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ANALYSES OF BIMODAL DISTRIBUTIONS[0]

(ON THE DECOMPOSITION OF A BIMODAL DISTRIBUTION INTO TWO NORMAL CURVES)

By

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§1. Introduction

As well known, if a random variable z be a sum of two independent normal variables x and y, so also the former shall be normal, and vice versa. Our present problem differs from this, and rather relates to the so-called general normal distribution.[1] In the operation of convolution the problem is to add independent variables $x_1+x_2=x$, so to speak, while our problem is concerned with the super-

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1) Just as the general Poisson's distribution is defined as $F(\lambda)=\int\frac{\lambda^x e^{-\lambda}}{x!}p(\lambda)d\lambda$, e.g. K. Kunizawa, Modern Theories of Probabilities, (Japanese), 1951, p. 75, we may conceive the general normal distribution $F(x)=\iint\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}p(a,\sigma)\,da\,d\sigma$.

positions $y_1+y_2=y$. The independent variable belongs to one or the other of the two normal distributions $N(x_1, a_1, \sigma_1)$ and $N(x_2, a_2, \sigma_2)$ with certain probabilities p_1 and $p_2(=1-p_1)$, so that the resulting statistics consists in a mixture of x_1 and x_2 with rates r_1 and r_2, each proportional to p_1 and p_2. From a given actual statistics we need to estimate those unknowns a_1, a_2, σ_1, σ_2, r_1, r_2 and in particular, when the given distribution is bimodal, although this is not sometimes apparently disclosed, if the difference $|a_1-a_2|$ is small enough, or one of r_1, r_2 quite large compared with the other.

§2. Preliminary Computations

In actual statistics frequently the distribution appears to be a superposition of two unimodal curves.[2] In the present note we shall mainly treat of the case, where two components are normal. So the presumed representation is of the form

$$y=Nf(x)=\frac{n_1}{\sqrt{2\pi}\sigma_1}\exp\left\{-\frac{(x-a_1)^2}{2\sigma_1^2}\right\}+\frac{n_2}{\sqrt{2\pi}\sigma_2}\exp\left\{-\frac{(x-a_2)^2}{2\sigma_2^2}\right\} \tag{1}$$

$$=n_1f_1(x)+n_2f_2(x),$$

where n_1, n_2 (unknown) and $n_1+n_2=$ the whole frequency N (known) denote the nomber of each component and the all, respectively. Since we are mainly concerned with the bimodal distribution, it shall be understood that $a_1 \neq a_2$. Also we are interested in the case when n_1, n_2 both >0, since, otherwise, the problem becomes an algebraical superposition, as difference, we exclude such sorts of representations. To estimate parameters, following Pearson's method of moments, it is usually said that we shall have only an enough number of moments equations to determine parameters—thus in the present case, up to the fifth moment, since there are six unknowns in (1). However, to speak more exactly, further (higher) moments equations should be also satisfied by thus determined values of parameters. Hence we ought to tackle necessarily to solve many moments equations by the method of least squares. Indeed, usual Pearson like treatment is nothing but its a first approximation. Nevertheless, since these calculations are intricate enough in the present state that culculating machines are of still lower capacity, we are obliged to put up with the first approximation.

Let ν_k be the k-th moment of (1) about $x=0$, thus

$$\nu_k=\sum_{i=1,2}\int_{-\infty}^{\infty}n_ix^kf_i(x)dx=\sum_{i=1,2}n_i\int_{-\infty}^{\infty}(x-a_i+a_i)^kf_i(x)dx=\sum_{i=1,2}n_i\sum_{h=0}^{k}\binom{k}{h}a_{i\,k-h}\,\mu_{ih},$$

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2) Y. Watanabe, Bimodal Distributions, this Journal, vol. V (1954), p. 29.

where $\mu_{ih}=\int_{-\infty}^{\infty}(x-a_i)^h f_i(x)\,dx$, so that,

$$\mu_{i0}=1,\quad \mu_{i1}=0,\quad \mu_{i2}=\sigma_i^2,\quad \mu_{i3}=0,\ \mu_{i4}=3\sigma_i^4\quad \mu_{i5}=0.$$

More in detail

$$\nu_0=n_1+n_2=N,\ \nu_1=n_1a_1+n_2a_2,\ \nu_2=n_1(a_1^2+\sigma_1^2)+n_2(a_2^2+\sigma_2^2),\ \nu_3=\sum_{i=1,2}n_i(a_i^3+3a_i\sigma_i^2),$$

$$\nu_4=\sum n_i(a_i^4+6a_i^2\sigma_i^2+3\sigma_i^4),\quad \nu_5=\sum_{i=1,2}n_ia_i(a_i^4+10a_i^2\sigma_i^2+15\sigma_i^4).$$

Further on setting $n_i/N=r_i$ and assuming that the mean of the whole distribution is taken as origin, the k-th moment about mean, μ_k, would be

$$\left.\begin{aligned}&\mu_0=r_1+r_2=1,\quad \mu_1=r_1a_1+r_2a_2=0,\quad \mu_2=r_1(a_1^2+\sigma_1^2)+r_2(a_2^2+\sigma_2^2),\\&\mu_3=r_1(a_1^3+3a_1\sigma_1^2)+r_2(a_2^3+3a_2\sigma_2^2),\quad \mu_4=\sum_{i=1,2}r_i(a_i^4+6a_i^2\sigma_i^2+3\sigma_i^4),\\&\mu_5=\sum_{i=1,2}r_ia_i(a_i^4+10a_i^2\sigma_i^2+15\sigma_i^4).\end{aligned}\right\}\tag{2}$$

Case I. If it happens that μ_3 and μ_5 are nearly zero, we may put simply $r_1=r_2=\frac{1}{2}$, $a_1=-a_2=a$, and $\sigma_1=\sigma_2=\sigma$ by symmetry. We have only to solve $\mu_2=a^2+\sigma^2$, $\mu_4=a^4+6a^2\sigma^2+3\sigma^4$, which yield immediately

$$a^4=\tfrac{3}{2}\mu_2^2-\tfrac{1}{2}\mu_4,\quad \sigma^2=\mu_2-a^2.\tag{3}$$

Case II. If the assumption $\sigma_1=\sigma_2=\sigma$ be still granted, but not symmetry ($r_1\neq r_2$), we have five unknowns, and equations (2) degenerate into

$$\mu_2=\sigma^2+r_1a_1^2+r_2a_2^2,\quad \mu_3=r_2a_1^3+r_2a_2^3,\quad \mu_4=r_1a_1^4+r_2a_2^4+6\mu_2\sigma^2-3\sigma^4,$$

besides $r_1+r_2=1$, $r_1a_1+r_2a_2=0$. In this case, firstly the variance σ^2 shall be found from the cubic equation

$$2(\mu_2-\sigma^2)^3-(3\mu_2^2-\mu_4)(\mu_2-\sigma^2)-\mu_3^2=0,\tag{4}$$

i.e.

$$(\sigma^2)^3-3\mu_2(\sigma^2)^2+\tfrac{1}{2}(\mu_4+3\mu_2^2)\sigma^2+\tfrac{1}{2}(\mu_3-\mu_2\mu_4+\mu_2^3)=0,\tag{5}$$

and secondly the proportion ratio $r_1/r_2=q(>0)$ from the quadratic equation

$$\frac{(q-1)^2}{q}=\frac{\mu_3^2}{(\mu_2-\sigma^2)^3},\tag{6}$$

and consequently r_1, r_2, a_1, a_2 can be all determined. Observing that the left-hand side of (6) is positive, the right-hand side must be the same, and accordingly $\mu_2>\sigma^2$ should hold, and whence by (4) it follows that

$$(\mu_2-\sigma^2)^2>\tfrac{1}{2}(3\mu_2^2-\mu_4).\tag{7}$$

If this inequality does not hold. such σ^2 should be abandoned. As to signs of a_1, a_2 we must choose such a pair as makes the equality in regard to μ_3 consistent.

Case III. In general $\sigma_1 \neq \sigma_2$. From the first two equations of (2) we get

$$r_1 = \frac{-a_1}{a_1 - a_2}, \quad r_2 = \frac{a_1}{a_1 - a_2} \qquad (a_1 \neq a_2). \tag{8}$$

Whence, putting

$$a_1 + a_2 = s, \quad a_1 a_2 = p, \tag{9}$$

yield

$$r_1 a_1^2 + r_2 a_2^2 = -p, \quad r_1 a_1^3 + r_1 a_2^3 = -sp, \quad r_1 a_1^4 + r_2 a_2^4 = -p(s^2 - p),$$

and

$$r_1 a_1^5 + r_2 a_2^5 = -sp(s^2 - 2p).$$

Further, upon writing

$$a_i^2 + \sigma_i^2 = b_i, \qquad (i = 1, 2) \tag{10}$$

the remaining equations of (2) reduce to

$$\left.\begin{aligned} &\mu_2 = r_1 b_1 + r_2 b_2. \ \mu_3 = 3(r_1 a_1 b_1 + r_2 a_2 b_2) + sp, \ \mu_4 = 3(r_1 b_1^2 + r_2 b_2^2) + p(s^2 - p), \\ &\mu_5 = 15(r_1 a_1 b_1^2 + r_2 a_2 b_2^2) - 20(r_1 a_1^3 + r_2 a_2^3) - 6sp(s^2 - 2p). \end{aligned}\right\} \tag{11}$$

From the first two of (11), we obtain

$$b_i = \mu_2 - \frac{1}{3a_j}(\mu_3 - 2sp), \qquad i, j = 1, 2 \quad (i \neq j) \tag{12}$$

and on substituting these in the last two equations of (11),

$$\left.\begin{aligned} &6p^3 - 2s^2 p^2 + (3\mu_4 - 9\mu_2^2 - 4s\mu_3)p + \mu_3^2 = 0, \\ &4sp^3 - (2s^3 + 20\mu_3)p^2 + (3\mu_5 - 30\mu_2\mu_3)p + 5\mu_3^2 s = 0. \end{aligned}\right\} \tag{13}$$

First, eliminating p^3 from (13) and second, eliminating p^0, we have

$$\left.\begin{aligned} &\alpha p^2 + \beta p + \gamma \equiv 2(s^3 + 30\mu_3)p^2 - (8\mu_3 s^2 + 6Bs - 9C)p - 13\mu_3^2 s = 0, \\ &\alpha' p^2 + \beta' p + r' \equiv 26sp^2 - 4(2s^3 - 5\mu_3)p - (20\mu_3 s^2 + 15Bs - 3C) = 0. \end{aligned}\right\} \tag{14}$$

where

$$B = 3\mu_2^2 - \mu_4 \text{ (Biquadratic)}. \quad C = 10\mu_2\mu_3 - \mu_5 \text{ (Cinq)}. \tag{15}$$

Third, eliminating p^2 between (14), also p^0, respectively, we get

$$(\alpha\beta' - \alpha'\beta)p = \gamma\alpha' - \gamma'\alpha, \quad (\gamma\alpha' - \gamma\alpha')p^2 = (\beta\gamma' - \beta'\gamma)p, \tag{16}$$

and whence finally

$$(\alpha\beta' - \alpha'\beta)(\beta\gamma' - \beta'\gamma) = (r\alpha' - \gamma'\alpha)^2. \tag{17}$$

If $\alpha, \beta, \cdots, \gamma'$, coefficients in (14) be fully written up, (16) yields

$$p=\frac{20\mu_3 s^5+15Bs^4-3Cs^3+431\mu_3^2 s^2+450B\mu_3 s-90C\mu_3}{-[8s^6+116\mu_3 s^3-78Bs^2+117Cs-600\mu_3^2]}$$
$$=\frac{56\mu_3^2 s^4+240\mu_3 Bs^3+(90B^2-204C\mu_3)s^2+(260\mu_3^3-153BC)s+27C^2}{2(\text{the expression in numerator of first fraction})}, \tag{18}$$

which must be negative, since a_1, a_2 are to have different signs.

From (18) we obtain, as the detailed form of (17),

$$s=0 \quad \text{and} \quad \sum_{m=0}^{9} A_m s^{9-m}=0, \tag{19}$$

an equation of ninth degree in s, where

$$\left.\begin{aligned}
&A_0=312\mu_3^2, \quad A_1=780\mu_3 B, \quad A_2=292.5B^2-468\mu_3 C,\\
&A_3=10764\mu_3^3-351BC, \quad A_4=21333B\mu_3^2+58.5C^2,\\
&A_5=4680B^2\mu_3-7371C\mu_3^2 \quad A_6=92020.5\mu_3^4+3861BC\mu_3-1755B^3,\\
&A_7=152880B\mu_3^3+5616B^2C-4914\mu_3 C^2,\\
&A_8=87750B^2\mu_3^2-585C\mu_3^2-5001.75BC^2,\\
&A_9=789.75C^3-17550\mu_3^2 BC-39000\mu_3^5.
\end{aligned}\right\} \tag{20}$$

Here all A_m are homogeneous expressions of degree $m+6$, because B and C are defined as (15). To compute these coefficients, it will be convenient to tabulate the requisite values of $B^l C^m \mu_3^n$ for $l, m, n=0, 1, 2, \cdots$, by preliminary calculations.

When a value of s is found from (19) and the corresponding values of p from (18), and if p be negative, a_1 and a_2 could be calculated by (9), and whence r_1, r_2 by (8); further b_1, b_2 by (12) and finally σ_1^2,σ_2^2 by (10). Thus all unknowns would be completely determind.

Lastly we should try the χ^2– or ω^2– test to examine the goodness of fit; these are illustrated in §5, §6 by examples. We have met $s=0$ at (19), namely $a_1+a_2=0$. Hence we have a special

Case IV. $r_1=r_2$, $a_1=-a_2=a$, $\sigma_1 \neq \sigma_2$.

In this case we obtain from (2)

$$\mu_2=a^2+\tfrac{1}{2}(\sigma_1^2+\sigma_2^2), \quad \mu_3=\tfrac{3}{2}a(\sigma_1^2-\sigma_2^2), \quad \mu_4=a^4+3a^2(\sigma_1^2+\sigma_2^2)+\tfrac{3}{2}(\sigma_1^4+\sigma_2^4). \tag{21}$$

Consequently $\sigma_1^2+\sigma_2^2=2(\mu_2-a_2)$, $\sigma_1^2-\sigma_2^2=\frac{2}{3}(\mu_3/a)$, and therefore

$$\sigma_1^2=\mu_2-a^2+\mu_3/3a, \quad \sigma_2^2=\mu_2-a^2-\mu_3/3a. \tag{22}$$

These being substituted in μ_4, we get an equation, cubic in a^2:

$$a^6+\tfrac{1}{2}(\mu_4-3\mu_2^2)a^2-\tfrac{1}{6}\mu_3^2=0, \tag{23}$$

from which a can be always found, and whence σ_1^2, σ_2^2 by (22).

§3. Alternative Formulas

We may also alternatively proceed as follows: Let

$$\left.\begin{aligned} &\frac{r_1}{r_2}=q\ (>0),\ \text{so that}\ r_1=\frac{q}{1+q},\ r_2=\frac{1}{1+q}, \\ \text{and}\qquad &a_1=a,\quad a_2=-aq. \end{aligned}\right\} \tag{24}$$

These being substituted in μ_2, μ_3 of (2), we have

$$\mu_2=qa^2+\frac{q\sigma_1^2+\sigma_2^2}{1+q},\quad \mu_3=q(1-q)a^3+\frac{3aq}{1+q}(\sigma_1^2-\sigma_2^2).$$

Whence

$$\sigma_1^2=\mu_2+\frac{\mu_3}{3aq}-\frac{a^2}{3}(1+2q), \tag{25}$$

and

$$\sigma_2^2=\mu_2-\frac{\mu_3}{3a}-\frac{a^2}{3}(2+q)q. \tag{26}$$

Again these being substituted in μ_4 of (2), we obtain

$$2q^2(1+q+q^2)a^6-4q(1-q)\mu_3a^3+3q(\mu_4-3\mu_2^2)a^2-\mu_3^2=0, \tag{27}$$

and $$2q(1-q)(1+q^2)a^6-15q(1-q)\mu_2a^4-20\mu_3a^3-3(10\mu_2\mu_3-\mu_5)a-5(1-q)\mu_3^2=0. \tag{28}$$

If we eliminate q between (27), (28), we shall obtain, besides $a=0$ an equation of 30-th degree in a, while, if a be eliminated, besides $q=0$, an equation of 27-th degree in q; thus both are impracticable, unless by means of electronic computer & *c*. However, from above we may deduce some special cases.

Case V. When q is known. In this case a can be found from (27), and consequently σ_1, σ_2 from (25), (26); of course r_1, r_2 from (24). It is noteworthy that there is no need of μ_5 here.

Case VI. When $a_1=a$ is known. Rewriting (27) in the form

$$q^4+q^3+\left(1+\frac{2\mu_3}{a^3}\right)q^2-\left(\frac{2\mu_3}{a^3}+\frac{3(3\mu_2^2-\mu_4)}{2a^4}\right)q-\frac{\mu_3^2}{2a^6}=0,$$

which permits at least one positive root. The remaining calculations are the same as Case V.

Case VII. When one of S.D. e.g. σ_2 is known. In this case, we may eliminate q between (26), (27) and obtain, after easy but somewhat lengthy calculations, an equation of 10-th degree in $a(=a_1)$:

$$a^{10}-\frac{\mu_3}{Q}a^9+\frac{11}{7}Qa^8-2\mu_3a^7+\frac{1}{28}\left[25Q^2-24B+\frac{26\mu_3^2}{Q}\right]a^6-\frac{25}{14}\mu_3Qa^5+\left(\frac{5}{4}\mu_3^2-\frac{45}{14}BQ\right)a^4$$
$$-\frac{3}{7}\left(\frac{\mu_3}{4Q}-12B\right)\mu_3a^3+\frac{9}{4}B\left(\frac{B}{28}-\frac{\mu_3^2}{Q}\right)a^2-\frac{9B^2\mu_3}{112Q}a-\frac{\mu_3^4}{112Q}=0,$$

where $Q=3(\mu_2-\sigma_2^2)$, $B=3\mu_2^2-\mu_4$. This equation has surely one positive and one negative root at least. With a thus obtained, q can be computed from (26), accordingly σ_1 from (25) and r_1, r_2, a_2 from (24). In general, formulas in this section are rather intricate except that in Case V, which is effective in some special example (cf. Ex. 7 in §6). But before we apply the above methods to actual examples, we shall still discuss the Case VIII, that $a_1=a_2$.

§4. The Case with Common Mean

In the foregoing we have assumed that $a_1 \neq a_2$. Now let us treat briefly the case where two normal components have the same mean $a_1=a_2=a$, but with different variances. Taking the common mean as origin, the superposed one becomes

$$y=y_1+y_2=\frac{r_1}{\sqrt{2\pi}\sigma_1}\exp\left\{-\frac{x^2}{2\sigma_1^2}\right\}+\frac{r_2}{\sqrt{2\pi}\sigma_2}\exp\left\{-\frac{x^2}{2\sigma_2^2}\right\}, \tag{29}$$

where $r_1+r_2=1$ and $\sigma_1 \neq \sigma_2$. The central moments become

$$\mu_{2k}=(2k-1)(2k-3)\cdots 3\cdot 1\,(r_1\sigma_1^{2k}+r_2\sigma_2^{2k}), \qquad \mu_{2k+1}=0.$$

Hence, to find r_1, r_2, σ_1, σ_2 we ought to utilize the first four even ordered moments:

$$r_1+r_2=\mu_0=1, \quad r_1\sigma_1^2+r_2\sigma_2^2=\mu_2, \quad r_1\sigma_1^4+r_2\sigma_2^4=\frac{1}{3}\mu_4, \quad r_1\sigma_1^6+r_2\sigma_2^6=\frac{1}{15}\mu_6.$$

Or. putting $\frac{r_1}{r_2}=q$, we have

$$r_1=\frac{q}{1+q}, \quad r_2=\frac{1}{1+q}; \tag{30}$$

and $$q\sigma_1^2+\sigma_2^2=(1+q)\mu_2, \quad q\sigma_1^4+\sigma_2^4=\frac{1+q}{3}\mu_4, \quad q\sigma_1^6+\sigma_2^6=\frac{1+q}{15}\mu_6.$$

Whence

$$q=\frac{\mu_2-\sigma_2^2}{\sigma_1^2-\mu_2}=\frac{\mu_4/3-\sigma_2^4}{\sigma_1^4-\mu_4/3}=\frac{\mu_6/15-\sigma_2^6}{\sigma_1^6-\mu_6/15}, \tag{31}$$

where $q \neq 0, \infty$, so that either numerator or denominator cannot vanish separately. and also simultaneously, since then $\sigma_1=\sigma_2$ contradictory to hyporhesis. Hence

$$(\sigma_1^2-\mu_2)\sigma_2^4-(\sigma_1^4-\frac{\mu_4}{3})\sigma_2^2+\mu_2\sigma_1^4-\frac{1}{3}\mu_4\sigma_1^2=0, \tag{32}$$

and $$\left(\sigma_1^4-\frac{\mu_4}{3}\right)\sigma_2^4-\left[\sigma_1^6+\mu_2\sigma_1^4-\frac{1}{3}\mu_4\sigma_1^2-\frac{1}{15}\mu_6\right]\sigma_2^2+\mu_2\sigma_1^6-\frac{1}{15}\mu_6\sigma_1^2=0. \tag{33}$$

If we eliminate σ_2^4 between (32), (33), we shall obtain

$$(\sigma_1^2-\sigma_2^2)\left[\left(\mu_2^2-\frac{1}{3}\mu_4\right)\sigma_1^4+\left(\frac{1}{15}\mu_6-\frac{1}{3}\mu_2\mu_4\right)\sigma_1^2+\frac{1}{9}\mu_4^2-\frac{1}{15}\mu_2\mu_6\right]=0. \tag{34}$$

Since $\sigma_1 \neq \sigma_2$, the secoud factor must vanish. Again (32) can be written as

$$(\sigma_2^2-\mu_2)\sigma_1^4-\left(\sigma_2^4-\frac{\mu_4}{3}\right)\sigma_1^2+\mu_2\sigma_2^4-\frac{1}{3}\mu_4\sigma_2^2=0. \tag{35}$$

Now eliminating σ_1^4 between (34) and (35), we obtain

$$(\sigma_1^2-\mu_2)\left[\left(\mu_2^2-\frac{1}{3}\mu_4\right)\sigma_2^4-\left(\frac{1}{3}\mu_2\mu_4-\frac{1}{15}\mu_6\right)\sigma_2^2+\frac{1}{9}\mu_4^2-\frac{1}{15}\mu_2\mu_6\right]=0. \tag{36}$$

But, since $\sigma_1^2 \neq \mu_2$ by (31), and we see by (34) and (36) that σ_1 as well as σ_2 should be 2 roots of the same equation

$$\left(\mu_2^2-\frac{1}{3}\mu_4\right)\sigma^4+\left(\frac{1}{15}\mu_6-\frac{1}{3}\mu_2\mu_4\right)\sigma^2+\frac{1}{9}\mu_4^2-\frac{1}{15}\mu_2\mu_6=0. \tag{37}$$

Specially, provided every coefficient in (37) vanishes, then it follows that $\mu_4=3\mu_2^2$ and $\mu_6=15\mu_2^3$, which implies that the given distribution is already normal as a whole, and there is no need to be decomposed.

Ex. 1. In a certain sampling distribution of means, the moments were obtained as in the following table (odd ordered moments are known to be zero). To decompose it into two normal distributions.

$\bar{x}-\bar{\bar{x}}$ central value	f in %	$u=\dfrac{\bar{x}-\bar{\bar{x}}}{0.5}$	fu^2	fu^4	fu^6
0	21.94	0	0	0	0
± 0.5	17.97	± 1	17.97	17.97	17.97
± 1.0	10.63	± 2	42.52	170.08	680.32
± 1.5	5.48	± 3	49.32	443.88	3994.92
± 2.0	2.78	± 4	44.48	711.68	11386.88
± 2.5	1.33	± 5	33.25	831.25	20781.25
± 3.0	0.56	± 6	20.16	725.76	26127.36
± 3.5	0.20	± 7	9.80	480.20	23529.80
± 4.0	0.06	± 8	3.84	245.76	15728.64
± 4.5	0.02	± 9	1.62	131.22	10628.82
sum	N=100.00		222.96×2	3757.80×2	112875.96×2
÷N	$\mu_0'=1$		$\mu_2'=4.4592$	$\mu_4'=75.1560$	$\mu_6'=2257.5192$

Performing Sheppard's corrections, we get

$$\mu_2=\mu_2'-0.0833=4.3759, \quad \sqrt{\mu_2}=2.0919,$$

$$\mu_4=\mu_4'-\frac{1}{2}\mu_2-0.0125=72.9566,$$

$$\mu_6=\mu_6'-\frac{5}{4}\mu_4-\frac{3}{4}\mu_2-\frac{1}{448}=2166.3213.$$

Substituting these values in (37) we obtain

$$5.1704\sigma^4-38.0043\sigma^2+26.1226=0, \text{ or } \sigma^4-7.3504\sigma^2+5.0523=0,$$

whence $\sigma_1^2=6.651$, $\sigma_2^2=0.6984$ and $\sigma_1=2.579$, $\sigma_2=0.8357$.

Hence by (31) $q=\frac{\mu_2-\sigma_2^2}{\sigma_1^2-\mu_2}=1.616$ and by (30) $r_1=0.618$, $r_2=0.382$.

Thus the given distribution seems to be a mixture of two samples whose proportion is about 3 : 2 and with different variances, $\sigma_1^2:\sigma_2^2=1:10$.

To test its legitimacy, we try e.g. χ^2-test. The above result gives as its representation

$$\tilde{y}=\frac{61.8}{2.579\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{u}{2.579}\right)^2\right\}+\frac{38.2}{0.8357\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{u}{0.8357}\right)^2\right\}$$
$$=2.396\,\varphi(t_1)+4.571\,\varphi(t_2)=\tilde{y}_1+\tilde{y}_2,$$

where $t_1=\frac{u}{2.579}$, $t_2=\frac{u}{0.8357}$ and $\varphi(t)=\frac{1}{\sqrt{\pi 2}}e^{-t^2/2}$. By use of the $\varphi(t)$-Table we computed the values of $\tilde{y}_{1j}$, $\tilde{y}_{2j}$, and $\tilde{y}_j$ for $u=j=0, \pm1, \pm2, \cdots$,

u	0	± 1	± 2	± 3	± 4	± 5	± 6	± 7	± 8	± 9	total
obs. y	21.94	17.97	10.63	5.48	2.78	1.33	0.56	0.20	0.06	0.02	100.00
cal. $\tilde{y}$	27.80	17.82	8.12	4.89	2.87	1.46	0.64	0.24	0.08	0.02	100.08

Whence it is found that $\chi^2=\sum|y-\tilde{y}|^2/\tilde{y}=3.270$. Here degrees of freedom being $10-4=6$, $Pr(\chi^2\geqq 3.270)>Pr(\chi^2\geqq 3.83)=0.7>0.05$, and the representation is not to be rejected.

To speak more precisely, we ought to use the Table of normal integral $\int_{-\infty}^{t}\varphi(t)dt=\Phi(t)$ and to calculate $Nr_i\left[\Phi\left(t_{ij}+\frac{1}{2\sigma_i}\right)-\Phi\left(t_{ij}-\frac{1}{2\sigma_i}\right)\right]$ as the correct value of $\tilde{y}_{ij}$. But, assuming that the width $1/\sigma_i$ is small, this is nearly equal to $\frac{Nr_i}{\sigma_i}\varphi(t_{ij})$, and thus it will do merely to put $u=u_j$ in $\tilde{y}_i$.

§5. Applications to Pedagogical Statistics

Ex. 2. A result of certain estimation test for students in some middle school is given as the two first columns in the following table, in which x and y denote the respective mark and the percentage of number of the corresponding students,[3] $\tilde{y}$ in the last column being theoretical values calculated afterwards from the representation that we shall obtain below. The distribution being bimodal we try representation (1).

For the sake of convenience, instead of central values x we have taken $u=\frac{1}{5}(x-67.5)$ and worked out as usual:

[3] Those numbers falling on ends of subintervals were bisected, and each half counted into both neighbouring subintervals.

c.v. x	y	u	yu	yu^2	yu^3	yu^4	yu^5	$\tilde{y}$
37.5	1	−6	−6	36	−216	1296	−7776	0.6
42.5	2	−5	−10	50	−250	1250	−6250	2.2
47.5	6	−4	−24	96	−384	1536	−6144	5.5
52.5	8	−3	−24	72	−216	648	−1944	9.6
57.5	13	−2	−26	52	−104	208	−416	12.2
62.5	12	−1	−12	12	−12	12	−12	12.0
67.5	11	0	0	0	0	0	0	11.0
72.5	12	1	12	12	12	12	12	11.5
77.5	13	2	26	52	104	208	416	12.4
82.5	10	3	30	90	270	810	2430	11.0
87.5	6	4	24	96	384	1536	6144	7.2
92.5	4	5	20	100	500	2500	12500	3.3
97.5	2	6	12	72	432	2592	15552	1.1
sum	N=100		22	740	520	12608	14512	99.6

Reducing the total to unity on dividing by $N=100$, we get the tabular moments about $u=0$ to be $\nu'_0=1$, $\nu'_1=d=0.22$, $\nu'_2=7.40$, $\nu'_3=5.20$, $\nu'_4=126.08$, $\nu'_5=145.12$; whence central moments (moments about mean $\bar{u}=d=0.22$, i.e. $v=u-d=0$) were obtained as $\mu'_0=1$, $\mu'_1=0$, $\mu'_2=\nu'_2-d^2=7.3156$, $\mu'_3=\nu'_3=3d\nu'_2+2d^3=0.3373$, $\mu'_4=\nu'_4-4d\nu'_3+6d^2\nu'_2-3d^4=123.6459$, $\mu'_5=\nu'_5-5d\nu'_4+10d^2\nu'_3-10d^3\nu'_2+4d^5=7.9499$. Finally Sheppard's corrections being made, they become[4] $\mu_0=1$, $\mu_1=0$, $\mu_2=\mu'_2-\frac{1}{12}=7.2683$, $\sqrt{\mu_2}=2.960$, $\mu_3=0.3373$, $\mu_4=\mu'_4-\frac{\mu_2}{2}-\frac{1}{80}=119.9992$, $\mu_5=\mu'_5-\frac{5}{6}\mu_3=7.6688$.

Here moments of odd order being comparatively small, Case I may be applied, and we get by (3) $a^4=\frac{1}{2}(3\mu_2^2-\mu_4)=19.23$, so that $a=\sqrt{4.385}=2.094$ and $\sigma^2=\mu_2-a^2=2.883$, $\sigma=1.698$. Therefore, the required representation becomes

$$\tilde{y}=\frac{50}{\sqrt{2\pi}\sigma}\left[\exp\left\{-\frac{(v-a)^2}{2\sigma^2}\right\}+\exp\left\{-\frac{(v+a)^2}{2\sigma^2}\right\}\right]$$

with $\sigma=\sigma_u=1.698$ and $a=2.094$. Or, on writing $v=u-d=\frac{1}{5}(x-67.5)-0.22=\frac{1}{5}(x-68.6)$,

$$\tilde{y}=\frac{50}{\sqrt{2\pi}\sigma_u}\left[\exp\left\{-\frac{(x-79)^2}{2\sigma_x^2}\right\}+\exp\left\{-\frac{(x-58)^2}{2\sigma_x^2}\right\}\right]\ \text{nearly},$$

where $\sigma_x=5\sigma_u=8.49$. Or, setting $\frac{x-79}{\sigma_x}=\frac{u-2.314}{\sigma_u}=t_1$, $\frac{x-58}{\sigma_x}=\frac{u+1.874}{\sigma_u}=t_2$,

4) These procedures are the usual way of calculations, namely first to compute central moments from tabular moments and second to make Sheppard's corrections. This way is mach more simple in calculations than the reverse procedures, i.e. to correct tabular moments by Sheppard at first, and then transform them into central moments. Though both manners give the same result, the former is preferable, because, even when Sheppard's corrections are found to be inapplicable after determination of representation, the calculated uncorrected central moments shall be still of use (compare §9)

$$\tilde{y}=\frac{50}{\sigma_u}\{\varphi(t_1)+\varphi(t_2)\},$$

where $\varphi(t)=\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$, the standard normal density function. Using Table of $\varphi(t)$, we found the values of $\tilde{y}$ written in the above table, and whence $\chi^2=\sum(y-\tilde{y})^2 \div\tilde{y}=1.68$. As the number of degrees of freedom equals $12-2=10$, and Pr $(\chi^2\geqq 4.87)$ is $0.9>0.05$, our representation fits the given data very well enough.

Ex. 3. A similar estimation test as in Ex. 2, gave the following table, here y being the actual numbers, find its representation.

x (central value)	32.5	37.5	42.5	47.5	52.5	57.5
y (frequency)	3	14	60	161	263	264

62.5	67.5	72.5	77.5	82.5	87.5	92.5	97.5	total
171	102	127	187	114	42	10	2	1520

Transformed x into $u=\frac{1}{5}(x-62.5)=-6, -5, \cdots, 6, 7$ and reduced the total number to unity, tabular moments about $u=0$ are found to be $\nu_0'=1$, $\nu_1'=0.03684$, $\nu_2'=6.4526$, $\nu_3'=5.6618$, $\nu_4'=89.4658$, $\nu_5'=153.0566$; whence moments about mean $u=d$ (uncorrected and corrected by Sheppard) are obtained as $\mu_0'=1$, $\mu_1'=0$, $\mu_2'=6.4513$, $\mu_3'=4.9488$, $\mu_4'=88.6838$, $\mu_5'=136.6536$; as well as $\mu_0=1$, $\mu_1=0$, $\mu_2=6.3680$, $\mu_3=4.9488$, $\mu_4=85.4873$, $\mu_5=132.5296$.

Here μ_3, μ_5 being not so small, but dispersions somewhat alike, let us apply Case II. Equation (4) becomes $X^3-18.0835X-12.2453=0$ $(X=\mu_2-\sigma^2>0)$, which has only one positive root 4.5438. So that we obtain $\sigma^2=1.8242$, which satisfies inequality (7) in fact. The corresponding proportions ratio equation (6) becomes $q^2-2.2611\,q+1=0$, that gives $q=1.6579$ or 0.6032. Hence $r_1=0.6238$ or 0.3762 whereas $r_2=1-r_1$ and $a_2=-qa_1$. Combining the last equation with $r_1a_1^2+r_2a_2^2=4.5438$, we obtain $a_1=\pm1.6554$, $a_2=\mp2.7445$, or else $a_1=\pm2.7469$, $a_2=\mp1.6569$. But the inequality $r_1a_1^3+r_2a_2^3=q(1-q)a_1^3=\mu_3>0$ repuires $a_1\gtrless0$ according as $q\gtrless1$. Hence we have either (i) $a_1=-1.6554$, $a_2=2.7445$, or (ii) $a_1=2.7445$, $a_2=-1.6554$. Consequently we obtain the following two representation:

$$\tilde{y}=\frac{1}{\sqrt{2\pi}\sigma}\left[n_1\exp\left\{-\frac{(u+1.6186)^2}{2\sigma^2}\right\}+n_2\exp\left\{-\frac{(u-2.7813)^2}{2\sigma^2}\right\}\right],$$

where $\sigma=1.3506$ and (i) $n_1=948.12$ or (ii) 571.88 while $n_2=1520-n_1$. Which one will do, shall be decided by the χ^2– or ω^2– test.

However, having evaluated $\tilde{y}$, ordinates at $u=-6, -5, \cdots, 7$, and computed $\chi^2=\sum(y-\tilde{y})^2/\tilde{y}$, similarly as in Ex. 2, we found extraordinarily large values $\chi^2=139.5$ and 416.1 for (i) and (ii) respectively, and as this shows that (i) is preferable to (ii), thereby, however, the acceptability of (i) is never ascertained. Hitherto we have consulted with ordinate values only as rough approximations of frequencies. To speak more exactly, we should compute the area under the normal density curve in every subclass. To do this we have to refer to the Table of $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt$, taking for x, the end values of u: $-6.5, -5.5, \cdots, 7.5$. Thus obtained values of frequencies, $\tilde{Y}$ say, are as in the following table:

u	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	total
obs. frequency	3	14	60	161	263	264	171	102	127	187	114	42	10	2	1520
cal. fr. by (i)	1.7	13.8	61.8	149.2	281.0	251.5	159.6	111.3	155.6	164.1	111.9	45.5	11.0	1.5	1519.5
cal. fr. by (ii)	1.0	8.3	37.3	89.9	170.0	155.7	118.2	136.9	248.5	271.0	185.4	75.4	18.3	2.6	1518.5

Whence $\chi^2=14.32$ for (i) and 372.9 for (ii). For 8 degrees of freedom it is $\Pr(\chi^2 \geqq 14.32) > \Pr(\chi^2 \geqq 15.51)=0.05$. Hence we may adopt the representation (i).

Ex. 4. A percertage result of entrance examination for mathematics held in some school was informed to have been as follows:

mark x	0~10	10~20	20~30	30~40	40~50	50~60	60~70	70~80	80~90	90~100	tatal
fr. y	0.5	2.8	9.4	17.3	17.3	10.6	12.2	19.2	9.5	1.2	100.00

Those falling to end marks were bisected, and each half distributed to the neighbouring subintervals.

Taking central values of subintervals $x=5, 15, \cdots, 95$ and putting $u=\frac{x-55}{10}$, the tabular moments about $u=0$ are found to be $\nu_0'=1$, $\nu_1'=-0.099=d$, $\nu_2'=4.221$, $\nu_3'=-1.521$, $\nu_4'=34.809$, $\nu_5'=-31.299$ and whence the central moments, corrected by Sheppard, $\mu_0=1$, $\mu_1=0$, $\mu_2=4.1279$, $\sqrt{\mu_2}=2.3172$, $\mu_3=-0.27221$, $\mu_4=32.8721$, $\mu_5=-12.8603$ (uncorrected μ_k' being $\mu_2'=4.2112$, $\mu_4'=34.9486$, &c.). Here μ_5 is not small, while $\sigma_1 \neq \sigma_2$ since two subranges $5<x<35$ and $75<x<95$ appear different in magnitude. So we have no choice but to solve equation (19) straightforwardly. It runs now

$$s^9-167.58s^8+4221.31s^7-459.206s^6+1254.28s^5-1838.47s^4$$
$$-462.492s^3+129040s^2+83231.68s-1517.79=0.$$

This equation has a root $s(=a_1+a_2)=0.5072$, which being substituted in (18), $p=a_1a_2=-3.358$ follows. Therefore $a_1, a_2=2.10, -1.60$ nearly. Accordingly by (8) $r_1=0.4324$, $r_2=0.5676$ and by (12) $b_1=0.6529$, $b_2=0.4975$ and by (9) $\sigma_1^2=0.3708$,

$\sigma_2^2 = 1.0704$. Hence we obtain

$$\tilde{y} = \frac{43.24}{\sigma_1} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(v-2.10)^2}{2\sigma_1^2}\right\} + \frac{56.76}{\sigma_2} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(v+1.60)^2}{2\sigma_2^2}\right\},$$

where $\sigma_1 = 0{,}6089$, $\sigma_2 = 1.0346$ and $v = u - d = 0.1(x - 54.01)$. Or, since $v - 2.10 = u - 2.001 = 0.1(x - 75)$; $v - 1.60 = u + 1.699 = 0.1(x - 38)$, we have

$$\tilde{y} = \frac{71.01}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-75}{6.089}\right)^2\right\} + \frac{54.86}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-38}{10.346}\right)^2\right\}.$$

To evaluate $\tilde{y}$, we set $t_1 = (u - 2.001)/\sigma_1$, $t_2 = (u + 1.699)/\sigma_2$ and

$$\tilde{y} = 71.01\varphi(t_1) + 54.86\varphi(t_2), \quad \text{where} \quad \varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}.$$

On using Table of $\varphi(t)$, we find $\tilde{y}_i$ as the 4-th column in the following table:

i	obs. y_i	S_i	$\tilde{y}_i$	$\tilde{S}_i$	$\lvert S_i - \tilde{S}_i \rvert$	$\lvert S_i - \tilde{S} \rvert^2$	$\tilde{y}_i \lvert S_i - \tilde{S}_i \rvert^2$
−5	0.5	0.5	0.1	0.1	0.4	0.16	0.02
−4	2.8	3.3	1.9	2.0	1.2	1.44	2.74
−3	9.4	12.7	9.9	11.9	0.8	0.64	6.34
−2	17.3	30.0	21.0	32.9	2.9	8.41	176.61
−1	17.3	47.3	17.4	50.3	3.0	9.00	156.60
0	10.6	57.9	5.8	56.1	1.8	3.24	18.79
1	12.2	70.1	7.9	64.0	6.1	37.21	293.96
2	19.2	89.3	28.0	92.0	2.7	7.29	204.12
3	9.5	98.8	7.6	99.6	0.8	0.64	4.80
4	1.2	100.0	0.1	99.7	0.3	0.09	0.01
$N = 100.00$							$\delta^2 = 864.05$

To try the ω^2-test,[5] we proceed as follows: Since ω^2 is defined as

$$\omega^2 = \frac{1}{N} \int_{-\infty}^{\infty} \lvert S(u) - NF(u) \rvert^2 f(u)\, du,$$

where $f(u)$ and $F(u)$ are the probability density function and cumulative distribution function, while $S(u)$ denotes the observed accumulated number, the approximate value of ω^2 is given by

$$\omega^2 = \frac{h}{N} \sum_i \lvert S_i - \tilde{S}_i \rvert^2 \frac{\tilde{y}_i}{N} = \frac{h\delta^2}{N^2}, \tag{38}$$

where h denotes the width of one u-subinterval, (usually $h = 1$), and

$$\delta^2 = \sum_i \lvert S - \tilde{S}_i \rvert^2 \tilde{y}_i, \tag{39}$$

where $S_i = \sum_{j=-5}^{i} y_j$, $\tilde{S}_i = \sum_{j=-5}^{i} \tilde{y}_j$. These being calculated as in the above table, we get

5) Y. Watanabe, On the ω^2-Distributions, this Journal vol. II (1952), p. 21; also T. Kondō, Evaluation of some ω_n^2-Distribution, this Journal vol. III (1954) p. 46.

$$\omega^2 = 864.05/100^2 = 0.0864.$$

Entering Table of the $\Phi(\omega_\infty^2)$ loc. cit., we find that $\Phi(\omega_\infty^2=0.0864)=0.3440$, and $\Pr(\omega_\infty^2 \geqq 0.0864) = 1-\Phi(\omega_n^2=0.0864) = 0.6560 > 0.05$. Or, if referred to Kondō's Table of $\Phi(\omega_9^2)$, more approximately $\Phi(\omega_9^2=0.0865) == 0.3602$, so that $\Pr(\omega_9^2 \geqq 0.0864) = 1-\Phi(\omega_9^2) = 0.6398 > 0.05$. Hence our representation is not to be rejected.

Here $\sigma_1=0.6089$ being somewhat small, we may compute $\tilde{Y}_j$ as remarked at the end of §4, and obtain $\omega^2=0.0553$, and correspondingly $Pr(\omega_9^2 \geqq 0.0553)=0.8146$. However with these $\tilde{Y}_j$, still χ^2-test does deny the above representation, since even when pooled at ends two by two, χ^2 amounts to 6.882, and with 2 degrees of freedom, $\Pr\{\chi^2 \geqq 6.882\} < 0.05$.

For later comparison we shall add one more example, which seems rather inadequate to be expressed by (1).

Ex. 5. A similar estimation test as in Ex. 2 gave the following result in percentage:

x	0~10	10~20	20~30	30~40	40~50	50~60	60~70	70~80	80~90	90~100	total
c.v.	5	15	25	35	45	55	65	75	85	95	
u	−5	−4	−3	−2	−1	0	1	2	3	4	
y	0	3	11	8	6	10	28	31	3	0	100

For a later use, now we shall calculate moments about $u=0$, and first correct them by Sheppard as: $\nu_0=1$, $\nu_1=d=\bar{u}=0.32$, $\nu_2=3.55667$, $\nu_3=-2.10$, $\nu_4=23.80917$, $\nu_5=-40.85$, $\nu_6=220.8194$, $\nu_7=-618.9531$ (those moments of higher order shall be used later in §11). Whence central moments $\mu_2=3.4543$, $\mu_3=-5.4489$, $\mu_4=28.6824$, $\mu_5=-82.2471$, $\mu_6=336.7033$, $\mu_7=-1172.9612$, while uncorrected moments are $\mu_2'=3.5376$, $\mu_3'=-5.4489$, $\mu_4'=30.3220, \cdots$ (these shall be used in §9).

Here approximately we may apply Case II: $\sigma_1=\sigma_2=\sigma$. Taking equation (4), we have to solve $X^3-3.5570X-14.8453=0$. It has only one positive root $X=\mu_2-\sigma^2=2.953$, so that $\sigma^2=0.5190$ and $\sigma=0.7204$. Hence by (6) $q^2-3.1740q+1=0$, which gives $q=0.3547$ (or 1.5870), and whence $r_1=0.262$, $r_2=0.738$. Also $a_1^2=X/q=8.2754$, so that $a_1=\pm 2.8767$, $a_2=-qa_1=\mp 10.203$. Describing given data in a graph, it is seen that $a_1=-2.8767$, $a_2=1.0203$ are to be taken preferably. Therefore we obtain, as a rough representation,

$$\tilde{y}=\frac{100}{0.7204\sqrt{2\pi}}\left[0.262\exp\left\{-\frac{1}{2}\left(\frac{v+2.88}{0.7204}\right)^2\right\}+0.738\exp\left\{-\frac{1}{2}\left(\frac{v-1.02}{0.7204}\right)^2\right\}\right].$$

Or, setting

$$t_1=\frac{(u+2.88)}{\sigma}=\frac{u+2.88-0.32}{0.7204}=1.388u+3.554,$$

and

$$t_2 = \frac{v-1.02}{\sigma} = \frac{u-1.02-0.32}{0.7204} = 1.388u - 1.860,$$

we obtain

$$\tilde{y} = 36.37\varphi(t_1) + 102.44\varphi(t_2) = \tilde{y}_1 + \tilde{y}_2,$$

where $\varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$. Whence, by use of $\varphi(t)$–Table we get $\tilde{y}$:

u	-5	-4	-3	-2	-1	0	1	2	3	4	total
obs. y	0	3	11	8	6	10	28	31	3	0	100
cal. $\tilde{y}$	0.05	1.98	12 05	10.72	1.60	7.27	36.56	26.81	2.88	0.04	99.96

Just as done in Ex. 4, we find $\delta^2 = \sum\tilde{y}(\sum y - \sum\tilde{y})^2 = 678$, so that $\omega^2 = \delta^2/N^2 = 0.0678$ and $\Phi(\omega_9^2 = 0.0678) = 0.2247$. Hence $\Pr(\omega_9^2 \geqq 0.0678) = 0.7753 > 0.05$. Thus our result is already not to be rejected. However, χ^2 amounts to 16.42 even when pooled 2 by 2 at both onds, and for 2 degrees of freedom, $\Pr(\chi^2 \geqq 16.42) < 0.01$, and the representation is to be rejected, We will endeavour to obtain a more elaborate representation in §11.

§6. Biometrical Applications

Ex. 6. Prof. Yoshikane OKA measured sizes of some sea–ears (Japanese abalone, *Haliotis gigantea*) and obtained the result as in the following table. It seems that there are two classes, and one class is in average one period older than the other, each class being somewhat normally distributed. Therefore it is required to decompose the whole distribution into two normal curves:

length x (cm)	2.0~2.4	2.5~2.9	3.0~3.4	3.5~3.9	4.0~4.4	4.5~4.9	5.0~5.4	5.5~5.9
central value	2.2	2.7	3.2	3.7	4.2	4.7	5.2	5.7
frequency y	1	2	2	11	24	25	16	14

6.0~6.4	6.5~6.9	7.0~7.4	7.5~7.9	8.0~8.4	8.5~8.9	9.0~9.4	9.5~9.9	10.0~10.4	total
6.2	6.7	7.2	7.7	8.2	8.7	9.2	9.7	10.2	
18	30	39	41	28	14	1	2	2	270

First taking the central value 6.2 of the middle subinterval 6.0~6.4 as origin, and putting $u = \frac{x-6.2}{0.5} = -8, -7, \cdots, 8$, we calculated tabular moments about $u=0$, namely $\sum u^k y$, $k=0, 1, 2, 3, 4$ and 5. Dividing them by $N=270$, we get tabular moments about $u=0$: $\nu_1' = 1$, $\nu_1' = 0.52593 = \bar{u} = d$, $\nu_2' = 10.41111$, $\nu_3' = 5.77407$, $\nu_4' = 225.87778$, $\nu_5' = 159.64074$. Further setting $v = u - d = u - 0.52593$, we computed the central moments about $v=0$ $(u=d)$, Sheppard's corrections being made at the same time :

$$\mu_0 = 1,\quad \mu_1 = 0,\quad \mu_2 = \nu_2' - d^2 - \frac{1}{12} = 10.04727,\quad \sqrt{\mu_2} = 3.16974,\quad \mu_3 = \nu_3' - 3d\nu_2' + 2d^3$$

$$= -10.47091,\quad \mu_4 = \nu_4' - 4\nu_3' d + 6\nu_2' d^2 - 3d^4 - \frac{1}{2}\mu_2 - \frac{1}{80} = 225.89552,$$

and $$\mu_5 = \nu_5' - 5\nu_4' d + 10\nu_3' d^2 - 10\nu_2' d^3 + 4d^5 - \frac{5}{6}\mu_3 = -428.8956.$$

We have to start with these moments about $v=0$.

Although here μ_3 and μ_5 are never small, because of easy calculation, let us assume Case I. In the same way as worked in Ex. 2, we obtain $a=2.491$, $\sigma^2=3.852$, $\sigma=1.963$ and hence as its representation

$$\tilde{y} = \frac{135}{\sqrt{2\pi}\sigma}\left[\exp\left\{-\frac{(v-a)^2}{2\sigma^2}\right\} + \exp\left\{-\frac{(v+a)^2}{2\sigma^2}\right\}\right], \quad a=2.491, \ \sigma=\sigma_u=1.953.$$

Or, since $v=u-d=2(x-6.2)-0.53=2(x-6.45)$ nearly,

$$\tilde{y} = \frac{135}{\sqrt{2\pi}\sigma_u}\left[\exp\left\{-\frac{(x-7.7)^2}{2\sigma_x^2}\right\} + \exp\left\{-\frac{(x-5.2)^2}{2\sigma_x^2}\right\}\right], \quad \sigma_x = \frac{1}{2}\sigma_u = 0.98.$$

Further putting $t_{1j}, t_{2j} = \frac{u_j \mp a}{\sigma}$, $u_j = -8, -7, \cdots, 8$ and computing by use of the Table of $\varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ the values of $\tilde{y}_j = \frac{270}{2\sigma}\{\varphi(t_{1j}) + \varphi(t_{2j})\}$, we obtain the following result:

u	−8	−7	−6	−5	−4	−3	−2	−1	0
obs. y	1	2	2	11	24	25	16	14	18
cal. $\tilde{y}$	0.27	1.12	3.57	8.80	16.71	24.58	28.40	26.95	23.86
u	1	2	3	4	5	6	7	8	total
obs. y	30	39	41	28	14	1	2	2	270
cal. $\tilde{y}$	23.74	26.76	28.44	24.96	17.22	9.21	3.80	1.21	269.6

Whence $\chi^2 = \sum(y-\tilde{y})^2/\tilde{y} = 39.01$. Here degrees of freedom being 12, the χ^2–Table affords $\Pr(\chi^2 \geqq 24.69) = 0.01$, and thus the above representation must be rejected with significant level 0.01. Or, we may apply the ω^2–test as done in Ex. 4. In the present example $N=270$ and δ^2 amounts to 35406, so that by (38), $\omega^2 = 35406/270^2 = 0.9714$. Entering the $\Phi(\omega_\infty^2)$ Table, we find that $\Phi(0.9714) = 0.9971$ and therefore $\Pr\{\omega_\infty^2 \geqq 0.9714\} = 1 - \Phi(9714) = 0.0029 < 0.05$. Thus again by the ω^2–test the above representation is to be rejected.

To obtain more legitimate solution, we are obliged to solve equation (19). Now the coefficients (20) are found to be as follows[6]:

$$A_0 = 273660.8085, \quad A_1 = -502\ 7621, \quad A_2 = 1057\ 4233, \quad A_3 = 3578\ 2783,$$
$$A_4 = 16\ 2153\ 4113, \quad A_5 = 17\ 0759\ 9010, \quad A_6 = 179\ 6075\ 1646,$$
$$A_5 = 1139\ 6555\ 1875, \quad A_8 = -7432\ 2465\ 0018, \quad A_9 = -7514\ 0444\ 7513.$$

The equation (19) with these coefficients, still divided by A_0, reduces to

6) To avoid decimals as possible, all A_m in (20) were multiplied by 8.

$$s^9-18.37173s^8-38.63992s^7+130.7560s^6+5925.343s^5+6339.838s^4$$
$$+65631.44s^3-416448.2s^2-271\ 5861s-274\ 5751=0.$$

Solving this equation by Horner's method we obtain three real roots -2.577, -1.354 and 19.08. Since the whole sample range is 17 units in u, the third root is evidently useless. Also the second root makes the value of p in (18) positive, so that a_1, a_2 have the same sign and the ratio $r_1:r_2=q$ becomes negative, which does not give proper superposition. Hence the first root -2.577 is only promising. Really substituting this root in (18), we get $p=-6.677$. On solving $s=a_1+a_2=-2.577$, $p=a_1a_2=-6.677$ we obtain a_1, $a_2=-4.176$, 1.599, and whence by (10) $\sigma_1^2=0.7231$, $\sigma_2^2=3.906$, so that $\sigma_1=1.403$, $\sigma_2=1.976$. But, since $v=u-d=u-0.526$ nearly, it results that $v-a_1=u+3.650$, $v-a_2=u-2.125$. Hence we obtain

$$\tilde{y}=\frac{53.31}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{u+3.650}{1.403}\right)^2\right\}+\frac{99.34}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{u-2.125}{1.976}\right)^2\right\}.$$

Or, since $u=2(x-6.2)$ and $\sigma_x=\sigma_u/2$,

$$\tilde{y}=\frac{53.31}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-4.375}{0.7015}\right)^2\right\}+\frac{99.34}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-7.262}{0.988}\right)^2\right\}.$$

In order to examine the above representation by the ω^2-test, we proceed as in Ex. 4, and find that $\delta^2=3622$, so that $\omega^2=3622/270^2=0.050$, and $\Phi(\omega_\infty^2=0.050)=0.1240$. Hence Pr $(\omega_\infty^2\geqq 0.050)=0.8760=0.8760>0.05$. Or, entering the $\Phi(\omega_q^2)$-Table, we get $\Phi(\omega_9^2=0.050)=0.1562$, and still Pr $(\omega_9^2\geqq 0.050)=0.8438>0.05$. Thus the above representation can be asserted.

Again, to try the χ^2-test, the requisite χ^2 is calculated as follows:

u	y	$\tilde{y}$	$\lvert y-\tilde{y}\rvert$	$\lvert y-\tilde{y}\rvert^2$	$\lvert y-\tilde{y}\rvert^2/\tilde{y}$
−8	1 }	0.23 }	0.77 }	0.593 }	2.58 }
−7	2 } 5	1.23 } 6.70	0.77 } 1.70	0.593 } 2.89	0.48 } 0.43
−6	2 }	5.24 }	3.24 }	10.498 }	2.00 }
−5	11	13.45	2.45	6.002	0.45
−4	24	20.94	3.06	9.364	0.45
−3	25	20.48	4.52	20.521	1.00
−2	16	15.13	0.89	0.757	0.01
−1	14	14.93	0.93	0.865	0.06
0	18	22.96	4.96	24.602	1.07
1	30	33.79	3.79	14.364	0.43
2	39	39.56	0.56	0.314	0.01
3	41	35.92	5.08	25.804	0.72
4	28	25.26	2.74	7.508	0.30
5	14	13.75	0.25	0.062	0.00
6	1 }	5.79 }	4.79 }	22.944 }	3.96 }
7	2 } 5	1.89 } 8.16	0.11 } 3.16	0.012 } 9.986	0.01 } 1.22
8	2 }	0.48 }	1.52 }	2.310 }	4.81 }

$N=270$. If each frequency be as it stands, $\chi^2=18.34$;

Or, on pooling the frequencies at ends, $\chi^2=6.15$.

When unpooled, degrees of freedom being $17-6=11$, we have $\Pr(\chi^2 \geqq 18.34) < 0.05$. But, when pooled at ends as shown above, degrees of freedom reduces to 7, for which $\Pr(\chi^2 \geqq 6.15) > 0.50 > 0.05$. Thus even with rather severe χ^2-criterion the adequacy of the above representation cannot be denied.

Ex. 7. At the same time as the length measurement of sea-ears in Ex. 6 Prof. Oka made also their breadth estimation, which runs as follows:

breadth x(c.v.)	1.2	1.7	2.2	2.7	3.2	3.7	4.2	4.7	5.2	5.7	6.2	6.7	7.2	7.7	total
frequency y	1	1	8	26	31	18	20	33	56	45	26	3	1	1	270

As it is very probable that this distribution should be similarly distributed as in Ex. 6, we might apply Case V, using the known value of $q=r_1/r_2$.

Now putting $u=\dfrac{x-4.7}{0.5}$, we get the first moment $\nu_1'=-0.2185=\bar{u}=d$ and the moments about $u=d(v=0)$ to be $\mu_0=1$, $\mu_1=0$, $\mu_2=5.9838$, $\sqrt{\mu_2}=2.4462$, $\mu_3=-5.5421$, $\mu_4=78.3470$. Utilizing the value of q obtained in Ex. 6, i.e. $q=\dfrac{r_1}{r_2}=\dfrac{0.2769}{0.7231}=0.3829$, and substituting it in (27), we obtain an equation of sixth degree in $a_1=a$

$$a^6+11.6627a^3-74.3784a^2-68.3769=0,$$

which has two real roots 2.701 and -3.315. But we have chosen a_1 in Ex. 6 to be negative. Therefore $a_1=-3.315$, and consequently by (24) $a_2=1.270$. Further, by (25) (26), $\sigma_1^2=0.9687$, $\sigma_2^2=2.0822$, so that $\sigma_1=0.9842$, $\sigma_2=1.4430$. Also, since $v=u+0.2185$, so $v-a_1=u+3.533$ and $v-a_2=u-1.052$, and thus the required representation is obtained to be

$$\tilde{y}=270\left[\frac{0.282}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{u+353}{0.984}\right)^2\right\}+\frac{0.506}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{u-1.052}{1.443}\right)^2\right\}\right].$$

Or, as $u=2(x-4.7)$ and $\sigma_u=2\sigma_x$

$$\tilde{y}=\frac{76.14}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-2.934}{0.492}\right)^2\right\}+\frac{136.6}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-5.226}{0.7215}\right)^2\right\}.$$

Now to try the ω^2-test, we compute the values $\tilde{y}$ for every $u=j$ by

$$\tilde{y}=76.14\varphi(t_1)+136.6\varphi(t_2),$$

where $t_1=(u+3.53)/0.984$, $t_2=(u-1.052)/1.443$ and $\varphi(t)=\dfrac{1}{\sqrt{2\pi}}e^{-t^2/2}$. Then, performing S_i, $\tilde{S}_i$ as in Ex. 4, we find that $\delta^2=3516$, and by (38) $\omega^2=\delta^2/N^2=3516/270^2=0.04824$. We see that $\Phi(\omega_\infty^2=0.04824)=0.1132$, so that $\Pr(\omega_\infty^2 \geqq 0.04824)=0.8867$

<0.05. Or, by the $\Phi(\omega_9^2)$–Table, $\Phi(\omega_9^2=0.04824)=0.1467$ and $\Pr(\omega_9^2 \geqq 0.04824) =0.8533$. Hence the above representation can be asserted with large probability.

For the χ^2–test we obtain, similarly as done in Ex. 6, $\chi^2=28.52$ if each sub-interval be held as it stands, and degrees of freedom being $18-6=10$, $\Pr(\chi^2 > 28.52) < 0.01$, so that this χ^2–test denies the above representation. However, the frequencies at the ends of distribution being so small we may lump together them, 3 by 3 at ends, and now we get $\chi^2=5.39$. This time with 6 degrees of freedom, $\Pr(\chi^2 \geqq 5.39)$ is nearly 0.5. Thus even with χ^2–test the affirmation remains the same as got by ω^2–test.

Ex. 8. Prof. Oka measured also length and breadth concerning another certain class of sea-ears. The result about length was as follows:

length x (c.v.)	2.7	3.2	3.7	4.2	4.7	5.2	5.7	6.2	6.7	7.2	7.7	8.2	8.7	9.2	9.7	10.2	total
frequency y	1	4	10	6	10	6	18	30	34	54	28	26	10	5	1	2	245

Setting $u=\frac{x-6.7}{0.5}$ and computing tabular moments $\nu'_k=\sum yu^k/N$, we get $\nu'_0=1$, $\nu'_1=d=\bar{u}=0.0938776$, $\nu'_2=7.77551$, $\nu'_3=-10.70612$, $\nu'_4=196.5918$, $\nu'_5=-548.1102$; whence moments about mean $u=d$ (uncorrected) $\mu'_0=1$, $\mu'_1=0$, $\mu'_2=7.76670$, $\mu'_3=-12.89431$, $\mu'_4=201.0233$, $\mu'_5=641.3956$; and finally making Sheppard's corrections $\mu_0=1$, $\mu_1=0$, $\mu_2=7.68337$, $\mu_3=-12.89431$, $\mu_4=197.1589$, $\mu_5=-630.6504$. With these values the coefficients (20) become $A_0=51874.1279$, $A_1=201718.2443$, $A_2=-2055178.012$, $A_3=-25611194$, $A_4=-63553402$, $A_5=416997325$, $A_6=2198556637$, $A_7=73140000940$, $A_8=18422138080$, $A_9=-44038078090$. Thus equation (19) is found, further divided by A_0, to be

$$s^9+3.8886098s^8-39.6185545s^7-493.718062s^6-1225.146423s^5+8038.63779s^4$$
$$+42382.52722s^2+1409951.433s^2+3551150.8012s-848941.0789=0.$$

The real roots are found to be $s=0.6542$ -0.9241, -8.32607. But for the latter two roots we get negative variances, so that they should be given up. Only for the remaining root $s=0.6542$, we obtain $p=-2.4464$, so that a_1, a_2 are the roots of quadratic $z^2+0.6542z-2.4464=0$. Solving this equation, we get $a_1=-1.27085$, $a_2=1.92505$ and whence $r_1=0.60235$, $r_2=0.39765$ by (8). Further by (12) we find $b_1=9.36184$, $b_2=5.14086$, and this time $\sigma_1^2=7.26875$, $\sigma_2^2=1.43504$, so that $\sigma_1=2.69606$, $\sigma_2=1.19793$. Hence we have, as the required bimodal representation

$$\tilde{y}=\frac{245\times 0.60235}{2.69606\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{v+1.27085}{2.69606}\right)^2\right\}+\frac{245\times 0.39765}{1.19793\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{v-1.92505}{1.19793}\right)^2\right\}.$$

Or, since $v=u-d=u-0.09388$, on putting $\frac{v+1.27085}{\sigma_1}=0.37091u+0.43655=t_1$, and

$\frac{v-1.92505}{\sigma_2}=0.83477u-1.68535=t_2$, we get

$$\tilde{y}=54.7376\varphi(t_1)+81.3272\varphi(t_2)=\tilde{y}_1+\tilde{y}_2,$$

where $\varphi(t)=\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$, the standard normal density function. On calculating $\tilde{y}_1$, $\tilde{y}_2$ and $\tilde{y}_1+\tilde{y}_2$ by use of $\varphi(t)$, we obtain the following result[7]

obs. y	1	4	10	6	10	6	18	30	34
$\tilde{y}_1$	0.89	2.12	4.41	7.99	12.62	17.37	20.84	21.79	19.85
$\tilde{y}_2$	0	0	0	0	0	0.01	0.02	1.36	7.84
cal. $\tilde{y}$	0.89	2.12	4.41	7.99	12.62	17.39	20.96	23.15	27.69

obs. y	54	28	26	10	5	1	2	$245.00=N$
$\tilde{y}_1$	15.76	10.91	6.58	3.46	1.58	0.63	0.22	147.02 (as sum)
$\tilde{y}_2$	22.60	32.44	23.20	8.27	1.47	0.13	0.01	97.45 (as sum)
cal. $\tilde{y}$	38.36	43.35	29.78	11.72	3.05	0.76	0.23	244.47 (as sum)

Trying the ω^2-test as in Ex. 6, it was found that $\delta^2=\sum\tilde{y}(\sum y-\sum\tilde{y})^2=17192$, so that $\omega^2=\delta^2/N^2=17192/245^2=0.2864$. Entering the $\Phi(\omega^2_\infty)$ Table, we find $\Phi(\omega^2_\infty=0.2864)=0.8524$, and hence $\Pr(\omega^2_\infty\geqq 0.2864)=0.1476>0.05$. Thus the ω^2-test does not deny the above representation. However, $\chi^2=\sum(y-\tilde{y})^2/\tilde{y}$ amounts to 37.21, even after pooling the frequencies at ends, and degrees of freedom being $13-6=7$, we find $\Pr(\chi^2\geqq 37.21)<0.005$. Thus the above representation is now to be rejected with significant level 0.005. Indeed, we have tried in § 8 later on to find if adequate corrections of parameters be possible, the result of which, however, being still negative, it seems that the above bimodal representation does not fit the given data suitably (cf. §7 Ex. 11 and §10 Ex. 24.).

Ex. 9. Prof. Oka's measurment for width of the sea ears in Ex. 8 runs as follows:

c.v. x	2.2	2.7	3.2	3.7	4.2	4.7	5.2	5.7	6.2	6.7	7.2	total
u	−5	−4	−3	−2	−1	0	1	2	3	4	5	
d.f. y	4	16	10	10	38	33	72	32	20	8	2	245

For a purpose of later use, at first we have computed moments about $u=0$ with Sheppard's corrections: $\nu_0=1$, $\nu_1=d=\bar{u}=0.26939$, $\nu_2=4.33299$, $\nu_3=-1.21837$, $\nu_4=51.31488$, $\nu_5=-45.01786$ and whence the central moments $\mu_2=4.3438$, $\mu_3=-4.6810$, $\mu_4=56.6413$, $\mu_5=-119.7469$. Just as Ex. 7 was solved by use of the ratio $q=r_1/r_2$ obtained in Ex. 6, we may proceed according to method of Case V as follows:

7) We have computed to some more decimal places than written in the table and hence there occur apparently some discrepancies in sum.

Availing the result of Ex. 8, we assume that $r_1=0.60235$, $r_2=0.39765$, so $q=r_1/r_2=1.51477$. Hence, substituting these values and the central moments above obtained, equation (27) becomes

$$22.07017a^6+14.6003a^3+0.16614a^2-21.91204=0,$$

or, dividing by the coefficient of a^6

$$a^6+0.66154a^3+0.0075278a^2-0.992835=0.$$

This equation has only two real roots, one positive and one negative. Really by Horner's method we find the two roots to be (i) 0.8947 and (ii) -1.1123. Making use of formulas (24), (25), (26), we get in succession (i) $a_1=0.8947$, $a_2=-1.3553$, $\sigma_1=1.4551$, $\sigma_2=2.1603$, and remembering that $d=0.2694$

$$\tilde{y}=101.42\varphi(t_1)+43.85\varphi(t_2),$$

where $t_1=\frac{1}{\sigma_1}(u-d-a_1)=0.6872u-0.7999$, $t_2=\frac{1}{\sigma_2}(u-d-a_2)=0.4629u+0.5027$, and $\varphi(t)=\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$. Also from the second root we get (ii) $a_1=-1.1123$, $a_2=1.6849$, $\sigma_1=1.5338$, $\sigma_2=0.8633$, so that

$$\tilde{y}=96.22\varphi(t_1)+130.72\varphi(t_2),$$

where $t_1=0.620u+0.5495$, $t_2=1.3419u-2.6224$, and $\varphi(t)=\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$.

Which one of the above two results is preferable shall be decided by the ω^2–test. In fact for (i) and (ii) we find that $\delta^2=12196.54$ and 32531.66 and hence each ω^2 becomes 0.2032 and 0.5420 respectively. Correspondingly for (i) $\Phi(\omega^2_\infty=0.2032)=0.7594$ and $\Pr(\omega^2_\infty\geqq0.2032)=0.2606>0.05$, while for (ii) $\Phi(\omega^2_\infty=0.5420)=0.9688$ and $\Pr(\omega^2_\infty\geqq0.5420)=0.0312<0.05$. Thus we have to take (i) only. However, as χ^2–test again denies even representation (i), we shall ponder over still furthermore.

§7. Trimodal Representations

The graph of Ex. 9 presents three maxima, and suggests its trimodal representation. The analysis becomes naturally more complex than bimodal. However, if e.g. the modes a_i be assumed beforehand, it goes rather simple. Let the representation be

$$y=N\sum_{i=1,2,3}\frac{r_i}{\sqrt{2\pi}\sigma_i}\exp\left\{-\frac{1}{2}\left(\frac{u-a_i}{\sigma_i}\right)^2\right\}. \tag{40}$$

Reducing to density distribution by dividing by N, and taking moments about $u=0$, we have

$$\nu_0 = r_1 + r_2 + r_3 = 1, \quad \nu_1 = r_1 a_1 + r_2 a_2 + r_3 a_3 = d, \tag{41}$$

$$\nu_2 = \sum r_i(\sigma_i^2 + a_i^2), \quad \nu_3 = \sum r_i a_i(3\sigma_i^2 + a_i^2), \tag{42}$$

$$\nu_4 = \sum r_i(3\sigma_i^4 + 6a_i^2\sigma_i^2 + a_i^4), \quad \nu_5 = \sum r_i a_i(15\sigma_i^4 + 10a_i^2\sigma_i^2 + a_i^4). \tag{43}$$

If the values $a_i (i=1, 2, 3)$ be assumed, we are able to solve (41) with respect to r_1, r_2

$$r_1 = \frac{a_2 - d + (a_3 - a_2)r_3}{a_2 - a_1}, \quad r_2 = \frac{d - a_1 + (a_1 - a_3)r_3}{a_2 - a_1}. \tag{44}$$

Also we get from (42)

$$\left.\begin{aligned} r_1\sigma_1^2 &= \frac{1}{a_2 - a_1}\left[(a_3 - a_2)r_3\sigma_3^2 + a_2\left(\nu_2 - \sum' r_i a_i^2\right) - \frac{1}{3}\left(\nu_3 - \sum' r_i a_i^3\right)\right], \\ r_2\sigma_2^2 &= \frac{1}{a_2 - a_1}\left[(a_1 - a_3)r_3\sigma_3^2 - a_1\left(\nu_2 - \sum' r_i a_i^2\right) + \frac{1}{3}\left(\nu_3 - \sum' r_i a_i^3\right)\right], \end{aligned}\right\} \tag{45}$$

as well as from (43)

$$\left.\begin{aligned} r_1\sigma_1^4 &= \frac{1}{a_2 - a_1}\Big[(a_3 - a_2)r_3\sigma_3^4 + \frac{2}{3}\sum' r_i a_i^3\sigma_i^2 - 2a_2\sum' r_i a_i^2\sigma_i^2 \\ &\qquad + \frac{1}{3}a_2\left(\nu_4 - \sum' r_i a_i^4\right) - \frac{1}{15}\left(\nu_5 - \sum' r_i a_i^5\right)\Big], \\ r_2\sigma_2^4 &= \frac{1}{a_2 - a_1}\Big[(a_1 - a_3)r_3\sigma_3^4 - \frac{2}{3}\sum' r_i a_i^3\sigma_i^3 + 2a_1\sum' r_i a_i^2\sigma_i^2 \\ &\qquad - \frac{1}{3}a_1\left(\nu_4 - \sum' r_i a_i^4\right) - \frac{1}{15}\left(\nu_5 - \sum' r_i a_i^5\right)\Big]. \end{aligned}\right\} \tag{46}$$

Eliminating σ_1, σ_2 between (45) and (46), we obtain two equations involving σ_3, r_3, and in fact biquadratic in σ_3, and further elimination of σ_3 yields a biquadratic equation in r_3.

Ex. 10. For a mingled group of girls aged 9, 10 and 11 stature measurements were made, and the result was as following table. Taking moments about $u=0$, and making Sheppard's corrections, we get $\nu_0=1$, $\nu_1=-0.5011=d$, $\nu_2=5.7796$, $\nu_3=-7.0604$, $\nu_4=95.2217$, $\nu_5=-158.2007$. If the trimodal representation (40) with $a_1=-2.5$, $a_2=-0.5$ and $a_3=1.5$ be assumed, we obtain from (44), (45) and (46)

middle values x (c.m.)	u	frequency y in percentage
110.5	−9	0.01
113.5	−8	0.06
116.5	−7	0.28
119.5	−6	0.99
122.5	−5	2.74
125.5	−4	5.90
128.5	−3	10.11
131.5	−2	14.16
134.5	−1	16.54
137.5	0	16.09
140.5	1	135.5
143.5	2	9.38
146.5	3	5.58
149.5	4	2.92
152.5	5	1.25
155.5	6	0.46
158.5	7	0.14
161.5	8	0.03
164.5	9	0.01
	sum	100.00

$$r_2 = 1.0006 - 2r_1, \quad r_3 = -0.0006 + r_1;$$

$$r_2\sigma_2^2 = -2r_1\sigma_1^2 - 8r_1 + 5.30365, \quad r_3\sigma_3^2 = r_1\sigma_1^2 + 0.22715;$$

$$r_2\sigma_2^4 = -2r_1\sigma_1^4 - 16r_1\sigma_1^2 - 10.66668r_1 + 26.34177, \quad r_3\sigma_3^4 = r_1\sigma_1^4 + 1.704962.$$

Eliminating σ_2 from these

$$2.0012r_1\sigma_1^4 - 5.2050r_1\sigma_1^2 + 42.666664r_1^2 - 21.501788r_1 + 1.771126 = 0,$$

also eliminating σ_3

$$\sigma_1^4 + 757.2\sigma_1^2 - 2841.6 + 87.7/r_1 = 0,$$

and finally eliminating σ_1 from the above two equations, we have

$$\psi(r_1) = r_1^4 - 27262.5145r_1^3 - 8333.8992r_1^2 + 420.2321r_1 + 16.5893 = 0.$$

This equation has positive roots 0.1059, 0.2253, and from the latter we obtain

$$\tilde{y} = \frac{22.53}{1.7958}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{u+2.5}{1.7958}\right)^2\right\} + \frac{55.00}{1.9297}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{u+0.5}{1.9297}\right)^2\right\}$$
$$+ \frac{22.47}{2.0602}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{u-1.5}{2.0602}\right)^2\right\}$$
$$= 12.5455\varphi(t_1) + 28.5021\varphi(t_2) + 10.9065\varphi(t_3),$$

where $t_1 = \dfrac{u+2.5}{1.7958}$, $t_2 = \dfrac{u+0.5}{1.9297}$, $t_3 = \dfrac{u-1.5}{2.0602}$ and $\varphi(t) = \dfrac{1}{\sqrt{2\pi}}e^{-t^2/2}$.

Calculating $\tilde{y}_j$ for $u=j=0, \pm 1, \cdots, \pm 9$, we get

u	−9	−8	−7	−6	−5	−4	−3	−2	−1	0
$y_{obs.}$	0.01	0.06	0.28	0.99	2.74	5.90	10.11	14.16	16.54	16.09
$\tilde{y}_{cal.}$	0.008	0.052	0.257	0.950	2.680	5.838	10.128	14.249	16.610	16.233
u	1	2	3	4	5	6	7	8	9	total
$y_{obs.}$	13.35	9.38	5.58	2.92	1.25	0.46	0.14	0.03	0.01	100.00
$\tilde{y}_{cal.}$	13.380	9.355	5.579	2.841	1.224	0.439	0.118	0.031	0.006	99.978

Whence we see that $\delta^2 = 1.5676$, $\omega^2 = 1.5676/100^2 = 0.0002$ and $\Phi(\omega^2 = 0.0005)$ is almost zero, so that $\Pr(\omega^2 \geqq 0.0002)$ is almost unity. Also with χ^2–test, χ_0^2 amounts only to 0.0154, and for degrees of freedom $16-7=9$, $\Pr(\chi^2 \geqq \chi_0^2) > 0.995$. Thus the above trimodal representation cannot be denied with almost certainty.

With the former value $r_1 = 0.1059$, the matter does not go so good and it shall be abandoned.

Ex. 11. (Ex. 9). Now we shall try to obtain a trimodal representation for Ex. 9. Using the values of ν_n's in Ex. 9, and assuming that $a_1 = -4$, $a_2 = -1$ and $a_3 = 1$, we get from (44), (45) and (46)

$$r_1 = -0.4231 + \frac{2}{3} r_3, \quad r_2 = 1.4231 - \frac{5}{3} r_3;$$

$$r_1\sigma_1^2 = \frac{9}{10} r_3\sigma_3^2 - \frac{9}{10} r^3 - 0.2405, \quad r_2\sigma_2^2 = -\frac{5}{3} r_3\sigma_3^2 - \frac{80}{9} r_3 + 9.9204;$$

$$r_1\sigma_1^4 = \frac{2}{3} r_3\sigma_3^4 - \frac{20}{9} r_3\sigma_3^2 + \frac{34}{39} r_3 - 1.7178, \quad r_2\sigma_2^4 = -\frac{5}{3} \nu_3\sigma_3^4 - \frac{160}{9} r_3\sigma_3^2 - \frac{64}{9} r_3 + 42.310.$$

The elimination of σ_1 yields

$$0.28209 r_3\sigma_3^4 - 1.2609 r_3\sigma_3^2 - 1.2840 r_3^2 + 3.2781 r_3 - 0.6690 = 0,$$

while the elimination of σ_2 gives

$$2.3719 r_3\sigma_3^4 - 7.7679 r_3\sigma_3^2 + 67.1605 r_3^2 - 95.7264 r_3 + 38.2021 = 0.$$

Finally eliminating σ_3 between the above two equations, we get

$$\psi(r) \equiv r_3^4 - 3.0065 r_3^3 + 3.3839 r_3^2 - 1.6900 r_3 + 0.3161 = 0.$$

This equation, however, has no root between 0 and 1; thus our problem seems to have no solution. But, this might be due to misestimates of modes: Indeed, if some mode were estimated only a little differently, then the corresponding equation could have a certain adoptable root between 0 and 1. Now the function $\psi(r)$ becomes extreem, taking minimum and maximum alternately at $r=0.7073$, 0.7361 and 0.8114 and thereabout $\psi(r)$ is small enough. We may therefore assume $r_3=0.750$ on trial, so that $r_1=0.077$, $r_2=0.173$ by (44). Also solving biquardratic equation of σ_3, we find $\sigma_3 = 1.5010$ and hence $\sigma_1=0.8278$, $\sigma_2 = 1.5897$ by (45). Consequently the required trimodal representation shall be

$$\tilde{y} = 22.7522\varphi(t_1) + 26.6817\varphi(t_2) + 122.4194\varphi(t_3) = \tilde{y}_1 + \tilde{y}_2 + \tilde{y}_3,$$

where $\varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$, the standard normal function and $t_1 = 1.2081u + 4.8324$, $t_2 = 0.6290(u+1)$, $t_3 = 0.6662(u-1)$. Their values are obtained as the following table:

u	t_1	$\varphi(t_1)$	$\tilde{y}_1$	t_2	$\varphi(t_2)$	$\tilde{y}_2$	t_3	$\varphi(t_3)$	$\tilde{y}_3$	$\tilde{y}_{cal.}$	$y_{obs.}$
−5	−1.2081	0.19230	4.375	−2.5160	0.01684	0.449	−3.9972	0.00013	0.016	4.84	4
−4	0	0.39894	9.077	−1.8870	0.06725	1.794	−3.3310	0.00156	0.191	11.06	16
−3	1.2081	0.19230	4.375	−1.2580	0.18083	4.825	−2.6648	0.01145	1.402	10.60	10
−2	2.4162	0.02154	0.490	−0.6290	0.32733	8.734	−1.9986	0.05415	6.629	15.85	10
−1	3.6243	0.00056	0.013	0	0.39894	10.644	−1.3324	0.16422	20.104	30.76	38
0	4.8324			0.6290	0.32733	8.734	−0.6662	0.31955	39·119	47.85	33
1				1.2580	0.18083	4.825	0	0.39894	48.838	53.66	72
2				1.8870	0.06725	1.794	0.6662	0.31955	39.119	40.91	32
3				2.5160	0.01684	0.449	1.3324	0.16422	20.104	20.55	20
4				3.1450	0.00284	0.076	1.9986	0.05415	6.629	6.71	8
5				3.7740	0.00032	0.009	2.6648	0.01145	1.402	1.41	2
									Sum	244.20	245

Calculating ω^2 as in Ex. 6, we get $\delta^2 = 23974$, $\omega^2 = \delta^2/N^2 = 23974/245^2 = 0.0666$,

$\Phi(\omega^2=0.0666)=0.2274$, so that $\Pr\{\omega^2 \geqq 0.0666\}=0.7726>0.05$. Thus the ω^2–test permits the above trimodal representation with stronger basis than the bimodal representation. However, the χ^2–test gives $\chi^2=19.58$ and for degrees of freedom $11-7=4$, $\Pr\{\chi^2 \geqq 19.58\}<0.0005$, so that the representation is still denied.

To obtain still more exact values we may proceed as follows: Let the corrections of a_i, σ_i, r_i be ξ_i, η_i, ζ_i ($i=1, 2, 3$), which we assume to be small, and substituting these in equations (41), (42) and (43) and besides

$$\left.\begin{aligned} \nu_6 &= \sum_{i=1,2,3} r_i(15\sigma_i^6+45a_i^2\sigma_i^4+15a_i^4\sigma_i^2+a_i^6), \\ \nu_7 &= \sum_{i=1,2,3} r_i a_i(105\sigma_i^6+105a_i^2\sigma_i^4+21a_i^4\sigma_i^2+a_i^6), \\ \nu_8 &= \sum_{i=1,2,3} r_i(105\sigma_i^8+420a_i^2\sigma_i^6+42a_i^4\sigma_i^4+28a_i^6\sigma_i^2+a_i^8 \end{aligned}\right\} \tag{47}$$

and neglecting terms of higher order than the first, we obtain nine linear equations in ξ_i, η_i, ζ_i, which, being solved, give the required corrections.

When the corrected values of r_1, r_2, and r_3 in Ex. 9 were thus determined, the trimodal representation of Ex. 8 could be obtained by substituting these values in (41), (42), (43) and the first of (47); thus we have six simultaneous equations, say (48), containing six unknowns. However the task being somewhat lengthy, we postpone its treatment as a future work. Or else, we may further assume the approximate values of a_i from the given data ($a_1=-6$, $a_2=-4$, $a_3=1$ say) and hence compute by (45) (46) the approximate values of σ_i. Putting the corrections of a_i, σ_i and r_i (in Ex. 8) to be ξ_i', η_i' and ζ_i, as before, we may solve the resulting nine linear equations (say, (49)). However, since the ratios $r_1:r_2:r_3$ must remain the same in Ex. 8 and 9, we had better solve these 18 equations, (48) and (49) altogether, in which r_1, r_2, r_3 are assumed to be the same, and consequently containing 15 unknowns, by the method of least squares.

§8. Corrections of Estimates by Method of Least Squares

Next we shall consider the method of successive approximations, which are to be available when a rough estimation of $\tilde{y}$ is obtained, even when, solved by method of general Case III, the calculated values $\tilde{y}$ differ largely from observed y, and the ω^2– or χ^2–test shows that the obtained representation is to be rejected, say on 1% level of significance. For this purpose we may utilize the old fashoned, yet still powerful, method of least squares, although the calculations are enough troublesome.

First for exactitude, let us consider the cumlative frequency

$$\begin{aligned}\tilde{F}(u) &= N\sum_i r_i F_i(u) = N\sum_i \frac{r_i}{\sqrt{2\pi}}\int_{-\infty}^{u}\exp\left\{-\frac{1}{2}\left(\frac{u-d-a_i}{\sigma_i}\right)^2\right\}\frac{du}{\sigma_i} \\ &= N\sum_i \frac{r_i}{\sqrt{2\pi}}\int_{-\infty}^{t_i}\exp\left(-\frac{t_i^2}{2}\right)dt_i = N\sum_i r_i\Phi(t_i) = \tilde{Y}(u) \text{ say} \\ &= G(a_1, a_2, \sigma_1, \sigma_2, r_1, u) \qquad (r_2 = 1-r_1),\end{aligned}$$

and therefore

$$\begin{aligned}\tilde{Y}_j &= N\sum_t r^j\int_{j-1/2}^{j+1/2}\exp\left\{-\frac{1}{2}\left(\frac{u-d-a_i}{\sigma_i}\right)^2\right\}\frac{du}{\sigma_i\sqrt{2\pi}} \\ &= N\sum_i r_i\left[\Phi\left(t_{ij}+\frac{1}{2\sigma_i}\right)-\Phi\left(t_{ij}-\frac{1}{2\sigma_i}\right)\right],\end{aligned}$$

where $\Phi(t_{ij})=\int_{-\infty}^{t_{ij}}\varphi(t)dt$ denotes the normal distribution function and $t_{ij}=\frac{u_j-d-a_i}{\sigma_i}=\frac{v_i-a_i}{\sigma_i}$. These calculated values do not agree with the corresponding observed values; thus $y_j-\tilde{Y}_j=\Delta y_j\neq 0$. Denoting the corrections of a_1, a_2, σ_1, σ_2 and r_1 by ξ_1, ξ_2, η_1, η_2 and ζ_1, we have, if the corrections be small enough,

$$\begin{aligned}\Delta y_j &= G(a_1+\xi_1, a_2+\xi_2, \sigma_1+\eta_1, \sigma_2+\eta_2, r_1+\zeta_1, j)-G(a_1, a_2, \sigma_1, \sigma_2, j) \\ &= \sum_i\left(\frac{\partial G}{\partial a_i}\xi_i+\frac{\partial G}{\partial\sigma_i}\eta_i\right)+\frac{\partial G}{\partial r_1}\zeta_1 \text{ nearly} \\ &= N\sum_i r_i\int_{j-1/2}^{j+1/2}\left\{\left(\frac{v-a_i}{\sigma_i}\right)\frac{\xi_i}{\sigma_i}+\left[\left(\frac{v-a_i}{\sigma_i}\right)^2-1\right]\frac{\eta_i}{\sigma_i}\right\}\exp\left\{-\frac{1}{2}\left(\frac{v-a_i}{\sigma_i}\right)^2\right\}\frac{du}{\sigma_i\sqrt{2\pi}} \\ &\quad + N\zeta_1\left[\int_{j-1/2}^{j+1/2}\exp\left\{-\frac{1}{2}\left(\frac{v-a_1}{\sigma_1}\right)^2\right\}\frac{du}{\sqrt{2\pi}\sigma_1}\right. \\ &\qquad\left.-\int_{j-1/2}^{j+1/2}\exp\left\{-\frac{1}{2}\left(\frac{v-a_2}{\sigma_2}\right)^2\right\}\frac{du}{\sqrt{2\pi}\sigma_2}\right] \quad (v=u-d) \\ &= N\sum_{i=1,2} r_i\int_{t_{ij}-1/2\sigma_i}^{t_{ij}+1/2\sigma_i}[t_i\xi_i+(t_i^2-1)\eta_i]\frac{\varphi(t_i)}{\sigma_i}dt_i \\ &\quad + N\zeta_1\left[\int_{t_{ij}-1/2\sigma_2}^{t_{ij}+1/2\sigma_2}\varphi(t_1)dt_1-\int_{t_{2j}-1/2\sigma_2}^{t_{2j}+1/2\sigma_2}\varphi(t_2)dt_2\right] \quad (v-a_i=\sigma_i t_i).\end{aligned} \tag{50}$$

Or, if the breadth $1/\sigma_i$ be small enough, then $\tilde{Y}_j$ coincides with $\tilde{y}_j$ and we have approximately

$$\begin{aligned}\Delta y_j &= N\sum_i[t_{ij}\xi_i+(t_{ij}^2-1)]\frac{r_i\varphi(t_{ij})}{\sigma_i^2}+N\zeta_1\left[\frac{\varphi(t_{1j})}{\sigma_1}-\frac{\varphi(t_{2j})}{\sigma_2}\right] \\ &= \frac{t_{1j}}{\sigma_1}\tilde{y}_{1j}\xi_1+\frac{t_{2j}}{\sigma_2}\tilde{y}_{2j}\xi_2+\frac{t_{1j}^2-1}{\sigma_1}\tilde{y}_{1j}\eta_1+\frac{t_{2j}^2-1}{\sigma_2}\tilde{y}_{2j}\eta_2+\left(\frac{\tilde{y}_{1j}}{r_1}-\frac{\tilde{y}_{2j}}{r_2}\right)\zeta_1 \\ &= A_{1j}\xi_1+A_{2j}\xi_2+B_{1j}\eta_1+B_{2j}\eta_2+C_j\zeta_1.\end{aligned} \tag{51}$$

All the coefficients could be evaluated conveniently utilizing every term t_{ij}, $\varphi(t_{ij})$ and $\tilde{y}_{ij}$, which have been already obtained during calculations of $\tilde{y}_j$.

But, to be more exact, we should treat upon (50). It is easily seen that (50) reduces to

$$\begin{aligned}\Delta y_j &= N\sum_i r_i[\varphi(t_{i,j-\frac{1}{2}})-\varphi(t_{i,j+\frac{1}{2}})]\xi_i/\sigma_i + N\sum_i r_i[t_{i,j-\frac{1}{2}}\varphi(t_{i,j-\frac{1}{2}}) \\ &\qquad - t_{i,j+\frac{1}{2}}\varphi(t_{i,j+\frac{1}{2}})]\eta_i/\sigma_i + [\tilde{Y}_{1j}/r_1 - \tilde{Y}_{2j}/r_2]\zeta \\ &= A_{1j}\xi_1 + A_{2j}\xi_2 + B_{1j}\eta_1 + B_{2j}\eta_2 + C_j\zeta_1. \end{aligned} \tag{52}$$

We have already computed $\tilde{y}_{ij} = Nr_i\varphi(t_{ij})/\sigma_i$ to obtain $\tilde{y}_j$. Now, obtain similarly $\tilde{y}_{i,j\pm\frac{1}{2}} = \frac{Nr_i}{\sigma_i}\varphi(t_{i,j\pm\frac{1}{2}})$. Also to obtain $\tilde{Y}_j$ we should compute $\tilde{Y}_{ij} = Nr_i\Big[\Phi\Big(t_{ij}+\frac{1}{2\sigma_i}\Big) - \Phi\Big(t_{ij}-\frac{1}{2\sigma_i}\Big)\Big]$. If $\tilde{Y}_{ij}$ and $\tilde{y}_{ij}$ differ only insignificantly, then (51) would almost coincide with (52). Otherwise, we should proceed with (52) as observation equations, whose coefficients are

$$A_{ij} = \tilde{y}_{i,j-\frac{1}{2}} - \tilde{y}_{i,j+\frac{1}{2}},\ B_{ij} = t_{i,j-\frac{1}{2}}\varphi(t_{i,j-\frac{1}{2}}) - t_{i,j+\frac{1}{2}}\varphi(t_{i,j+\frac{1}{2}}),\ C_j = \tilde{Y}_{1j}/r_1 - \tilde{Y}_{2j}/r_2. \tag{53}$$

Ex. 12. (Ex. 8). Actually we have calculated every values of (51) for Ex. 8, and further obtained Gaussian sums $[AA]$, $[AB]$, $\cdots$, $[SS]$, as follows

156.6029,	−21.5476,	0.3406,	−99.7924,	−998.1984,	67.4807,	−895.1142,
	778.7573,	224.6250,	0.01872,	−659.1514,	−405.8534,	−83.1514,
		234.0938,	−122.7879,	−848.2896,	−81.9684,	−593.9865,
			1168.5542,	2388.8562,	463.9100,	3798.7588,
				11386.9098,	542.9410,	11813.0677,
					775.3049,	1361.8148,
						15401.3892.

On solving normal equations, corrections are found to be

$$\xi_1 = 0.9720,\ \xi_2 = -0.6507,\ \eta_1 = 0.7038,\ \eta_2 = 0.4414,\ \zeta = 0.05505.$$

However, these corrections, except ζ, being so large, our previous assumption that their powers are enough small to be neglected, is not satisfied. Really the corrected results become $a_1' = -0.2988$, $a_2' = 1.2744$, $\sigma_1' = 3.3999$, $\sigma_2' = 1.6394$, $r_1 = 0.6574$, $r_2 = 0.3426$, and the values $\tilde{y}_j'$, recomputed using these new parameters, fit no better than before. It would have been better to have used rather (52), (53).

Ex. 13. (Ex. 1). On the otherhand we have obtained a successful correction with Ex. 1 by least squares. We have already found its representation in §3 in the form

$$\tilde{y} = \frac{Nr_1}{\sqrt{2\pi}\sigma_1}\exp\Big\{-\frac{1}{2}\Big(\frac{u}{\sigma_1}\Big)^2\Big\} + \frac{Nr_2}{\sqrt{2\pi}\sigma_2}\exp\Big\{-\frac{1}{2}\Big(\frac{u}{\sigma_2}\Big)^2\Big\} = \tilde{y}_1 + \tilde{y}_2.$$

Now putting the corrections of σ_1, σ_2, r_1 and r_2 to be ξ, η, ζ and $-\zeta$, we have

$$\Big[\Big(\frac{u_j}{\sigma_1}\Big)^2 - 1\Big]\frac{\tilde{y}_{1j}}{\sigma_1}\xi + \Big[\Big(\frac{u_j}{\sigma_2}\Big)^2 - 1\Big]\frac{\tilde{y}_{2j}}{\sigma_2}\eta + \Big[\frac{\tilde{y}_{1j}}{r_1} - \frac{\tilde{y}_{2j}}{r_2}\Big]\zeta = y_j - \tilde{y}_j,$$

or

$$a_j\xi + b_j\eta + c_j\zeta = d_j \qquad (j = 0, \pm 1, \pm 2, \cdots).$$

And in fact Gaussian coefficients are obtained to be

$[aa]=47.104$,	$[ab]=41.556$,	$[ac]=187.864$,	$[ad]=15.206$,	$[as]=291.730$,
	$[bb]=588.532$,	$[bc]=729.899$,	$[bd]=159.166$,	$[bs]=1519.147$,
		$[cc]=1531.610$,	$[cd]=237.670$,	$[cs]=2687.064$,
			$[dd]=47.733$,	$[ds]=465.291$,
				$[ss]=4957.740$.

On solving the normal equations we get $\zeta=0.136$, $\eta=0.125$, $\xi=-0.331$, and hence the improved parameters become $r_1'=0.754$, $r_2'=0.246$, $\sigma_1'=2.248$, $\sigma_2'=0.960$. With these parameters we recomputed the new representation $\tilde{y}'$. Indeed, this time χ^2 amounts to only 0.675, so that $\Pr(\chi^2 \geqq 0.675)$ becomes >0.995 for 6 degrees of freedom, thus the agreement becomes much more better than before corrections.

§9. Analysed as Pearson's Unimodal Distributions

In Pearsonian school almost all problems of curve fitting had been treated with method of unimodal analysis by means of Pearson's $\beta\kappa$ criterion.[8] So also all the foregoing examples might be computed in that way, which will be described below.

Ex. 14 (Ex. 1). We obtained $\mu_2=4.3759$, $\mu_3=0$, $\mu_4=72.9566$. Hence $\beta_1=\frac{\mu_3^2}{\mu_2^3}=0$, $\beta_2=\frac{\mu_4}{\mu_2^2}=\frac{72.9566}{19.1485}=3.8100>3$, so that $\kappa=\frac{\beta_1(\beta_1+3)}{4(4\beta_2-3\beta_1)(2\beta_2-3\beta_1-6)}=0$, and consequently it belongs to Pearson's symmetrical Type VII with unlimited ends: $\eta=\eta_0\left(1+\frac{u^2}{A^2}\right)^{-p}$. After Pearson's method the constants are determined as follows:

$$\rho=\frac{3(\beta_2-1)}{\beta_2-3}=10.407.\ p=\frac{1}{2}(\rho+2)=6.2035,\ A^2=(\rho-1)\mu_2=41.164,$$

and $\eta_0=\frac{2^{2p-2}\Gamma(p)^2}{A\pi\Gamma(2p-1)}$, where p being a little large, we may use Stirling's asymptotic formula $\Gamma(p)\cong\sqrt{2\pi}p^{p-\frac{1}{2}}e^{-p}$. Thus by logarithmic computation we get $\eta_0=0.2049$. Therefore

$$\tilde{y}=N\eta_0\left(1+\frac{u^2}{41.164}\right)^{-p}=20.49\left[1+\frac{(\bar{x}-\tilde{x})^2}{10.29}\right]^{-6.2035}.$$

Calculating the values of $\tilde{y}$ for $u=0, \pm1, \pm2, \cdots$, we obtain the following result[9]:

u	0	±1	±2	±3	±4	±5	±6	±7	±8	±9	total
obs. y	21.94	17.97	10.63	5.48	2.78	1.33	0.56	0.20	0.06	0.02	100.00
cal. $\tilde{y}$	20.49	17.66	11.53	6.01	2.67	1.08	0.42	0.16	0.06	0.02	99.71

8) Cf. e.g. W. P. Elderton, Frequency Curve and Correlation, 1938; or, Y. Watanabe, Saisho Zizyôhô oyobi Tôkei (Japanese), 1935 (Maruzen).

9) Cal. $\tilde{y}$ had been obtained informally by ordinates, not by areas, so that the total does not coincide with the observed.

Whence we get $\chi^2=\sum(y-\tilde{y})^2/\tilde{y}=1.026$ and degrees of freedom being $10-3=7$, $\Pr(\chi^2>1.026)$ lies between 0.990 and 0.995. However this means, by no means, that the unimodal representation fits better than bimodal. Indeed, the result improved by least squares as obtained in the end of §8 is mudh better than the representation obtained just now.

Ex. 15 (Ex. 2). It was $\mu_1=7.2683$, $\mu_3=0.3373$, $\mu_4=119.9274$, so that $\beta_1=\frac{\mu_3^2}{\mu_2^3}=0.0002963$, $\beta_2=\frac{\mu_4}{\mu_2^2}=2.27014<3$ and $\kappa=\frac{\beta_1(\beta_2+3)}{4(4\beta_2-3\beta_1)(2\beta_2-3\beta_1-6)}=-0.0^3296\doteqdot 0$. Hence it belongs to Pearson's symmetrical Type with limited ends: $\eta=\eta_0\left(1-\frac{v^2}{c^2}\right)^p$. Here $p=\frac{5\beta_2-9}{2(3-\beta_2)}=1.6102$, $c^2=\frac{2p+5}{3}\frac{\mu_4}{\mu_2}=45.212$, $c=6.7238$ and the required representation becomes $\eta=\eta_0\left(1-\frac{v^2}{45.212}\right)^p$, $\tilde{y}=100\eta$, where $\eta_0=\frac{\Gamma(2p+2)}{2^{2p+1}c\Gamma(p+1)^2}=0.12926$. As it was $v=u-0.22$ and $u=\frac{1}{5}(x-67.5)$, so we get $v=\frac{1}{5}(x-68.6)$ and

$$\tilde{y}=12.926\left(1-\frac{u-0.22}{45.212}\right)^p=12.926\left(1-\frac{x-68.6}{1130.3}\right)^{1.6102}.$$

Here χ^2 amounts to 1.985, and degrees of freedom being $13-3=10$, $\Pr(\chi^2\geqq 1.985)>0.995$, so that we see that even such an unimodal representation would give already a sufficiently good fitting.

Strictly speaking, Sheppard's correction is only correct, so far as the density function $f(u)$ behaves at finite ends, so that $f(u)=f'(u)=f''(u)=f'''(u)=0$, or else $u^3f(u)$, $u^4f'(u)$, $u^3f''(u)$, $u^4f'''(u)$ tend zero as $u\to\pm\infty$. Hence, if the representation we have found does not satisfy these conditions, we must recompute the parameter values by taking the original uncorrected moments. Thus the above solution needs a recomputation. However, when we use the uncorrected moments in Ex. 2 in §4: $\mu_2'=7.3516$, $\mu_3'=0.3373$, $\mu_4'=123.6459$ and repeat the above made computations, we obtain an almost same representation $\tilde{y}=12.914\left(1-\frac{u-0.22}{47.23}\right)^{1.712}$, which only slightly differs from the before obtained.

Ex. 16. (Ex. 3). This example being similar to the foregoing, it appears better to use the uncorrected moments: $\mu_2'=6.4513$, $\mu_3'=4.9488$, $\mu_4'=85.4873$. Here $\beta_1=0.091213$, $\beta_2=2.05403<3$, $\kappa=-0.0088\doteqdot 0$, so it belongs still to the same type as before. However the constants become now $p=0.67125$, $c^2=28.106$, $c=5.293$, $\eta_0=0.1280$, so that the unimodal representation reduces to

$$\eta=0.1280\left(1-\frac{v^2}{28.02}\right)^{0.67135},\qquad \tilde{y}=1520\eta=194.6\left[1-\frac{x-62.68}{700.5}\right]^{0.67135}.$$

Thus the contact of $\tilde{y}$-curve to x-axis being slight, the uncorrected moments were legitimately used. However, its ω^2 becomes extraordinarily large, and the above representation is to be rejected.

Ex. 17 (Ex. 4). Here also beginning with uncorrected moments $\mu_2'=4.2112$, $\mu_3'=-0.2722$, $\mu_4'=34.9489$, we obtain $\beta_1=0.0009913$, $\beta_2=1.9707<3$, and $\kappa=-0.00008\doteqdot 0$. Thus we get once more again the same type, and $p=0.4146$, $c^2=16.125$, $c=4.0156$. $\eta_0=0.1532$, so that $\tilde{y}=15.32\left[1-\frac{u-0.099}{4.0156}\right]^{0.4146}$. Thus, theoretically $-3.9166<u<4.1146$ and we get

u	-5	-4	-3	-2	-1	0	1	2	3	4	total
obs. y	0.5	2.8	9.4	17.3	17.3	10.6	12.2	19.2	9.5	1.2	100.0
cal. $\tilde{y}$	0	4.65	11.28	13.79	15.00	15.28	14.84	13.42	10.52	0	98.78

Whence we get $\delta^2=751$ and $\omega_9^2=751/100^2=0.0751$, $\Phi(\omega_9^2=0.0751)=0.3389$, $\Pr\{\omega_9^2\geqq 0.0751\}=0.6611>0.05$. Hence the above unimodal representation is not to be rejected.

Ex. 18 (Ex. 5). Using uncorrected moments $\mu_2'=3.5376$, $\mu_3'=-5.4489$, $\mu_4'=30.4220$, we have $\beta_1=0.67063$, $\beta_2=-2.43092$, $\kappa=-0.03748<0$. Hence it belongs to Pearson's asymmetrical Type I: $\eta=\eta_0\left|1-\frac{\xi}{c_1}\right|^{-p_1}\left|1-\frac{\xi}{c_2}\right|^{p_2}$. The constants are found in succession as follows: $r=\frac{6(\beta_2-\beta_1-1)}{3\beta_1-2\beta_2+6}=1.4482$, $t=\sqrt{16(r+1)+\beta_1(r+2)^2}=6.8662$, $q_1, q_2=\frac{r}{2}\left[1\pm\frac{r+2}{t}\sqrt{\beta_1}\right]=1.0203, 0.4279$, where μ_3 being negative, $q_1>q_2$ and $-p_1=q_1-1=0.0203$, $p_2=q_2-1=-0.5721$. Further $b=\frac{1}{2}\sqrt{\mu_2'}t=6.4575$, $\nu=\frac{b}{r-2}=-11.7014$, whence $c_1=\nu p_1=0.2375$ as well as $c_2=\nu p_2=6.6944$, and lastly $\eta_0=\frac{|p_2|^{p_2}|p_1|^{-p_1}}{b|p_2-p_1|^{p_2-p_1}}\frac{\Gamma(p_2-p_1+2)}{\Gamma(1-p_1)\Gamma(1+p_2)}=0.06136$. Therefore

$$\eta=0.06136\left(\frac{\xi}{0.2375}-1\right)^{0.0203}\left(1-\frac{\xi}{6.6944}\right)^{-0.5721}, \qquad \tilde{y}=100\eta.$$

Thus we obtain a *J*-shaped distribution. To express it by u, we need further calculations. We have originally determined ξ axis by translating origin into mode on u axis. Thus $m_0=0$ and $\bar{\xi}$ is given by $\bar{\xi}-m_0=\frac{1}{2}\frac{\mu_3'}{\mu_2'}\frac{r+2}{r-2}$, so that $\bar{\xi}=4.8126$. On the other hand it was $\bar{u}=d=0.32$. Hence $u=0$ corresponds to $\xi=4.8126-0.32=4.4926$, and in general $\xi=4.4926+u$. Thus finally

$$\tilde{y}=6.136\left[\frac{u+4.4926}{0.2375}-1\right]^{0.0203}\left[1-\frac{u+4.4926}{6.6944}\right]^{-0.5721}$$
$$=18.747(u+4.2551)^{0.0203}(2.2018-u)^{-0.5721}.$$

Using the last expression, we compute $\tilde{y}$ for $u=-5, -4, \cdots, 2$, and obtain the following result:

u	-4	-3	-2	-1	0	1	2	3	total
obs. y	3	11	8	6	10	28	31	3	100
cal. $\tilde{y}$	6	7	8	10	12	17	48	0	108

Whence, $\omega_9^2=0.0688$ and $\Pr\{\omega_9^2\geqq 0.0688\}=1-\Phi(\omega_9^2=0.0688)=0.7589>0.05$. Hence the unimodal representation is not to be rejected.

Ex. 19 (Ex. 6). In this biometrical example it will also be found that Pearson's unimodal representation does not have higher contact with x-axis. Hence we have to start with uncorrected central moments: $\mu_2'=10.13337$, $\mu_3'=-10.4709$, $\mu_4'=230.9316$. Accordingly $\beta_1=\frac{\mu_3'^2}{\mu_2'^3}=0.10664$, $\beta_2=\frac{\mu_4'}{\mu_2'^2}=2.2489<3$ and $\kappa=-0.00895<0$. Hence, if κ be assumed nearly zero, we shall obtain just the same Type as in Ex. 15: $\eta=\eta_0\left(1-\frac{v^2}{c^2}\right)^p$. But, more exactly, it may be classified into Pearson's asymmetrical Type I, as in Ex. 18: $\eta=\eta_0\left|1-\frac{\xi}{c_1}\right|^{-p_1}\left|1-\frac{\xi}{c_2}\right|^{p_2}$. The parameters are computed successively as follows:

$$r=\frac{6(\beta_2-\beta_1-1)}{3\beta_1-2\beta_2+6}=3.7615,\quad t=\sqrt{16(r+1)+\beta_1(r+2)^2}=3.3404,$$

$$q_1,\, q_2=\frac{r}{2}\left[1\pm\frac{r+2}{t}\sqrt{\beta_1}\right]=2.2770,\ 1.4844,$$

where $q_1>q_2$ since $\mu_3'<0$. Consequently $-p_1=q_1-1=1.2771$; $p_2=q_2-1=0.4844$. Further $b=\frac{1}{2}\sqrt{\mu_2'}t=14.2115$, $\nu=\frac{b}{r-2}=8.0678$, $c_1=\nu p_1=-10.3031$, $c_2=\nu p_2=3.9084$. Therefore the required representation for density function becomes

$$\eta=\eta_0\left(1+\frac{\xi}{10.5031}\right)^{1.2771}\left(1-\frac{\xi}{3.9084}\right)^{0.4844}\quad\text{and } \tilde{y}=270\eta,$$

where $\eta_0=\frac{|p_1|^{-p_1}p_2^{p_2}}{b(p_2-p_1)^{p_2-p_1}}\frac{\Gamma(p_2-p_1+2)}{\Gamma(1-p_1)\Gamma(1+p_2)}=0.04154$[10] which was evaluated by use of Legendre's Table of $\log\Gamma(p)$ for $1<p<2$.

The origin ξ was measured from mode on u-axis, and $\bar{\xi}=\frac{1}{2}\left(\frac{r-2}{r+2}\right)\frac{\mu_3}{\mu_2}=-1.6869$. On the other hand it was $\bar{u}=\mathrm{d}=0.5259$. Hence $\xi=u-0.5259-1.6864$ $=u-2.2128$, and the representation becomes

$$\tilde{y}=0.7414(u+8.091)^{1.2771}(6.1212-u)^{0.4844}.$$

Calculating $\tilde{y}$ for each u, we get

10) This was evaluated by use of the Table of $\log\Gamma(p)$. However, if p be large, we may utilize Stirling's asymptotic formula. Also we may simply compute relative values z of η for $u=0,\pm1,\pm2,\cdots$, and obtain $\eta_0=1/\sum z_j$.

u	−8	−7	−6	−5	−4	−3	−2	−1	0
obs. y	1	2	2	11	24	25	16	14	18
cal. $\tilde{y}$	0.13	2.92	6.44	10.18	13.91	17.52	20.78	23.64	25.98
u	1	2	3	4	5	6	7	8	total
obs. y	30	39	41	28	14	1	2	2	270
cal. $\tilde{y}$	27.63	31.34	30.60	28.05	22.60	8.31	0	0	270.03

Whence we obtain by (39) $\delta^2=19726$ and $\omega^2=\delta^2/N^2=19726/270^2=0.2706$, $\Phi(\omega^2)=0.8362$, so that $\Pr\{\omega^2\geqq 0.2706\}=0.1638>0.05$. Thus the above unimodal representation is not to be rejected, although, compared with bimodal representation obtained in Ex. 6, the probability reduces far less.

Ex. 20 (Ex. 7). Here still using uncorrected moments $\mu_2'=6.0671$, $\mu_3'=-5.5421$, $\mu_4'=81.3514$, we obtain $\beta_1=0.13753$, $\beta_2=2.21005$, and $\kappa=-0.00862<0$, so that it belongs again to Pearson's Type I. Computing parameters in the same way as before, we get $r=3.2297$, $t=8.4520$, $q_1, q_2=1.9855$, 1.2442. Hence $-p_1=0.9855$, $p_2=0.2442$. Further $b=10.1690$, $\nu=8.2697$, $c_1=-8.1500$, $c_2=2.0191$ and $\eta_0=0.14783$. The required unimodal representation is, therefore,

$$\tilde{y}=270\times 0.14783\left[1+\frac{\xi}{8.1500}\right]^{0.9855}\left[1-\frac{\xi}{2.0191}\right]^{0.2442}.$$

The mean ξ is found to be -1.94243, while $\bar{u}=-0.2185$. Hence $\xi=u+0.2185-1.9424=u-1.7239$. Substituting this in the above, we get

$$\tilde{y}=4.2527(u+6.4261)^{0.9855}(3.7430-u)^{0.2442},$$

from which we obtain $\tilde{y}$ for every u as follows:

u	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	total
obs. y	1	1	8	26	31	18	20	33	56	45	26	3	1	1	270
cal. $\tilde{y}$	0	3.2	10.2	16.8	22.8	28.2	33.6	36.7	42.2	39.8	36.1	0	0	0	269.6

and whence the squared deviation in sum $\delta^2=17333$, so that $\omega^2=\dfrac{17333}{270^2}=0.2378$. Entering the ω^2–Table we find that $\Phi(\omega^2)=0.7952$. Hence $\Pr\{\omega^2\geqq 0.2378\}=0.2048>0.05$. Consequently the above representation is also not to be rejected.

Ex. 21 (Ex. 8). Starting with the corrected moments $\mu_2=7.68337$, $\mu_3=-12.8943$, $\mu_4=197.1589$, and proceeding similarly as foregoing, we get

$$\tilde{y}=245\eta_0\left[1+\frac{\xi}{33.2350}\right]^{22.1842}\left[1-\frac{\xi}{5.9795}\right]^{3.9913},$$

where

$$\eta_0=\frac{|p_1|^{p_1}p_2^{p_2}}{b|p_2-p_1|^{p_2-p_1}}\frac{\Gamma(p_2-p_1+2)}{\Gamma(1-p_1)\Gamma(1+p_2)},\quad b=39.2146,\quad -p_1=22.1842,\quad p_2=3.9913.$$

For smaller value of p we may put $\Gamma(p+1)=p\Gamma(p)=p(p-1)\Gamma(p-1)=\cdots$ and finally refer to Legendre's Tables of log $\Gamma(p)$ for $1<p<2$. But for larger p, it is more convenient to use Stirling's formula $\Gamma(p)=\sqrt{2\pi}p^{p-\frac{1}{2}}e^{-p}$. Thus in the above we find $\eta_0=0.1472$. Furthermore we get $\bar{\xi}=-0.9673$ and it was $\bar{u}=-0.2185$, so that $\xi=u-0.7508$. Using these substitution we get a more convenient form:

$$\tilde{y}=10^{-36}\times 5.0318(u+32.4842)^{22.1842}(6.7303-u)^{3.9913} \qquad \text{(i)}$$

from which $\tilde{y}$ can be found for every $u=-8, -7, \cdots, 7$, by logarithmic computations. We see that the above curve has a strong contact with u–axis at the left end, but at the right $\frac{d^4y}{du^4}$ does not vanish, so seems apparently Sheppard's correction inadequate. If, however, on taking uncorrected moments $\mu'_2=7.7667$, $\mu'_3=-12.8943$, $\mu'_4=201.0233$, and recomputing, we find only a little different result:

$$\tilde{y}=32.7467\left[1+\frac{\xi}{34.3888}\right]^{23.4760}\left[1-\frac{\xi}{6.1787}\right]^{4.2184},$$

or $$\tilde{y}=10^{-38}\times 1.2880(u+33.6573)^{23.4760}(6.9102-u)^{4.2184} \qquad \text{(ii)}$$

and now $\frac{d^4y}{du^4}$ vanishes at the right end also, so that Sheppard's correction becomes applicable. To decide this dilemma, we have only to compare the goodness of fitting by ω^2–test. Really these two give the following results:

u	-8	-7	-6	-5	-4	-3	-2	-1	0
obs. y	1	4	10	6	10	6	18	30	34
cal. $\tilde{y}$ (i)	1.5	2.8	4.8	7.8	12.1	17.7	24.0	30.2	34.8
cal. $\tilde{y}$ (ii)	1.4	2.5	4.4	7.2	11.2	16.2	22.0	27.6	31.7

u	1	2	3	4	5	6	7	total
obs. y	54	28	26	10	5	1	2	245
cal. $\tilde{y}$ (i)	35.9	32.1	23.4	12.5	3.7	0.2	0	243.6
cal. $\tilde{y}$ (ii)	32.6	29.1	21.3	11.5	3.6	0.3	0	222.6

and it is found that ω_0^2 amount to 0.4318 and 0.5740 for (i) and (ii), so that Pr $\{\omega^2\geqq\omega_0^2\}=0.0595$ and 0.0258 respectively. Hence (ii) is to be rejected, while (i) is hardly not to be rejectet.

Ex. 22 (Ex. 9). Using the corrected moments $\mu_2=4.26042$, $\mu_3=-4.68103$, $\mu_4=54.49860$, we obtain by the same way as before,

$$\tilde{y}=39.43275\left[1+\frac{\xi}{9.2229}\right]^{5.5642}\left[1-\frac{\xi}{7.6912}\right]^{4.6402}, \quad \xi=u-0.6641,$$

in which both exponents being enough large, Sheppard's corrections are correctly done, and we have no more to recalculate. Now, writing it as

$$\tilde{y}=10^{-8}\times 1.3058(u+8.5588)^{5.5642}(8.3553-u)^{4.6402},$$

we have calculated $\tilde{y}$ for $u=j=0, \pm 1, \pm 2, \cdots$ and whence computed $\delta^2=\sum(\sum y_j - \sum\tilde{y}_j)^2\tilde{y}_j$, $\omega^2=\delta^2/N^2$. But we obtain $\omega_9^2 \doteqdot 1$ and $\Phi(\omega_9^2)=0.9976$. Thus $\Pr\{\omega^2 \geqq 1\} = 0.0024 < 0.05$. Heoce the above Pearson's representation is to be rejected.

Ex. 23 (Ex. 10). From ν_k's the moments about $\bar{u}=d=-0.5011$ are found to be $\mu_2=5.5285$, $\sqrt{\mu_2}=2.3512$, $\mu_3=1.3764$, $\mu_4=89.5887$. Hence $\beta_1=0.011212$, $\beta_2=2.39115$ and $\kappa=-0.0830<0$, so that it belongs still to Pearson's Type I: $\eta=\eta_0\left|1-\frac{\xi}{c_1}\right|^{-p_1}\left|1-\frac{\xi}{c_2}\right|^{p_2}$. Calculating in a similar way as before we obtain

$$\tilde{y}=245\eta, \quad \eta=\eta_0\left[1+\frac{\xi}{18.138}\right]^{36.725}\left[1-\frac{\xi}{31.375}\right]^{63.525},$$

where $\eta_0=\frac{(-p_1)^{-p_1}p_2^{p_2}}{b(p_2-p_1)^{p_2-p_1}}\frac{\Gamma(p_2-p_1+2)}{\Gamma(1-p_1)\Gamma(1+p_2)}$ and since $-p_1$ and p_2 both large, we may use Stirling's formula and obtain $\eta_0=\frac{p_2-p_1+2}{b}\sqrt{\frac{p_2-p_1}{2\pi(-p_1)p_2}}=0.1691$. Also the mean $\xi=0.1196$. But, as $\bar{u}=d=-0.5011$, we have $\xi=u+0.6107$, and consequently

$$\tilde{y}=16.91\left[1+\frac{u+0.6107}{18.138}\right]^{36.725}\left[1-\frac{u+0.6107}{31.375}\right]^{63.525}.$$

Whence calculating $\tilde{y}_j$ for $u=j$, we get

u	−9	−8	−7	−6	−5	−4	−3	−2	−1	0
obs. y	0.01	0.06	0.28	0.99	2.74	5.90	10.11	14.16	16.54	16.09
cal. $\tilde{y}$	0.01	0.05	0.26	0.95	2.64	5.74	10.00	14.21	16.70	16.38

u	1	2	3	4	5	6	7	8	9	total
obs. y	13.35	9.38	5.58	2.92	1.25	0.46	0.14	0.03	0.01	100.00
cal. $\tilde{y}$	13.54	9.46	5.46	2.86	1.23	0.45	0.14	0.04	0.01	100.15

Hence χ^2 amounts to only 0.0336. The degrees of freedom being $19-6=13$, $\Pr\{\chi^2 \geqq 0.0336\}>0.9$. Also all $|\sum y-\sum\tilde{y}|$ being less than 1, we have $\delta^2=\sum|\sum y - \sum\tilde{y}|^2\tilde{y}<\sum\tilde{y}\doteqdot 100$. Hence $\omega^2<\frac{100}{100^2}=0.01$, $\Phi(\omega^2)<0.0001$, $\Pr\{\omega^2\geqq 0.01\}>0.9999$. Thus either χ^2– or ω^2–test does not reject the above unimodal representation.

§10. Gram-Charlier's Representation

This method is frequently recommended because of its easy calculation. It is nothing but a single normal representation with additional corrections

$$y_C=\frac{N}{\sigma}[\varphi_0(t)+A_3\varphi_3(t)+A_4\varphi_4(t)], \tag{54}$$

where $\sigma=\sqrt{\mu_2}$, $t=\frac{u-d}{\sigma}=\frac{x-\bar{x}}{\sigma_x w}$ and $\varphi_0(t)$, $\varphi_3(t)$, $\varphi_4(t)$ are the standard normal density function $\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$ as well as its 3–rd and 4–th derivatives, respectively, while

$$A_3 = -\frac{\mu_3}{6\sqrt{\mu_2}^3},\quad A_4 = \frac{\mu_4}{24\mu_2^2} - \frac{1}{8}\,. \tag{55}$$

Ex. 24. We shall calculate y_C of (54) for every value of u in Ex. 1–10 and examine ω^2–test in regard to acceptability of Gram-Charlier's representation. The normal representation, i.e. the first single term alone in (54), denoted by y_N, shall be incidentally considered.

Ex. 1. Here $d=0$, $\sigma=2.092$, $N=100$, $A_3=0$, $A_4=0.03375$.

u	0	±1	±2	±3	±4	±5	±6	±7	±8	±9	total
y	21.94	17.97	10.63	5.48	2.78	1.33	0.56	0.20	0.06	0.02	100.00
y_N	19.07	17.01	12.07	6.82	3.06	1.09	0.31	0.07	0.01	0.00	99.95
y_C	20.69	17.98	11.41	5.65	2.49	1.14	0.54	0.22	0.01	0.00	99.57

Whence we get $\omega_0^2=0.0032$ and 0.0263 for y_C and y_N, so that both are nearly 0, and therefore $\Pr(\omega^2\geqq\omega_0)$ nearly 1. Thus y_C as well as y_N are both acceptable as representations. Moreover, if we try χ^2–test, the calculated value y_C becomes nearly 0 at the end interval $u=\pm 9$, so that $|y-y_C|^2 \div y_C=\infty$. However, if these be lumped to $u=\pm 8$, χ^2 amounts to 1.3209 nearly. The degrees of freedom n being 9–4=5, $\Pr(\chi^2\geqq 1.3209)>0.9$. Similarly for y_N, $\chi^2=3.4497$, and for $n=5$, $\Pr\{x^2 \geqq 3.4497\}>0.5$. Thus even with severe χ^2–test both repersentations are not to be rejected.

Ex. 2. $d=0.22$, $\sigma=2.696$, $N=100$, $A_3=-0.0029$, $A_4=-0.0304$.

u	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	total
y	1	2	6	8	13	12	11	12	13	10	6	4	2	100
y_N	1.0	2.3	4.4	7.3	10.6	13.4	14.8	14.2	11.9	8.7	5.6	3.1	1.5	98.8
y_C	1.0	2.6	5.2	8.1	10.8	12.7	13.4	13.1	11.6	9.2	6.4	3.7	1.7	99.5

For y_C, $\omega_0^2=0.0106$, $\Pr\{\omega^2\geqq\omega_0^2\}$ is nearly 1, thus surely acceptable. Also for y_N, $\omega_0^2=0.055$, $\Pr\{\omega^2\geqq\omega_0^2\}=0.845$, and thus still not to be rejected, though less acceptable than y_C.

Ex. 3. $d=0.03684$, $\sigma=2.5235$, $N=1520$, $A_3=-0.0513$, $A_4=-0.0339$.

u	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	total
y	3	14	60	161	263	264	171	102	127	187	114	42	10	2	1520
y_N	13.8	32.5	66.8	117.0	173.1	220.9	240.3	224.6	177.3	121.2	70.1	34.5	14.8	5.2	1512
y_C	8.6	34.8	82.4	141.3	192.9	219.0	216.2	195.8	162.0	122.1	80.6	43.7	19.2	4.7	1523.3

here with y_C the ω^2 becomes 0.2 and $\Pr\{\omega^2\geqq\omega_0^2 =0.266>0.05$. Hence y_C is not to be rejected, but for y_N not so.

Ex. 4. $d=-0.099$, $\sigma=2.3172$, $N=100$, $A_3=0.1246$, $A_4=-0.0446$.

u	−5	−4	−3	−2	−1	0	1	2	3	4	total
y	0.5	2.8	9.4	17.3	17.3	10.6	12.2	19.2	9.5	1.2	100.0
y_N	1.1	3.1	7.1	12.6	17.8	19.6	17.0	11.6	6.2	3.6	98.7
y_C	0.3	4.4	7.5	10.4	13.6	17.4	19.0	15.6	8.5	3.4	99.1

For y_C, $\omega^2=0.228$ and $\Phi(\omega^2)=0.7953$, $1-\Phi(\omega^2)=0.2047>0.05$, so it is not to be rejected.

Ex. 5. $d=0.32$, $\sigma=1.8586$, $N=100$, $A_3=0.1415$, $A_4=-0.0248$.

u	−5	−4	−3	−2	−1	0	1	2	3	4	total
y	0	3	11	8	6	10	28	31	3	0	100
y_N	0.4	1.4	3.6	9.9	16.7	21.1	20.1	14.3	7.6	3.0	98.1
y_C	1.1	2.6	3.9	7.5	12.5	19.5	21.9	18.3	9.0	2.3	98.6

For y_C and y_N we get $\omega^2=0.7687$ and 0.8398, and $1-\Phi(\omega^2)=0.008$, $0.006<0.05$, so that y_C and y_N are both to be rejected.

Ex. 6. $d=0.52593$, $\sigma=3.1697$, $N=270$, $A_3=0.0548$, $A_4=-0.0318$.

u	−8	−7	−6	−5	−4	−3	−2	−1	0
y	1	2	2	11	24	25	16	14	18
y_N	0.91	2.05	4.07	7.48	12.22	18.35	24.68	30.28	33.55
y_C	1.21	2.70	5.22	8.92	13.28	18.05	22.47	26.46	29.63

u	1	2	3	4	5	6	7	8	total
y	30	39	41	28	14	1	2	2	270.00
y_N	33.60	30.57	25.07	18.56	12.58	7.61	4.24	2.10	267.92
y_C	31.36	30.99	27.85	22.21	15.55	9.07	4.32	1.36	270.65

Here with y_C, $\omega^2=0.322$, $\Phi(\omega^2=0.322)=0.8829$, $\Pr\{\omega^2\geqq 0.822\}=0.1171>0.05$, thus not to be rejected; while with y_N, $\omega^2=0.794$, $\Phi(\omega^2=0.794)=0.9925$, $\Pr\{\omega^2\geqq 0.794\}=0.0075<0.05$, and so to be rejected.

Ex. 7. $d=0.0893$, $\sigma=2.4462$, $N=270$, $A_3=0.0631$, $A_4=-0.00338$.

u	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	total
y	1	1	8	26	31	18	20	33	56	45	26	3	1	1	270
y_N	1.1	2.7	6.8	13.4	23.0	33.7	41.8	43.9	38.9	29.1	18.4	10.0	4.6	1.8	269.2
y_C	1.2	3.8	8.7	15.2	22.6	29.8	36.0	40.2	40.2	34.0	23.1	12.1	4.2	0.4	271.5

For y_C it is $\omega^2=0.4208$, $\Phi(\omega^2=0.4208)=0.9364$, $\Pr\{\omega^2>0.4208\}=0.0636>0.05$, thus the y_C-representation is not to be rejected. However, for y_N, we get $\omega^2=0.9408$, $\Phi(\omega^2=0.9408)=0.9966$, $\Pr\{\omega^2\geqq 0.9408\}=0.0034<0.05$, hence y_N-representation is to be rejected.

Ex. 8. $d=-0.2185$, $\sigma=2.7719$, $N=245$, $A_3=0.1009$, $A_4=0.0142$.

u	−8	−7	−6	−5	−4	−3	−2	−1	0
y	1	4	10	6	10	6	18	30	34
y_N	0.5	1.3	3.2	6.5	11.8	18.9	26.5	32.6	35.2
y_C	1.5	2.7	4.3	6.4	9.5	14.3	21.5	30.0	36.4

u	1	2	3	4	5	6	7	total
y	54	28	26	10	5	1	2	245
y_N	33.4	27.8	20.4	13.1	7.4	3.6	1.6	243.6
y_C	37.7	32.8	23.7	14.0	6.6	2.3	0.4	244.1

Whence with y_C, $\omega_0^2=0.1446$, $\Phi(\omega_0^2)=0.5980$, $1-\Phi(\omega_0^2)=0.4020>0{\cdot}05$, hence it is not to be rejected. But with y_N, $\omega_0^2=0.490$, $\Phi(\omega_0)=0.9579$, $1-\Phi(\omega_0^2)=0.0421<0.05$, so it is to be rejected.

Ex. 9. $d=0.2694$, $\sigma=2.0641$, $N=245$, $A_3=0.0887$, $A_4=0.0001$.

u	-5	-4	-3	-2	-1	0	1	2	3	4	5	total
y	4	16	10	10	38	33	72	32	20	8	2	245
y_N	1.8	5.6	13.5	25.9	39.2	47.0	44.5	33.3	19.7	9.2	3.4	243.2
y_C	3.3	6.9	12.6	21.3	33.6	45.3	48.5	39.0	22.6	8.8	1.9	243.8

Here for y_C, $\omega^2=0.2582$, $\Phi(\omega^2=0.2582)=0.8220$, $\Pr\{\omega^2\geqq 0.2582\}=0.1780>0.05$, not to be rejected, but for y_N, $\omega^2=0.5150$, $\Phi(\omega^2=0.515)=0.9636$, $\Pr\{\omega^2\geqq 0.515\}=0.0364<0.05$, to be rejected.

Ex. 10. $d=-0.5011$, $\sigma=2.3512$, $N=100$, $A_3=-0.016036$, $A_4=-0.002873$.

u	-9	-8	-7	-6	-5	-4	-3	-2	-1	0
y	0.01	0.06	0.28	0.99	2.74	5.90	10.11	14.16	16.54	16.09
y_N	0.02	0.10	0.37	1.10	2.72	5.61	9.65	13.85	16.59	16.59
y_C	0.00	0.05	0.28	1.00	2.71	5.80	10.02	14.19	16.63	16.29
u	1	2	3	4	5	6	7	8	9	total
y	13.35	9.38	5.58	2.92	1.25	0.46	0.14	0.03	0.01	100.00
y_N	13.84	9.64	5.60	2.72	1.10	0.37	0.10	0.02	0.00	99.99
y_C	13.44	9.40	5.58	2.81	1.20	0.43	0.13	0.03	0.01	100.00

Here $\omega^2=0.0028$ so that $\Phi(\omega^2)$ nearly zero, $1-\Phi(\omega^2)$ nearly unity. Also $\chi^2=0.0230$ and degrees of freedom being $17-10=7$, $\Pr\{\chi^2\geqq 0.023\}>0.995$. Both acceptable.

§11. Watanabe's Representation

Watanabe proposed some bimodal representations, either by those of his new types, or by superposition of curves belonging to Pearson's types.[11] One case of the latter has been thoroughly developed in the present note. The former shall be illustrated below by treating Ex. 5 as example.

Ex. 25 (Ex. 5). It was found in Ex. 5, that $d=\bar{u}=0.32$ and $\mu_2=3.4543$, $\mu_3=-5.4489$, $\mu_4=28.6824$, $\mu_5=-82.2471$, $\mu_6=-336.7033$, $\mu_7=-1172.9612$. We shall represent this distribution by the genuine bimodal curve

$$y=y_0 \exp \varphi(v)=y_0 \exp \{c_1v+c_2v^2+c_3v^3+c_4v^4\}, \qquad v=u-d. \tag{56}$$

To determine parameters by Pearson's method of moments, as a first approximation, we have to solve the following linear equations:

$$\left.\begin{aligned} 0+c_2'\mu_2+c_3'\mu_3+c_4'\mu_4&=1,\\ c_1'\mu_1+c_2'\mu_3+c_3'\mu_4+c_4'\mu_5&=0,\\ c_1'\mu_2+c_2'\mu_4+c_3'\mu_5+c_4'\mu_6&=-3\mu_2,\\ c_1'\mu_3+c_2'\mu_5+c_3'\mu_6+c_4'\mu_7&=-4\mu_3. \end{aligned}\right\} \qquad c_i'=\frac{c_i}{i} \tag{57}$$

11) Y. Watanabe, Bimodal Distributions, this Journal vol. V (1954), p. 30.

On substituting μ_n's values in equations (57) and solving them, we get

$$c_4=4c_4'=-0.0482,\ c_3=3c_3'=-0.2004,\ c_2=2c_2'=0.1733,\ c_1=c_1'=0.9487.$$

Therefore

$$\varphi(v)=0.9487v+0.1733v^2-0.2004v_3-0.0482v^4,$$

and we obtain, as the required representation expessed in $u=v+0.32$

$$\tilde{y}=k\exp\{0.7825u+0.3361u^2-0.1387u^3-0.0482u^4\}=k\exp\varphi(u).$$

To determine k we have only to integrate the above expression numerically. But, with a later purpose. we have calculated the frequencies in each subclass, i.e. the areas of every subclass: $u_j-\frac{1}{2}<u<u_j+\frac{1}{2}$ $(j=0, \pm 1, \pm 2, \cdots)$, (i) roughly from ordinate values $\exp\varphi(u)$, and (ii) by means of Simpson's formula, lastly (iii) using Gauss' method of 5 selected ordinates, —— partly for the sake of comparison —— the results are as follows:

$u=j$	(i) $\exp\varphi(u)$	(ii) Simpson	(iii) Gauss	(iv) cal. $\tilde{y}$	(v) obs. y
−5	0.00025	0.0027	0.00275	0.0270	0
−4	0.29677	0.3923	0.39274	3.8510	3
−3	1.67815	1.5501	1.53205	15.0226	11
−2	1.12498	1.1455	1.14515	11.2288	8
−1	0.70054	0.7208	0.73136	7.1714	6
0	1	1.0568	1.05365	10.3316	10
1	2.53876	2.5254	2.52550	24.7639	28
2	2.79715	2.5592	2.57838	28.2825	31
3	0.10266	0.2364	0.23654	2.3194	3
4	(0.000003)	0.0002	0.00017	0.0017	0
sum	10.239263	10.1894	10.19829	99.9999	100

Taking the sum of (iii) we have $k=100/10.19829=9.8056$, and on multiplying this value to column (iii) we obtained column (iv). Thus the required representation is given as the first approximation, by

$$\tilde{y}=9.8056\exp\{0.7825u+0.3361u^2-0.1381u^3-0.0482u^4\}.$$

Using above table, we get $\chi^2=\sum(y-\tilde{y})^2/\tilde{y}=4.31$ on pooling at both ends. Degrees of freedom being 3, $\Pr\{\chi^2>4.31\}>0.2>0.05$. Also $\omega^2=\sum_{i=-5}^{4}[\sum_{-5}^{i}y_j-\sum_{-5}^{i}\tilde{y}_j]^2\tilde{y}_i/100^2=0.0371$ and $\Phi(\omega_9^2=0.0371)=0.0519$, so that $\Pr\{\omega_9^2\geq 0.0371\}=0.9481>0.05$. Thus, by either test the representation is not to be rejected.

To obtain a more elaborate result we proceed by method of least squares. Let the corrections of c_i and k be ξ_i $(i=1, 2, 3, 4)$ and ξ_5. The corrected ordinate becomes

$$y^*=\left(1+\frac{\xi_5}{k}\right)\tilde{y}\exp\{\xi_1u+\xi_2u^2+\xi_3u^3+\xi_4u^4\}.$$

Corrections being assumed to be small, we have approximately

$$y^* = \tilde{y}[1+\xi_1 u+\xi_2 u^2+\xi_3 u^3+\xi_4 u^4+\xi_5/k],$$

and consequently

$$\Delta y = y-\tilde{y} = \sum_{i=1}^{4}\tilde{y}\xi_i u^i+\tilde{y}\xi_5/k. \tag{58}$$

The total frequency should be always equal to 100, so that

$$\int y^* du = 100, \qquad \text{as well as} \qquad \int \tilde{y} du = 100.$$

Hence

$$\sum_i \xi_i \int \tilde{y} u^i du + 100\xi_5/k = 0. \tag{59}$$

But $\int \tilde{y} u^i du = 100\nu_i$ are approximately known by given statistics (cf. Ex. 5 in §5). Hence the above residual equation (58) becomes

$$(u-\nu_1)\tilde{y}\xi_1 + (u^2-\nu_2)\tilde{y}\xi_2 + (u^3-\nu_3)\tilde{y}\xi_3 + (u^4-\nu_4)\tilde{y}\xi_4 = \Delta y,$$

and ξ_5 is eliminated. Putting

$$(u_j-\nu_1)\tilde{y}_j = a_j, \quad (u_j^2-\nu_2)\tilde{y}_j = b_j, \quad (u_j^3-\nu_3)\tilde{y}_j = c_j, \quad (u_j^4-\nu_4)\tilde{y}_j = d_j \quad \text{and} \quad \Delta y_j = e_j,$$

we obtain, as observation equations,

$$a_j\xi_1+b_j\xi_2+c_j\xi_3+d_j\xi_4 = e_j \qquad (j=-5, -4, \cdots, 3, 4), \tag{60}$$

whose coefficients are computed as follows:

j	a_j	b_j	c_j	d_j	e_j	s_j
−5	−0.1436	0.5790	−3.3183	16.2321	−0.0270	13.2222
−4	−16.6355	47.9178	−238.3645	894.1173	−0.8508	686.1843
−3	−49.8724	81.7727	−374.0428	859.0967	−4.0218	512·9324
−2	−26.0494	4.9808	−66.2464	−87.6922	−3.2282	−178.2354
−1	−9.4657	−18.3341	7.8881	−163.5705	−1.1710	−184.6532
0	−3.3059	−36.7443	21.6951	−245.9811	−0.3310	−264.6672
1	16.8385	−63.3103	76.7638	−564.8326	3.2375	−531.3031
2	42.4721	11.2147	255.3381	−197.4446	5.7190	117.2993
3	6.2157	12.6253	67.4916	132.6408	0.6807	219.6541
4	0.0062	0.2115	0.1124	0.3947	−0.0017	0.7231
sum	−39.9400	40.9131	−252.6829	642.9606	0.0057	391.2565

Whence Gaussian coefficients are obtained as follows

$[aa]=5669.17$, $[ab]=-5221.41$, $[ac]=36756.33$, $[ad]=-70148.17$, $[ae]=612.65$,
$[bb]=14987.72$, $[bc]=-44426.34$, $[bd]=159924.89$, $[be]=-484.34$,
$[cc]=277303.50$, $[cd]=-620157.82$, $[ce]=3659.40$,
$[dd]=2008322.27$, $[de]=-6527.76$,
$[ee]=72.4534$.

Solving the normal equations, we find

$$\xi_1=0.1853,\ \xi_2=0.0176,\ \xi_3=-0.0145,\ \xi_4=-0.00264,$$

and the corrected coefficients become

$$c_1=0.9678,\ c_2=0.3537,\ c_3=-0.1532,\ c_4=-0.05084.$$

With these new values we recomputed the following integrals again by Gauss' method of 5 selected ordinates:

$$A_j=\int_{u_j-1/2}^{u_j+1/2}\exp\{\sum_{i=1}^{4}c_iu^i\}du,\qquad J=\sum A_j$$

u	A_j	$\tilde{y}=kA_j$……area	k exp $\varphi(u)$……ord.	obs. y
−5	0.00213	0.0197	0.0002	0
−4	0.31882	2.9467	2.2287	3
−3	1.24467	11.5041	12.4551	11
−2	0.91500	8.4570	8.2915	8
−1	0.62542	5.7806	5.5408	6
0	0.97933	9.0516	9.2427	10
1	3.06513	28.3300	28.2567	28
2	3.22078	29.7686	34.3029	31
3	0.44797	4.1404	1.0576	3
4	0.00013	0.0012	0.0000	0
	J=10.81938	99.9999	101.3462	100

Therefore $k=100/J=9.24267$ and $\tilde{y}=kA_j$ are obtained, as above. Also remark that the values of central ordinates $k\exp\varphi(u)$ differ from theoretical frequencies $\tilde{y}$ untolerably.[12] Now $\chi^2=\sum|y-\tilde{y}|^2/\tilde{y}$ becomes only 0.4240 and for 3 degrees of freedom Pr $\{\chi^2>0.4240\}>0.9$. Also $\omega_9^2=\delta^2/N^2=52.25/100^2=0.0052$ and $\Phi(\omega_9^2=0.0052)$ being nearly zero, Pr $\{\omega_9^2>0.0052\}$ is almost unity. Thus our improved repersentation fits the given data utterly good.

§12. Concluding Remark

(A further Scheme for the case when Correlation Table is given)

Although bi- or tri-modal distributions could be somehow represented by Pearson's unimodal curves or those of Gram-Charlier, our multimodal distributions fit far better, especially when the existence of modes is distinct, as exhibited by the χ^2- or ω^2-test. This is a matter of course since other representations do not pay attension to existence of modes, whereas our method has taken special account of it purposely. In general the χ^2-test denies more frequently than the ω^2-test does. This is partly due to the fact that the former is heavily affected by those data with less probabilities, while the latter puts stress on those with larger probabilities. In-

12) Even for these ordinates representation we get $\omega^2=0.0394$ and $\Phi(\omega_9^2=0.0394)=0.0609$, so that Pr$\{\omega_9^2>0.0394\}=0.9391>0.05$. Thus it is already not to be rejected. However, this probability is less than that corresponding to $\tilde{y}=kA_j$.

deed, without ω^2–test our task should have been much more troublesome, in order to make results pass the stubborn χ^2–test. The parameters could be always determined by Pearson's method of moments. However, this being only a first approximation, we should appeal to method of least squares to obtain good representations, although it is frequently enough intricate with the present common calculating machines.

However, the frontal attack made e.g. in §3, Case III, to solve an equation of ninth degree or suchlike might have been too much tedious. Rather some method of successive approximation in Ex. 11, §7 would be more recommendable.

To generalize the method describeed in this note to the case of many variables, one may suppose that a Correlation Table for two variates x, y say, length in Ex. 6 and width in Ex. 7, is reported, and that the density distribution $f(x, y)$ is likely a superposition of two normal surfaces, such that

$$f(x, y) = r_1 f_1(x, y) + r_2 f_2(x, y), \qquad r_1 + r_2 = 1, \tag{61}$$

where $f_i(x, y)$ $(i = 1, 2)$ denotes a normal density function of two variates x, y i.e.

$$f_i(x, y) = \frac{1}{2\pi\sigma_i\tau_i\sqrt{1-\rho_i^2}} \exp\left[-\frac{1}{2} Q(x, y)\right], \tag{62}$$

$$Q(x, y) = \frac{1}{1-\rho_i^2}\left\{\frac{(x-a_i)^2}{\sigma_i^2} + \frac{(y-b_i)^2}{\tau_i^2} - \frac{2\rho_i(x-a_i)(y-b_i)}{\sigma_i\tau_i}\right\}, \tag{63}$$

where a_i, b_i, σ_i, τ_i are respective mean and S.D. of x and y and ρ_i is their correlation coefficient. By a similar treatment as in §2, we can calculate the moments about origin $(0, 0)$ of order k, l

$$\nu_{k,l} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^k y^l f(x, y)\,dx\,dy, \tag{64}$$

and in particular

$$\nu_{0,0} = r_1 + r_2 = 1, \quad \nu_{1,0} = r_1 a_1 + r_2 a_2, \quad \nu_{0,1} = r_1 b_1 + r_2 b_2,$$

$$\nu_{2,0} = r_1(a_1^2 + \sigma_1^2) + r_2(a_2^2 + \sigma_2^2), \quad \nu_{0,2} = r_1(b_1^2 + \tau_1^2) + r_2(b_2^2 + \tau_2^2),$$

$$\nu_{1,1} = r_1(a_1 b_1 + \rho_1\sigma_1\tau_1) + r_2(a_2 b_2 + \rho_2\sigma_2\tau_2),$$

$$\nu_{3,0} = \sum_{i=1,2} r_i[a_i^3 + 3a_i\sigma_i^2], \quad \nu_{2,1} = \sum_{i=1,2} r_i[(a_i^2 + \sigma_i^2)b_i + 2a_i\sigma_i\tau_i\rho_i,],$$

$$\nu_{1,2} = \sum_{i=1,2} r_i[(b_i^2 + \tau_i^2)a_i + 2b_i\sigma_i\tau_i\rho_i], \quad \nu_{0,3} = \sum_{i=1,2} r_i(b_i^3 + 3b_i\tau_i^2),$$

$$\nu_{4,0} = \sum_{i=1,2} r_i(a_i^4 + 6a_i^2\sigma_i^2 + 3\sigma_i^4), \quad \nu_{0,4} = \sum_{i=1,2} r_i(b_i^4 + 6b_i^2\tau_i^2 + 3\tau_i^4),$$

$$\nu_{3,1} = \sum_{i=1,2} r_i[a_i^3 b_i + 3a_i^2\rho_i\sigma_i\tau_i + 3a_i b_i\sigma_i^2 + 3\rho_i\sigma_i^3\tau_i],$$

$$\nu_{1,3} = \sum_{i=1,2} r_i[a_i b_i^3 + 3b_i^2\rho_i\sigma_i\tau_i + 3a_i b_i\tau_i^2 + 3\rho_i\sigma_i\tau_i^3],$$

$$\nu_{2,2} = \sum_{i=1,2} r_i[(a_i^2 + \sigma_i^2)(b_i^2 + \tau_i^2) + 4a_i b_i\sigma_i\tau_i\rho_i + 3\sigma_i^2\tau_i^2\rho_i^2].$$

Of course, any moment of further order could be computed: e.g.

$$\nu_{5,0}=\sum_{i=1,2} r_i a_i(a_i^4+10a_i^2\sigma_i^2+15\sigma_i^4), \text{ \&c.}$$

Thus there being 12 unknowns r_i, a_i, b_i, σ_i, τ_i, ρ_i $(i=1, 2)$, we may obtain sufficient number or more of equations to determine these parameters. Therefore it reduces naturally to a problem of least squares.

From the given correlation table we can compute each moment $\nu'_{k,l}$ about origin, and whence the moments $\mu_{k,l}$ about center. On writing $\mu_{k,l}$ in place of $\nu_{k,l}$ above, we obtain observation equations, and by solving 12 equations among them, we are able to estimate 12 unknowns.

However, this method of moments (Pearson) is only a first approximation to obtain a rough estimation of parameters. To get a more minute result we should necessarily proceed to find their corrections by method of least squares.

In fact the values of r_i $(i=1, 2)$ are determined from those of a_j or b_j while the values σ_i (or τ_i) could be found from μ_{20}, μ_{30}, (or $\mu_{0,2}$, $\mu_{0,3}$) in terms of a_j, and on their substitution in μ_{40} (or μ_{04}), we obtain a relation between a_1, a_2 (or b_1, b_2) and thus we get two equations between a_1, a_2, b_1, b_2. Similar substitutions in ν_{11}, ν_{12}, ν_{21} and ν_{22} yield four equations between a_1, a_2, b_1, b_2, ρ_1, ρ_2. These six equation being combined, they are sufficient to determine six unknowns. Analysis may go enough complex, yet we need not here the fifth moment, and consequently the procedure might be carried out more simply than the treatment described in this note.

However, the usual method of likelihood cannot be applied here, because the joint probability that x_k, y_k $(k=1, 2, \cdots, n)$ take place, is now

$$P=\prod_{k=1}^{n} f(x_k, y_k)=\prod_{k=1}^{n}[r_1 f_1(x_k, y_k)+r_2 f_2(x_k, y_k)].$$

Thus it is a product of several binomials, so that $\frac{\partial}{\partial a_i}\log P$, $\frac{\partial}{\partial b_i}\log P$ do not reduce to simple forms as in the ordinary case of a product of monomials:

$$P=\prod_{k=1}^{n}\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_k-a)^2}{\sigma_x^2}+\frac{(y_k-b)^2}{\sigma_y^2}-\frac{2\rho(x_k-a)(y_k-b)}{\sigma_x\sigma_y}\right]\right\}.$$