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NOTES ON GENERAL ANALYSIS (V)

Singular subspaces

By

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In the preceding paper,[1] we discussed the isolated singular point of an analytic function in complex Banach spaces. The states of analytic functions at singular points are very complicated in complex Banach spaces. The isolated singular point does not exist generally in complex Banach spaces. If the set of singular points of an analytic function in complex Banach spaces is a subspace*), then we call the subspace "*the singular subspace*" of an analytic function. In this paper, we investigate mainly the characters of functions which have the singular subspaces.

In the chapter 1, we discuss homogeneous functions and reciprocal homogeneous functions of degree n which are analytic on whole spaces except their singular subspaces. The conditions, under which homogeneous functions of degree n are homogeneous polynomials of degree n, are stated.

In the chapter 2, removable singular subspaces of analytic functions and another theorems of functions which have singular subspaces are stated. Finally, some of the general theorems in complex Banach spaces is applied to the case of functions of several complex variables.

§ 1. Homogeneous functions and reciprocal homogeneous functions

Let $E_1, E_2, E_3, \cdots$ be complex Banach spaces and L_0 be a subspace of E_1.

Definition 1. *Let an E_2–valued function $f_n(x)$ defined in the outside of L_0 in E_1 be analytic and satisfy $f_n(\alpha x)=\alpha^n f(x)$ in the outside of L_0 in E_1, where α is an arbitrary complex number. $f_n(x)$ is called a homogeneous function of degree n, if n is a positive integer. $f_n(x)$ is called a reciprocal homogeneous function of degree $-n$, if n is a negative integer. L_0 is called their singular subspaces.*

Definition 2. *If x_0 and y_0 do not belong to L_0 and $y_0 \neq \alpha x_0+\beta y$ for any complex number α, any complex number β and any y in L_0, then x_0 and y_0 are called independent mutually of L_0. That is, y_0 does not belong to the subspace $L(x_0, L_0)$ which is spun by x_0 and L_0.*

Theorem 1. *If there exist two vectors at least which are independent mutually of L_0,* *a homogeneous function $f_n(x)$ of degree n is a homogeneous polynomial of degree n, where L_0 is a singular subspace of $f_n(x)$.*

Proof. Let x_0 be an arbitrary point which does not belong to L_0. Since $f_n(x)$ is analytic at x_0, we have

$$f_n(x) = \frac{1}{2\pi i}\int_C \frac{f_n(x_0+\alpha(x-x_0))}{\alpha-1}\,d\alpha = \sum_{m=0}^{\infty} h_m(x),$$

where $h_m(x) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f_n(x_0+\alpha(x-x_0))}{\alpha^{m+1}}d\alpha$ for $m=0, 1, 2, \cdots$ and C is a circle whose radius $\rho>1$ and $\rho\|x-x_0\|<d$, which is the distance between x_0 and L_0. Since $f_n(\alpha x)=\alpha^n f_n(x)$,

$$h_m(x) = \frac{1}{2\pi i}\int_C \frac{f_n(x_0+\alpha(x-x_0))}{\alpha^{m+1}}\,d\alpha = \frac{1}{2\pi i}\int_C \frac{f_n\left(\frac{1}{\alpha}x_0+x-x_0\right)}{\alpha^{m-n+1}}\,d\alpha.$$

Put $\dfrac{1}{\alpha}=\beta$, then $d\alpha=-\dfrac{1}{\beta^2}d\beta$ and we have

$$h_m(x) = \frac{1}{2\pi i}\int_{C'} f_n(\beta x_0+x-x_0)\,\beta^{m-n-1}\,d\beta,$$

where C' is a circle whose radius is $\dfrac{1}{\rho}$.

Let $L(x_0, L_0)$ be the subspace which is spun by x_0 and L_0. By the assumption, there exists at least a point x which does not belong to $L(x_0, L_0)$. If x does not belong to $L(x_0, L_0)$, βx_0+x-x_0 does not belong to L_0, because, if $\beta x_0+x-x_0\in L_0$, put $\beta x_0+x-x_0=y$, then $x=y+(1-\beta)x_0$, contradicting to that x does not belong to $L(x_0, L_0)$. Therefore, $f_n(\beta x_0+x-x_0)$ is analytic in $|\beta|\leqq\dfrac{1}{\rho}$, and we have

$$h_m(x) = \frac{1}{2\pi i}\int_{C'} f_n(\beta x_0+x-x_0)\,\beta^{m-1-n}d\beta=0, \quad \text{for } m\geqq n+1.$$

On the other hand, since $h_m(x)=h_m(x_0, x-x_0)$, $h_m(x)$ is a homogeneous polynomial of degree n with respect to $x-x_0$ and we see that $h_m(x)$ is a polynomial of degree m. As a polynomial of degree m is continuous, $h_m(x)=0$, even if $x\in L(x_0, L_0)$. Then we have

$$f_n(x) = \sum_0^n h_m(x).$$

This shows that $f_n(x)$ is a polynomial of degree n and we see that $f_n(x)$ is analytic on whole spaces. On the other hand, since $f_n(x)$ is homogeneous, that is, $f_n(\alpha x)=\alpha^n f_n(x)$ for $x\overline{\in} L_0$, we have $f_n(\alpha x)=\alpha^n f_n(x)$ for every x, because $f_n(x)$ is continuous.

Thus we see that $f_n(x)$ is a homogeneous polynomial of degree n[2].

Theorem 2. *If there exist two vectors x_0 and y_0 at least which are independent mutually of L_0, then there does not exist a reciprocal homogeneous function $f_{-n}(x)$ of degree n, whose singular subspace is L_0, where $n=1, 2, 3, \cdots$.*

Proof. Let x_0 be an arbitrary point which does not belong to L_0. Since $f_{-n}(x)$ is analytic at x_0, we have

$$f_{-n}(x) = \sum_{m=0}^{\infty} h_m(x_0, x-x_0),$$

where

$$h_m(x_0, x-x_0) = \frac{1}{2\pi i}\int_C \frac{f_{-n}(x_0+\alpha(x-x_0))}{\alpha^{m+1}}\,d\alpha, \quad \text{for} \quad m=0, 1, 2, \cdots,$$

and C is a circle whose radius $\rho>1$ and satisfies $\rho\|x-x_0\|<d$, which is the distance between x_0 and L_0.

$$h_m(x_0, x-x_0) = \frac{1}{2\pi i}\int_C \frac{f_{-n}\left(\frac{1}{\alpha}x_0+x-x_0\right)}{\alpha^{n+m+1}}\,d\alpha$$

$$= \frac{1}{2\pi i}\int_{C'} f_{-n}(\beta x_0 + x - x_0)\,\beta^{n+m-1}\,d\beta,$$

where C' is a circle whose radius is $\frac{1}{\rho}$ and $\beta=\frac{1}{\alpha}$. Then

$$h_m(x_0, x-x_0) = 0,$$

for $m=0, 1, 2, \cdots$, if $x\in L(x_0, L_0)$.

On the other hand, there exists at least two vectors which are independent mutually of L_0, $h_m(x_0, x-x_0)\equiv 0$, from its continuity. Then we have $f_{-n}(x)\equiv 0$. That is, there do not exist reciprocal homogeneous functions in our cases.

Let $L(x_0, y_0, L_0)$ be a subspace spun by x_0, y_0 and L_0, where x_0 and y_0 are independent mutually of L_0. If the space $E_1\supseteq L(x_0, y_0, L_0)$, then there do not exist reciprocal homogenous functions which have L_0 as their singular subspaces and homogeneous functions which have L_0 as their singular subspaces are homogeneous polynomials. But, when there do not exist two vectors which are independent mutually of a subspace L_0, these theorems are false as the following examples show.

Put $x=(x_1, x_2)$, whose norm $\|x\|=\max(|x_1|, |x_2|)$. Then we have the complex Banach spaces of 2 dimensions with respect to complex numbers. Put

$$h(x) = x_2^n e^{\frac{x_1}{x_2}}.$$

$h(x)$ is analytic at outside points of the closed linear subspace L_1 which is defined by $x_2=0$. Since $\alpha x=(\alpha x_1, \alpha x_2)$,

$$h(\alpha x) = (\alpha x_2)^n e^{\frac{\alpha x_1}{\alpha x_2}}$$
$$= \alpha^n x_2^n e^{\frac{x_1}{x_2}}$$
$$= \alpha^n h(x).$$

Thus, we see that $h(x)$ is a homogeneous function of degree n which has a singular subspace L_1. But, $h(x)$ is not a homogeneous polynomial of degree n.

From now on, let L_1 be a proper closed linear subspace of E_1 such that $E_1 = L(x_0, L_1)$ for an arbitrary outside point x_0 of L_1.

Theorem 3. *Let $h(x)$ be a homogeneous function of degree n whose singular subspace is L_1,*

(1) *If $y \overline{\in} L_1$ and $h(y) \neq 0$, $\|h(x+\alpha y)\| = |\alpha|^n(0)$ as $|\alpha| \to \infty$, for an arbitrary point x.*

(2) *If $x \in L_1$ and $y \overline{\in} L_1$, $h_m(y, x) = h_{-(m-n)}(x, y)$ and $h_m(y, \alpha x) = \alpha^m h_m(y, x)$, $h_m(\alpha y, x) = \alpha^{n-m} h_m(y, x)$.*

Proof of (1). $\lim_{|\alpha| \to \infty} \frac{\|h(x+\alpha y)\|}{|\alpha|^n} = \lim_{|\alpha| \to \infty} \left\| h\left(\frac{x}{\alpha} + y\right) \right\|$, since $h(x)$ is analytic at $x=y$.

Proof of (2). Since $y \overline{\in} L_1$, $h(x)$ is analytic at $x=y$ and we have

$$h(y + \alpha x) = \sum_{m=0}^{\infty} h_m(y, x) \alpha^m,$$

where $h_m(y, x) = \frac{1}{2\pi i} \int_{C'} \frac{h(y+\alpha x)}{\alpha^{m+1}} d\alpha$, for $m = 0, 1, 2, \cdots$ $h_m(y, x)$ is a homogeneous polynomial of degree m with respect to x. Clearly, $h_m(y, \alpha x) = \alpha^m h_m(y, x)$.

$$h_m(\beta y, \beta x) = \frac{1}{2\pi i} \int_C \frac{h(\beta y + \alpha \beta x)}{\alpha^{m+1}} d\alpha$$
$$= \frac{1}{2\pi i} \int_C \beta^n \frac{h(y+\alpha x)}{\alpha^{m+1}} d\alpha$$
$$= \beta^n h_m(y, x).$$

On the other hand, $h_m(\beta y, \beta x) = \beta^n h_m(\beta y, x)$. Then we have

$$\beta^m h_m(\beta y, x) = \beta^n h_m(y, x).$$

Dividing by β^m, we have $h_m(\beta y, x) = \beta^{n-m} h_m(y, x)$.

Since $h(y+\alpha x)$ is an analytic function of y lying in the outside of L_1, $h_m(y, x)$ is an analytic function of y lying on the outside of L_1 by uniform convergence of the integral. Then $h_m(y, x)$ is a homogeneous function of degree $n-m$ whose singular subspace is L_1, if $n \geqslant m$. If $n < m$, $h_m(y, x)$ is a reciprocal homogeneous function of degree $m-n$ whose singular subspace is L_1.

$$h_m(y,x) = \frac{1}{2\pi i}\int_C \frac{h(y+\alpha x)}{\alpha^{m+1}}\,d\alpha$$
$$= \frac{1}{2\pi i}\int_C \frac{h\left(\frac{1}{\alpha}y+x\right)}{\alpha^{m-n+1}}\,d\alpha$$
$$= \frac{1}{2\pi i}\int_{C'} h(\beta y+x)\beta^{m-n-1}\,d\beta,$$

where C' is a circle whose radius is $\frac{1}{|\alpha|}$ and $\beta=\frac{1}{\alpha}$, then

$$h_m(y,x) = h_{-(m-n)}(x,y).$$

This completes the proof.

Theorem 4. *Let $h(x)$ be a homogeneous function of degree n whose singular subspace is L_1. The necessary and sufficient condition that $h(x)$ should be a homogeneous polynomial is that*

$$\|h(x+\alpha y)\| \leqq K(x,y), \quad \textit{as } |\alpha| \textit{ tends to } 0,$$

for an arbitrary point x in L_1 and an arbitrary point y lying on the outside of L_1, where $K(x,y)$ is a positive constant with respect to α and is defined by x and y.

Proof. If $f(x)$ is a homogeneous polynomial of degree n, $h(x)$ is continuous and we have $\lim_{\alpha\to 0}\|h(x+\alpha y)\| = \|h(x)\|$, for arbitrary points x and y.

Suppose that $\|h(x+\alpha y)\| \leqq K(x,y)$ as $|\alpha|$ tends to 0, where x is an arbitrary point of L_1 and y is an arbitrary point which lies on the outside of L_1 and $K(x,y)$ is a constant with respect to α being defined by x and y. Let f^* be an arbitrary complex valued bounded linear functional in the conjugate space $E_1{}^*$ of E_1,

$$|f^*(h(x+\alpha y))| \leqq M\|h(x+\alpha y)\|, \quad \text{where } M = \|f^*\|.$$

For an arbitrary positive number ε, there exists a positive number δ such that $\|h(x+\alpha y)\| \leqq K(x,y)+\varepsilon$, for $|\alpha|<\delta$. Then we have $|f^*(h(x+\alpha y))| \leqq M(K(x,y)+\varepsilon)$ for $|\alpha|<\delta$. On the other hand, if $|\alpha|>0$, $x+\alpha y \overline{\in} L_1$ and $h(x+\alpha y)$ is an analytic function of α for $|\alpha|>0$ and we see that $f^*(h(x+\alpha y))$ is regular for $|\alpha|>0$. Thus we see that $\alpha=0$ is a removable singular point and $f^*(h(x+\alpha y))$ is regular at $\alpha=0$. Since f^* is an arbitrary point of the conjugate space $E_1{}^*$, we see that $h(x+\alpha y)$ is analytic at $\alpha=0$ [3] that is $h(x+\alpha y)$ is G-differentiable at x on L_1, if $y\overline{\in} L_1$.

Now, if x and y are arbitrary points lying on the outside of L_1, there exists only one complex number α_0 which satisfies $y+\alpha_0 x \in L_1$. Since $E_1 = L(y, L_1)$, there exists x' in L_1 which satisfies $x=\beta' y+\alpha' x'$, where α', β' are complex numbers. Put $-\frac{1}{\beta'}=\alpha_0$, $y+\alpha_0 x = -\frac{\alpha'}{\beta'}x' \in L_1$. If $y+\alpha_1 x\in L_1$ for $\alpha_1 \neq \alpha_0$, $y+\alpha_1 x-(y+\alpha_0 x)=$

$(\alpha_1-\alpha_0)x\in L_1$ and we have $x\in L_1$ contradicting to the assumption $x\overline{\in}L_1$. Then

$$h(y+\alpha x)=h(y+\alpha_0 x+(\alpha-\alpha_0)x).$$

Put $y+\alpha_0 x=x_0$ which belongs to L_1. $h(y+\alpha x)=h(x_0+(\alpha-\alpha_0)x)$. This shows that $h(y+\alpha x)$ is an analytic function of α for $|\alpha|<\infty$. If $y\in L_1$ and $x\in L_1$, $y+\alpha x\overline{\in}L_1$ for $|\alpha|<\infty$ and we see that $h(y+\alpha x)$ is an analytic function of α for $|\alpha|<\infty$ if y does not belong to L_1. Then we see that $h(y+\alpha x)$ is an analytic function of α, if only $y\in L_1$, and we have

$$h(y+\alpha x)=\sum_{m=0}^{\infty}h_m(y,x)\,\alpha^m,$$

where $h_m(y,x)$ is a homogeneous polynomial of degree m with respect to x and satisfies

$$h_m(y,x)=\frac{1}{2\pi i}\int_C\frac{h(y+\alpha x)}{\alpha^{m+1}}\,d\alpha,\quad \text{for}\quad m=0,1,2,\cdots.$$

Since $h(y+\alpha x)$ is analytic for $|\alpha|<\infty$, the radius of the circle C can be taken as large as we like. Then we have

$$\|h_m(y,x)\|\leqq\frac{1}{2\pi}\int_0^{2\pi}\frac{\|h(y+re^{i\theta}x)\|}{r^m}\,d\theta.$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{\|h(\frac{e^{-i\theta}}{r}y+x\|)}{r^{m-n}}\,d\theta$$

If $m>n$,

$$\|h_m(y,x)\|\leqq\overline{\lim_{r\to\infty}}\frac{1}{2\pi}\int_0^{2\pi}\frac{\|h(\frac{e^{-i\theta}}{r}y+x)\|}{r^{m-n}}\,d\theta$$

$$\leqq\frac{1}{2\pi}\int_0^{2\pi}\overline{\lim_{r\to\infty}}\frac{\|h(\frac{e^{-i\theta}}{r}y+x)\|}{r^{m-n}}\,d\theta\quad{}^{4)}$$

$$=0.$$

Because, $\lim_{r\to\infty}\|h(\frac{e^{-i\theta}}{r}y+x)\|=\|h(x)\|$, if $x\overline{\in}L_1$ and $\overline{\lim_{r\to\infty}}\|h(\frac{e^{-i\theta}}{r})y+x\|\leqq K(y,x)$, if $x\in L_1$. Since x is an arbitrary point, we have $h_m(y,x)\equiv 0$ for $m>n$.

Therefore, $h(y+\alpha x)=\sum_0^n h_m(y,x)$.

$\sum_0^n h_m(y,x)$ is a polynomial of degree n. This shows that $h(x)$ is analytic on whole spaces. If $x\overline{\in}L_1$, $h(\alpha x)=\alpha^n h(x)$. Since $h(x)$ is analytic, $\lim_{x\to x'}h(\alpha x)=\lim_{x\to x'}\alpha^n h(x)$ for $x'\in L_1$ and we have

$$h(\alpha x')=\alpha^n h(x').$$

Thus we see that $h(x)$ is a homogeneous polynomial of degree n.

Theorem 5. *Let $h(x)$ be a homogeneous function of degree n whose singular subspace is L_1. The necessary and sufficient condition that $h(x)$ should be a homogeneous polynomial of degree n is that $\overline{\lim}_{\|x\|\to\infty} \frac{\|h(x)\|}{\|x\|^n} \leqq K$, where K is a constant.*

Rroof. If $h(x)$ is a homogeneous polynomial of degree n, we have $\sup_{\|x\|=1} \|h(x)\| < \infty$. Then

$$\overline{\lim_{\|x\|\to\infty}} \frac{\|h(x)\|}{\|x\|^n} = \overline{\lim_{\|x\|\to\infty}} \left\| h\left(\frac{x}{\|x\|}\right)\right\| \leqq \sup_{\|x\|=1} \|h(x)\| < \infty.[5)]$$

Suppose that $\overline{\lim}_{\|x\|\to\infty} \frac{\|h(x)\|}{\|x\|^n} \leqq K$, where K is a constant. Let x be an arbitrary point of L_1 and y be an arbitrary point which does not belong to L_1. Then, $x+\alpha y \overline{\in} L_1$ and we have

$$\begin{aligned}\overline{\lim_{\alpha\to 0}} \|h(x+\alpha y)\| &= \overline{\lim_{\alpha\to 0}} |\alpha|^n \left\| h\left(\frac{1}{\alpha}x+y\right)\right\| \\ &\leqq \overline{\lim_{\alpha\to 0}} |\alpha|^n \cdot \frac{\left\| h\left(\frac{1}{\alpha}x+y\right)\right\|}{\left\|\left(\frac{1}{\alpha}x+y\right)\right\|^n} \left\|\frac{1}{\alpha}x+y\right\|^n \\ &= \overline{\lim_{\alpha\to 0}} \frac{\left\| h\left(\frac{1}{\alpha}x+y\right)\right\|}{\left\|\left(\frac{1}{\alpha}x+y\right)\right\|^n} \cdot \|x+\alpha y\|^n \\ &= K\|x\|^n,\end{aligned}$$

since $\lim_{\alpha\to 0} \|\frac{1}{\alpha}x+\alpha y\| = +\infty$. Then Theorem 4 is applicable and we see that the condition $\overline{\lim}_{\|x\|\to\infty} \frac{\|h(x)\|}{\|x\|^n} \leqq K$ is sufficient.

Theorem 6. *If $h_n(x)$ is an E_1-valued homogeneous polynomial of degree n defined on E_1 and $h_m(x)$ is an E_1-valued homogeneous polynomial of degree m defined on E_1, then $h_n(h_m(x))$ and $h_m(h_n(x))$ is a homogeneous polynomial of degree mn, but $h_n(h_m(x)) \neq h_m(h_n(x))$ generally.*

Proof. $h_n(h_m(x))$ is clearly an analytic function.

$$h_n(h_m(\alpha x)) = h_n(\alpha^m h_m(x)) = \alpha^{nm} h_n(h_m(x)).$$

This shows that $h_n(h_m(x))$ is a homogeneous polynomial of degree mn. On the same way, $h_n(h_m(x))$ is a homogeneous polynomial of degree mn.

Let $X=\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{pmatrix}$ be a matrix of 2–2–types of complex numbers, and $\|x\| =$ max $(|x_{11}|, |x_{12}, |x_{21}|, |x_{22}|)$. Then the set of such X is complex Banach spaces. Let $f(X)=\begin{pmatrix} 1 & 2 \\ 3 & 0\end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{pmatrix}$ and $g(X)=\begin{pmatrix} 0 & 1 \\ 2 & 5\end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{pmatrix}$. Then

$$f(g(X))=\begin{pmatrix} 4 & 11 \\ 0 & 3\end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{pmatrix}, \quad g(f(X))=\begin{pmatrix} 3 & 0 \\ 17 & 4\end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{pmatrix}.$$

This shows that generally $f(g(x)) \neq g(f(x))$.

Theorem 7. *Let $R(x)$ be a reciprocal homogeneous function whose singular subspace is L_1. If $\overline{\lim}_{|\alpha|\to 0} \|R(x+\alpha y)\| \cdot |\alpha|^n \leqq K(x, y)$, for an arbitrary point x on L_1 and an arbitrary point y which does not belong to L_1, then $R(x+y)=R(y)$.*

Proof. For an arbitrary x on L_1 and an arbitrary y which does not belong to L_1, $R(x+\alpha y)$ is analytic when $|\alpha|>0$. Then we have

$$R(x+y) = \sum_{-\infty}^{\infty} R_m(x, y),$$

as well as the Laurent expansion of the complex valued function of complex variables, where

$$R_m(x, y) = \frac{1}{2\pi i}\int_C \frac{R(x+\alpha y)}{\alpha^{m+1}} d\alpha, \quad \text{for } m=0, \pm 1, \pm 2, \cdots.$$

$$R_m(x, y) = \frac{1}{2\pi i}\int_C \frac{R\left(\frac{1}{\alpha}x+y\right)}{\alpha^{n+m+1}} d\alpha$$

$$= \frac{1}{2\pi i}\int_{C'} R(\zeta x + y)\,\zeta^{n+m-1} d\zeta,$$

where $\zeta=\frac{1}{\alpha}$ and C' is a circle whose radius is $\frac{1}{|\alpha|}$. Since clearly $\zeta x + y \overline{\in} L_1$, $R(\zeta x+y)$ is analytic with respect to ζ for $|\zeta|<\infty$. Then

$$R_m(x, y) = 0, \quad \text{when} \quad n+m-1 \geqq 0.$$

Since $R_m(x, y)=0$ for an arbitrary y which does not belong to L_1, by the analytic continuation $R_m(x, y)\equiv 0$ for all y in E_1, where x is arbitrarily fixed. Since x is arbitrary, $R_m(x, y)\equiv 0$ for $m\geqq -n+1$.

Now, since

$$R_m(x, y) = \frac{1}{2\pi i}\int_C R(x+\alpha y)\,\alpha^{-m-1} d\alpha,$$

$$\|R_m(x, y)\| \leqq \frac{1}{2\pi}\int_0^{2\pi} \|R(x+re^{i\theta}y)\|\, r^{-m} d\theta$$

where $\alpha=re^{i\theta}$. Thus we have

$$\|R_m(x, y)\| \leqq \overline{\lim_{\alpha\to 0}} \int_0^{2\pi} \|R(x+\alpha y)\|\, r^{-m} d\theta$$

$$\leqq \int_0^{2\pi} \overline{\lim_{\alpha\to 0}} \|R(x+\alpha y)\| \cdot |\alpha|^n \cdot r^{-m-n} d\theta$$

$$\leqq \int_0^{2\pi} K(x, y) \lim_{\alpha\to 0} r^{-m-n} d\theta$$

$$= 0, \quad \text{if } -n > m.$$

As well as the above case, $R_m(x, y)\equiv 0$ for $m<-n$. Thus we have

$$R(x+y) = R_{-n}(x, y).$$

Since $x+y\overline{\in}L_1$, $R(\alpha(x+y))=\frac{1}{\alpha^n}R(x+y)$.

On the other hand, $R(\alpha(x+y))=R(\alpha x+\alpha y)=R_{-n}(\alpha x, \alpha y)=\frac{1}{\alpha^n}R_{-n}(\alpha x, y)$.

Then we have $R(x+y)=R_{-n}(\alpha x, y)=R(\alpha x+y)$. Since $R(\alpha x+y)$ is analytic as to α, we have

$$R(x+y) = \lim_{\alpha\to 0} R(x+y) = \lim_{\alpha\to 0} R(\alpha x+y) = R(y).$$

This completes the proof.

From this theorem,

$$\begin{aligned}\lim_{\alpha\to 0}\|R(x+\alpha y)\| &= \lim_{\alpha\to 0}\|R(\alpha y)\| \\ &= \lim_{\alpha\to 0}\frac{1}{|\alpha|^n}\|R(y)\| \\ &= +\infty,\end{aligned}$$

since $R(y)\neq 0$.**) The order of infinity of $R(x)$ is n.

Let $x=(x_1, x_2)$ and $\|x\|=\max(|x_1|, |x_2|)$. Then the set of x is a complex Banach spaces Ω. The Ω-valued reciprocal homogeneous function whose singular subspace is $x_1=0$, defined on Ω

$$f(x) = \left(\frac{1}{x_1^n}, 0\right)$$

satisfies the condition of Theorem 7. The complex valued reciprocal homogeneous function of degree n whose singular subspace is $x_2=0$, defined on Ω, $\frac{1}{x_2^n}e^{\frac{x_1}{x_2}}$ does not satisfy the condition of Theorem 7.

§ 2. Analytic functions

Let L_0 be a linear subspace of E_1.

Theorem 8. *If there exist at least two vectors which independent mutually of L_0 and an E_2-valued function $f(x)$ is analytic on the outside of L_0 in E_1, then $f(x)$ is analytic on whole space E_1.*

Proof. For an arbitrary point x which does not belong to L_0, $f(\alpha x)$ is analytic when $|\alpha|>0$. As well as the Laurent expansion of the complex valued function of complex variables, we have

$$f(\alpha x) = \sum_{-\infty}^{+\infty} f_m(x)\alpha^m,$$

where

$$f_m(x) = \frac{1}{2\pi i}\int_C \frac{f(\alpha x)}{\alpha^{m+1}}\, d\alpha, \quad \text{for } m = 0, \pm 1, \pm 2, \cdots.$$

By the uniformity of the integral, we see that $h_m(x)$ is analytic if x lies on the outside of L_0. Moreover, we can easily see that

$$f_m(\beta x) = \beta^m f_m(x), \quad \text{for } m = 0, \pm 1, \pm 2, \cdots.$$

This shows that $h_m(x)$ is a homogeneous function of degree m, whose singular subspace is L_0, when m is positive, and $h_m(x)$ is a reciprocal homogeneous function of degree $(-m)$, whose singular subspace is L_0, when m is a negative integer.

Appealing to Theorem 2, $f_m(x) \equiv 0$ if $m < 0$. Then wee have

$$f(x) = \sum_0^\infty f_m(x).$$

Appealing to Theorem 1, $f_m(x)$ is a homogeneous polynomial of degree m. Put $f_m(x) = h_m(x)$. Thus we see that $f(x)$ is a power series, that is $f(x) = \sum_0^\infty h_m(x)$.

Let x_0 be an arbitrary point which does not belong to L_0, and $d = \text{dis.}(x_0, L_0)$. Since $f(x)$ is analytic at x_0, for an arbitrary positive number ε there exists a positive number δ which satisfies

$$\|f(x) - f(x_0)\| < \varepsilon, \quad \text{if} \quad \|x - x_0\| < \delta (< d).$$

Let $U(x_0, \delta)$ be a set of x which satisfies $\|x - x_0\| < \delta$. On the same way, we have

$$\|f(x) - f(e^{i\theta} x_0)\| < \varepsilon, \quad \text{if } x \in U(e^{i\theta} x_0, \delta_\theta),$$

where $U(e^{i\theta}x_0, \delta_\theta) \frown L_0 = 0$. Appealing to the covering theorem of Borel, we have $\theta_1, \theta_2, \cdots, \theta_k$, such that the set $\sum_1^k U\left(e^{i\theta_j}x_0, \frac{\delta_{\theta_j}}{2}\right)$ includes the set $x_0 e^{i\theta}$ $(0 \leqq \theta \leqq 2\pi)$. Put $M = \max_{1 \le j \le k} (\|f(e^{i\theta_j}x_0)\| + \varepsilon)$, then if x lies in $\sum_1^k U(e^{i\theta_j} x_0, \delta_{\theta_j})$,

$$\|f(x)\| \leqq M.$$

When δ_0 is a small positive number such that $0 < \delta_0 \ll \operatorname*{Min}_{1 \le j \le m}\left(\frac{\delta_{\theta_j}}{2}\right)$, we have

$$e^{i\theta} U(x_0, \delta_0) \subset \sum_1^k U(x_0 e^{i\theta_j}, \delta_{\theta_j}), \quad \text{for } 0 \leqq \theta \leqq 2\pi.$$

Then

$$\begin{aligned} \|h_m(x)\| &= \left\| \frac{1}{2\pi i}\int_C \frac{f(\alpha x)}{\alpha^{m+1}}\, d\alpha \right\| \\ &= \left\| \frac{1}{2\pi}\int_0^{2\pi} \frac{f(e^{i\theta}x)}{e^{im\theta}}\, d\theta \right\| \\ &\leqq M, \end{aligned}$$

where C is a circle whose radius is 1, for $m=0, 1, 2, \cdots$ and $x \in U(x_0, \delta_0)$. Appealing to the lemma of Zorn[6], we see that

$$\|h_m(x)\| \leqq M, \quad \text{when} \quad \|x\| < \delta_0, \quad \text{for } m = 0, 1, 2, 3, \cdots.$$

Thus we have

$$\begin{aligned} &\sup_{\|y\|=1} \overline{\lim_{m\to\infty}} \sqrt[m]{\|h_m(y)\|}\ ^{7)} \\ &= \sup_{\|y\|=1} \overline{\lim_{m\to\infty}} \sqrt[m]{\left\|h_m\left(\frac{\delta y}{\delta}\right)\right\|}, \quad \text{for } 0<\delta<\delta_0, \\ &= \frac{1}{\delta} \sup_{\|y\|=1} \overline{\lim_{m\to\infty}} \sqrt[m]{\|h_m(\delta y)\|}, \\ &\leqq \frac{1}{\delta} \sup_{\|y\|=1} \overline{\lim_{m\to\infty}} \sqrt[m]{M}, \quad \text{because} \quad \|\delta y\| = \delta < \delta_0, \\ &= \frac{1}{\delta}. \end{aligned}$$

This shows that the radius of analyticity of $f(x)$ is not smaller than δ and we see that $f(x)$ is analytic in the neighbourhood of 0. On the same method, we see that $f(x)$ is analytic at an arbitrary point of L_0. This completes the proof.

Corollary. *If a complex valued function $f(z_1, z_2, \cdots, z_n)$ of n-complex variables is regular on the outside of the subspace $L(z_1, z_2, \cdots, z_{n-2})$ of $(n-2)$-dimensions, then $f(z_1, z_2, \cdots, z_n)$ is regular on whole spaces.*[8]

Proof. Since $f(z_1, z_2, \cdots, z_n)$ is regular on the outside of L, $f(z_1, z_2, \cdots, z_n)$ is continuous at the point of the outside of L. Let $z=(z_1, z_2, \cdots, z_n)$ be an arbitrary point in the outside of L and $w=(w_1, w_2, \cdots, w_n)$ be an arbitrary point.

$$\begin{aligned} &\lim_{\alpha\to 0} \frac{f(z+\alpha w)-f(z)}{\alpha} \\ &= \lim_{\alpha\to 0} \sum_{i=1}^{n} \frac{f(z_1, \cdots, z_{i-1}, z_i+\alpha w_i, \cdots, z_n+\alpha w_n) - f(z_1, \cdots, z_i, z_{i+1}+\alpha w_{i+1}, z_n+\alpha w_n)}{\alpha} \\ &= \sum_{i=1}^{n} \frac{\partial f(z_1, \cdots, z_n)}{\partial z_i} w_i. \end{aligned}$$

This shows that $f(z_1, z_2, \cdots, z_n)$ is G-differentiable on the outside of L. Appealing to Theorem 8, $f(z_1, z_2, \cdots, z_n)$ is analytic on whole spaces. Then $f(z_1, z_2, \cdots, z_n)$ is partially differentiable, because it is G-differentiable, and we see that $f(z_1, z_2, \cdots, z_n)$ is regular on whole spaces. If the dimension of L is smaller than n-2, this theorem is clearly true.

Let exist only one vector which is independent of a subspace L_1 in E_1, that is, $E_1=L(x, L_1)$ for an arbitrary point x in the outside of L_1.

Theorem 9. *If an E_2-valued function $f(x)$ defined on the outside of L_1 in E_1 is analytic in E_1 removing L_1 and*

$$\overline{\lim_{|\alpha|\to\infty}} \|f(\alpha x+y)\| \leqq K(x, y),$$

for an arbitrary point x of L_1 and an arbitrary y in the outside of L_1 in E_1, where $K(x, y)$ is a constant as to α, then

$$f(x+y)=f(y).$$

Proof. Since y lies in the outside of L_1, $f(x)$ is analytic at y and so we have

$$f(y+\alpha x) = \sum_0^\infty h_n(y, x)\alpha^n,$$

$$h_n(y, x) = \frac{1}{2\pi i}\int_C \frac{f(y+\alpha x)}{\alpha^{n+1}} d\alpha, \quad \text{for } n=0, 1, 2, \cdots$$

Clearly, $y+\alpha x \in L_1$ and we see that $f(y+\alpha x)$ is analytic for $|\alpha|<\infty$. By the assumption, $\overline{\lim_{|\alpha|\to\infty}} \|f(y+\alpha x)\| \leqq K(x, y)$, we have

$$\|f(y+\alpha x)\| \leqq K(x, y)+\varepsilon, \quad \text{for } |\alpha|>R,$$

where ε is an arbitrary positive number and a positive number R is determined by ε. Since $f(y+\alpha x)$ is continuous on $|\alpha|\leqq R$, $\|f(y+\alpha x)\|$ is bounded on $|\alpha|\leqq R$. That is, for a suitable positive number M, we have

$$\|f(y+\alpha x)\| \leqq M, \quad \text{for } |\alpha| \leqq R.$$

Then we have

$$\|f(y+\alpha x)\| \leqq \max(M, K(x, y)+\varepsilon) \quad \text{when } |\alpha|<\infty.$$

Appealing to the extended theorem of Liouville, $f(y+\alpha x)=c(x, y)$, where $c(x, y)$ is a constant as to α. Then, for $\alpha=0$ and $\alpha=1$, we have $f(y+x)=f(y)$. Since x and y are arbitrary, this completes the proof.

Theorem 10. *If an E_2-valued function $f(x)$ defined on the outside of L_1 is analytic there and satisfies the following inequality*

$$\overline{\lim_{|\alpha|\to\infty}} \|f(y+\alpha x)\| \leqq K,$$

where K is a constant and x is an arbitrary point in L_1 and y is an arbitrary outside point of L_1, then $f(y)$ is a constant.

Proof. Appealing to Theorem 9, we have $f(y+x)=f(y)$, for an arbitrary x in L_1 and an arbitrary y in the outside of L_1. Then

$$\|f(y)\| = \lim_{|\alpha|\to\infty} \|f(y)\| = \lim_{|\alpha|\to\infty} \|f(y+\alpha x)\| \leqq K.$$

That is, $\|f(y)\|\leqq K$. This inequality is true for an arbitrary y in the outside of L_1. Since $f(\beta y)$ is analytic for $|\beta|>0$ and $\|f(\beta y)\|\leqq K$ for $|\beta|<\infty$, $\beta=0$ is a removable singular point. Appealing to the extended theorem of Liouville, we see that $f(\beta y)=c(y)$, where $c(y)$ is a constant with respect to β. On the same way, since $\alpha y+x\overline{\in}L_1$, for $\alpha\neq 0$, $\|f(\alpha y+x)\|\leqq K$ and then we see that $f(\alpha y+x)$ is a constant with respect to α. Let y_1 and y_2 be arbitrary points in the outside of L_1. If $y_1=y_2+\beta x$ for a suitable point x in L_1 and a suitable complex number β, $f(y_1)=f(y_2+\beta x)=f(y_2)$. If $y_1\neq y_2+\beta x$, since $E_1=L(y_2, L_1)$, $y_1=\alpha y_2+\beta x$ for suitable complex number α, β and a suitable x in L_1, where $\alpha\neq 1$. Then, $y_2+\gamma(y_1-y_2)$ $=y_2+\gamma(\alpha y_2+\beta x-y_2)=\gamma\beta x+(1+\gamma(\alpha-1))y_2$. For $\gamma_0=\frac{1}{1-\alpha}$, $y_2+\gamma_0(y_1-y_2)=$ $\alpha\beta x\in L_1$. Put $y_2+\gamma_0(y_1-y_2)=x_0$, then $y_2+\gamma(y_1-y_2)-x_0=(\gamma-\gamma_0)(y_1-y_2)$ and we have $y_2+\gamma(y_1-y_2)=x_0+(\gamma-\gamma_0)(y_1-y_2)$. Since $y_1-y_2\overline{\in}L_1$, $f(y_2+\gamma(y_1-y_2))=$ $f(x_0+(\gamma-\gamma_0)(y_1-y_2))$ is constant with respect to $\gamma-\gamma_0$ and we have $f(y_2)=f(y_1)$, for $\gamma=0$ and $\gamma=1$. From this we can easily see that $f(y)$ is a constant if $y\overline{\in}L_1$. By the analytic continuation, $f(y)$ in a constant on E_1.

Corollary. *If an E_2-valued function $f(x)$ defined on the outside of L_1 is analytic there and satisfies the following inequality*

$$\|f(y+\alpha x)\|\leqq K,$$

for an arbitrary x in L_1 and an arbitrary y in the outside of L_1, where K is a constant, then $f(y)$ is a contant.

References

1). I. Shimoda: Notes on general analysis (IV), Journal of Gakugei, Tokushima University, Vol. V, 1954.

2) I. Shimoda: Notes on general analysis (II), Journal of Gakugei, Tokushima Univ. Vol. III. Thorem 1. The necessary and sufficient conditions that $P(x)$ should de a homogeneous polynomial of degree n are that it is analytic on E and satisfies $p(\alpha x)=\alpha^n p(x)$.

3) Dunford Nelson: Uniformity in linear spaces, Trans. Amer. Soc. 44 (1938). Let $f(\alpha)$ be a function defined on a domain D of α-plane to E. If the numerical function $f^*(f(\alpha))$ for every functional f^* is differentiable on D, $f(\alpha)$ is differentiable on D.

4) Lebesgue's theorem on term by term integration.

5) A. E. Taylor: Addition to the theory of polynomials in normed linear spaces, Tohoku Math. Jour. 44, 1938. page 307.

6) M. A. Zorn: Characterization of analytic functions in Banach spaces. Annals of Math. (2) 46 (1945), p. 590. (4.1): Let $P(x)$ satisfy the following conditions: (1) it is G-differentiable on x, (2) for $|\zeta|=1$, $\|p(\zeta x)\|=\|p(x)\|$, (3) there exists an x in X and real numbers $M\sigma$ with $\sigma<0$, such that for $\|x-x_0\|\leqq\sigma$ we have $\|p(x)\|\leqq M$. Then $\|p(x)\|\leqq M$ for $\|x\|\leqq\sigma$.

7) E. Hille: Functional analysis and semi-groups, 1948.
I. Shimoda: On power series in abstract tpaces, 1948.

8) See, Osgood: Lehrbuch der Funktionentheorie. If $n=2$, this is a trivial case of Hartogs's theorem. If L' is transformed analytically to $L(z_1, z_2, \cdots, z_{n-2})$, this theorem is also true for L'.

*) A subspace is, of course, closed and linear.

**) Let $R(y_1)=0$ for a y_1, which lies in the outside of L_1. Since there is an element which is linearly independent mutually of L_1, an arbitrary point $z=\beta x_1+\alpha y_1$, if $z\overline{\in}L_1$, for a suitable point x_1

in L_1 and suitable complex numbers α, β. Clearly $\beta x_1 \in L_1$, then we have

$$R(\beta x_1 + \alpha y_1) = R(\alpha y_1) = \frac{1}{\alpha^n} R(y_1) = 0.$$

Thus we see that $R(y)=0$ on the outside of L_1 and we have $R(x) \equiv 0$ on L_1 by the analytic continuation, contradicting to the fact that $R(x)$ is not a constant.