

we obtain

$$\tilde{y}=36.37\varphi(t_1)+102.44\varphi(t_2)=\tilde{y}_1+\tilde{y}_2,$$

where $\varphi(t)=\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$. Whence, by use of $\varphi(t)$–Table we get $\tilde{y}$:

u	-5	-4	-3	-2	-1	0	1	2	3	4	total
obs. y	0	3	11	8	6	10	28	31	3	0	100
cal. $\tilde{y}$	0.05	1.98	12 05	10.72	1.60	7.27	36.56	26.81	2.88	0.04	99.96

Just as done in Ex. 4, we find $\delta^2=\sum\tilde{y}(\sum y-\sum\tilde{y})^2=678$, so that $\omega^2=\delta^2/N^2=0.0678$ and $\Phi(\omega_9^2=0.0678)=0.2247$. Hence $\Pr(\omega_9^2\geqq 0.0678)=0.7753>0.05$. Thus our result is already not to be rejected. However, χ^2 amounts to 16.42 even when pooled 2 by 2 at both onds, and for 2 degrees of freedom, $\Pr(\chi^2\geqq 16.42)<0.01$, and the representation is to be rejected, We will endeavour to obtain a more elaborate representation in §11.

§6. Biometrical Applications

Ex. 6. Prof. Yoshikane OKA measured sizes of some sea-ears (Japanese abalone, *Haliotis gigantea*) and obtained the result as in the following table. It seems that there are two classes, and one class is in average one period older than the other, each class being somewhat normally distributed. Therefore it is required to decompose the whole distribution into two normal curves:

length x (cm)	2.0~2.4	2.5~2.9	3.0~3.4	3.5~3.9	4.0~4.4	4.5~4.9	5.0~5.4	5.5~5.9
central value	2.2	2.7	3.2	3.7	4.2	4.7	5.2	5.7
frequency y	1	2	2	11	24	25	16	14

6.0~6.4	6.5~6.9	7.0~7.4	7.5~7.9	8.0~8.4	8.5~8.9	9.0~9.4	9.5~9.9	10.0~10.4	total
6.2	6.7	7.2	7.7	8.2	8.7	9.2	9.7	10.2	
18	30	39	41	28	14	1	2	2	270

First taking the central value 6.2 of the middle subinterval 6.0~6.4 as origin, and putting $u=\frac{x-6.2}{0.5}=-8,-7,\cdots,8$, we calculated tabular moments about $u=0$, namely $\sum u^k y$, $k=0, 1, 2, 3, 4$ and 5. Dividing them by $N=270$, we get tabular moments about $u=0$: $\nu_1'=1$, $\nu_1'=0.52593=\bar{u}=d$, $\nu_2'=10.41111$, $\nu_3'=5.77407$, $\nu_4'=225.87778$, $\nu_5'=159.64074$. Further setting $v=u-d=u-0.52593$, we computed the central moments about $v=0$ $(u=d)$, Sheppard's corrections being made at the same time:

$$\mu_0=1,\ \ \mu_1=0,\ \ \mu_2=\nu_2'-d^2-\frac{1}{12}=10.04727,\ \ \sqrt{\mu_2}=3.16974,\ \ \mu_3=\nu_3'-3d\nu_2'+2d^3$$

$$=-10.47091,\ \mu_4=\nu_4'-4\nu_3'd+6\nu_2'd^2-3d^4-\frac{1}{2}\mu_2-\frac{1}{80}=225.89552,$$

and $$\mu_5 = \nu_5' - 5\nu_4' d + 10\nu_3' d^2 - 10\nu_2' d^3 + 4d^5 - \frac{5}{6}\mu_3 = -428.8956.$$

We have to start with these moments about $v=0$.

Although here μ_3 and μ_5 are never small, because of easy calculation, let us assume Case I. In the same way as worked in Ex. 2, we obtain $a=2.491$, $\sigma^2=3.852$, $\sigma=1.963$ and hence as its representation

$$\tilde{y} = \frac{135}{\sqrt{2\pi}\sigma}\left[\exp\left\{-\frac{(v-a)^2}{2\sigma^2}\right\} + \exp\left\{-\frac{(v+a)^2}{2\sigma^2}\right\}\right], \quad a=2.491, \quad \sigma=\sigma_u=1.953.$$

Or, since $v=u-d=2(x-6.2)-0.53=2(x-6.45)$ nearly,

$$\tilde{y} = \frac{135}{\sqrt{2\pi}\sigma_u}\left[\exp\left\{-\frac{(x-7.7)^2}{2\sigma_x^2}\right\} + \exp\left\{-\frac{(x-5.2)^2}{2\sigma_x^2}\right\}\right], \quad \sigma_x = \frac{1}{2}\sigma_u = 0.98.$$

Further putting $t_{1j}, t_{2j} = \frac{u_j \mp a}{\sigma}$, $u_j = -8, -7, \cdots, 8$ and computing by use of the Table of $\varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ the values of $\tilde{y}_j = \frac{270}{2\sigma}\{\varphi(t_{1j}) + \varphi(t_{2j})\}$, we obtain the following result:

u	− 8	− 7	− 6	− 5	− 4	− 3	− 2	− 1	0
obs. y	1	2	2	11	24	25	16	14	18
cal. $\tilde{y}$	0.27	1.12	3.57	8.80	16.71	24.58	28.40	26.95	23.86

u	1	2	3	4	5	6	7	8	total
obs. y	30	39	41	28	14	1	2	2	270
cal. $\tilde{y}$	23.74	26.76	28.44	24.96	17.22	9.21	3.80	1.21	269.6

Whence $\chi^2 = \sum(y-\tilde{y})^2/\tilde{y} = 39.01$. Here degrees of freedom being 12, the χ^2–Table affords $\Pr(\chi^2 \geqq 24.69) = 0.01$, and thus the above representation must be rejected with significant level 0.01. Or, we may apply the ω^2–test as done in Ex. 4. In the present example $N=270$ and δ^2 amounts to 35406, so that by (38), $\omega^2 = 35406/270^2 = 0.9714$. Entering the $\Phi(\omega_\infty^2)$ Table, we find that $\Phi(0.9714) = 0.9971$ and therefore $\Pr\{\omega_\infty^2 \geqq 0.9714\} = 1 - \Phi(9714) = 0.0029 < 0.05$. Thus again by the ω^2–test the above representation is to be rejected.

To obtain more legitimate solution, we are obliged to solve equation (19). Now the coefficients (20) are found to be as follows[6]:

$$A_0 = 273660.8085, \quad A_1 = -502\ 7621, \quad A_2 = 1057\ 4233, \quad A_3 = 3578\ 2783,$$
$$A_4 = 16\ 2153\ 4113, \quad A_5 = 17\ 0759\ 9010, \quad A_6 = 179\ 6075\ 1646,$$
$$A_5 = 1139\ 6555\ 1875, \quad A_8 = -7432\ 2465\ 0018, \quad A_9 = -7514\ 0444\ 7513.$$

The equation (19) with these coefficients, still divided by A_0, reduces to

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[6] To avoid decimals as possible, all A_m in (20) were multiplied by 8.

$$s^9-18.37173s^8-38.63992s^7+130.7560s^6+5925.343s^5+6339.838s^4$$
$$+65631.44s^3-416448.2s^2-271\ 5861s-274\ 5751=0.$$

Solving this equation by Horner's method we obtain three real roots -2.577, -1.354 and 19.08. Since the whole sample range is 17 units in u, the third root is evidently useless. Also the second root makes the value of p in (18) positive, so that a_1, a_2 have the same sign and the ratio $r_1:r_2=q$ becomes negative, which does not give proper superposition. Hence the first root -2.577 is only promising. Really substituting this root in (18), we get $p=-6.677$. On solving $s=a_1+a_2=-2.577$, $p=a_1a_2=-6.677$ we obtain a_1, $a_2=-4.176$, 1.599, and whence by (10) $\sigma_1^2=0.7231$, $\sigma_2^2=3.906$, so that $\sigma_1=1.403$, $\sigma_2=1.976$. But, since $v=u-d=u-0.526$ nearly, it results that $v-a_1=u+3.650$, $v-a_2=u-2.125$. Hence we obtain

$$\tilde{y}=\frac{53.31}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{u+3.650}{1.403}\right)^2\right\}+\frac{99.34}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{u-2.125}{1.976}\right)^2\right\}.$$

Or, since $u=2(x-6.2)$ and $\sigma_x=\sigma_u/2$,

$$\tilde{y}=\frac{53.31}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-4.375}{0.7015}\right)^2\right\}+\frac{99.34}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-7.262}{0.988}\right)^2\right\}.$$

In order to examine the above representation by the ω^2–test, we proceed as in Ex. 4, and find that $\delta^2=3622$, so that $\omega^2=3622/270^2=0.050$, and $\Phi(\omega_\infty^2=0.050)=0.1240$. Hence $\Pr(\omega_\infty^2\geqq 0.050)=0.8760=0.8760>0.05$. Or, entering the $\Phi(\omega_q^2)$–Table, we get $\Phi(\omega_9^2=0.050)=0.1562$, and still $\Pr(\omega_9^2\geqq 0.050)=0.8438>0.05$. Thus the above representation can be asserted.

Again, to try the X^2–test, the requisite X^2 is calculated as follows:

u	y	$\tilde{y}$	$\lvert y-\tilde{y}\rvert$	$\lvert y-\tilde{y}\rvert^2$	$\lvert y-\tilde{y}\rvert^2/\tilde{y}$
−8	1	0.23	0.77	0.593	2.58
−7	2 } 5	1.23 } 6.70	0.77 } 1.70	0.593 } 2.89	0.48 } 0.43
−6	2	5.24	3.24	10.498	2.00
−5	11	13.45	2.45	6.002	0.45
−4	24	20.94	3.06	9.364	0.45
−3	25	20.48	4.52	20.521	1.00
−2	16	15.13	0.89	0.757	0.01
−1	14	14.93	0.93	0.865	0.06
0	18	22.96	4.96	24.602	1.07
1	30	33.79	3.79	14.364	0.43
2	39	39.56	0.56	0.314	0.01
3	41	35.92	5.08	25.804	0.72
4	28	25.26	2.74	7.508	0.30
5	14	13.75	0.25	0.062	0.00
6	1	5.79	4.79	22.944	3.96
7	2 } 5	1.89 } 8.16	0.11 } 3.16	0.012 } 9.986	0.01 } 1.22
8	2	0.48	1.52	2.310	4.81

$N=270$. If each frequency be as it stands, $\chi^2=18.34$;

Or, on pooling the frequencies at ends, $\chi^2=6.15$.

When unpooled, degrees of freedom being $17-6=11$, we have $\Pr(\chi^2 \geqq 18.34) < 0.05$. But, when pooled at ends as shown above, degrees of freedom reduces to 7, for which $\Pr(\chi^2 \geqq 6.15) > 0.50 > 0.05$. Thus even with rather severe χ^2-criterion the adequacy of the above representation cannot be denied.

Ex. 7. At the same time as the length measurement of sea-ears in Ex. 6 Prof. Oka made also their breadth estimation, which runs as follows:

breadth x(c.v.)	1.2	1.7	2.2	2.7	3.2	3.7	4.2	4.7	5.2	5.7	6.2	6.7	7.2	7.7	total
frequency y	1	1	8	26	31	18	20	33	56	45	26	3	1	1	270

As it is very probable that this distribution should be similarly distributed as in Ex. 6, we might apply Case V, using the known value of $q=r_1/r_2$.

Now putting $u=\frac{x-4.7}{0.5}$, we get the first moment $\nu_1'=-0.2185=\bar{u}=d$ and the moments about $u=d(v=0)$ to be $\mu_0=1$, $\mu_1=0$, $\mu_2=5.9838$, $\sqrt{\mu_2}=2.4462$, $\mu_3=-5.5421$, $\mu_4=78.3470$. Utilizing the value of q obtained in Ex. 6, i.e. $q=\frac{r_1}{r_2}=\frac{0.2769}{0.7231}=0.3829$, and substituting it in (27), we obtain an equation of sixth degree in $a_1=a$

$$a^6+11.6627a^3-74.3784a^2-68.3769=0,$$

which has two real roots 2.701 and -3.315. But we have chosen a_1 in Ex. 6 to be negative. Therefore $a_1=-3.315$, and consequently by (24) $a_2=1.270$. Further, by (25) (26), $\sigma_1^2=0.9687$, $\sigma_2^2=2.0822$, so that $\sigma_1=0.9842$, $\sigma_2=1.4430$. Also, since $v=u+0.2185$, so $v-a_1=u+3.533$ and$v-a_2=u-1.052$, and thus the required representation is obtained to be

$$\tilde{y}=270\left[\frac{0.282}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{u+353}{0.984}\right)^2\right\}+\frac{0.506}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\frac{u-1.052}{1.443}\Big)^2\right\}\right].$$

Or, as $u=2(x-4.7)$ and $\sigma_u=2\sigma_x$

$$\tilde{y}=\frac{76.14}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-2.934}{0.492}\right)^2\right\}+\frac{136.6}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-5.226}{0.7215}\right)^2\right\}.$$

Now to try the ω^2-test, we compute the values $\tilde{y}$ for every $u=j$ by

$$\tilde{y}=76.14\varphi(t_1)+136.6\varphi(t_2),$$

where $t_1=(u+3.53)/0.984, t_2=(u-1.052)/1.443$ and $\varphi(t)=\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$. Then, performing S_i, $\tilde{S}_i$ as in Ex. 4, we find that $\delta^2=3516$, and by (38) $\omega^2=\delta^2/N^2=3516/270^2=0.04824$. We see that $\Phi(\omega_\infty^2=0.04824)=0.1132$, so that $\Pr(\omega_\infty^2 \geqq 0.04824)=0.8867$

<0.05. Or, by the $\Phi(\omega_9^2)$-Table, $\Phi(\omega_9^2 = 0.04824) = 0.1467$ and $\Pr(\omega_9^2 \geqq 0.04824) = 0.8533$. Hence the above representation can be asserted with large probability.

For the χ^2-test we obtain, similarly as done in Ex. 6, $\chi^2 = 28.52$ if each subinterval be held as it stands, and degrees of freedom being $18-6=10$, $\Pr(\chi^2 > 28.52) < 0.01$, so that this χ^2-test denies the above representation. However, the frequencies at the ends of distribution being so small we may lump together them, 3 by 3 at ends, and now we get $\chi^2 = 5.39$. This time with 6 degrees of freedom, $\Pr(\chi^2 \geqq 5.39)$ is nearly 0.5. Thus even with χ^2-test the affirmation remains the same as got by ω^2-test.

Ex. 8. Prof. Oka measured also length and breadth concerning another certain class of sea-ears. The result about length was as follows:

length x (c.v.)	2.7	3.2	3.7	4.2	4.7	5.2	5.7	6.2	6.7	7.2	7.7	8.2	8.7	9.2	9.7	10.2	total
frequency y	1	4	10	6	10	6	18	30	34	54	28	26	10	5	1	2	245

Setting $u = \frac{x-6.7}{0.5}$ and computing tabular moments $\nu_k' = \sum yu^k/N$, we get $\nu_0' = 1$, $\nu_1' = d = \bar{u} = 0.0938776$, $\nu_2' = 7.77551$, $\nu_3' = -10.70612$, $\nu_4' = 196.5918$, $\nu_5' = -548.1102$; whence moments about mean $u=d$ (uncorrected) $\mu_0' = 1$, $\mu_1' = 0$, $\mu_2' = 7.76670$, $\mu_3' = -12.89431$, $\mu_4' = 201.0233$, $\mu_5' = 641.3956$; and finally making Sheppard's corrections $\mu_0 = 1$, $\mu_1 = 0$, $\mu_2 = 7.68337$, $\mu_3 = -12.89431$, $\mu_4 = 197.1589$, $\mu_5 = -630.6504$. With these values the coefficients (20) become $A_0 = 51874.1279$, $A_1 = 201718.2443$, $A_2 = -2055178.012$, $A_3 = -25611194$, $A_4 = -63553402$, $A_5 = 416997325$, $A_6 = 2198556637$, $A_7 = 73140000940$, $A_8 = 18422138080$, $A_9 = -44038078090$. Thus equation (19) is found, further divided by A_0, to be

$$s^9 + 3.8886098s^8 - 39.6185545s^7 - 493.718062s^6 - 1225.146423s^5 + 8038.63779s^4$$
$$+ 42382.52722s^2 + 1409951.433s^2 + 3551150.8012s - 848941.0789 = 0.$$

The real roots are found to be $s = 0.6542$ -0.9241, -8.32607. But for the latter two roots we get negative variances, so that they should be given up. Only for the remaining root $s = 0.6542$, we obtain $p = -2.4464$, so that a_1, a_2 are the roots of quadratic $z^2 + 0.6542z - 2.4464 = 0$. Solving this equation, we get $a_1 = -1.27085$, $a_2 = 1.92505$ and whence $r_1 = 0.60235$, $r_2 = 0.39765$ by (8). Further by (12) we find $b_1 = 9.36184$, $b_2 = 5.14086$, and this time $\sigma_1^2 = 7.26875$, $\sigma_2^2 = 1.43504$, so that $\sigma_1 = 2.69606$, $\sigma_2 = 1.19793$. Hence we have, as the required bimodal representation

$$\tilde{y} = \frac{245 \times 0.60235}{2.69606\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{v+1.27085}{2.69606}\right)^2\right\} + \frac{245 \times 0.39765}{1.19793\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{v-1.92505}{1.19793}\right)^2\right\}.$$

Or, since $v = u - d = u - 0.09388$, on putting $\frac{v+1.27085}{\sigma_1} = 0.37091u + 0.43655 = t_1$, and

$\frac{v-1.92505}{\sigma_2}=0.83477u-1.68535=t_2$, we get

$$\tilde{y}=54.7376\varphi(t_1)+81.3272\varphi(t_2)=\tilde{y}_1+\tilde{y}_2,$$

where $\varphi(t)=\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$, the standard normal density function. On calculating $\tilde{y}_1$, $\tilde{y}_2$ and $\tilde{y}_1+\tilde{y}_2$ by use of $\varphi(t)$, we obtain the following result[7]

obs. y	1	4	10	6	10	6	18	30	34
y_1	0.89	2.12	4.41	7.99	12.62	17.37	20.84	21.79	19.85
$\tilde{y}_2$	0	0	0	0	0	0.01	0.02	1.36	7.84
cal. $\tilde{y}$	0.89	2.12	4.41	7.99	12.62	17.39	20.96	23.15	27.69
obs. y	54	28	26	10	5	1	2	$245.00=N$	
$\tilde{y}_1$	15.76	10.91	6.58	3.46	1.58	0.63	0.22	147.02	
$\tilde{y}_2$	22.60	32.44	23.20	8.27	1.47	0.13	0.01	97.45	as sum
cal. $\tilde{y}$	38.36	43.35	29.78	11.72	3.05	0.76	0.23	244.47	

Trying the ω^2–test as in Ex. 6, it was found that $\delta^2=\sum'\tilde{y}(\sum'y-\sum'\tilde{y})^2=17192$, so that $\omega^2=\delta^2/N^2=17192/245^2=0.2864$. Entering the $\Phi(\omega^2_\infty)$ Table, we find $\Phi(\omega^2_\infty=0.2864)=0.8524$, and hence $\Pr(\omega^2_\infty\geqq 0.2864)=0.1476>0.05$. Thus the ω^2–test does not deny the above representation. However, $\chi^2=\sum'(y-\tilde{y})^2/\tilde{y}$ amounts to 37.21, even after pooling the frequencies at ends, and degrees of freedom being $13-6=7$, we find $\Pr(\chi^2\geqq 37.21)<0.005$. Thus the above representation is now to be rejected with significant level 0.005. Indeed, we have tried in § 8 later on to find if adequate corrections of parameters be possible, the result of which, however, being still negative, it seems that the above bimodal representation does not fit the given data suitably (cf. §7 Ex. 11 and §10 Ex. 24.).

Ex. 9. Prof. Oka's measurment for width of the sea ears in Ex. 8 runs as follows:

c.v. x	2.2	2.7	3.2	3.7	4.2	4.7	5.2	5.7	6.2	6.7	7.2	total
u	−5	−4	−3	−2	−1	0	1	2	3	4	5	
d.f. y	4	16	10	10	38	33	72	32	20	8	2	245

For a purpose of later use, at first we have computed moments about $u=0$ with Sheppard's corrections: $\nu_0=1$, $\nu_1=d=\bar{u}=0.26939$, $\nu_2=4.33299$, $\nu_3=-1.21837$, $\nu_4=51.31488$, $\nu_5=-45.01786$ and whence the central moments $\mu_2=4.3438$, $\mu_3=-4.6810$, $\mu_4=56.6413, \mu_5=-119.7469$. Just as Ex. 7 was solved by use of the ratio $q=r_1/r_2$ obtained in Ex. 6, we may proceed according to method of Case V as follows:

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[7] We have computed to some more decimal places than written in the table and hence there occur apparently some discrepancies in sum.

Availing the result of Ex. 8, we assume that $r_1=0.60235$, $r_2=0.39765$, so $q=r_1/r_2=1.51477$. Hence, substituting these values and the central moments above obtained, equation (27) becomes

$$22.07017a^6 + 14.6003a^3 + 0.16614a^2 - 21.91204 = 0,$$

or, dividing by the coefficient of a^6

$$a^6 + 0.66154a^3 + 0.0075278a^2 - 0.992835 = 0.$$

This equation has only two real roots, one positive and one negative. Really by Horner's method we find the two roots to be (i) 0.8947 and (ii) -1.1123. Making use of formulas (24), (25), (26), we get in succession (i) $a_1=0.8947$, $a_2=-1.3553$, $\sigma_1=1.4551$, $\sigma_2=2.1603$, and remembering that $d=0.2694$

$$\tilde{y} = 101.42\varphi(t_1) + 43.85\varphi(t_2),$$

where $t_1=\frac{1}{\sigma_1}(u-d-a_1)=0.6872u-0.7999$, $t_2=\frac{1}{\sigma_2}(u-d-a_2)=0.4629u+0.5027$, and $\varphi(t)=\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$. Also from the second root we get (ii) $a_1=-1.1123$, $a_2=1.6849$, $\sigma_1=1.5338$, $\sigma_2=0.8633$, so that

$$\tilde{y} = 96.22\varphi(t_1) + 130.72\varphi(t_2),$$

where $t_1=0.620u+0.5495$, $t_2=1.3419u-2.6224$, and $\varphi(t)=\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$.

Which one of the above two results is preferable shall be decided by the ω^2–test. In fact for (i) and (ii) we find that $\delta^2=12196.54$ and 32531.66 and hence each ω^2 becomes 0.2032 and 0.5420 respectively. Correspondingly for (i) $\Phi(\omega_\infty^2=0.2032)=0.7594$ and $\Pr(\omega_\infty^2\geqq 0.2032)=0.2606>0.05$, while for (ii) $\Phi(\omega_\infty^2=0.5420)=0.9688$ and $\Pr(\omega_\infty^2\geqq 0.5420)=0.0312<0.05$. Thus we have to take (i) only. However, as χ^2–test again denies even representation (i), we shall ponder over still furthermore.

§7. Trimodal Representations

The graph of Ex. 9 presents three maxima, and suggests its trimodal representation. The analysis becomes naturally more complex than bimodal. However, if e.g. the modes a_i be assumed beforehand, it goes rather simple. Let the representation be

$$y = N\sum_{i=1,2,3}\frac{r_i}{\sqrt{2\pi}\sigma_i}\exp\left\{-\frac{1}{2}\left(\frac{u-a_i}{\sigma_i}\right)^2\right\}. \tag{40}$$

Reducing to density distribution by dividing by N, and taking moments about $u=0$, we have

$$\nu_0 = r_1 + r_2 + r_3 = 1, \quad \nu_1 = r_1 a_1 + r_2 a_2 + r_3 a_3 = d, \tag{41}$$

$$\nu_2 = \sum r_i(\sigma_i^2 + a_i^2), \quad \nu_3 = \sum r_i a_i (3\sigma_i^2 + a_i^2), \tag{42}$$

$$\nu_4 = \sum r_i(3\sigma_i^4 + 6a_i^2\sigma_i^2 + a_i^4), \quad \nu_5 = \sum r_i a_i(15\sigma_i^4 + 10a_i^2\sigma_i^2 + a_i^4). \tag{43}$$

If the values $a_i (i=1, 2, 3)$ be assumed, we are able to solve (41) with respect to r_1, r_2

$$r_1 = \frac{a_2 - d + (a_3 - a_2)r_3}{a_2 - a_1}, \quad r_2 = \frac{d - a_1 + (a_1 - a_3)r_3}{a_2 - a_1}. \tag{44}$$

Also we get from (42)

$$\left.\begin{aligned} r_1\sigma_1^2 &= \frac{1}{a_2 - a_1}\left[(a_3 - a_2)r_3\sigma_3^2 + a_2(\nu_2 - \sum r_i a_i^2) - \frac{1}{3}(\nu_3 - \sum r_i a_i^3)\right], \\ r_2\sigma_2^2 &= \frac{1}{a_2 - a_1}\left[(a_1 - a_3)r_3\sigma_3^2 - a_1(\nu_2 - \sum r_i a_i^2) + \frac{1}{3}(\nu_3 - \sum r_i a_i^3)\right], \end{aligned}\right\} \tag{45}$$

as well as from (43)

$$\left.\begin{aligned} r_1\sigma_1^4 = \frac{1}{a_2 - a_1}\Big[&(a_3 - a_2)r_3\sigma_3^4 + \frac{2}{3}\sum r_i a_i^3\sigma_i^2 - 2a_2\sum r_i a_i^2\sigma_i^2 \\ &+ \frac{1}{3}a_2(\nu_4 - \sum r_i a_i^4) - \frac{1}{15}(\nu_5 - \sum r_i a_i^5)\Big], \\ r_2\sigma_2^4 = \frac{1}{a_2 - a_1}\Big[&(a_1 - a_3)r_3\sigma_3^4 - \frac{2}{3}\sum r_i a_i^3\sigma_i^3 + 2a_1\sum r_i a_i^2\sigma_i^2 \\ &- \frac{1}{3}a_1(\nu_4 - \sum r_i a_1^4) - \frac{1}{15}(\nu_5 - \sum r_i a_i^5)\Big]. \end{aligned}\right\} \tag{46}$$

Eliminating σ_1, σ_2 between (45) and (46), we obtain two equations involving σ_3, r_3, and in fact biquadratic in σ_3, and further elimination of σ_3 yields a biquadratic equation in r_3.

middle values x (c.m.)	u	frequency y in percentage
110.5	−9	0.01
113.5	−8	0.06
116.5	−7	0.28
119.5	−6	0.99
122.5	−5	2.74
125.5	−4	5.90
128.5	−3	10.11
131.5	−2	14.16
134.5	−1	16.54
137.5	0	16.09
140.5	1	135.5
143.5	2	9.38
146.5	3	5.58
149.5	4	2.92
152.5	5	1.25
155.5	6	0.46
158.5	7	0.14
161.5	8	0.03
164.5	9	0.01
	sum	100.00

Ex. 10. For a mingled group of girls aged 9, 10 and 11 stature measurements were made, and the result was as following table. Taking moments about $u=0$, and making Sheppard's corrections, we get $\nu_0=1$, $\nu_1=-0.5011=d$, $\nu_2=5.7796$, $\nu_3=-7.0604$, $\nu_4= 95.2217$, $\nu_5 = -158.2007$. If the trimodal representation (40) with $a_1=-2.5$, $a_2= -0.5$ and $a_3=1.5$ be assumed, we obtain from (44), (45) and (46)

$$r_2 = 1.0006 - 2r_1, \quad r_3 = -0.0006 + r_1;$$

$$r_2\sigma_2^2 = -2r_1\sigma_1^2 - 8r_1 + 5.30365, \quad r_3\sigma_3^2 = r_1\sigma_1^2 + 0.22715;$$

$$r_2\sigma_2^4 = -2r_1\sigma_1^4 - 16r_1\sigma_1^2 - 10.66668r_1 + 26.34177, \quad r_3\sigma_3^4 = r_1\sigma_1^4 + 1.704962.$$

Eliminating σ_2 from these

$$2.0012r_1\sigma_1^4 - 5.2050r_1\sigma_1^2 + 42.666664r_1^2 - 21.501788r_1 + 1.771126 = 0,$$

also eliminating σ_3

$$\sigma_1^4 + 757.2\sigma_1^2 - 2841.6 + 87.7/r_1 = 0,$$

and finally eliminating σ_1 from the above two equations, we have

$$\psi(r_1) = r_1^4 - 27262.5145r_1^3 - 8333.8992r_1^2 + 420.2321r_1 + 16.5893 = 0.$$

This equation has positive roots 0.1059, 0.2253, and from the latter we obtain

$$\begin{aligned}\tilde{y} = \frac{22.53}{1.7958}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{u+2.5}{1.7958}\right)^2\right\} + \frac{55.00}{1.9297}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{u+0.5}{1.9297}\right)^2\right\} \\ + \frac{22.47}{2.0602}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{u-1.5}{2.0602}\right)^2\right\} \\ = 12.5455\varphi(t_1) + 28.5021\varphi(t_2) + 10.9065\varphi(t_3),\end{aligned}$$

where $t_1 = \dfrac{u+2.5}{1.7958}$, $t_2 = \dfrac{u+0.5}{1.9297}$, $t_3 = \dfrac{u-1.5}{2.0602}$ and $\varphi(t) = \dfrac{1}{\sqrt{2\pi}}e^{-t^2/2}$.

Calculating $\tilde{y}_j$ for $u=j=0, \pm 1, \cdots, \pm 9$, we get

u	-9	-8	-7	-6	-5	-4	-3	-2	-1	0
$y_{\text{obs.}}$	0.01	0.06	0.28	0.99	2.74	5.90	10.11	14.16	16.54	16.09
$\tilde{y}_{\text{cal.}}$	0.008	0.052	0.257	0.950	2.680	5.838	10.128	14.249	16.610	16.233

u	1	2	3	4	5	6	7	8	9	total
$y_{\text{obs.}}$	13.35	9.38	5.58	2.92	1.25	0.46	0.14	0.03	0.01	100.00
$\tilde{y}_{\text{cal.}}$	13.380	9.355	5.579	2.841	1.224	0.439	0.118	0.031	0.006	99.978

Whence we see that $\delta^2 = 1.5676$, $\omega^2 = 1.5676/100^2 = 0.0002$ and $\Phi(\omega^2 = 0.0005)$ is almost zero, so that $\Pr(\omega^2 \geqq 0.0002)$ is almost unity. Also with χ^2–test, χ_0^2 amounts only to 0.0154, and for degrees of freedom $16-7=9$, $\Pr(\chi^2 \geqq \chi_0^2) > 0.995$. Thus the above trimodal representation cannot be denied with almost certainty.

With the former value $r_1 = 0.1059$, the matter does not go so good and it shall be abandoned.

Ex. 11. (Ex. 9). Now we shall try to obtain a trimodal representation for Ex. 9. Using the values of ν_n's in Ex. 9, and assuming that $a_1 = -4$, $a_2 = -1$ and $a_3 = 1$, we get from (44), (45) and (46)

$$r_1 = -0.4231 + \frac{2}{3} r_3, \quad r_2 = 1.4231 - \frac{5}{3} r_3;$$

$$r_1\sigma_1^2 = \frac{9}{10} r_3\sigma_3^2 - \frac{9}{10} r^3 - 0.2405, \quad r_2\sigma_2^2 = -\frac{5}{3} r_3\sigma_3^2 - \frac{80}{9} r_3 + 9.9204;$$

$$r_1\sigma_1^4 = \frac{2}{3} r_3\sigma_3^4 - \frac{20}{9} r_3\sigma_3^2 + \frac{34}{39} r_3 - 1.7178, \quad r_2\sigma_2^4 = -\frac{5}{3} \nu_3\sigma_3^4 - \frac{160}{9} r_3\sigma_3^2 - \frac{64}{9} r_3 + 42.310.$$

The elimination of σ_1 yields

$$0.28209 r_3\sigma_3^4 - 1.2609 r_3\sigma_3^2 - 1.2840 r_3^2 + 3.2781 r_3 - 0.6690 = 0,$$

while the elimination of σ_2 gives

$$2.3719 r_3\sigma_3^4 - 7.7679 r_3\sigma_3^2 + 67.1605 r_3^2 - 95.7264 r_3 + 38.2021 = 0.$$

Finally eliminating σ_3 between the above two equations, we get

$$\psi(r) \equiv r_3^4 - 3.0065 r_3^3 + 3.3839 r_3^2 - 1.6900 r_3 + 0.3161 = 0.$$

This equation, however, has no root between 0 and 1; thus our problem seems to have no solution. But, this might be due to misestimates of modes: Indeed, if some mode were estimated only a little differently, then the corresponding equation could have a certain adoptable root between 0 and 1. Now the function $\psi(r)$ becomes extreem, taking minimum and maximum alternately at $r=0.7073$, 0.7361 and 0.8114 and thereabout $\psi(r)$ is small enough. We may therefore assume $r_3=0.750$ on trial, so that $r_1=0.077$, $r_2=0.173$ by (44). Also solving biquardratic equation of σ_3, we find $\sigma_3 = 1.5010$ and hence $\sigma_1=0.8278$, $\sigma_2 = 1.5897$ by (45). Consequently the required trimodal representation shall be

$$\tilde{y} = 22.7522\varphi(t_1) + 26.6817\varphi(t_2) + 122.4194\varphi(t_3) = \tilde{y}_1 + \tilde{y}_2 + \tilde{y}_3,$$

where $\varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$, the standard normal function and $t_1 = 1.2081u + 4.8324$, $t_2 = 0.6290(u+1)$, $t_3 = 0.6662(u-1)$. Their values are obtained as the following table:

u	t_1	$\varphi(t_1)$	$\tilde{y}_1$	t_2	$\varphi(t_2)$	$\tilde{y}_2$	t_3	$\varphi(t_3)$	$\tilde{y}_3$	$\tilde{y}_{cal.}$	$y_{obs.}$
−5	−1.2081	0.19230	4.375	−2.5160	0.01684	0.449	−3.9972	0.00013	0.016	4.84	4
−4	0	0.39894	9.077	−1.8870	0.06725	1.794	−3.3310	0.00156	0.191	11.06	16
−3	1.2081	0.19230	4.375	−1.2580	0.18083	4.825	−2.6648	0.01145	1.402	10.60	10
−2	2.4162	0.02154	0.490	−0.6290	0.32733	8.734	−1.9986	0.05415	6.629	15.85	10
−1	3.6243	0.00056	0.013	0	0.39894	10.644	−1.3324	0.16422	20.104	30.76	38
0	4.8324			0.6290	0.32733	8.734	−0.6662	0.31955	39·119	47.85	33
1				1.2580	0.18083	4.825	0	0.39894	48.838	53.66	72
2				1.8870	0.06725	1.794	0.6662	0.31955	39.119	40.91	32
3				2.5160	0.01684	0.449	1.3324	0.16422	20.104	20.55	20
4				3.1450	0.00284	0.076	1.9986	0.05415	6.629	6.71	8
5				3.7740	0.00032	0.009	2.6648	0.01145	1.402	1.41	2
									Sum	244.20	245

Calculating ω^2 as in Ex. 6, we get $\delta^2 = 23974$, $\omega^2 = \delta^2/N^2 = 23974/245^2 = 0.0666$,

$\Phi(\omega^2=0.0666)=0.2274$, so that $\Pr\{\omega^2 \geqq 0.0666\}=0.7726>0.05$. Thus the ω^2–test permits the above trimodal representation with stronger basis than the bimodal representation. However, the χ^2–test gives $\chi^2=19.58$ and for degrees of freedom $11-7=4$, $\Pr\{\chi^2 \geqq 19.58\}<0.0005$, so that the representation is still denied.

To obtain still more exact values we may proceed as follows: Let the corrections of a_i, σ_i, r_i be ξ_i, η_i, ζ_i $(i=1, 2, 3)$, which we assume to be small, and substituting these in equations (41), (42) and (43) and besides

$$\left.\begin{aligned}\nu_6 &= \sum_{i=1,2,3} r_i(15\sigma_i^6+45a_i^2\sigma_i^4+15a_i^4\sigma_i^2+a_i^6),\\ \nu_7 &= \sum_{i=1,2,3} r_i a_i(105\sigma_i^6+105a_i^2\sigma_i^4+21a_i^4\sigma_i^2+a_i^6),\\ \nu_8 &= \sum_{i=1,2,3} r_i(105\sigma_i^8+420a_i^2\sigma_i^6+42a_i^4\sigma_i^4+28a_i^6\sigma_i^2+a_i^8\end{aligned}\right\} \tag{47}$$

and neglecting terms of higher order than the first, we obtain nine linear equations in ξ_i, η_i, ζ_i, which, being solved, give the required corrections.

When the corrected values of r_1, r_2, and r_3 in Ex. 9 were thus determined, the trimodal representation of Ex. 8 could be obtained by substituting these values in (41), (42), (43) and the first of (47); thus we have six simultaneous equations, say (48), containing six unknowns. However the task being somewhat lengthy, we postpone its treatment as a future work. Or else, we may further assume the approximate values of a_i from the given data ($a_1=-6$, $a_2=-4$, $a_3=1$ say) and hence compute by (45) (46) the approximate values of σ_i. Putting the corrections of a_i. σ_i and r_i (in Ex. 8) to be ξ_i', η_i' and ζ_i, as before, we may solve the resulting nine linear equations (say, (49)). However, since the ratios $r_1 : r_2 : r_3$ must remain the same in Ex. 8 and 9, we had better solve these 18 equations, (48) and (49) altogether, in which r_1, r_2, r_3 are assumed to be the same, and consequently containing 15 unknowns, by the method of least squares.

§8. Corrections of Estimates by Method of Least Squares

Next we shall consider the method of successive approximations, which are to be available when a rough estimation of $\tilde{y}$ is obtained, even when, solved by method of general Case III, the calculated values $\tilde{y}$ differ largely from observed y, and the ω^2– or χ^2–test shows that the obtained representation is to be rejected, say on 1% level of significance. For this purpose we may utilize the old fashoned, yet still powerful, method of least squares, although the calculations are enough troublesome.

First for exactitude, let us consider the cumlative frequency

$$\begin{aligned}\tilde{F}(u) &= N\sum_i r_i F_i(u) = N\sum_i \frac{r_i}{\sqrt{2\pi}}\int_{-\infty}^{u} \exp\left\{-\frac{1}{2}\left(\frac{u-d-a_i}{\sigma_i}\right)^2\right\}\frac{du}{\sigma_i} \\ &= N\sum_i \frac{r_i}{\sqrt{2\pi}}\int_{-\infty}^{t_i} \exp\left(-\frac{t_i^2}{2}\right)dt_i = N\sum_i r_i \Phi(t_i) = \tilde{Y}(u) \text{ say} \\ &= G(a_1, a_2, \sigma_1, \sigma_2, r_1, u) \qquad (r_2 = 1 - r_1),\end{aligned}$$

and therefore

$$\begin{aligned}\tilde{Y}_j &= N\sum_t r^j \int_{j-1/2}^{j+1/2} \exp\left\{-\frac{1}{2}\left(\frac{u-d-a_i}{\sigma_i}\right)^2\right\}\frac{du}{\sigma_i\sqrt{2\pi}} \\ &= N\sum_i r_i\left[\Phi\left(t_{ij}+\frac{1}{2\sigma_i}\right) - \Phi\left(t_{ij}-\frac{1}{2\sigma_i}\right)\right],\end{aligned}$$

where $\Phi(t_{ij}) = \int_{-\infty}^{t_{ij}} \varphi(t)dt$ denotes the normal distribution function and $t_{ij} = \frac{u_j-d-a_i}{\sigma_i} = \frac{v_i-a_i}{\sigma_i}$. These calculated values do not agree with the corresponding observed values; thus $y_j - \tilde{Y}_j = \varDelta y_j \neq 0$. Denoting the corrections of a_1, a_2, σ_1, σ_2 and r_1 by ξ_1, ξ_2, η_1, η_2 and ζ_1, we have, if the corrections be small enough,

$$\begin{aligned}\varDelta y_j &= G(a_1+\xi_1, a_2+\xi_2, \sigma_1+\eta_1, \sigma_2+\eta_2, r_1+\zeta_1, j) - G(a_1, a_2, \sigma_1, \sigma_2, j) \\ &= \sum_i\left(\frac{\partial G}{\partial a_i}\xi_i + \frac{\partial G}{\partial \sigma_i}\eta_i\right) + \frac{\partial G}{\partial r_1}\zeta_1 \text{ nearly} \\ &= N\sum_i r_i \int_{j-1/2}^{j+1/2}\left\{\left(\frac{v-a_i}{\sigma_i}\right)\frac{\xi_i}{\sigma_i} + \left[\left(\frac{v-a_i}{\sigma_i}\right)^2 - 1\right]\frac{\eta_i}{\sigma_i}\right\}\exp\left\{-\frac{1}{2}\left(\frac{v-a_i}{\sigma_i}\right)^2\right\}\frac{du}{\sigma_i\sqrt{2\pi}} \\ &\quad + N\zeta_1\left[\int_{j-1/2}^{j+1/2}\exp\left\{-\frac{1}{2}\left(\frac{v-a_1}{\sigma_1}\right)^2\right\}\frac{du}{\sqrt{2\pi}\sigma_1}\right. \\ &\qquad \left. - \int_{j-1/2}^{j+1/2}\exp\left\{-\frac{1}{2}\left(\frac{v-a_2}{\sigma_2}\right)^2\right\}\frac{du}{\sqrt{2\pi}\sigma_2}\right] \quad (v = u-d) \\ &= N\sum_{i=1,2} r_i \int_{t_{ij}-1/2\sigma_i}^{t_{ij}+1/2\sigma_i}[t_i\xi_i + (t_i^2-1)\eta_i]\frac{\varphi(t_i)}{\sigma_i}dt_i \\ &\quad + N\zeta_1\left[\int_{t_{ij}-1/2\sigma_2}^{t_{ij}+1/2\sigma_2}\varphi(t_1)dt_1 - \int_{t_2 j-1/2\sigma_2}^{t_2 j+1/2\sigma_2}\varphi(t_2)dt_2\right] \quad (v-a_i = \sigma_i t_i).\end{aligned} \tag{50}$$

Or, if the breadth $1/\sigma_i$ be small enough, then $\tilde{Y}_j$ coincides with $\tilde{y}_j$ and we have approximately

$$\begin{aligned}\varDelta y_j &= N\sum_i [t_{ij}\xi_i + (t_{ij}^2 - 1)]\frac{r_i\varphi(t_{ij})}{\sigma_i^2} + N\zeta_1\left[\frac{\varphi(t_{1j})}{\sigma_1} - \frac{\varphi(t_{2j})}{\sigma_2}\right] \\ &= \frac{t_{1j}}{\sigma_1}\tilde{y}_{1j}\xi_1 + \frac{t_{2j}}{\sigma_2}\tilde{y}_{2j}\xi_2 + \frac{t_{1j}^2-1}{\sigma_1}\tilde{y}_{1j}\eta_1 + \frac{t_{2j}^2-1}{\sigma_2}\tilde{y}_{2j}\eta_2 + \left(\frac{\tilde{y}_{1j}}{r_1} - \frac{\tilde{y}_{2j}}{r_2}\right)\zeta_1 \\ &= A_{1j}\xi_1 + A_{2j}\xi_2 + B_{1j}\eta_1 + B_{2j}\eta_2 + C_j\zeta_1.\end{aligned} \tag{51}$$

All the coefficients could be evaluated conveniently utilizing every term t_{ij}, $\varphi(t_{ij})$ and $\tilde{y}_{ij}$, which have been already obtained during calculations of $\tilde{y}_j$.

But, to be more exact, we should treat upon (50). It is easily seen that (50) reduces to

$$\begin{aligned}\Delta y_j &= N\sum_i r_i[\varphi(t_{i,\,j-\frac{1}{2}})-\varphi(t_{i,\,j+\frac{1}{2}})]\xi_i/\sigma_i + N\sum_i r_i[t_{i,\,j-\frac{1}{2}}\varphi(t_{i,\,j-\frac{1}{2}}) \\ &\qquad - t_{i,\,j+\frac{1}{2}}\varphi(t_{i,\,j+\frac{1}{2}})]\eta_i/\sigma_i + [\tilde{Y}_{1j}/r_1 - \tilde{Y}_{2j}/r_2]\zeta \\ &= A_{1j}\xi_1 + A_{2j}\xi_2 + B_{1j}\eta_1 + B_{2j}\eta_2 + C_j\zeta_1. \end{aligned} \tag{52}$$

We have already computed $\tilde{y}_{ij}=Nr_i\varphi(t_{ij})/\sigma_i$ to obtain $\tilde{y}_j$. Now, obtain similarly $\tilde{y}_{i,\,j\pm\frac{1}{2}}=\frac{Nr_i}{\sigma_i}\varphi(t_{i,\,j\pm\frac{1}{2}})$. Also to obtain $\tilde{Y}_j$ we should compute $\tilde{Y}_{ij}=Nr_i\left[\Phi\left(t_{ij}+\frac{1}{2\sigma_i}\right)-\Phi\left(t_{ij}-\frac{1}{2\sigma_i}\right)\right]$. If $\tilde{Y}_{ij}$ and $\tilde{y}_{ij}$ differ only insignificantly, then (51) would almost coincide with (52). Otherwise, we should proceed with (52) as observation equations, whose coefficients are

$$A_{ij}=\tilde{y}_{i,\,j-\frac{1}{2}}-\tilde{y}_{i,\,j+\frac{1}{2}},\ B_{ij}=t_{i,\,j-\frac{1}{2}}\varphi(t_{t,\,j-\frac{1}{2}})-t_{i,\,j+\frac{1}{2}}\varphi(t_{i,\,j+\frac{1}{2}}),\ C_j=\tilde{Y}_{1j}/r_1-\tilde{Y}_{2j}/r_2. \tag{53}$$

Ex. 12. (Ex. 8). Actually we have calculated every values of (51) for Ex. 8, and further obtained Gaussian sums $[AA]$, $[AB]$, $\cdots$, $[SS]$, as follows

156.6029,	−21.5476,	0.3406,	−99.7924,	−998.1984,	67.4807,	−895.1142,
	778.7573,	224.6250,	0.01872,	−659.1514,	−405.8534,	−83.1514,
		234.0938,	−122.7879,	−848.2896,	−81.9684,	−593.9865,
			1168.5542,	2388.8562,	463.9100,	3798.7588,
				11386.9098,	542.9410,	11813.0677,
					775.3049,	1361.8148,
						15401.3892.

On solving normal equations, corrections are found to be

$$\xi_1=0.9720,\ \xi_2=-0.6507,\ \eta_1=0.7038,\ \eta_2=0.4414,\ \zeta=0.05505.$$

However, these corrections, except ζ, being so large, our previous assumption that their powers are enough small to be neglected, is not satisfied. Really the corrected results become $a_1'=-0.2988$, $a_2'=1.2744$, $\sigma_1'=3.3999$, $\sigma_2'=1.6394$, $r_1=0.6574$, $r_2=0.3426$, and the values $\tilde{y}_j'$, recomputed using these new parameters, fit no better than before. It would have been better to have used rather (52), (53).

Ex. 13. (Ex. 1). On the otherhand we have obtained a successful correction with Ex. 1 by least squares. We have already found its representation in §3 in the form

$$\tilde{y}=\frac{Nr_1}{\sqrt{2\pi}\sigma_1}\exp\left\{-\frac{1}{2}\left(\frac{u}{\sigma_1}\right)^2\right\}+\frac{Nr_2}{\sqrt{2\pi}\sigma_2}\exp\left\{-\frac{1}{2}\left(\frac{u}{\sigma_2}\right)^2\right\}=\tilde{y}_1+\tilde{y}_2.$$

Now putting the corrections of σ_1, σ_2, r_1 and r_2 to be ξ, η, ζ and $-\zeta$, we have

$$\left[\left(\frac{u_j}{\sigma_1}\right)^2-1\right]\frac{\tilde{y}_{1j}}{\sigma_1}\xi+\left[\left(\frac{u_j}{\sigma_2}\right)^2-1\right]\frac{\tilde{y}_{2j}}{\sigma_2}\eta+\left[\frac{\tilde{y}_{1j}}{r_1}-\frac{\tilde{y}_{2j}}{r_2}\right]\zeta=y_j-\tilde{y}_j,$$

or $$a_j\xi+b_j\eta+c_j\zeta=d_j \qquad (j=0,\ \pm1,\ \pm2,\ \cdots).$$

And in fact Gaussian coefficients are obtained to be

$[aa]=47.104$,	$[ab]=41.556$,	$[ac]=187.864$,	$[ad]=15.206$,	$[as]=291.730$,
	$[bb]=588.532$,	$[bc]=729.899$,	$[bd]=159.166$,	$[bs]=1519.147$,
		$[cc]=1531.610$,	$[cd]=237.670$,	$[cs]=2687.064$,
			$[dd]=47.733$,	$[ds]=465.291$,
				$[ss]=4957.740$.

On solving the normal equations we get $\zeta=0.136$, $\eta=0.125$, $\xi=-0.331$, and hence the improved parameters become $r_1'=0.754$, $r_2'=0.246$, $\sigma_1'=2.248$, $\sigma_2'=0.960$. With these parameters we recomputed the new representation $\tilde{y}'$. Indeed, this time X^2 amounts to only 0.675, so that $\Pr(X^2 \geqq 0.675)$ becomes >0.995 for 6 degrees of freedom, thus the agreement becomes much more better than before corrections.

§9. Analysed as Pearson's Unimodal Distributions

In Pearsonian school almost all problems of curve fitting had been treated with method of unimodal analysis by means of Pearson's $\beta\kappa$ criterion.[8] So also all the foregoing examples might be computed in that way, which will be described below.

Ex. 14 (Ex. 1). We obtained $\mu_2=4.3759$, $\mu_3=0$, $\mu_4=72.9566$. Hence $\beta_1=\frac{\mu_3^2}{\mu_2^3}=0$, $\beta_2=\frac{\mu_4}{\mu_2^2}=\frac{72.9566}{19.1485}=3.8100>3$, so that $\kappa=\frac{\beta_1(\beta_1+3)}{4(4\beta_2-3\beta_1)(2\beta_2-3\beta_1-6)}=0$, and consequently it belongs to Pearson's symmetrical Type VII with unlimited ends: $\eta=\eta_0\left(1+\frac{u^2}{A^2}\right)^{-p}$. After Pearson's method the constants are determined as follows:

$$\rho=\frac{3(\beta_2-1)}{\beta_2-3}=10.407.\ p=\frac{1}{2}(\rho+2)=6.2035,\ A^2=(\rho-1)\mu_2=41.164,$$

and $\eta_0=\frac{2^{2p-2}\Gamma(p)^2}{A\pi\Gamma(2p-1)}$, where p being a little large, we may use Stirling's asymptotic formula $\Gamma(p)\cong\sqrt{2\pi}p^{p-\frac{1}{2}}e^{-p}$. Thus by logarithmic computation we get $\eta_0=0.2049$. Therefore

$$\tilde{y}=N\eta_0\left(1+\frac{u^2}{41.164}\right)^{-p}=20.49\left[1+\frac{(\bar{x}-\tilde{x})^2}{10.29}\right]^{-6.2035}.$$

Calculating the values of $\tilde{y}$ for $u=0$, ± 1, ± 2, …,we obtain the following result[9]:

u	0	±1	±2	±3	±4	±5	±6	±7	±8	±9	total
obs. y	21.94	17.97	10.63	5.48	2.78	1.33	0.56	0.20	0.06	0.02	100.00
cal. $\tilde{y}$	20.49	17.66	11.53	6.01	2.67	1.08	0.42	0.16	0.06	0.02	99.71

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[8] Cf. e.g. W. P. Elderton, Frequency Curve and Correlation, 1938; or, Y. Watanabe, Saisho Zizyôhô oyobi Tôkei (Japanese), 1935 (Maruzen).

[9] Cal. $\tilde{y}$ had been obtained informally by ordinates, not by areas, so that the total does not coincide with the observed.

Whence we get $\chi^2=\sum(y-\tilde{y})^2/\tilde{y}=1.026$ and degrees of freedom being $10-3=7$, $\Pr(\chi^2>1.026)$ lies between 0.990 and 0.995. However this means, by no means, that the unimodal representation fits better than bimodal. Indeed, the result improved by least squares as obtained in the end of §8 is mudh better than the representation obtained just now.

Ex. 15 (Ex. 2). It was $\mu_1=7.2683$, $\mu_3=0.3373$, $\mu_4=119.9274$, so that $\beta_1=\frac{\mu_3^2}{\mu_2^3}=0.0002963$, $\beta_2=\frac{\mu_4}{\mu_2^2}=2.27014<3$ and $\kappa=\frac{\beta_1(\beta_2+3)}{4(4\beta_2-3\beta_1)(2\beta_2-3\beta_1-6)}=-0.0^3296$ $\doteqdot 0$. Hence it belongs to Pearson's symmetrical Type with limited ends: $\eta=\eta_0\left(1-\frac{v^2}{c^2}\right)^p$. Here $p=\frac{5\beta_2-9}{2(3-\beta_2)}=1.6102$, $c^2=\frac{2p+5}{3}\frac{\mu_4}{\mu_2}=45.212$, $c=6.7238$ and the required representation becomes $\eta=\eta_0\left(1-\frac{v^2}{45.212}\right)^p$, $\tilde{y}=100\eta$, where $\eta_0=\frac{\Gamma(2p+2)}{2^{2p+1}c\Gamma(p+1)^2}$ $=0.12926$. As it was $v=u-0.22$ and $u=\frac{1}{5}(x-67.5)$, so we get $v=\frac{1}{5}(x-68.6)$ and

$$\tilde{y}=12.926\left(1-\frac{u-0.22}{45.212}\right)^p=12.926\left(1-\frac{x-68.6}{1130.3}\right)^{1.6102}.$$

Here χ^2 amounts to 1.985, and degrees of freedom being $13-3=10$, $\Pr(\chi^2\geqq 1.985)$ >0.995, so that we see that even such an unimodal representation would give already a sufficiently good fitting.

Strictly speaking, Sheppard's correction is only correct, so far as the density function $f(u)$ behaves at finite ends, so that $f(u)=f'(u)=f''(u)=f'''(u)=0$, or else $u^3f(u)$, $u^4f'(u)$, $u^3f''(u)$, $u^4f'''(u)$ tend zero as $u\to\pm\infty$. Hence, if the representation we have found does not satisfy these conditions, we must recompute the parameter values by taking the original uncorrected moments. Thus the above solution needs a recomputation. However, when we use the uncorrected moments in Ex. 2 in §4: $\mu_2'=7.3516$, $\mu_3'=0.3373$, $\mu_4'=123.6459$ and repeat the above made computations, we obtain an almost same representation $\tilde{y}=12.914\left(1-\frac{u-0.22}{47.23}\right)^{1.712}$, which only slightly differs from the before obtained.

Ex. 16. (Ex. 3). This example being similar to the foregoing, it appears better to use the uncorrected moments: $\mu_2'=6.4513$, $\mu_3'=4.9488$, $\mu_4'=85.4873$. Here $\beta_1=0.091213$, $\beta_2=2.05403<3$, $\kappa=-0.0088\doteqdot 0$, so it belongs still to the same type as before. However the constants become now $p=0.67125$, $c^2=28.106$, $c=5.293$, $\eta_0=0.1280$, so that the unimodal representation reduces to

$$\eta=0.1280\left(1-\frac{v^2}{28.02}\right)^{0.67135}, \qquad \tilde{y}=1520\eta=194.6\left[1-\frac{x-62.68}{700.5}\right]^{0.67135}.$$

Thus the contact of $\tilde{y}$-curve to x-axis being slight, the uncorrected moments were legitimately used. However, its ω^2 becomes extraordinarily large, and the above representation is to be rejected.

Ex. 17 (Ex. 4). Here also beginning with uncorrected moments $\mu_2' = 4.2112$, $\mu_3' = -0.2722$, $\mu_4' = 34.9489$, we obtain $\beta_1 = 0.0009913$, $\beta_2 = 1.9707 < 3$, and $\kappa = -0.00008 \doteqdot 0$. Thus we get once more again the same type, and $p = 0.4146$, $c^2 = 16.125$, $c = 4.0156$. $\eta_0 = 0.1532$, so that $\tilde{y} = 15.32\left[1 - \frac{u-0.099}{4.0156}\right]^{0.4146}$. Thus, theoretically $-3.9166 < u < 4.1146$ and we get

u	-5	-4	-3	-2	-1	0	1	2	3	4	total
obs. y	0.5	2.8	9.4	17.3	17.3	10.6	12.2	19.2	9.5	1.2	100.0
cal. $\tilde{y}$	0	4.65	11.28	13.79	15.00	15.28	14.84	13.42	10.52	0	98.78

Whence we get $\delta^2 = 751$ and $\omega_9^2 = 751/100^2 = 0.0751$, $\Phi(\omega_9^2 = 0.0751) = 0.3389$, $\Pr\{\omega_9^2 \geqq 0.0751\} = 0.6611 > 0.05$. Hence the above unimodal representation is not to be rejected.

Ex. 18 (Ex. 5). Using uncorrected moments $\mu_2' = 3.5376$, $\mu_3' = -5.4489$, $\mu_4' = 30.4220$, we have $\beta_1 = 0.67063$, $\beta_2 = -2.43092$, $\kappa = -0.03748 < 0$. Hence it belongs to Pearson's asymmetrical Type I: $\eta = \eta_0 \left|1 - \frac{\xi}{c_1}\right|^{-p_1} \left|1 - \frac{\xi}{c_2}\right|^{p_2}$. The constants are found in succession as follows: $r = \frac{6(\beta_2 - \beta_1 - 1)}{3\beta_1 - 2\beta_2 + 6} = 1.4482$, $t = \sqrt{16(r+1) + \beta_1(r+2)^2} = 6.8662$, $q_1, q_2 = \frac{r}{2}\left[1 \pm \frac{r+2}{t}\sqrt{\beta_1}\right] = 1.0203,\ 0.4279$, where μ_3 being negative, $q_1 > q_2$ and $-p_1 = q_1 - 1 = 0.0203$, $p_2 = q_2 - 1 = -0.5721$. Further $b = \frac{1}{2}\sqrt{\mu_2'}\,t = 6.4575$, $\nu = \frac{b}{r-2} = -11.7014$, whence $c_1 = \nu p_1 = 0.2375$ as well as $c_2 = \nu p_2 = 6.6944$, and lastly $\eta_0 = \frac{|p_2|^{p_2}|p_1|^{-p_1}}{b|p_2 - p_1|^{p_2 - p_1}} \frac{\Gamma(p_2 - p_1 + 2)}{\Gamma(1 - p_1)\Gamma(1 + p_2)} = 0.06136$. Therefore

$$\eta = 0.06136\left(\frac{\xi}{0.2375} - 1\right)^{0.0203}\left(1 - \frac{\xi}{6.6944}\right)^{-0.5721}, \qquad \tilde{y} = 100\eta.$$

Thus we obtain a *J*-shaped distribution. To express it by u, we need further calculations. We have originally determined ξ axis by translating origin into mode on u axis. Thus $m_0 = 0$ and $\bar{\xi}$ is given by $\bar{\xi} - m_0 = \frac{1}{2}\frac{\mu_3'}{\mu_2'}\frac{r+2}{r-2}$, so that $\bar{\xi} = 4.8126$. On the other hand it was $\bar{u} = d = 0.32$. Hence $u = 0$ corresponds to $\xi = 4.8126 - 0.32 = 4.4926$, and in general $\xi = 4.4926 + u$. Thus finally

$$\tilde{y} = 6.136\left[\frac{u + 4.4926}{0.2375} - 1\right]^{0.0203}\left[1 - \frac{u + 4.4926}{6.6944}\right]^{-0.5721}$$
$$= 18.747(u + 4.2551)^{0.0203}(2.2018 - u)^{-0.5721}.$$

Using the last expression, we compute $\tilde{y}$ for $u = -5, -4, \cdots, 2$, and obtain the following result:

u	-4	-3	-2	-1	0	1	2	3	total
obs. y cal. $\tilde{y}$	3 6	11 7	8 8	6 10	10 12	28 17	31 48	3 0	100 108

Whence, $\omega_9^2=0.0688$ and $\Pr\{\omega_9^2 \geqq 0.0688\}=1-\Phi(\omega_9^2=0.0688)=0.7589>0.05$. Hence the unimodal representation is not to be rejected.

Ex. 19 (Ex. 6). In this biometrical example it will also be found that Pearson's unimodal representation does not have higher contact with x-axis. Hence we have to start with uncorrected central moments: $\mu_2'=10.13337$, $\mu_3'=-10.4709$, $\mu_4'=230.9316$. Accordingly $\beta_1=\frac{\mu_3'^2}{\mu_2'^3}=0.10664$, $\beta_2=\frac{\mu_4'}{\mu_2'^2}=2.2489<3$ and $\kappa=-0.00895<0$. Hence, if κ be assumed nearly zero, we shall obtain just the same Type as in Ex. 15: $\eta=\eta_0\left(1-\frac{v^2}{c^2}\right)^p$. But, more exactly, it may be classified into Pearson's asymmetrical Type I, as in Ex. 18: $\eta=\eta_0\left|1-\frac{\xi}{c_1}\right|^{-p_1}\left|1-\frac{\xi}{c_2}\right|^{p_2}$. The parameters are computed successively as follows:

$$r=\frac{6(\beta_2-\beta_1-1)}{3\beta_1-2\beta_2+6}=3.7615,\quad t=\sqrt{16(r+1)+\beta_1(r+2)^2}=3.3404,$$

$$q_1, q_2=\frac{r}{2}\left[1\pm\frac{r+2}{t}\sqrt{\beta_1}\right]=2.2770,\ 1.4844,$$

where $q_1>q_2$ since $\mu_3'<0$. Consequently $-p_1=q_1-1=1.2771$; $p_2=q_2-1=0.4844$. Further $b=\frac{1}{2}\sqrt{\mu_2'}t=14.2115$, $\nu=\frac{b}{r-2}=8.0678$, $c_1=\nu p_1=-10.3031$, $c_2=\nu p_2=3.9084$. Therefore the required representation for density function becomes

$$\eta=\eta_0\left(1+\frac{\xi}{10.5031}\right)^{1.2771}\left(1-\frac{\xi}{3.9084}\right)^{0.4844}\quad\text{and } \tilde{y}=270\eta,$$

where $\eta_0=\frac{|p_1|^{-p_1}p_2^{p_2}}{b(p_2-p_1)^{p_2-p_1}}\frac{\Gamma(p_2-p_1+2)}{\Gamma(1-p_1)\Gamma(1+p_2)}=0.04154$[10] which was evaluated by use of Legendre's Table of log $\Gamma(p)$ for $1<p<2$.

The origin ξ was measured from mode on u-axis, and $\tilde{\xi}=\frac{1}{2}\left(\frac{r-2}{r+2}\right)\frac{\mu_3}{\mu_2}=-1.6869$. On the other hand it was $\bar{u}=\mathrm{d}=0.5259$. Hence $\xi=u-0.5259-1.6864=u-2.2128$, and the representation becomes

$$\tilde{y}=0.7414(u+8.091)^{1.2771}(6.1212-u)^{0.4844}.$$

Calculating $\tilde{y}$ for each u, we get

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[10] This was evaluated by use of the Table of log $\Gamma(p)$. However, if p be large, we may utilize Stirling's asymptotic formula. Also we may simply compute relative values z of η for $u=0, \pm 1, \pm 2, \cdots$, and obtain $\eta_0=1/\sum z_j$.

u	-8	-7	-6	-5	-4	-3	-2	-1	0
obs. y	1	2	2	11	24	25	16	14	18
cal. $\tilde{y}$	0.13	2.92	6.44	10.18	13.91	17.52	20.78	23.64	25.98
u	1	2	3	4	5	6	7	8	total
obs. y	30	39	41	28	14	1	2	2	270
cal. $\tilde{y}$	27.63	31.34	30.60	28.05	22.60	8.31	0	0	270.03

Whence we obtain by (39) $\delta^2=19726$ and $\omega^2=\delta^2/N^2=19726/270^2=0.2706$, $\Phi(\omega^2)=0.8362$, so that $\Pr\{\omega^2\geqq 0.2706\}=0.1638>0.05$. Thus the above unimodal representation is not to be rejected, although, compared with bimodal representation obtained in Ex. 6, the probability reduces far less.

Ex. 20 (Ex. 7). Here still using uncorrected moments $\mu_2'=6.0671$, $\mu_3'=-5.5421$, $\mu_4'=81.3514$, we obtain $\beta_1=0.13753$, $\beta_2=2.21005$, and $\kappa=-0.00862<0$, so that it belongs again to Pearson's Type I. Computing parameters in the same way as before, we get $r=3.2297$, $t=8.4520$, $q_1, q_2=1.9855$, 1.2442. Hence $-p_1=0.9855$, $p_2=0.2442$. Further $b=10.1690$, $\nu=8.2697$, $c_1=-8.1500$, $c_2=2.0191$ and $\eta_0=0.14783$. The required unimodal representation is, therefore,

$$\tilde{y}=270\times 0.14783\left[1+\frac{\xi}{8.1500}\right]^{0.9855}\left[1-\frac{\xi}{2.0191}\right]^{0.2442}.$$

The mean ξ is found to be -1.94243, while $\bar{u}=-0.2185$. Hence $\xi=u+0.2185-1.9424=u-1.7239$. Substituting this in the above, we get

$$\tilde{y}=4.2527(u+6.4261)^{0.9855}(3.7430-u)^{0.2442},$$

from which we obtain $\tilde{y}$ for every u as follows:

u	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	total
obs. y	1	1	8	26	31	18	20	33	56	45	26	3	1	1	270
cal. $\tilde{y}$	0	3.2	10.2	16.8	22.8	28.2	33.6	36.7	42.2	39.8	36.1	0	0	0	269.6

and whence the squared deviation in sum $\delta^2=17333$, so that $\omega^2=\dfrac{17333}{270^2}=0.2378$. Entering the ω^2–Table we find that $\Phi(\omega^2)=0.7952$. Hence $\Pr\{\omega^2\geqq 0.2378\}=0.2048>0.05$. Consequently the above representation is also not to be rejected.

Ex. 21 (Ex. 8). Starting with the corrected moments $\mu_2=7.68337$, $\mu_3=-12.8943$, $\mu_4=197.1589$, and proceeding similarly as foregoing, we get

$$\tilde{y}=245\eta_0\left[1+\frac{\xi}{33.2350}\right]^{22.1842}\left[1-\frac{\xi}{5.9795}\right]^{3.9913},$$

where

$$\eta_0=\frac{|p_1|^{p_1}p_2^{p_2}}{b|p_2-p_1|^{p_2-p_1}}\frac{\Gamma(p_2-p_1+2)}{\Gamma(1-p_1)\Gamma(1+p_2)},\quad b=39.2146,\quad -p_1=22.1842,\quad p_2=3.9913.$$

For smaller value of p we may put $\Gamma(p+1)=p\Gamma(p)=p(p-1)\Gamma(p-1)=\cdots$ and finally refer to Legendre's Tables of log $\Gamma(p)$ for $1<p<2$. But for larger p, it is more convenient to use Stirling's formula $\Gamma(p)=\sqrt{2\pi}p^{p-\frac{1}{2}}e^{-p}$. Thus in the above we find $\eta_0=0.1472$. Furthermore we get $\bar{\xi}=-0.9673$ and it was $\bar{u}=-0.2185$, so that $\xi=u-0.7508$. Using these substitution we get a more convenient form:

$$\tilde{y}=10^{-36}\times 5.0318(u+32.4842)^{22.1842}(6.7303-u)^{3.9913} \qquad \text{(i)}$$

from which $\tilde{y}$ can be found for every $u=-8, -7, \cdots, 7$, by logarithmic computations. We see that the above curve has a strong contact with u–axis at the left end, but at the right $\frac{d^4y}{du^4}$ does not vanish, so seems apparently Sheppard's correction inadequate. If, however, on taking uncorrected moments $\mu_2'=7.7667$, $\mu_3'=-12.8943$, $\mu_4'=201.0233$, and recomputing, we find only a little different result:

$$\tilde{y}=32.7467\left[1+\frac{\xi}{34.3888}\right]^{23.4760}\left[1-\frac{\xi}{6.1787}\right]^{4.2184},$$

$$\text{or} \qquad \tilde{y}=10^{-38}\times 1.2880(u+33.6573)^{23.4760}(6.9102-u)^{4.2184} \qquad \text{(ii)}$$

and now $\frac{d^4y}{du^4}$ vanishes at the right end also, so that Sheppard's correction becomes applicable. To decide this dilemma, we have only to compare the goodness of fitting by ω^2–test. Really these two give the following results:

u	-8	-7	-6	-5	-4	-3	-2	-1	0
obs. y	1	4	10	6	10	6	18	30	34
cal. $\tilde{y}$ (i)	1.5	2.8	4.8	7.8	12.1	17.7	24.0	30.2	34.8
cal. $\tilde{y}$ (ii)	1.4	2.5	4.4	7.2	11.2	16.2	22.0	27.6	31.7
u	1	2	3	4	5	6	7	total	
obs. y	54	28	26	10	5	1	2	245	
cal. $\tilde{y}$ (i)	35.9	32.1	23.4	12.5	3.7	0.2	0	243.6	
cal. $\tilde{y}$ (ii)	32.6	29.1	21.3	11.5	3.6	0.3	0	222.6	

and it is found that ω_0^2 amount to 0.4318 and 0.5740 for (i) and (ii), so that Pr $\{\omega^2 \geqq \omega_0^2\}=0.0595$ and 0.0258 respectively. Hence (ii) is to be rejected, while (i) is hardly not to be rejectet.

Ex. 22 (Ex. 9). Using the corrected moments $\mu_2=4.26042$, $\mu_3=-4.68103$, $\mu_4=54.49860$, we obtain by the same way as before,

$$\tilde{y}=39.43275\left[1+\frac{\xi}{9.2229}\right]^{5.5642}\left[1-\frac{\xi}{7.6912}\right]^{4.6402}, \qquad \xi=u-0.6641,$$

in which both exponents being enough large, Sheppard's corrections are correctly done, and we have no more to recalculate. Now, writing it as

$$\tilde{y}=10^{-8}\times 1.3058(u+8.5588)^{5.5642}(8.3553-u)^{4.6402},$$

we have calculated $\tilde{y}$ for $u=j=0, \pm 1, \pm 2, \cdots$ and whence computed $\delta^2=\sum(\sum y_j - \sum\tilde{y}_j)^2\tilde{y}_j$, $\omega^2=\delta^2/N^2$. But we obtain $\omega_9^2 \doteqdot 1$ and $\Phi(\omega_9^2)=0.9976$. Thus $\Pr\{\omega^2 \geqq 1\} =0.0024<0.05$. Heoce the above Pearson's representation is to be rejected.

Ex. 23 (Ex. 10). From ν_k's the moments about $\bar{u}=d=-0.5011$ are found to be $\mu_2=5.5285$, $\sqrt{\mu_2}=2.3512$, $\mu_3=1.3764$, $\mu_4=89.5887$. Hence $\beta_1=0.011212$, $\beta_2=2.39115$ and $\kappa=-0.0830<0$, so that it belongs still to Pearson's Type I: $\eta=\eta_0\left|1-\frac{\xi}{c_1}\right|^{-p_1}\left|1-\frac{\xi}{c_2}\right|^{p_2}$. Calculating in a similar way as before we obtain

$$\tilde{y}=245\eta, \quad \eta=\eta_0\left[1+\frac{\xi}{18.138}\right]^{36.725}\left[1-\frac{\xi}{31.375}\right]^{63.525},$$

where $\eta_0=\dfrac{(-p_1)^{-p_1}p_2^{p_2}}{b(p_2-p_1)^{p_2-p_1}}\dfrac{\Gamma(p_2-p_1+2)}{\Gamma(1-p_1)\Gamma(1+p_2)}$ and since $-p_1$ and p_2 both large, we may use Stirling's formula and obtain $\eta_0=\dfrac{p_2-p_1+2}{b}\sqrt{\dfrac{p_2-p_1}{2\pi(-p_1)p_2}}=0.1691$. Also the mean $\xi=0.1196$. But, as $\bar{u}=d=-0.5011$, we have $\xi=u+0.6107$, and consequently

$$\tilde{y}=16.91\left[1+\frac{u+0.6107}{18.138}\right]^{36.725}\Bigg]\left[1-\frac{u+0.6107}{31.375}\right]^{63.525}.$$

Whence calculating $\tilde{y}_j$ for $u=j$, we get

u	-9	-8	-7	-6	-5	-4	-3	-2	-1	0
obs. y	0.01	0.06	0.28	0.99	2.74	5.90	10.11	14.16	16.54	16.09
cal. $\tilde{y}$	0.01	0.05	0.26	0.95	2.64	5.74	10.00	14.21	16.70	16.38
u	1	2	3	4	5	6	7	8	9	total
obs. y	13.35	9.38	5.58	2.92	1.25	0.46	0.14	0.03	0.01	100.00
cal. $\tilde{y}$	13.54	9.46	5.46	2.86	1.23	0.45	0.14	0.04	0.01	100.15

Hence χ^2 amounts to only 0.0336. The degrees of freedom being 19–6=13, $\Pr\{\chi^2\geqq 0.0336\}>0.9$. Also all $|\sum y-\sum\tilde{y}|$ being less than 1, we have $\delta^2=\sum|\sum y -\sum\tilde{y}|^2\tilde{y}<\sum\tilde{y}\doteqdot 100$. Hence $\omega^2<\dfrac{100}{100^2}=0.01$, $\Phi(\omega^2)<0.0001$, $\Pr\{\omega^2\geqq 0.01\}>0.9999$. Thus either χ^2– or ω^2–test does not reject the above unimodal representation.

§10. Gram-Charlier's Representation

This method is frequently recommended because of its easy calculation. It is nothing but a single normal representation with additional corrections

$$y_C=\frac{N}{\sigma}[\varphi_0(t)+A_3\varphi_3(t)+A_4\varphi_4(t)], \tag{54}$$

where $\sigma=\sqrt{\mu_2}$, $t=\dfrac{u-d}{\sigma}=\dfrac{x-\bar{x}}{\sigma_x w}$ and $\varphi_0(t)$, $\varphi_3(t)$, $\varphi_4(t)$ are the standard normal density function $\dfrac{1}{\sqrt{2\pi}}e^{-t^2/2}$ as well as its 3–rd and 4–th derivatives, respectively, while

$$A_3 = -\frac{\mu_3}{6\sqrt{\mu_2}^3}, \quad A_4 = \frac{\mu_4}{24\mu_2^2} - \frac{1}{8}. \tag{55}$$

Ex. 24. We shall calculate y_C of (54) for every value of u in Ex. 1–10 and examine ω^2–test in regard to acceptability of Gram-Charlier's representation. The normal representation, i.e. the first single term alone in (54), denoted by y_N, shall be incidentally considered.

Ex. 1. Here $d=0$, $\sigma=2.092$, $N=100$, $A_3=0$, $A_4=0.03375$.

u	0	±1	±2	±3	±4	±5	±6	±7	±8	±9	total
y	21.94	17.97	10.63	5.48	2.78	1.33	0.56	0.20	0.06	0.02	100.00
y_N	19.07	17.01	12.07	6.82	3.06	1.09	0.31	0.07	0.01	0.00	99.95
y_C	20.69	17.98	11.41	5.65	2.49	1.14	0.54	0.22	0.01	0.00	99.57

Whence we get $\omega_0^2=0.0032$ and 0.0263 for y_C and y_N, so that both are nearly 0, and therefore $\Pr(\omega^2 \geqq \omega_0)$ nearly 1. Thus y_C as well as y_N are both acceptable as representations. Moreover, if we try χ^2–test, the calculated value y_C becomes nearly 0 at the end interval $u=\pm 9$, so that $|y-y_C|^2 \div y_C=\infty$. However, if these be lumped to $u=\pm 8$, χ^2 amounts to 1.3209 nearly. The degrees of freedom n being 9–4=5, $\Pr(\chi^2 \geqq 1.3209)>0.9$. Similarly for y_N, $\chi^2=3.4497$, and for $n=5$, $\Pr\{x^2 \geqq 3.4497\}>0.5$. Thus even with severe χ^2–test both repersentations are not to be rejected.

Ex. 2. $d=0.22$, $\sigma=2.696$, $N=100$, $A_3=-0.0029$, $A_4=-0.0304$.

u	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	total
y	1	2	6	8	13	12	11	12	13	10	6	4	2	100
y_N	1.0	2.3	4.4	7.3	10.6	13.4	14.8	14.2	11.9	8.7	5.6	3.1	1.5	98.8
y_C	1.0	2.6	5.2	8.1	10.8	12.7	13.4	13.1	11.6	9.2	6.4	3.7	1.7	99.5

For y_C, $\omega_0^2=0.0106$, $\Pr\{\omega^2 \geqq \omega_0^2\}$ is nearly 1, thus surely acceptable. Also for y_N, $\omega_0^2=0.055$, $\Pr\{\omega^2 \geqq \omega_0^2\}=0.845$, and thus still not to be rejected, though less acceptable than y_C.

Ex. 3. $d=0.03684$, $\sigma=2.5235$, $N=1520$, $A_3=-0.0513$, $A_4=-0.0339$.

u	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	total
y	3	14	60	161	263	264	171	102	127	187	114	42	10	2	1520
y_N	13.8	32.5	66.8	117.0	173.1	220.9	240.3	224.6	177.3	121.2	70.1	34.5	14.8	5.2	1512
y_C	8.6	34.8	82.4	141.3	192.9	219.0	216.2	195.8	162.0	122.1	80.6	43.7	19.2	4.7	1523.3

here with y_C the ω^2 becomes 0.2 and $\Pr\{\omega^2 \geqq \omega_0^2 =0.266>0.05$. Hence y_C is not to be rejected, but for y_N not so.

Ex. 4. $d=-0.099$, $\sigma=2.3172$, $N=100$, $A_3=0.1246$, $A_4=-0.0446$.

u	−5	−4	−3	−2	−1	0	1	2	3	4	total
y	0.5	2.8	9.4	17.3	17.3	10.6	12.2	19.2	9.5	1.2	100.0
y_N	1.1	3.1	7.1	12.6	17.8	19.6	17.0	11.6	6.2	3.6	98.7
y_C	0.3	4.4	7.5	10.4	13.6	17.4	19.0	15.6	8.5	3.4	99.1

For y_C, $\omega^2=0.228$ and $\Phi(\omega^2)=0.7953$, $1-\Phi(\omega^2)=0.2047>0.05$, so it is not to be rejected.

Ex. 5. $d=0.32$, $\sigma=1.8586$, $N=100$, $A_3=0.1415$, $A_4=-0.0248$.

u	-5	-4	-3	-2	-1	0	1	2	3	4	total
y	0	3	11	8	6	10	28	31	3	0	100
y_N	0.4	1.4	3.6	9.9	16.7	21.1	20.1	14.3	7.6	3.0	98.1
y_C	1.1	2.6	3.9	7.5	12.5	19.5	21.9	18.3	9.0	2.3	98.6

For y_C and y_N we get $\omega^2=0.7687$ and 0.8398, and $1-\Phi(\omega^2)=0.008$, $0.006<0.05$, so that y_C and y_N are both to be rejected.

Ex. 6. $d=0.52593$, $\sigma=3.1697$, $N=270$, $A_3=0.0548$, $A_4=-0.0318$.

u	-8	-7	-6	-5	-4	-3	-2	-1	0
y	1	2	2	11	24	25	16	14	18
y_N	0.91	2.05	4.07	7.48	12.22	18.35	24.68	30.28	33.55
y_C	1.21	2.70	5.22	8.92	13.28	18.05	22.47	26.46	29.63

u	1	2	3	4	5	6	7	8	total
y	30	39	41	28	14	1	2	2	270.00
y_N	33.60	30.57	25.07	18.56	12.58	7.61	4.24	2.10	267.92
y_C	31.36	30.99	27.85	22.21	15.55	9.07	4.32	1.36	270.65

Here with y_C, $\omega^2=0.322$, $\Phi(\omega^2=0.322)=0.8829$, Pr $\{\omega^2\geqq 0.822\}=0.1171>0.05$, thus not to be rejected; while with y_N, $\omega^2=0.794$, $\Phi(\omega^2=0.794)=0.9925$, Pr $\{\omega^2\geqq 0.794\}$ $=0.0075<0.05$, and so to be rejected.

Ex. 7. $d=0.0893$, $\sigma=2.4462$, $N=270$, $A_3=0.0631$, $A_4=-0.00338$.

u	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	total
y	1	1	8	26	31	18	20	33	56	45	26	3	1	1	270
y_N	1.1	2.7	6.8	13.4	23.0	33.7	41.8	43.9	38.9	29.1	18.4	10.0	4.6	1.8	269.2
y_C	1.2	3.8	8.7	15.2	22.6	29.8	36.0	40.2	40.2	34.0	23.1	12.1	4.2	0.4	271.5

For y_C it is $\omega^2=0.4208$, $\Phi(\omega^2=0.4208)=0.9364$, Pr $\{\omega^2>0.4208\}=0.0636>0.05$, thus the y_C-representation is not to be rejected. However, for y_N, we get $\omega^2=0.9408$, $\Phi(\omega^2=0.9408)=0.9966$, Pr $\{\omega^2\geqq 0.9408\}=0.0034<0.05$, hence y_N-representation is to be rejected.

Ex. 8. $d=-0.2185$, $\sigma=2.7719$, $N=245$, $A_3=0.1009$, $A_4=0.0142$.

u	-8	-7	-6	-5	-4	-3	-2	-1	0
y	1	4	10	6	10	6	18	30	34
y_N	0.5	1.3	3.2	6.5	11.8	18.9	26.5	32.6	35.2
y_C	1.5	2.7	4.3	6.4	9.5	14.3	21.5	30.0	36.4

u	1	2	3	4	5	6	7	total
y	54	28	26	10	5	1	2	245
y_N	33.4	27.8	20.4	13.1	7.4	3.6	1.6	243.6
y_C	37.7	32.8	23.7	14.0	6.6	2.3	0.4	244.1

Whence with y_C, $\omega_0^2=0.1446$, $\Phi(\omega_0^2)=0.5980$, $1-\Phi(\omega_0^2)=0.4020>0{\cdot}05$, hence it is not to be rejected. But with y_N, $\omega_0^2=0.490$, $\Phi(\omega_0)=0.9579$, $1-\Phi(\omega_0^2)=0.0421<0.05$, so it is to be rejected.

Ex. 9. $d=0.2694$, $\sigma=2.0641$, $N=245$, $A_3=0.0887$, $A_4=0.0001$.

u	-5	-4	-3	-2	-1	0	1	2	3	4	5	total
y	4	16	10	10	38	33	72	32	20	8	2	245
y_N	1.8	5.6	13.5	25.9	39.2	47.0	44.5	33.3	19.7	9.2	3.4	243.2
y_C	3.3	6.9	12.6	21.3	33.6	45.3	48.5	39.0	22.6	8.8	1.9	243.8

Here for y_C, $\omega^2=0.2582$, $\Phi(\omega^2=0.2582)=0.8220$, $\Pr\{\omega^2\geqq 0.2582\}=0.1780>0.05$, not to be rejected, but for y_N, $\omega^2=0.5150$, $\Phi(\omega^2=0.515)=0.9636$, $\Pr\{\omega^2\geqq 0.515\}=0.0364<0.05$, to be rejected.

Ex. 10. $d=-0.5011$, $\sigma=2.3512$, $N=100$, $A_3=-0.016036$, $A_4=-0.002873$.

u	-9	-8	-7	-6	-5	-4	-3	-2	-1	0
y	0.01	0.06	0.28	0.99	2.74	5.90	10.11	14.16	16.54	16.09
y_N	0.02	0.10	0.37	1.10	2.72	5.61	9.65	13.85	16.59	16.59
y_C	0.00	0.05	0.28	1.00	2.71	5.80	10.02	14.19	16.63	16.29
u	1	2	3	4	5	6	7	8	9	total
y	13.35	9.38	5.58	2.92	1.25	0.46	0.14	0.03	0.01	100.00
y_N	13.84	9.64	5.60	2.72	1.10	0.37	0.10	0.02	0.00	99.99
y_C	13.44	9.40	5.58	2.81	1.20	0.43	0.13	0.03	0.01	100.00

Here $\omega^2=0.0028$ so that $\Phi(\omega^2)$ nearly zero, $1-\Phi(\omega^2)$ nearly unity. Also $\chi^2=0.0230$ and degrees of freedom being $17-10=7$, $\Pr\{\chi^2\geqq 0.023\}>0.995$. Both acceptable.

§11. Watanabe's Representation

Watanabe proposed some bimodal representations, either by those of his new types, or by superposition of curves belonging to Pearson's types.[11] One case of the latter has been thoroughly developed in the present note. The former shall be illustrated below by treating Ex. 5 as example.

Ex. 25 (Ex. 5). It was found in Ex. 5, that $d=\bar{u}=0.32$ and $\mu_2=3.4543$, $\mu_3=-5.4489$, $\mu_4=28.6824$, $\mu_5=-82.2471$, $\mu_6=-336.7033$, $\mu_7=-1172.9612$. We shall represent this distribution by the genuine bimodal curve

$$y=y_0\exp\varphi(v)=y_0\exp\{c_1v+c_2v^2+c_3v^3+c_4v^4\},\qquad v=u-d. \tag{56}$$

To determine parameters by Pearson's method of moments, as a first approximation, we have to solve the following linear equations:

$$\left.\begin{aligned}0+c_2'\mu_2+c_3'\mu_3+c_4'\mu_4&=1,\\ c_1'\mu_1+c_2'\mu_3+c_3'\mu_4+c_4'\mu_5&=0,\\ c_1'\mu_2+c_2'\mu_4+c_3'\mu_5+c_4'\mu_6&=-3\mu_2,\\ c_1'\mu_3+c_2'\mu_5+c_3'\mu_6+c_4'\mu_7&=-4\mu_3.\end{aligned}\right\}\qquad c_i'=\frac{c_i}{i} \tag{57}$$

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[11] Y. Watanabe, Bimodal Distributions, this Journal vol. V (1954), p. 30.

On substituting μ_n's values in equations (57) and solving them, we get

$$c_4=4c_4'=-0.0482,\ c_3=3c_3'=-0.2004,\ c_2=2c_2'=0.1733,\ c_1=c_1'=0.9487.$$

Therefore

$$\varphi(v)=0.9487v+0.1733v^2-0.2004v_3-0.0482v^4,$$

and we obtain, as the required representation expessed in $u=v+0.32$

$$\tilde{y}=k\exp\{0.7825u+0.3361u^2-0.1387u^3-0.0482u^4\}=k\exp\varphi(u).$$

To determine k we have only to integrate the above expression numerically. But, with a later purpose. we have calculated the frequencies in each subclass, i.e. the areas of every subclass: $u_j-\frac{1}{2}<u<u_j+\frac{1}{2}$ $(j=0, \pm 1, \pm 2, \cdots)$, (i) roughly from ordinate values $\exp\varphi(u)$, and (ii) by means of Simpson's formula, lastly (iii) using Gauss' method of 5 selected ordinates, —— partly for the sake of comparison —— the results are as follows:

$u=j$	(i) $\exp\varphi(u)$	(ii) Simpson	(iii) Gauss	(iv) cal. $\tilde{y}$	(v) obs. y
-5	0.00025	0.0027	0.00275	0.0270	0
-4	0.29677	0.3923	0.39274	3.8510	3
-3	1.67815	1.5501	1.53205	15.0226	11
-2	1.12498	1.1455	1.14515	11.2288	8
-1	0.70054	0.7208	0.73136	7.1714	6
0	1	1.0568	1.05365	10.3316	10
1	2.53876	2.5254	2.52550	24.7639	28
2	2.79715	2.5592	2.57838	28.2825	31
3	0.10266	0.2364	0.23654	2.3194	3
4	(0.000003)	0.0002	0.00017	0.0017	0
sum	10.239263	10.1894	10.19829	99.9999	100

Taking the sum of (iii) we have $k=100/10.19829=9.8056$, and on multiplying this value to column (iii) we obtained column (iv). Thus the required representation is given as the first approximation, by

$$\tilde{y}=9.8056\exp\{0.7825u+0.3361u^2-0.1381u^3-0.0482u^4\}.$$

Using above table, we get $\chi^2=\sum(y-\tilde{y})^2/\tilde{y}=4.31$ on pooling at both ends. Degrees of freedom being 3, $\Pr\{\chi^2>4.31\}>0.2>0.05$. Also $\omega^2=\sum_{i=-5}^{4}[\sum_{-5}^{i}y_j-\sum_{-5}^{i}\tilde{y}_j]^2\tilde{y}_i/100^2=0.0371$ and $\Phi(\omega_9^2=0.0371)=0.0519$, so that $\Pr\{\omega_9^2\geqq 0.0371\}=0.9481>0.05$. Thus, by either test the representation is not to be rejected.

To obtain a more elaborate result we proceed by method of least squares. Let the corrections of c_i and k be ξ_i $(i=1, 2, 3, 4)$ and ξ_5. The corrected ordinate becomes

$$y^*=\left(1+\frac{\xi_5}{k}\right)\tilde{y}\exp\{\xi_1u+\xi_2u^2+\xi_3u^3+\xi_4u^4\}.$$

Corrections being assumed to be small, we have approximately

$$y^{*}=\tilde{y}[1+\xi_1 u+\xi_2 u^2+\xi_3 u^3+\xi_4 u^4+\xi_5/k],$$

and consequently

$$\Delta y=y-\tilde{y}=\sum_{i=1}^{4}\tilde{y}\xi_i u^i+\tilde{y}\xi_5/k. \tag{58}$$

The total frequency should be always equal to 100, so that

$$\int y^{*}du=100, \qquad \text{as well as} \qquad \int \tilde{y}du=100.$$

Hence

$$\sum_i \xi_i \int \tilde{y}u^i du+100\xi_5/k=0. \tag{59}$$

But $\int \tilde{y}u^i du=100\nu_i$ are approximately known by given statistics (cf. Ex. 5 in §5). Hence the above residual equation (58) becomes

$$(u-\nu_1)\tilde{y}\xi_1+(u^2-\nu_2)\tilde{y}\xi_2+(u^3-\nu_3)\tilde{y}\xi_3+(u^4-\nu_4)\tilde{y}\xi_4=\Delta y,$$

and ξ_5 is eliminated. Putting

$$(u_j-\nu_1)\tilde{y}_j=a_j,\quad (u_j^2-\nu_2)\tilde{y}_j=b_j,\quad (u_j^3-\nu_3)\tilde{y}_j=c_j,\quad (u_j^4-\nu_4)\tilde{y}_j=d_j \ \text{ and } \ \Delta y_j=e_j,$$

we obtain, as observation equations,

$$a_j\xi_1+b_j\xi_2+c_j\xi_3+d_j\xi_4=e_j \qquad (j=-5,\ -4,\ \cdots,\ 3,\ 4), \tag{60}$$

whose coefficients are computed as follows:

j	a_j	b_j	c_j	d_j	e_j	s_j
−5	−0.1436	0.5790	−3.3183	16.2321	−0.0270	13.2222
−4	−16.6355	47.9178	−238.3645	894.1173	−0.8508	686.1843
−3	−49.8724	81.7727	−374.0428	859.0967	−4.0218	512·9324
−2	−26.0494	4.9808	−66.2464	−87.6922	−3.2282	−178.2354
−1	−9.4657	−18.3341	7.8881	−163.5705	−1.1710	−184.6532
0	−3.3059	−36.7443	21.6951	−245.9811	−0.3310	−264.6672
1	16.8385	−63.3103	76.7638	−564.8326	3.2375	−531.3031
2	42.4721	11.2147	255.3381	−197.4446	5.7190	117.2993
3	6.2157	12.6253	67.4916	132.6408	0.6807	219.6541
4	0.0062	0.2115	0.1124	0.3947	−0.0017	0.7231
sum	−39.9400	40.9131	−252.6829	642.9606	0.0057	391.2565

Whence Gaussian coefficients are obtained as follows

$$\begin{array}{lllll}
[aa]=5669.17, & [ab]=-5221.41, & [ac]=36756.33, & [ad]=-70148.17, & [ae]=612.65, \\
 & [bb]=14987.72, & [bc]=-44426.34, & [bd]=159924.89, & [be]=-484.34, \\
 & & [cc]=277303.50, & [cd]=-620157.82, & [ce]=3659.40, \\
 & & & [dd]=2008322.27, & [de]=-6527.76, \\
 & & & & [ee]=72.4534.
\end{array}$$

Solving the normal equations, we find

$$\xi_1=0.1853,\ \xi_2=0.0176,\ \xi_3=-0.0145,\ \xi_4=-0.00264,$$

and the corrected coefficients become

$$c_1=0.9678,\ c_2=0.3537,\ c_3=-0.1532,\ c_4=-0.05084.$$

With these new values we recomputed the following integrals again by Gauss' method of 5 selected ordinates:

$$A_j=\int_{u_j-1/2}^{u_j+1/2}\exp\{\sum_{i=1}^{4}c_iu^i\}du, \qquad J=\sum A_j$$

u	A_j	$\tilde{y}=kA_j$……area	$k\exp\varphi(u)$……ord.	obs. y
−5	0.00213	0.0197	0.0002	0
−4	0.31882	2.9467	2.2287	3
−3	1.24467	11.5041	12.4551	11
−2	0.91500	8.4570	8.2915	8
−1	0.62542	5.7806	5.5408	6
0	0.97933	9.0516	9.2427	10
1	3.06513	28.3300	28.2567	28
2	3.22078	29.7686	34.3029	31
3	0.44797	4.1404	1.0576	3
4	0.00013	0.0012	0.0000	0
	$J=10.81938$	99.9999	101.3462	100

Therefore $k=100/J=9.24267$ and $\tilde{y}=kA_j$ are obtained, as above. Also remark that the values of central ordinates $k\exp\varphi(u)$ differ from theoretical frequencies $\tilde{y}$ untolerably.[12] Now $\chi^2=\sum|y-\tilde{y}|^2/\tilde{y}$ becomes only 0.4240 and for 3 degrees of freedom $\Pr\{\chi^2>0.4240\}>0.9$. Also $\omega_9^2=\delta^2/N^2=52.25/100^2=0.0052$ and $\Phi(\omega_9^2=0.0052)$ being nearly zero, $\Pr\{\omega_9^2>0.0052\}$ is almost unity. Thus our improved repersentation fits the given data utterly good.

§12. Concluding Remark

(A further Scheme for the case when Correlation Table is given)

Although bi- or tri-modal distributions could be somehow represented by Pearson's unimodal curves or those of Gram-Charlier, our multimodal distributions fit far better, especially when the existence of modes is distinct, as exhibited by the χ^2- or ω^2-test. This is a matter of course since other representations do not pay attension to existence of modes, whereas our method has taken special account of it purposely. In general the χ^2-test denies more frequently than the ω^2-test does. This is partly due to the fact that the former is heavily affected by those data with less probabilities, while the latter puts stress on those with larger probabilities. In-

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[12] Even for these ordinates representation we get $\omega^2=0.0394$ and $\Phi(\omega_9^2=0.0394)=0.0609$, so that $\Pr\{\omega_9^2>0.0394\}=0.9391>0.05$. Thus it is already not to be rejected. However, this probability is less than that corresponding to $\tilde{y}=kA_j$.

deed, without ω^2–test our task should have been much more troublesome, in order to make results pass the stubborn χ^2–test. The parameters could be always determined by Pearson's method of moments. However, this being only a first approximation, we should appeal to method of least squares to obtain good representations, although it is frequently enough intricate with the present common calculating machines.

However, the frontal attack made e.g. in §3, Case III, to solve an equation of ninth degree or suchlike might have been too much tedious. Rather some method of successive approximation in Ex. 11, §7 would be more recommendable.

To generalize the method describeed in this note to the case of many variables, one may suppose that a Correlation Table for two variates x, y say, length in Ex. 6 and width in Ex. 7, is reported, and that the density distribution $f(x, y)$ is likely a superposition of two normal surfaces, such that

$$f(x, y) = r_1 f_1(x, y) + r_2 f_2(x, y), \qquad r_1 + r_2 = 1, \tag{61}$$

where $f_i(x, y)$ $(i=1, 2)$ denotes a normal density function of two variates x, y i.e.

$$f_i(x, y) = \frac{1}{2\pi\sigma_i\tau_i\sqrt{1-\rho_i^2}} \exp\left[-\frac{1}{2}Q(x, y)\right], \tag{62}$$

$$Q(x, y) = \frac{1}{1-\rho_i^2}\left\{\frac{(x-a_i)^2}{\sigma_i^2} + \frac{(y-b_i)^2}{\tau_i^2} - \frac{2\rho_i(x-a_i)\,(y-b_i)}{\sigma_i\tau_i}\right\}, \tag{63}$$

where a_i, b_i, σ_i, τ_i are respective mean and S.D. of x and y and ρ_i is their correlation coefficient. By a similar treatment as in §2, we can calculate the moments about origin $(0, 0)$ of order k, l

$$\nu_{k,l} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^k y^l f(x, y)\,dx\,dy, \tag{64}$$

and in particular

$$\nu_{0,0}=r_1+r_2=1, \quad \nu_{1,0}=r_1a_1+r_2a_2, \quad \nu_{0,1}=r_1b_1+r_2b_2,$$
$$\nu_{2,0}=r_1(a_1^2+\sigma_1^2)+r_2(a_2^2+\sigma_2^2), \quad \nu_{0,2}=r_1(b_1^2+\tau_1^2)+r_2(b_2^2+\tau_2^2),$$
$$\nu_{1,1}=r_1(a_1b_1+\rho_1\sigma_1\tau_1)+r_2(a_2b_2+\rho_2\sigma_2\tau_2),$$
$$\nu_{3,0}=\sum_{i=1,2} r_i[a_i^3+3a_i\sigma_i^2], \quad \nu_{2,1}=\sum_{i=1,2} r_i[(a_i^2+\sigma_i^2)b_i+2a_i\sigma_i\tau_i\rho_i,],$$
$$\nu_{1,2}=\sum_{i=1,2} r_i[(b_i^2+\tau_i^2)a_i+2b_i\sigma_i\tau_i\rho_i], \quad \nu_{0,3}=\sum_{i=1,2} r_i(b_i^3+3b_i\tau_i^2),$$
$$\nu_{4,0}=\sum_{i=1,2} r_i(a_1^4+6a_i^2\sigma_i^2+3\sigma_i^4), \quad \nu_{0,4}=\sum_{i=1,2} r_i(b_i^4+6b_i^2\tau_i^2+3\tau_i^4),$$
$$\nu_{3,1}=\sum_{i=1,2} r_i[a_i^3b_i+3a_i^2\rho_i\sigma_i\tau_i+3a_ib_i\sigma_i^2+3\rho_i\sigma_i^3\tau_i],$$
$$\nu_{1,3}=\sum_{i=1,2} r_i[a_ib_i^3+3b_i^2\rho_i\sigma_i\tau_i+3a_ib_i\tau_i^2+3\rho_i\sigma_i\tau_i^3],$$
$$\nu_{2,2}=\sum_{i=1,2} r_i[(a_i^2+\sigma_i^2)\,(b_i^2+\tau_i^2)+4a_ib_i\sigma_i\tau_i\rho_i+3\sigma_i^2\tau_i^2\rho_i^2].$$

Of course, any moment of further order could be computed: e.g.

$$\nu_{5,0}=\sum_{i=1,2} r_i a_i(a_i^4+10a_i^2\sigma_i^2+15\sigma_i^4), \text{ \&c.}$$

Thus there being 12 unknowns r_i, a_i, b_i, σ_i, τ_i, ρ_i $(i=1, 2)$, we may obtain sufficient number or more of equations to determine these parameters. Therefore it reduces naturally to a problem of least squares.

From the given correlation table we can compute each moment $\nu'_{k,l}$ about origin, and whence the moments $\mu_{k,l}$ about center. On writing $\mu_{k,l}$ in place of $\nu_{k,l}$ above, we obtain observation equations, and by solving 12 equations among them, we are able to estimate 12 unknowns.

However, this method of moments (Pearson) is only a first approximation to obtain a rough estimation of parameters. To get a more minute result we should necessarily proceed to find their corrections by method of least squares.

In fact the values of r_i $(i=1, 2)$ are determined from those of a_j or b_j while the values σ_i (or τ_i) could be found from μ_{20}, μ_{30}, (or $\mu_{0,2}$, $\mu_{0,3}$) in terms of a_j, and on their substitution in μ_{40} (or μ_{04}), we obtain a relation between a_1, a_2 (or b_1, b_2) and thus we get two equations between a_1, a_2, b_1, b_2. Similar substitutions in ν_{11}, ν_{12}, ν_{21} and ν_{22} yield four equations between a_1, a_2, b_1, b_2, ρ_1, ρ_2. These six equation being combined, they are sufficient to determine six unknowns. Analysis may go enough complex, yet we need not here the fifth moment, and consequently the procedure might be carried out more simply than the treatment described in this note.

However, the usual method of likelihood cannot be applied here, because the joint probability that x_k, y_k $(k=1, 2, \cdots, n)$ take place, is now

$$P=\prod_{k=1}^{n} f(x_k, y_k) = \prod_{k=1}^{n} [r_1 f_1(x_k, y_k)+r_2 f_2(x_k, y_k)].$$

Thus it is a product of several binomials, so that $\frac{\partial}{\partial a_i}\log P$, $\frac{\partial}{\partial b_i}\log P$ do not reduce to simple forms as in the ordinary case of a product of monomials:

$$P=\prod_{k=1}^{n}\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_k-a)^2}{\sigma_x^2}+\frac{(y_k-b)^2}{\sigma_y^2}-\frac{2\rho(x_k-a)(y_k-b)}{\sigma_x\sigma_y}\right]\right\}.$$