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ON THE LINEAR PARTIAL DIFFERENTIAL EQUATION OF SECOND ORDER IN N INDEPENDENT VARIABLES WITH CONSTANT COEFFICIENT

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§1. The proposed equation is of the form:

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial w}{\partial x_i} + a_0 w = f(x_1, x_2, \cdots, x_n), \tag{1.1}$$

where a_0, a_i, $a_{ij}(=a_{ji})$, $i=1, 2, \cdots, n$ are given real constant, and $f(x_1, x_2, \cdots, x_n)$ is a given integrable function.

First we intend to find the complementary function, *i.e.* the general integral of

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial w}{\partial x_i} + a_0 w = 0. \tag{1.2}$$

We commence with a particular case, such that the lefthanded side is resoluble into linear factors as

$$\left(\sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + b_0\right)\left(\sum_{j=1}^{n} c_j \frac{\partial}{\partial x_j} + c_0\right) w = 0, \tag{1.3}$$

where b's, c's are constants, and, since (1.1) is assumed to be really of second order, at least one among a_{ij} and accordingly one of b_i and c_j should be non-zero, so that conveniently let it be $b_1 c_1 \neq 0$.[1)]

Since the factors of product in (1.3) are commutative, the required complementary function shall be found by solving

$$\left(\sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + b_0\right) w = 0, \tag{1.4}$$

or

$$\left(\sum_{i=1}^{n} c_j \frac{\partial}{\partial x_i} + c_0\right) w = 0. \tag{1.5}$$

On writing the subsidiary equation of the partial differential equation of first order (1.4)

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1) If all $a_{ii}=0$, this assumption becomes absurd, to speak more we must say that some $b_i c_j \neq 0$. But the matter being trivial, only for the sake of brevity we have assumed as above.

$$\frac{dx_1}{b_1} = \frac{dx_2}{b_2} = \cdots = \frac{dx_n}{b_n} = \frac{dw}{-b_0 w},$$

where $b_1 \neq 0$, we see immediately that their solutions are

$$x_i - \frac{b_i}{b_1} x_1 = \text{const.}, \qquad i = 2, 3, \cdots, n,$$

and

$$w \exp\left\{\frac{b_0}{b_1} x_1\right\} = \text{const.},$$

so that the general integral of (1.4) is

$$w = \exp\left\{-\frac{b_0}{b_1} x_1\right\} \Phi\left(x_2 - \frac{b_2}{b_1} x_1, \cdots, x_n - \frac{b_n}{b_1} x_1\right), \tag{1.6}$$

where Φ denotes any arbitrary function. Quite similarly with (1.5) we get

$$w = \exp\left\{-\frac{c_0}{c_1} x_1\right\} \Psi\left(x_2 - \frac{c_2}{c_1} x_1, \cdots, x_n - \frac{c_n}{c_1} x_1\right). \tag{1.7}$$

Therefore the required general integral of (1.2) is given by

$$w = e^{-\frac{b_0}{b_1}x_1} \Phi\left(x_2 - \frac{b_2}{b_1} x_1, \cdots, x_n - \frac{b_n}{b_1} x_1\right) + e^{-\frac{c_0}{c_1}x_1} \Psi\left(x_2 - \frac{c_2}{c_1} x_1, \cdots, x_n - \frac{c_n}{c_1} x_1\right), \tag{1.8}$$

where Φ and Ψ are arbitrary functions.

In the case, that all $b_i = c_i$, however (1.3) becomes

$$\left(\sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + b_0\right)^2 w = 0, \tag{1.9}$$

and the corresponding solution (1.8) contains essentially only one arbitrary function, so that it ceases to be general. To obtain the general integral, let us put

$$\left(\sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + b_0\right) w = v,$$

and solve

$$\left(\sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + b_0\right) v = 0.$$

In view of (1.6) the latter's general integral is

$$v = \exp\left\{-\frac{b_0}{b_1} x_1\right\} \Phi\left(x_2 - \frac{b_2}{b_1} x_1, \cdots, x_n - \frac{b_n}{b_1} x_1\right),$$

and accordingly we have to solve

$$\left(\sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + b_0\right) w = \exp\left\{-\frac{b_0}{b_1} x_1\right\} \Phi\left(x_2 - \frac{b_2}{b_1} x_1, \cdots, x_n - \frac{b_n}{b_1} x_1\right).$$

With regard to this linear partial differential equation of first order the subsidiary equations become

$$\frac{dx_1}{b_1}=\frac{dx_2}{b_2}=\cdots=\frac{dx_n}{b_n}=\frac{dw}{\exp\left\{-\frac{b_0}{b_1}x_1\right\}\Phi\left(x_2-\frac{b_2}{b_1}x_1,\cdots,x_n-\frac{b_n}{b_1}x_1\right)-b_0w}\cdot$$

whose solutions are $n-1$ equations

$$x_i-\frac{b_i}{b_1}x_1=k_i \qquad (i=2,\ 3,\ \cdots,\ n),$$

where k_i are arbitrary constants, and one more equation that is obtainable from

$$\frac{dw}{dx_1}+\frac{b_0}{b_1}w=\frac{1}{b_1}\exp\left\{-\frac{b_0}{b_1}x_1\right\}\Phi(k_2,\ \cdots,\ k_n),$$

i. e.

$$w\exp\left\{\frac{b_0}{b_1}x_1\right\}-\frac{x_1}{b_1}\Phi(k_2,\ \cdots,\ k_n)=k_1,$$

where $k_i=x_i-\frac{b_i}{b_1}x_1$ $(i=2, 3, \cdots, n)$, and $\frac{1}{b_1}\Phi$ can be written simply Φ as an arbitrary function. Therefore the general integral of (1.9) is

$$w=\exp\left\{-\frac{b_0}{b_1}x_1\right\}\left[x\Phi\left(x_2-\frac{b_2}{b_1}x_1,\ \cdots,\ x_n-\frac{b_n}{b_1}x_1\right)+\psi\left(x_2-\frac{b_2}{b_1}x_1,\ \cdots,\ x_n-\frac{b_n}{b_1}x_1\right)\right], \tag{1.10}$$

where Φ and ψ are arbitrary functions.

Next we proceed to find a particular integral of (1.1). For this purpose we put again in view of (1.3)

$$\left(\sum_{i=1}^{n}c_i\frac{\partial}{\partial x_i}+c_0\right)w=u \tag{1.11}$$

and

$$\left(\sum_{i=1}^{n}b_i\frac{\partial}{\partial x_i}+b_0\right)u=f(x_1,\ x_2,\ \cdots,\ x_n)\,. \tag{1.12}$$

Now the subsidiary equations of the latter being

$$\frac{dx_1}{b_1}=\frac{dx_2}{b_2}=\cdots=\frac{dx_n}{b_n}=\frac{du}{f(x_1,\ \cdots,\ x_n)-b_0u},$$

their solutions are again

$$x_i-\frac{b_i}{b_1}x_1=k_i \qquad (i=2,\ 3,\ \cdots,\ n),$$

and the solution of

$$\frac{du}{dx_1}+\frac{b_0}{b_1}u=\frac{1}{b_1}f(x_1,\ \cdots,\ x_n),$$

i. e.

$$u \exp\left\{\frac{b_0}{b_1}x_1\right\} = \frac{1}{b_1}\int \exp\left\{\frac{b_0}{b_1}x_1\right\} f\left(x_1, \frac{b_2}{b_1}x_1+k_2, \cdots, \frac{b_n}{b_1}x_1+k_n\right) dx_1 + k_1 .$$

Hence, on setting the additional arbitrary constant $k_1=0$, we obtain as a particular solution of (1.12)

$$(1.13) \qquad u = \frac{1}{b_1}\exp\left\{-\frac{b_0}{b_1}x_1\right\}\int \exp\left\{\frac{b_0}{b_1}x_1\right\} f\left(x_1, \frac{b_2}{b_1}x_1+k_2, \cdots, \frac{b_n}{b_1}x_1+k_n\right) dx_1 ,$$

where constants k_i should be replaced by $x_i - \frac{b_i}{b_1}x_1$ after integration, so that it yields

$$u = v\,(x_1, x_2, \cdots, x_n).$$

Substituting this in (1.11) and solving it, we get a solution of (1.11), namely, a particular integral of the linear partial differential equation of second order (1.11)

$$(1.14) \qquad w = \frac{1}{c_1}\exp\left\{-\frac{c_0}{c_1}x_1\right\}\int \exp\left\{\frac{c_0}{c_1}x_1\right\} v\left(x_1, \frac{c_2}{c_1}x_1+l_2, \cdots, \frac{c_n}{c_1}x_1+l_n\right) dx_1 ,$$

where again l_i must be replaced by $x_i-\frac{c_i}{c_1}x_1$ after integration.

Example 1.

$$2\frac{\partial^2 w}{\partial x^2}+3\frac{\partial^2 w}{\partial y^2}-4\frac{\partial^2 w}{\partial z^2}-5\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 w}{\partial y\partial z}-2\frac{\partial^2 w}{\partial x\,\partial z}$$
$$+8\frac{\partial w}{\partial x}-10\frac{\partial w}{\partial y}-4\frac{\partial w}{\partial z}+8w = yze^{-2x}.$$

Factorizing the left-handed member, we get

$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}+\frac{\partial}{\partial z}+2\right)\left(2\frac{\partial}{\partial x}-3\frac{\partial}{\partial y}-4\frac{\partial}{\partial z}+4\right)w = yze^{-2x},$$

and thus $b_1=1$, $b_2=-1$, $b_3=1$, $b_0=2$, $c_1=2$, $c_2=-3$, $c_3=-4$, $c_0=4$. Hence the complementary function becomes by virtue of (1.8)

$$w = e^{-2x}\{\Phi(y+x, z-x)+\psi(2y+3x, z+2x)\},$$

while the particular integral is obtained by means of (1.13) and (1.14) as follows:

$$u = e^{-2x}\int (k_2-x)(k_3+x)dx = e^{-2x}\left\{k_2k_3x+\frac{1}{2}(k_2-k_3)x^2-\frac{x^3}{3}\right\}$$
$$= e^{-2x}\left[xyz-\frac{1}{2}x^2y+\frac{1}{2}x^2z-\frac{1}{3}x^3\right],$$

and

$$w = \frac{1}{2}e^{-2x}\int e^{2x}u\,dx$$
$$= \frac{1}{2}e^{-2x}\int e^{2x}e^{-2x}\left\{x(l_2-\frac{3}{2}x)(l_3-2x)-\frac{1}{2}x^2(l_2-\frac{3}{2}x)+\frac{1}{2}x^2(l_3-2x)-\frac{x^3}{3}\right\}dx$$

$$= \frac{1}{2}e^{-2x}\left[\frac{x^2}{2}l_2l_3 - \frac{x^3}{3}\left(\frac{5}{2}l_2 + l_3\right) + \frac{29}{48}x^4\right]$$

$$= \frac{1}{2}e^{-2x}\left[\frac{3}{32}x^4 + \frac{1}{12}x^3y + \frac{5}{24}x^3z + \frac{1}{4}x^2yz\right].$$

It is easy to check that

$$u = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + \frac{\partial}{\partial z} + 2\right)w = e^{-2x}\left[\frac{x^3}{2} + 2x^2y + \frac{3}{8}x^2z + \frac{1}{2}xyz\right],$$

$$\left(2\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y} - 4\frac{\partial}{\partial z} + 4\right)u = e^{-2x}yz.$$

Example 2. Our method might be repeatedly applied *e.g.* for a linear partial differential equation of third order as

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + \frac{\partial}{\partial z} + 2\right)\left(2\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y} - 4\frac{\partial}{\partial z} + 4\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} + 2\right)v = e^{-2x}yz.$$

Putting $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} + 2\right)v = w$, the problem reduces to *Ex.* 1, and w is rendered by the above result. Hence the particular integral is found similarly, as before,

$$v = e^{-2x}\int w\,dx \quad (y = x + h_2,\ z = x + h_3)$$

$$= e^{-2x}\left[\frac{1}{12}h_2h_3x^3 + \frac{1}{96}(8h_2 + 11h_3)x^4 + \frac{61}{480}x^5\right]$$

$$= e^{-2x}\left[\frac{x^5}{80} + \frac{7}{12}x^4y + \frac{1}{32}x^4z - \frac{1}{12}x^3yz\right],$$

while the complementary function is easily found to be

$$V = e^{-2x}\{\Phi(y+z,\ z-x) + \Psi(2y+3x,\ z+2x) + \Theta(y-x,\ z-x)\},$$

where Φ, Ψ, Θ are arbitrary functions.

§2. Next we shall treat the case that does not permit any factorization like (1.3). In this case we will write in a standard form the given linear partial differential equation of second order

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i\frac{\partial w}{\partial x_i} + c_0 w = f(x_1, x_2, \cdots, x_n), \tag{2.1}$$

where $a_{ij}(=a_{ji})$, b_i, c_0 $(i, j = 1, 2, \cdots, n)$ are given real constants.

As well known, the symmetric matrix $A=(a_{ij})$ can be brought into a diagonal matrix by operating a suitably chosen orthogonal matrix T:

$$\bar{T}AT = A' = \begin{pmatrix} a_1' & & 0 \\ & \ddots & \\ 0 & & a_n' \end{pmatrix}, \tag{2.2}$$

where

$$T=\begin{pmatrix} l_{11}\cdots\cdots\cdots l_{1n} \\ \vdots \quad \cdots\cdots\cdots\cdots \\ l_{n1}\cdots\cdots\cdots l_{nn} \end{pmatrix}, \quad \bar{T}=\begin{pmatrix} l_{11}\cdots\cdots\cdots l_{n1} \\ \vdots \quad \cdots\cdots\cdots\cdots \\ l_{1n}\cdots\cdots\cdots l_{nn} \end{pmatrix}$$

with $\bar{T}=T^{-1}$, $T\bar{T}=I$. If $\mathfrak{x}$ is transformed into $\mathfrak{x}'$ by T:

$$T\mathfrak{x}=\begin{pmatrix} l_{11}\cdots\cdots\cdots l_{1n} \\ \vdots \quad \cdots\cdots\cdots\cdots \\ l_{n1}\cdots\cdots\cdots l_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}=\begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}=\mathfrak{x}', \quad \text{or } \mathfrak{x}=\bar{T}\mathfrak{x}', \tag{2.3}$$

the binary quadratic form

$$Q=\sum_{i,\,j=1}^{n} a_{ij}x_ix_j=\bar{\mathfrak{x}}A\mathfrak{x} \tag{2.4}$$

$$=(x_1,\,\cdots,\,x_n)\begin{pmatrix} a_{11}\cdots\cdots a_{1n} \\ \vdots \\ a_{n1}\cdots\cdots a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

shall be transformed into the standard form

$$\sum_{i=1}^{r} a_i' x_i'^2 \tag{2.5}$$

where r denotes the rank of matrix A (or it may be written still $\sum_{i=1}^{n} a_i'x_i'^2$, but now some a_i' are allowed to be 0). By transformation (2.3) we get $x_j=\sum l_{ij}x_i'$, so that $\frac{\partial w}{\partial x_i'}=\sum_j \frac{\partial w}{\partial x_j}\frac{\partial x_j}{\partial x_i'}=\sum_j l_{ij}\frac{\partial w}{\partial x_j}$ and similarly $\frac{\partial w}{\partial x_j}=\sum_i l_{ij}\frac{\partial w}{\partial x_i'}$. Thus

$$\begin{cases} \dfrac{\partial}{\partial x_i'}=l_{j1}\dfrac{\partial}{\partial x_1}+l_{i2}\dfrac{\partial}{\partial x_2}+\cdots+l_{in}\dfrac{\partial}{\partial x_n} & (i=1,\,2,\,\cdots,\,n), \\ \dfrac{\partial}{\partial x_j}=l_{1j}\dfrac{\partial}{\partial x_1'}+l_{2j}\dfrac{\partial}{\partial x_2'}+\cdots+l_{nj}\dfrac{\partial}{\partial x_n'} & (j=1,\,2,\,\cdots,\,n). \end{cases} \tag{2.6}$$

By the transformations, the quadratic differential form $\sum_{i,\,j=1}^{n} a_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}w$ in (2.1) could be brought into $\sum_{i=1}^{n} a_i'\frac{\partial^2 w}{\partial x_i'^2}$. At the same time, the linear differential form in (2.1) would be transformed into

$$\sum_{j=1}^{n} b_j\frac{\partial w}{\partial x_j}=\sum_j b_j\sum_i l_{ij}\frac{\partial w}{\partial x_i'}=\sum_i\Big(\sum_j l_{ij}b_j\Big)\frac{\partial w}{\partial x_i'}=\sum_i b_i'\frac{\partial w}{\partial x_i'} \tag{2.7}$$

where $\mathfrak{b}'=T\mathfrak{b}$, *i.e.*

$$\begin{pmatrix} b_1' \\ \vdots \\ b_n' \end{pmatrix}=\begin{pmatrix} l_{11}\cdots\cdots\cdots l_{1n} \\ \vdots \quad \cdots\cdots\cdots\cdots \\ l_{n1}\cdots\cdots\cdots l_{nn} \end{pmatrix}\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \tag{2.8}$$

and lastly either $c_0'=c_0$, or it might be written as $c_0=\sum_{i=1}^{n} c_i'$, where c_i' are chosen arbitrariry, so far as their sum become c_0.

Thus, on performing T, the partial differential equation (2.1) reduces to

$$\sum_{i=1}^{n}\left(a_i'\frac{\partial^2 w}{\partial x_i'^2}+b_i'\frac{\partial w}{\partial x_i'}+c_i' w\right)=f_1(x_1, x_2, \cdots, x_n), \tag{2.9}$$

which form we have to treat below.

To find a complete integral, we write $f_1=0$:

$$\sum_{i=1}^{n}\left(a_i'\frac{\partial^2 w}{\partial x_i'^2}+b_i'\frac{\partial w}{\partial x_i'}+c_i' w\right)=0. \tag{2.10}$$

Now, as usually made in Harmonic Analysis, we assume that

$$w=\prod_{i=1}^{n} u_i(x_i), \tag{2.11}$$

where every u_i is a function of x_i only. On substituting (2.11) in (2.10), and dividing out by w, we obtain

$$\sum_{i=1}^{n}\frac{1}{u_i}\left(a_i'\frac{d^2 u_i}{dx_i'^2}+b_i'\frac{du_i}{dx_i'}+c_i' u_i\right)=0,$$

so that each summand ought to vanish reparately:

$$a_i'\frac{d^2 u_i}{dx_i'^2}+b_i'\frac{du_i}{dx_i'}+c_i' u_i=0, \qquad (i=1, 2, \cdots, n), \tag{2.12}$$

with auxiliary equation

$$a_i' m^2+b_i' m+c_i'=0. \tag{2.13}$$

If the rank of matrix A be n, namely the determinant $|A|=|A'|=\prod_{i=1}^{n} a_i' \neq 0$, no a_i' could be zero, and consequently (2.13) should have two roots;

$$\alpha_i, \beta_i=\frac{1}{2a_i'}\{b_i'\pm\sqrt{b_i'^2-4a_i'c_i'}\} \qquad (i=1, 2, \cdots, n)$$

and we obtain, as solutions

$$u_i(x_i)=A_i\exp\alpha_i x_i+B_i\exp\beta_i x_i \qquad \text{if} \quad \alpha_i\neq\beta_i,$$

or else

$$=(A_i x_i+B_i)\exp\alpha_i x_i \qquad \text{if} \quad \alpha_i=\beta_i.$$

Thus we get, as a complete integral of (2.9),

$$w=u_1(x_1)\,u_2(x_2)\cdots u_n(x_n),$$

wich contains $3n-1$ arbitrary constants A_i, B_i and c_i' with condition $\sum_{i=1}^{n} c_i'=c_0$.

If the rank of matrix A be $1\leqq r<n$, then equations (2.12) becomes

$$(2.14)\quad \begin{cases} \dfrac{1}{u_i}\left(a_i'\dfrac{d^2u_i}{dx_i'^2}+b_i'\dfrac{du_i}{dx_i'}+c_i'u_i\right)=0 & (i=1,\ 2,\ \cdots,\ r), \\ b_i'\dfrac{du_i}{dx_i'}+c_i'u_i=0 & (i=r+1,\ \cdots,\ n). \end{cases}$$

In the latter equations, every coefficients b_i' surely $\neq 0$, since, otherwise, the very variable x_i' does disappear, what contradicts our assumption of n independent variables, and those solutions are

$$u_i(x_i)=C_i\exp\left\{\frac{c_i'}{b_i'}x_i\right\}\qquad (i=r+1,\ \cdots,\ n).$$

However, in a complete integral $w=\prod_{i=1}^{n}u_i(x_i)$, the constants C_i may be mingled in some A_j, B_j, so it contains only $n+2r-1$ arbitrary constants.

In the present case, it is somewhat difficult to discuss in general how to obtain a particular integral. We ought to find it ingeneously by problem. However, if it occurs that

$$f(x_1,\ \cdots,\ x_n)\sim f(x_1^{(0)},\ \cdots,\ x_n^{(0)})+\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i^{(0)}}\right)(x_i-x_i^{(0)})=d_0+\sum_{i=1}^{n}d_ix_i,$$

as seen in the small oscillation about equilibrium position, then upon writing

$$d_0\delta_i^1+d_ix_i,\quad \text{with}\quad \delta_i^1=0\ (i=1)\ \text{or}\ =0\ (i\neq 1)$$

in the right-handed side of (2.12) or (2.14), we may find the required particular integral.

§3. To illustrate how the above mentioned transformation to be executed actually, let us consider the case n=3:

$$(3.1)\quad A'\frac{\partial^2 w}{\partial x^2}+B'\frac{\partial^2 w}{\partial y^2}+C'\frac{\partial^2 w}{\partial z^2}+2F\frac{\partial^2 w}{\partial y\partial z}+2G\frac{\partial^2 w}{\partial z\partial x}+2H\frac{\partial^2 w}{\partial x\partial y}$$
$$+2K\frac{\partial w}{\partial x}+2L\frac{\partial w}{\partial y}+2M\frac{\partial w}{\partial z}+Nw=f(x,\ y,\ z),$$

where $A', B', \cdots, N$ are given (real) constants, and $f(x, y, z)$ a given function. We conceive the problem of principal axes of the corresponding quadratic surface:

$$A'x^2+B'y^2+C'z^2+2Fyz+2Gzx+2Hxy=R.$$

To reduce this to the standard form we solve the characteristic equation

$$(3.2)\quad \Delta(\lambda)=\begin{vmatrix} A'-\lambda & H & G \\ H & B'-\lambda & F \\ G & F & C'-\lambda \end{vmatrix}=0,$$

and let its *3* characteristic roots be $\lambda_1, \lambda_2, \lambda_3$.

i) *When 3 roots are all different.* Then, among the simultaneous equations

$$(3.3)\qquad \begin{cases} (A'-\lambda_i)\,l_i+Hm_i+Gn_i=0 \\ Hl_i+(B'-\lambda_i)m_i+Fn_i=0 \qquad (i=1,2,3), \\ Gl_i+Fm_i+(C'-\lambda_i)\,n_i=0 \end{cases}$$

there only two being independent, we have to take

$$(A'-\lambda_i)\,l_i+Hm_i+Gn_i=0$$
$$(H+G)l_i+(B'+F-\lambda_i)m_i+(F+C'-\lambda_i)n_i=0.$$

Whence the ratios $l_i:m_i:n_i$, and further on combing them with $l_i^2+m_i^2+n_i^2=1$, the respective values l_i, m_i, n_i $(i=1,2,3)$ could be determined; moreover selecting the root-signs $\pm$ for l_i, m_i, n_i adequately, it is always possible to make the Jacobian

$$J=\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}=1.$$

ii) *When 2 roots of* (2.3) *are equal, say* $\lambda_2=\lambda_3$. We can determined l_1, m_1, n_1 as in i). As to λ_2, λ_3, we have

$$(A'-\lambda_2)\,l_2+Hm_2+Gn_2=0,$$
$$(A'-\lambda_3)\,l_3+Hm_3+Gn_3=0,$$

and

$$l_2l_3+m_2m_3+n_2n_3=0,$$

of which first two assure that the directions (l_2, m_2, n_2), (l_3, m_3, n_3) are perpendicular to (l_1, m_1, n_1). Here we ought to determine *4* ratio's $l_2:m_2:n_2$, $l_3:m_3:n_3$ from the above *3* equations, so that one unknown may be assumed at will. Hence, *e.g.* on taking $l_2=0$, we get $m_2:n_2=-G:H$, and consequently

$$(A'-\lambda_2)\,l_3+Hm_3+Gn_3=0, \quad -Gm_3+Hn_3=0,$$

whence

$$l_3:m_3:n_3=-(G^2+H^2):H(A'-\lambda_2):G(A'-\lambda_3).$$

Thus in the above two cases we have already found a triple orthogonal system with Jacobien $J=1$. Hence making transformations

$$(3.4)\qquad \begin{cases} \xi=l_1x+m_1y+n_1z \\ \eta=l_2x+m_2y+n_2z \\ \zeta=l_3x+m_3y+n_3z \end{cases} \quad \text{or} \quad \begin{cases} x=l_1\xi+l_2\eta+l_3\zeta \\ y=m_1\xi+m_2\eta+m_3\zeta \\ z=n_1\xi+n_2\eta+n_3\zeta \end{cases}$$

i.e.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = T \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \quad \text{with } T = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix}, \quad |T| = 1,$$

we have

$$A'x^2 + B'y^2 + C'z^2 + 2Fyz + 2Gzx + 2Hxy = \lambda_1\xi^2 + \lambda_2\eta^2 + \lambda_3\zeta^2.$$

Hence also by transformation

$$\begin{cases} \dfrac{\partial}{\partial x} = l_1 \dfrac{\partial}{\partial \xi} + l_2 \dfrac{\partial}{\partial \eta} + l_3 \dfrac{\partial}{\partial \zeta} \\ \dfrac{\partial}{\partial y} = m_1 \dfrac{\partial}{\partial \xi} + m_2 \dfrac{\partial}{\partial \eta} + m_3 \dfrac{\partial}{\partial \zeta} \\ \dfrac{\partial}{\partial z} = n_1 \dfrac{\partial}{\partial \xi} + n_2 \dfrac{\partial}{\partial \eta} + n_3 \dfrac{\partial}{\partial \zeta}, \end{cases} \qquad J = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = 1,$$

the linear partial differential equation (3.1) becomes

$$\lambda_1 \frac{\partial^2 w}{\partial \xi^2} + \lambda_2 \frac{\partial^2 w}{\partial \eta^2} + \lambda_3 \frac{\partial^2 w}{\partial \zeta^2} + 2K_1 \frac{\partial w}{\partial \xi} + 2L_1 \frac{\partial w}{\partial \eta} + 2M_1 \frac{\partial w}{\partial \zeta} + N_1 w = f_1(\xi, \eta, \zeta).$$

Remark. When (3.2) has *3* equal roots $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, we have $\Delta(\lambda) = 0$, $\Delta'(\lambda) = 0$, $\Delta''(\lambda) = 2(A' + B' + C') - 6\lambda = 0$. Hence $\lambda = \frac{1}{3}(A' + B' + C')$ and this being substituted in $\Delta'(\lambda) = 0$, we get

$$(A' - B')^2 + (B' - C')^2 + (C' - A')^2 + 2F^2 + 2G^2 + 2H^2 = 0,$$

so that, for real coefficients, we must have $A' = B' = C'$, $F = G = H = 0$. Therefore the quadratic differential form in (3.1) becomes

$$A' \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) w,$$

i.e. Laplace's form and there is no need of transformation.

Example 3.

$$6 \frac{\partial^2 w}{\partial y^2} - 18 \frac{\partial^2 w}{\partial y \partial z} - 6 \frac{\partial^2 w}{\partial x \partial z} + 2 \frac{\partial^2 w}{\partial x \partial y} - 9 \frac{\partial w}{\partial x} + 5 \frac{\partial w}{\partial y} - 5 \frac{\partial w}{\partial z} + w = 0.$$

Here

$$|A| = \begin{vmatrix} 0 & 1 & -3 \\ 1 & 6 & -9 \\ -3 & -9 & 0 \end{vmatrix} = 0, \text{ and the matrix } A \text{ is of rank } 2.$$

Also from

$$\Delta(\lambda) = \begin{vmatrix} -\lambda & 1 & -3 \\ 1 & 6-\lambda & -9 \\ -3 & -9 & -\lambda \end{vmatrix} = -\lambda(\lambda + 7)(\lambda - 13) = 0,$$

we get *3* differents roots *0*, *−7*, *13*, and correspondingly

$$l_1 : m_1 : n_1 = -9 : 3 : 1 \qquad\qquad \text{so} \quad T = \begin{pmatrix} -9/\sqrt{91} & 3/\sqrt{91} & 1/\sqrt{91} \\ 1/\sqrt{14} & 2/\sqrt{14} & 3/\sqrt{14} \\ 1/\sqrt{26} & 4/\sqrt{26} & -3/\sqrt{26} \end{pmatrix}.$$
$$l_2 : m_2 : n_2 = 1 : 2 : 3$$
$$l_3 : m_3 : n_3 = 1 : 4 : -3$$

Therefore by transformation

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = T \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{so also} \quad \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \bar{T} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{pmatrix}$$

the given partial differential equation is reduced to

$$7\frac{\partial^2 w}{\partial \eta^2} - 13\frac{\partial^2 w}{\partial \zeta^2} + \sqrt{91}\,\frac{\partial w}{\partial \xi} - \sqrt{14}\,\frac{\partial w}{\partial \eta} + \sqrt{26}\,\frac{\partial w}{\partial \zeta} + w = 0.$$

This becomes, on assuming $w = X(\xi)Y(\eta)Z(\zeta)$

$$\left(\frac{\sqrt{91}}{X}\frac{dX}{d\xi} + 1\right) + \frac{1}{Y}\left(\frac{d^2Y}{d\eta^2} - \sqrt{14}\frac{dY}{d\eta}\right) - \frac{1}{Z}\left(13\frac{d^2 Z}{d\zeta^2} - \sqrt{26}\frac{dZ}{d\zeta}\right) = 0.$$

Putting the expression under every bracket $=0$, we get

$$X = A_1 \exp\left\{-\frac{\xi}{\sqrt{91}}\right\}, \quad Y = A_2 \exp\left\{\sqrt{\frac{2}{7}}\,\eta\right\} + B_2, \quad Z = A_3 \exp\left\{\sqrt{\frac{2}{13}}\,\zeta\right\} + B_3,$$

so that a complete integral of the given partial differential equation is

$$w = \exp\left\{\frac{1}{91}(-9x + 3y + z)\right\}\left[A_2 \exp\left\{\frac{1}{7}\ (x + y + z)\right\} + B_2\right] \times$$
$$\left[A_3 \exp\left\{\frac{1}{13}(x + 4y - 3z)\right\} + B_3\right],$$

where A_2, B_2, A_3, B_3 are arbitrary constants.

Example 4.

$$\frac{\partial^2 w}{\partial z^2} = 3\frac{\partial w}{\partial x} + 4\frac{\partial w}{\partial y}$$

Here

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and its rank is *1*. Assuming $w = X(x)Y(y)Z(z)$, the given equation becomes

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = \frac{3}{X}\frac{dX}{dx} + \frac{4}{Y}\frac{dY}{dy}.$$

Putting

$$\frac{3}{X}\frac{dX}{dx}=c_1,\quad \frac{4}{Y}\frac{dY}{dy}=c_2,\quad \frac{1}{Z}\frac{d^2Z}{dz^2}=c_1+c_2=c,$$

we obtain

$$X=k_1\exp\left\{\frac{c_1}{3}x\right\},\quad Y=k_2\exp\left\{\frac{c_2}{4}y\right\},$$

$$\begin{aligned} Z &= A_1e^{\sqrt{c}z}+B_1e^{-\sqrt{c}z}, && \text{if}\quad c>0,\\ &= A_1\cos\sqrt{-c}z+B_1\sin\sqrt{-c}z, && \text{if}\quad c<0,\\ &= A_1+B_1z, && \text{if}\quad c=0,\end{aligned}$$

where $c=c_1+c_2$. Therefore, a complete integral of the given partial differential epuation is

$$w=Z\exp\left\{\frac{c_1}{3}x+\frac{c_2}{4}y\right\},$$

which contains 4 arbitrary constants A_1, B_1, c_1, c_2.

The writer closes his paper by expressing his hearty thanks to Professor Y. Watanabe for his interest in this work and valuable suggestions.